Recent advances on the quarter-plane problem

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1 Introduction
Consider an incident plane wave $u_{\text{in}}$ on a plane sector $\mathcal{P}$. Let us call $\mathcal{P}^+$ and $\mathcal{P}^-$ the upper and lower surface of the plane sector. The problem of scattering can be set as follows:

1. $u_{\text{in}} = e^{i(k_1 x_1 + k_2 x_2 - x_3 \sqrt{k_2^2 - k_1^2 - k_2^2})}$
2. $u_{\text{tot}} = u_{\text{in}} + u$
3. $\Delta u + k^2 u = 0$
4. Dirichlet conditions on $\mathcal{P}^+$ and $\mathcal{P}^-$
5. Edge ($O(\rho^{1/2})$) and vertex ($o(r^{-1/2})$) conditions (bounded energy)
6. Radiation condition

Open canonical problem in diffraction theory
Example of potential applications

- **Acoustic:** Better understanding of noisy open rotor engines

- **Electromagnetic:** Radar detection of “triangular” aeroplanes
Far-field behaviour and diffraction coefficient

- Using GTD

- Sommerfeld integrals
  [Lyalinov, Wave Motion, 2013, 2015]

- Computation of $f(\omega, \omega_0)$
  [Smyshlyaev, Wave Motion, 1990]
  [Shanin, Wave Motion, 2005]
  [Assier & Peake, Wave Motion, 2012]

Objective: Diffraction coefficient $f(\omega, \omega_0)$

$$u_{\text{sph}}(\omega, r) = 2\pi \frac{e^{ikr}}{kr} f(\omega, \omega_0) + \mathcal{O}((kr)^{-2})$$
A series expansion solution

Separation of physical variables

- Quarter-plane as degenerate elliptic cone
- Separation of variables in spherico-conal coordinates
- Series expansion involving Lamé functions

[Kraus & Levine, 61] [Satterwhite, 74]

In the far-field

- Poor convergence as $kr \to \infty$, not suitable for diffraction coefficient

In the near-field

- Corner behaviour: $u = O(r^{\sqrt{\lambda_1 + \frac{1}{4}} - \frac{1}{2}})$ as $r \to 0$ [Boersma & Jansen, 1990]
- $\lambda_1$: first eigenvalue of Laplace-Beltrami operator $\tilde{\Delta}$ on a sphere with a cut [Assier, Poon & Peake, QJMAM, 2016]
Some previous attempts

**Radlow’s ansatz**  

- Closed-form solution
- Wrong local physical corner behaviour (no sign of $\lambda_1$)
- The boundary conditions are not satisfied [Albani, Radio Sci., 2007]
- See [Assier & Abrahams, arXiv:1905.03863, 2019] for a constructive re-visit of this approach


**Operator theory approach**

- Infinite iteration of operators, see [Meister & Speck 1988]
Analytical continuation and additive crossing

[Assier & Shanin, QJMAM, 2019]
Double Fourier transforms and functional equation

Definition for any $\phi(x_1, x_2, x_3), \xi = (\xi_1, \xi_2), x = (x_1, x_2)$

\[
\mathcal{F}[\phi](\xi, x_3) = \int\int_{-\infty}^{\infty} \phi(x, x_3) e^{ix \cdot \xi} \, dx
\]

\[
\mathcal{F}_{1/4}[\phi](\xi, x_3) = \int\int_{Q_1} \phi(x, x_3) e^{ix \cdot \xi} \, dx
\]

\[
\mathcal{F}_{3/4}[\phi](\xi, x_3) = \int\int_{\bigcup_{i=2}^{4} Q_i} \phi(x, x_3) e^{ix \cdot \xi} \, dx
\]

Kernel: $K(\xi) = \frac{1}{\sqrt{k^2 - \xi_1^2 - \xi_2^2}}$

1/4-based and 3/4-based functions

A function $\Phi(\xi)$ is 1/4-based if there exists $\phi(x)$ s.t. $\Phi = \mathcal{F}_{1/4}[\phi]$.
A function $\Phi(\xi)$ is 3/4-based if there exists $\phi(x)$ s.t. $\Phi = \mathcal{F}_{3/4}[\phi]$. 

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Functional equation (Assume $\text{Im}(k) = \varepsilon > 0$ and $x_3 > 0$)

Green’s 2nd identity $\implies$ for any $w$ satisfying Helmholtz,

$$\iint_{\partial \Omega_{\varepsilon}} \left( w \frac{\partial u}{\partial n} - u \frac{\partial w}{\partial n} \right) \, dS = 0$$

Choose $w = e^{i(\xi \cdot x + x_3/K(\xi))}$

$$\iint_{\partial \Omega_{\varepsilon}} = \iint_{S^+_R} + \iint_{S^+_\varepsilon} + \iint_{\bigcup_{i=1}^4 Q'_i}$$

But

$$\iint_{\bigcup_{i=1}^4 Q'_i} = \iint_{\bigcup_{i=1}^4 Q'_i} \frac{\partial u}{\partial x_3} e^{i\xi \cdot x} \, dS - \frac{i}{K(\xi)} \iint_{\bigcup_{i=1}^4 Q'_i} u e^{i\xi \cdot x} \, dS,$$

which

$$\mathcal{F} \left[ \frac{\partial u}{\partial x_3} \right] [\xi, 0^+] - \frac{i}{K(\xi)} \mathcal{F}[u](\xi, 0^+) = W(\xi) - \frac{i}{K(\xi)} U(\xi)$$

**Functional equation:** $iU(\xi) = K(\xi)W(\xi)$
Some remarks

Note that \( W(\xi) = F[\partial u/\partial x_3][\xi,0^+] = F_{1/4}[\partial u/\partial x_3][\xi,0^+] \)
since \( \partial u/\partial x_3 \) only non-zero on \( Q_1 \), hence \( W \) is 1/4-based.

And due to Dirichlet BC,
\[
U(\xi) = F[u](\xi,0^+) = F_{3/4}[u](\xi,0^+) - \iint_{Q_1} u_{in}e^{i\xi \cdot x} dS = U'(\xi) + \frac{1}{(\xi_1+k_1)(\xi_2+k_2)},
\]
where \( U' \) is 3/4-based.

Hence the functional equation can be rewritten as
\[
U'(\xi) = -iK(\xi)W(\xi) - \frac{1}{(\xi_1+k_1)(\xi_2+k_2)} \quad (1)
\]

\( W \) being 1/4-based, we can prove that it is analytic in \( \text{UHP}_1 \times \text{UHP}_2 \).

Very difficult to say anything about domain of analyticity if \( U' \)...
Let the function $W(\xi_1, \xi_2)$ have the following properties:

1. $W(\xi_1, \xi_2)$ is $1/4$-based.
2. $U'(\xi_1, \xi_2)$, as defined by (1), is $3/4$-based.
3. (Spectral edge conditions)
   For fixed $\xi_1$, $|W(\xi_1, \xi_2)| = O(|\xi_2|^{-1/2})$ as $|\xi_2| \to \infty$ with $\text{Im}(\xi_2) > 0$.
   For fixed $\xi_2$, $|W(\xi_1, \xi_2)| = O(|\xi_1|^{-1/2})$ as $|\xi_1| \to \infty$ with $\text{Im}(\xi_1) > 0$.
4. (Spectral vertex conditions)
   $|W(\xi_1, \xi_2)| = O(\Lambda^{-1-\lambda})$ as $\Lambda \to \infty$, $\lambda > -1/2$
   where $\xi_{1,2} = \Lambda e^{i\psi_{1,2}}(\cos(\beta), \sin(\beta))$ and $\beta \in (0, \pi/2)$, $\psi_{1,2} \in (0, \pi)$.

Then the field $u(\mathbf{x}, x_3)$ defined by

$$u(\mathbf{x}, x_3) = -i\xi^{-1}[K W \exp\{i|x_3|/K\}](\mathbf{x}, x_3).$$

is the solution to the quarter-plane problem.
Let us define the function \( \gamma(\xi) = \sqrt{\sqrt{k^2 - \xi_1^2 + \xi_2}} \), which participates in the factorisation of \( K \) as follows:

\[
\frac{1}{K(\xi)} = \gamma(\xi_1, \xi_2) \gamma(\xi_1, -\xi_2) = \gamma(\xi_2, \xi_1) \gamma(\xi_2, -\xi_1)
\]

Primary integral formula 1

Within the thin product of strips \( |\text{Im}(\xi_{1,2})| < \kappa \)

\[
W(\xi) = \frac{i \gamma(\xi) \gamma(\xi_1, k_2)}{(\xi_1 + k_1)(\xi_2 + k_2)} + \frac{\gamma(\xi)}{4\pi^2} \int_{-\infty - i\kappa}^{\infty + i\kappa} d\zeta_2 \int_{-\infty - i\kappa}^{\infty + i\kappa} d\zeta_1 \frac{\gamma(\xi_1, -\zeta_2) K(\zeta) W(\zeta)}{(\zeta_1 - \xi_1)(\zeta_2 - \xi_2)}
\]

Obtained by applying 1D WH in the \( \xi_2 \)-plane

- It can be used to analytically continue \( W \) to a much wider domain!
- Then the contour of integration can be deformed
- And resulting formulae can be used for further analytical continuation
Real trace of the singularity sets of $W$ and $U'$ (for real $k > 0$)

\[ W(\xi) \]

\[ U'(\xi) \]

Polar 2-lines $\xi_{1,2} = -k_{1,2}$

Branch 2-lines $\xi_{1,2} = -k$

Branch 2-line $\xi_1^2 + \xi_2^2 = k^2$
Let $\hat{H}^+$ and $\hat{H}^-$ be the upper and lower-half planes of a $\xi$ variable.

We define the cuts $h^\pm$ as $h^\pm : \{\xi = \pm \sqrt{k^2 - \tau^2}, \tau \in \mathbb{R}\}$, and the cut domains $H^\pm$ as $H^\pm = \hat{H}^\pm \setminus h^\pm$. 
Analytical continuation of $W$ and $U'$

$W$ can be analytically continued to:

$$(\hat{H}^+ \times (\hat{H}^+ \cup H^- \setminus \{-k_2\})) \cup ((\hat{H}^+ \cup H^- \setminus \{-k_1\}) \times \hat{H}^+).$$

$U'$ can be analytically continued to:

$$(H^- \setminus \{-k_1\}) \times (H^- \setminus \{-k_2\})$$
Additive crossing along branch lines

Definition
We say that a function $f$ of the two complex variables $\eta_1$ and $\eta_2$ with branch lines at $\eta_1 = 0$ and $\eta_2 = 0$ with corresponding cuts $\chi_1$ and $\chi_2$ has the additive crossing property if

$$f(\eta_1^l, \eta_2^l) + f(\eta_1^r, \eta_2^r) = f(\eta_1^l, \eta_2^r) + f(\eta_1^r, \eta_2^l),$$

where the superscripts $^l$ and $^r$ correspond to the left and right shores of the cuts respectively.
An important property of $U'$

**Result**

It can be shown that the function $f(\eta_1, \eta_2)$ defined by

$$f(\eta_1, \eta_2) = U'(\eta_1 - k, \eta_2 - k)$$

with the cuts $\chi_{1,2} = h^- + k$ has the additive crossing property.

We say that $U'(\xi)$ has the additive crossing property for the branch 2-lines $\xi_{1,2} = -k$ associated to the cuts $h^-$. 

**Bypass interpretation**

$$U'(\xi^*) + U'(\xi^*; \sigma_1 \sigma_2) = U'(\xi^*; \sigma_1) + U'(\xi^*; \sigma_2)$$

Non-trivial note (bypass commutativity):

$$U'(\xi^*; \sigma_1 \sigma_2) = U'(\xi^*; \sigma_2 \sigma_1)$$
Let the function $W(\xi_1, \xi_2)$ have the following properties:

1’. $W(\xi_1, \xi_2)$ is analytic in the domain
$$\left(\hat{H}^+ \times (\hat{H}^+ \cup H^- \setminus \{-k_2\})\right) \cup ((\hat{H}^+ \cup H^- \setminus \{-k_1\}) \times \hat{H}^+).$$

1”. $W(\xi_1, \xi_2)$ has poles at $\xi_1 = -k_1, \xi_2 = -k_2$ with known residues.

2’. $U'(\xi_1, \xi_2)$, as defined by (1), is analytic in the domain
$$(H^- \setminus \{-k_1\}) \times (H^- \setminus \{-k_2\}).$$

2”. $U'(\xi_1, \xi_2)$ has the additive crossing property for the 2-lines $\xi_1 = -k$ and $\xi_2 = -k$ with the cuts $h^-$. 

3. (Spectral edge conditions)

4. (Spectral vertex conditions)

Then the field $u(x_1, x_2, x_3)$ defined by
$$u(x_1, x_2, x_3) = -i\mathfrak{S}^{-1}[K W \exp\{i|x_3|/K\}](x_1, x_2).$$
is the solution to the quarter-plane problem.
Conclusion

Summary

- Analytic continuation of the spectral unknown $W$ and $U'$ within $\mathbb{C}^2$
- New spectral formulation including additive crossing

Current/Future work

- Additive crossing is connected to Lamé’s equation and LBO eigenvalues
- It allows to recover the correct vertex behaviour
- Complexification and analytic continuation of the physical field

Thank you!
Some references

- R.C. Assier and A.V. Shanin (2019)
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