On Janashia-Lagvilava method of matrix spectral factorisation

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Factorisation of matrix functions: new techniques & applications
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Wiener Matrix Spectral Factorisation Theorem

Let

$$S(t) = \begin{pmatrix} s_{11}(t) & s_{12}(t) & \cdots & s_{1r}(t) \\ s_{21}(t) & s_{22}(t) & \cdots & s_{2r}(t) \\ \vdots & \vdots & \vdots & \vdots \\ s_{r1}(t) & s_{r2}(t) & \cdots & s_{rr}(t) \end{pmatrix}$$

$|t| = 1$, be a positive definite (a.e.) matrix function on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ with integrable entries, $0 < S \in L_1(\mathbb{T})^{r\times r}$, which satisfies the Paley-Wiener condition

$$\log\det S(t) \in L_1(\mathbb{T}).$$

Then S admits a spectral factorization

$$S(t) = S^+(t)\big(S^+(t)\big)^*,$$

where $S^+ \in H_2(\mathbb{T})^{r\times r}$ and $(S^+)^* = \overline{S^+}^T$ is tis Hermitian conjugate. The factorization is unique (up to a constant right unitary factor) if S^+ is *outer*, $S^+ \in H_2^O(\mathbb{T})^{r\times r}$

N. Wiener and P. Masani, *The prediction theory of multivariate stochastic processes*, I, Acta Math., vol. 98, pp. 111–150, 1957.

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V. Kuchera, *Factorization of rational spectral matrices: A survey of methods*, Proc. IEEE International Conference on Control, Edinburgh, 1991, 1074-1078.

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Some definitions

Hardy spaces H_p, $p > 0$:

$$H_p := \left\{ f \in \mathcal{A}(\mathbb{D}) : \sup_{r<1} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta < \infty \right\}$$

is naturally identified with $L_p^+(\mathbb{T})$. For $p \geq 1$,

$$L_p^+ := \{ f \in L_p : c_n(f) = 0 \text{ for } n < 0 \}$$

A function $f \in H_p$ is called outer, denoted $f \in H_p^O$, if

$$f(z) = c \cdot \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \left| f(e^{i\theta}) \right| d\theta \right), \qquad |c| = 1.$$

A matrix function $M \in H_2(\mathbb{T})^{r\times r}$ is called outer, denoted $M \in H_2^O(\mathbb{T})^{r\times r}$, if its determinant is outer.

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- Smirnov's theorem

If

$$f \in H_p \cap L_q(\mathbb{T}),$$

where $q > p$, then

$$f \in H^q.$$

- Smirnov's generalized theorem

If

$$L_q(\mathbb{T}) \ni f = \frac{g}{h},$$

where $g \in H_{p_1}$, $h \in H^O_{p_2}$, then

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Janashia-Lagvilava method

The proposed method does not improve the existing algorithms of scalar spectral factorization, but employs them to carry out the matrix spectral factorization.

Scalar spectral factorization:
If $0 < S \in L_1(\mathbb{T})$ and $\log S \in L_1(\mathbb{T})$, then

$$S^+(z) = \exp\left(\frac{1}{4\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log S(e^{i\theta})\,d\theta\right).$$

Scalar spectral factorization theorem can be reformulated in the following terms: If $0 < f \in L_2(\mathbb{T})$ and $\log f \in L_1$, then there exists $u \in L_\infty$ with absolute value 1, $u(t) = 1$ for a.e. $t \in \mathbb{T}$, such that

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Triangular factorization

First, a lower-upper triangular factorization of S is performed:

$$S(t) = M(t)M^*(t),$$

where

$$M(t) = \begin{pmatrix} f_1^+(t) & 0 & \cdots & 0 & 0 \\ \xi_{21}(t) & f_2^+(t) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \xi_{r-1,1}(t) & \xi_{r-1,2}(t) & \cdots & f_{r-1}^+(t) & 0 \\ \xi_{r1}(t) & \xi_{r2}(t) & \cdots & \xi_{r,r-1}(t) & f_r^+(t), \end{pmatrix}$$

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Note that $\det S(t)$ has already been factorised by this way

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Cholesky-like factorization

Two simple ways to make sure that above LU triangular factorization of S is possible

1) We use the same recursive formulas as for Cholesky factorization of numerical matrices:

$$\xi_{jj} = \sqrt{s_{jj} - \sum_{n=1}^{j-1} \xi_{jn}\overline{\xi_{jn}}}$$

$$\xi_{ij} = \left(s_{ij} - \sum_{n=1}^{j-1} \xi_{in}\overline{\xi_{jn}}\right) / \overline{\xi_{jj}}, \quad \text{for } i > j$$

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2) Perform pointwise Cholesky factorization for each $t \in \mathbb{T}$

$$S(t) = S_C(t) S_C^*(t),$$

$$S_C(t) = \begin{pmatrix} f_{11}(t) & 0 & \cdots & 0 & 0 \\ f_{21}(t) & f_{22}(t) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{r-1,1}(t) & f_{r-1,2}(t) & \cdots & f_{r-1,r-1}(t) & 0 \\ f_{r1}(t) & f_{r2}(t) & \cdots & f_{r,r-1}(t) & f_{rr}(t), \end{pmatrix}, \; f_{ii} > 0.$$

$f_{nn}\overline{f_{nn}} = |f_{nn}|^2 \leq \sum_{i=1}^n |f_{in}|^2 = s_{nn} \in L_1 \Longrightarrow f_{nn} \in L_2$
$L_1 \ni \log\det S_C(t) = \sum_{n=1}^r \log f_{nn}(t) \Longrightarrow \log f_{nn} \in L_1$
Therefore, there exists u_n with $|u_n| = 1$ such that $f_n u_n \in H_2^O$.
Let $U(t) = \mathrm{diag}[\mathrm{u_1(t), u_2(t), \ldots, u_r(t)}]$. Then

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We need $M(t)U(t) \in L_2^+(\mathbb{T})^{r\times r}$ and $det\big(\mathbf{U}(t)\big) = 1$.

Then $S^+(t) = M(t)U(t)$ since $S = S^+(S^+)^*$ and $\det S^+ \in H_p^O$

We construct $S^+(t) = M(t)\mathbf{U}_2(t)\mathbf{U}_3(t)\ldots\mathbf{U}_r(t)$

$\big[M(t)\mathbf{U}_2(t)\cdots\mathbf{U}_m(t)\big]_{m\times m}$ will be a spectral factor of $[S]_{m\times m}$

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Unitary matrix functions $U(t)$

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Each $\mathbf{U}_m$ is a block matrix function

$$\mathbf{U}_m(t) = \begin{pmatrix} U_m(t) & \mathbf{0}_{m\times(r-m)} \\ \mathbf{0}_{(r-m)\times m} & I_{r-m} \end{pmatrix},$$

where $U_m(t)$ is a unitary matrix function of the special form

$$U_m(t) = \begin{pmatrix} u_{11}(t) & u_{12}(t) & \cdots & u_{1m}(t) \\ u_{21}(t) & u_{22}(t) & \cdots & u_{2m}(t) \\ \vdots & \vdots & \vdots & \vdots \\ u_{m-1,1}(t) & u_{m-1,2}(t) & \cdots & u_{m-1,m}(t) \\ \overline{u_{m1}(t)} & \overline{u_{m2}(t)} & \cdots & \overline{u_{mm}(t)} \end{pmatrix} \quad u_{ij} \in L_\infty^+$$

$$\det U_m \equiv 1$$

L. Ephremidze and E. Lagvilava, *On compact wavelet matrices of rank m and of order and degree N*, J. Fourier Anal. Appl., vol. 20, no. 2, pp. 401–420, 2014

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A recurrent step from $m-1$ to m

$$\left[M(t)\mathbf{U}_2(t)\cdots\mathbf{U}_{m-1}(t)\right]_{m\times m}=$$

$$\begin{bmatrix} & & & 0\\ & [S]^+_{(m-1)\times(m-1)}(t) & & 0\\ & & & \vdots\\ & & & 0\\ \zeta_1(t) & \dots & \zeta_{m-1}(t) & f_m^+(t)\end{bmatrix}=$$

$$\begin{bmatrix} & & & 0\\ & [S]^+_{(m-1)\times(m-1)}(t) & & 0\\ & & & \vdots\\ & & & 0\\ 0 & \dots & 0 & 1\end{bmatrix}\begin{bmatrix} 1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0\\ \zeta_1(t) & \zeta_2(t) & \cdots & \zeta_{m-1}(t) & f_m^+(t)\end{bmatrix}$$

A crucial lemma

For each $m \times m$ matrix function F of the form

$$F(t) = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ \zeta_1(t) & \zeta_2(t) & \cdots & \zeta_{m-1}(t) & f_m^+(t) \end{bmatrix}$$

where $\zeta_j \in L_2(\mathbb{T})$ and $f_m^+ \in H_2^O$, there exists an $m \times m$ unitary matrix function U of the form

$$U(t) = \begin{pmatrix} u_{11}(t) & u_{12}(t) & \cdots & u_{1,m-1}(t) & u_{1m}(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{m-1,1}(t) & u_{m-1,2}(t) & \cdots & u_{m-1,m-1}(t) & u_{m-1,m}(t) \\ \overline{u_{m1}(t)} & \overline{u_{m2}(t)} & \cdots & \overline{u_{m,m-1}(t)} & \overline{u_{mm}(t)} \end{pmatrix}$$

where $u_{ij} \in L_\infty^+(\mathbb{T})$, with $\det U \equiv 1$ (a.e.), such that

$$FU \in H_2(\mathbb{T})^{m\times m} = L_2^+(\mathbb{T})^{m\times m} \iff F(t)U(t) = \Phi^+(t)$$

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For each $m \times m$ matrix function F of the form

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$$FU \in H_2(\mathbb{T})^{m\times m} = L_2^+(\mathbb{T})^{m\times m} \iff F(t)U(t) = \Phi^+(t)$$

In order to construct the unitary matrix function U approximately, we approximate the functions ζ_j, $j = 1, 2, \ldots, m-1$, by its Fourier series

$$\zeta(t) \approx \sum_{k=-N}^{\infty} c_k(\zeta) t^k,$$

for a large positive integer N, and obviously

$$\left\| \zeta(t) - \sum_{k=-N}^{\infty} c_k(\zeta) t^k \right\|_{L_2} \to 0 \quad \text{as } N \to \infty.$$

Let $\mathcal{P}_N$ be the set of polynomials of degree N:

$$\mathcal{P}_N = \left\{ p(t) = \sum_{k=0}^{N} c_k t^k : c_k \in \mathbb{C},\ k = 0, 1, \ldots, N \right\}$$

$$L_2^{N-} := \{ f \in L_2(\mathbb{T}) : c_n(f) = 0 \text{ for } n < -N \}$$

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An explicit construction

For each $m \times m$ matrix function F of the form

$$F(t) = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ \zeta_1(t) & \zeta_2(t) & \cdots & \zeta_{m-1}(t) & f_m^+(t), \end{bmatrix}$$

where $\zeta_j \in L_2^{N-}(\mathbb{T})$ and $f_m^+ \in H_2^O$, there exists an $m \times m$ unitary matrix function U of the form

$$U(t) = \begin{pmatrix} u_{11}(t) & u_{12}(t) & \cdots & u_{1,m-1}(t) & u_{1m}(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{m-1,1}(t) & u_{m-1,2}(t) & \cdots & u_{m-1,m-1}(t) & u_{m-1,m}(t) \\ \overline{u_{m1}(t)} & \overline{u_{m2}(t)} & \cdots & \overline{u_{m,m-1}(t)} & \overline{u_{mm}(t)} \end{pmatrix}$$

where $u_{ij} \in \mathcal{P}_N$, with $\det U \equiv 1$ (a.e.), such that

$$FU \in L_2^+(\mathbb{T})^{m\times m}$$

A crucial system of boundary values

The specific matrix boundary value problem (F is known, U and Φ^+ are unknowns):

$$F(t)U(t) = \Phi^+(t)$$

is equivalent to the following system (ζ_j and f_m^+ are known, x_i and ϕ_i^+ are unknowns):

$$\begin{cases} \zeta_1(t)x_m^+(t) - f_m^+(t)\overline{x_1^+(t)} = \phi_1^+(t), \\ \zeta_2(t)x_m^+(t) - f_m^+(t)\overline{x_2^+(t)} = \phi_2^+(t), \\ \vdots \\ \zeta_{m-1}(t)x_m^+(t) - f_m^+(t)\overline{x_{m-1}^+(t)} = \phi_{m-1}^+(t), \\ \zeta_1(t)x_1^+(t) + \zeta_2(t)x_2^+(t) + \ldots + \zeta_{m-1}(t)x_{m-1}^+(t) + f_m^+(t)\overline{x_m^+(t)} = \phi_m^+, \end{cases}$$

m independent solutions of the system are columns of U

$$\begin{cases} \zeta_1(t)x_m^+(t) - f_m^+(t)\overline{x_1^+(t)} \in L_2^+(\mathbb{T}) \\ \zeta_2(t)x_m^+(t) - f_m^+(t)\overline{x_2^+(t)} \in L_2^+(\mathbb{T}) \\ \vdots \\ \zeta_{m-1}(t)x_m^+(t) - f_m^+(t)\overline{x_{m-1}^+(t)} \in L_2^+(\mathbb{T}) \\ \zeta_1(t)x_1^+(t) + \ldots + \zeta_{m-1}(t)x_{m-1}^+(t) + f_m^+(t)\overline{x_m^+(t)} \in L_2^+(\mathbb{T}) \end{cases}$$

If the system has two solutions

$$\mathbf{u}^+(t) = \left(u_1^+(t), u_2^+(t), \ldots, \overline{u_m^+(t)}\right),\ u_j \in L_\infty^+(\mathbb{T})$$

$$\mathbf{v}^+(t) = \left(v_1^+(t), v_2^+(t), \ldots, \overline{v_m^+(t)}\right),\ v_j \in L_\infty^+(\mathbb{T})$$

then

$$\left\langle \mathbf{u}^+(t), \mathbf{v}^+(t) \right\rangle = \sum_{i=1}^{m-1} u_i^+(t)\overline{v_i^+(t)} + \overline{u_m^+(t)}v_m^+(t) = \text{const}\,.$$

Proof

$$\begin{cases} \zeta_1^- v_m^+ u_1^+ - f_m^+ \overline{v_1^+} u_1^+ \in L_2^+ \\ \zeta_2^- v_m^+ u_2^+ - f_m^+ \overline{v_2^+} u_2^+ \in L_2^+ \\ . \qquad\quad . \qquad\quad . \\ \zeta_{m-1}^- v_m^+ u_{m-1}^+ - f_m^+ \overline{v_{m-1}^+} u_{m-1}^+ \in L_2^+ \\ \zeta_1^- u_1^+ v_m^+ + \zeta_2^- u_2^+ v_m^+ + \ldots + \zeta_{m-1}^- u_{m-1}^+ v_m^+ + f_m^+ \overline{u_m^+} v_m^+ \in L_2^+ \end{cases}$$

$$f_m^+(t)\left(\sum_{i=1}^{m-1} u_i^+(t)\overline{v_i^+(t)} + \overline{u_m^+(t)} v_m^+(t)\right) \in L_2^+(\mathbb{T})$$

$$\sum_{i=1}^{m-1} u_i^+(t)\overline{v_i^+(t)} + \overline{u_m^+(t)} v_m^+(t) = \frac{\phi^+}{f_m^+} \in H_\infty \text{(by Smirnov theorem)}$$

$$\sum_{i=1}^{m-1} \overline{u_i^+(t)} v_i^+(t) + u_m^+(t)\overline{v_m^+(t)} \in H_\infty \cap \overline{H_\infty} \equiv \mathrm{Const}$$

Proof

$$\begin{cases} \zeta_1^- v_m^+ u_1^+ - f_m^+ \overline{v_1^+} u_1^+ \in L_2^+ \\ \zeta_2^- v_m^+ u_2^+ - f_m^+ \overline{v_2^+} u_2^+ \in L_2^+ \\ . \qquad . \qquad . \\ \zeta_{m-1}^- v_m^+ u_{m-1}^+ - f_m^+ \overline{v_{m-1}^+} u_{m-1}^+ \in L_2^+ \\ \zeta_1^- u_1^+ v_m^+ + \zeta_2^- u_2^+ v_m^+ + \ldots + \zeta_{m-1}^- u_{m-1}^+ v_m^+ + f_m^+ \overline{u_m^+} v_m^+ \in L_2^+ \end{cases}$$

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linearisation of the problem

If the functions $\zeta_j \in L_2^{N-}(\mathbb{T})$, then the system

$$\begin{cases} \zeta_1(t)x_m^+(t) - f_m^+(t)\overline{x_1^+(t)} \in L_2^+(\mathbb{T}) \\ \zeta_2(t)x_m^+(t) - f_m^+(t)\overline{x_2^+(t)} \in L_2^+(\mathbb{T}) \\ \vdots \\ \zeta_{m-1}(t)x_m^+(t) - f_m^+(t)\overline{x_{m-1}^+(t)} \in L_2^+(\mathbb{T}) \\ \zeta_1(t)x_1^+(t) + \ldots + \zeta_{m-1}(t)x_{m-1}^+(t) + f_m^+(t)\overline{x_m^+(t)} \in L_2^+(\mathbb{T}) \end{cases}$$

is transformed into the linear conditions.

If we equate the negative indexed Fourier coefficients of the above functions to 0, then we get a linear system of algebraic equations.

$$\text{If } x^+(t) = \sum_{k=0}^{N} c_k t^k, \text{ then } \overline{x^+(t)} = \sum_{k=0}^{N} \overline{c_k} t^{-k} \text{ for } t \in \mathbb{T}$$

m freedom of selections of 0-th coefficients provide m independent solutions of the system.

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m freedom of selections of 0-th coefficients provide m independent solutions of the system.

The linear system of algebraic equations

If $f_m^+(z)=\sum_{n=0}^{\infty} d_n z^n$ and $\zeta_i^-(t)=\sum_{n=0}^{N}\gamma_{in}t^{-n}$, then

$$\begin{cases}
\Gamma_1\cdot X_m - D\cdot\overline{X_1}=\mathbf{0}, & \mathbf{0}=(0,0,\dots,0)^T\in\mathbb{C}^{N+1}\\
\Gamma_2\cdot X_m - D\cdot\overline{X_2}=\mathbf{0}, & \\
\cdot\quad\cdot\quad\cdot & \\
\Gamma_j\cdot X_m - D\cdot\overline{X_j}=\mathbf{1}, & \mathbf{1}=(1,0,0,\dots,0)^T\in\mathbb{C}^{N+1}\\
\cdot\quad\cdot\quad\cdot & \\
\Gamma_{m-1}\cdot X_m - D\cdot\overline{X_{m-1}}=\mathbf{0}, & \\
\Gamma_1\cdot X_1+\Gamma_2\cdot X_2+\ldots+\Gamma_{m-1}\cdot X_{m-1}+D\cdot\overline{X_m}=\mathbf{0}\,. &
\end{cases}$$

$$D=\begin{pmatrix} d_0 & d_1 & d_2 & \cdots & d_N\\ 0 & d_0 & d_1 & \cdots & d_{N-1}\\ 0 & 0 & d_0 & \cdots & d_{N-2}\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ 0 & 0 & 0 & \cdots & d_0\end{pmatrix}\ \Gamma_i=\begin{pmatrix}\gamma_{i0} & \gamma_{i1} & \cdots & \gamma_{i,N-1} & \gamma_{iN}\\ \gamma_{i1} & \gamma_{i2} & \cdots & \gamma_{iN} & 0\\ \gamma_{i2} & \gamma_{i3} & \cdots & 0 & 0\\ \cdot & \cdot & \cdot & \cdots & \cdot\\ \gamma_{iN} & 0 & 0 & \cdots & 0\end{pmatrix}$$

Unknowns: $X_i=(x_{i0},x_{i1},\dots,x_{iN})^T$

Construction of the unitary matrix function

If we take m independent solutions of the system

$$\begin{cases} \zeta_1(t)x_m^+(t) - f_m^+(t)\overline{x_1^+(t)} = \phi_1^+(t), \\ \vdots \\ \zeta_{m-1}(t)x_m^+(t) - f_m^+(t)\overline{x_{m-1}^+(t)} = \phi_{m-1}^+(t), \\ \zeta_1(t)x_1^+(t) + \zeta_2(t)x_2^+(t) + \ldots + \zeta_{m-1}(t)x_{m-1}^+(t) + f_m^+(t)\overline{x_m^+(t)} = \phi_m^+, \end{cases}$$

as columns of the matrix function

$$V(t) = \begin{pmatrix} v_{11}^+(t) & v_{12}^+(t) & \cdots & v_{1,m-1}^+(t) & v_{1m}^+(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{m-1,1}^+(t) & v_{m-1,2}^+(t) & \cdots & v_{m-1,m-1}^+(t) & v_{m-1,m}^+(t) \\ \overline{v_{m1}^+(t)} & \overline{v_{m2}^+(t)} & \cdots & \overline{v_{m,m-1}^+(t)} & \overline{v_{mm}^+(t)} \end{pmatrix},$$

then

$$V^*(t)V(t) = \text{Const}$$

because of the Lemma, and $F(t)V(t) = \Phi^+(t)$.

$\det V(t) = \text{Const}$

Proof: $F(t)V(t) = \Phi^+(t) \Rightarrow f_m^+(t) \det V(t) = \det \Phi^+(t)$.

$L_\infty \ni \det V(t) = \dfrac{\det \Phi^+(t)}{f_m^+(t)} \in H_\infty$ by generalized Smirnov theorem

$$F^{-1} = \frac{1}{f_m^+} \begin{bmatrix} f_m^+ & 0 & \cdots & 0 & 0 \\ 0 & f_m^+ & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & f_m^+ & 0 \\ -\zeta_1 & -\zeta_2 & \cdots & -\zeta_{m-1} & 1, \end{bmatrix}$$

Direct computations show that

$$V^*(t)F^{-1}(t)f_m^+(t) = \Phi_1^+(t) \in L_2^+(\mathbb{T})^{m\times m}$$

$L_\infty \ni \overline{\det V(t)} = \dfrac{\det \Phi^+(t)}{\left(f_m^+(t)\right)^{m-1}} \in H_\infty$ by generalized Smirnov theorem

$\det V(t) = \text{Const}$

Proof: $F(t)V(t) = \Phi^+(t) \Rightarrow f_m^+(t)\det V(t) = \det \Phi^+(t).$

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$$V^*(t)F^{-1}(t)f_m^+(t) = \Phi_1^+(t) \in L_2^+(\mathbb{T})^{m\times m}$$

$L_\infty \ni \overline{\det V(t)} = \dfrac{\det \Phi^+(t)}{\left(f_m^+(t)\right)^{m-1}} \in H_\infty$ by generalized Smirnov theorem

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Therefore, after normalization,

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It also holds

$$\det U(t) = 1.$$

Thus, the following product belongs to $L_2^+(\mathbb{T})^{m\times m}$

$$\begin{bmatrix} 1 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ \zeta_1(t) & \cdots & \zeta_{m-1}(t) & f_m^+(t) \end{bmatrix} \begin{bmatrix} u_{11}(t) & \cdots & u_{1m}(t) \\ \vdots & \vdots & \vdots \\ u_{m-1,1}(t) & \cdots & u_{m-1,m}(t) \\ \overline{u_{m1}(t)} & \cdots & \overline{u_{mm}(t)} \end{bmatrix}$$

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Convergent properties

$$F(t)=\begin{bmatrix}1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0\\ \zeta_1(t) & \zeta_2(t) & \cdots & \zeta_{m-1}(t) & f_m^+(t)\end{bmatrix},\ \zeta_j\in L_2,\ f_m^+\in H_2^O.$$

$$F_N(t)=\begin{bmatrix}1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0\\ \zeta_1^{\{N\}}(t) & \zeta_2^{\{N\}}(t) & \cdots & \zeta_{m-1}^{\{N\}}(t) & f_m^+(t)\end{bmatrix}$$

$$\zeta^{\{N\}}(t)\approx\sum\nolimits_{k=-N}^{\infty}c_k t^k,$$

Obviously $\|F-F_N\|_{L_2}\to 0$ as $N\to\infty$.

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Proof (based on the uniqueness of spectral factorisation)

$F_N U_N$ is a spectral factor of $F_N F_N^*$ because

$$F_N U_N \in H_2(\mathbb{T})^{m \times m}$$
$$(F_N U_N)(F_N U_N)^* = F_N U_N U_N^* F_N^* = F_N F_N^*$$

and

$$\det(F_N U_N) = \det F_N \in H_2^O.$$

U_N is unique up to a constant right unitary factor.
WLOG we assume that $F_N U_N$ is positive definite at the origin

We prove that:

1) every convergent subsequence of $F_N U_N$ converges to the spectral factor of FF^* which is positive definite at 0.
2) every subsequence of $F_N U_N$ contains a convergent subsequence.

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$$\|F_N U_N \to \Psi\|_{L_2} \to 0$$

Since $F_N U_N \in H_2(\mathbb{T})^{m\times m}$, $\det(F_N U_N) = f_m^+ \in H_2^O$ and $F_N U_N(0) > 0$ the limiting function will have all these three properties:

$$\Psi \in H_2(\mathbb{T})^{m\times m}, \ \det\Psi = f_m^+, \quad \text{and } \Psi(0) > 0.$$

Since $\|F - F_N\|_{L_2} \to 0 \Longrightarrow \|F_N F_N^* - FF^*\|_{L_1} \to 0$ and $\|F_N U_N U_N^* F_N^* - \Psi\Psi^*\|_{L_1} \to 0$, we have

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$$\begin{cases} \zeta_1^{\{N\}}(t)u_m^{\{N\}}(t) - f_m^+(t)\overline{u_1^{\{N\}}(t)} = \phi_1^+(t), \\ \vdots \\ \zeta_{m-1}^{\{N\}}(t)u_m^{\{N\}}(t) - f_m^+(t)\overline{u_{m-1}^{\{N\}}(t)} = \phi_{m-1}^+(t), \\ \zeta_1^{\{N\}}(t)u_1^{\{N\}} + \zeta_2^{\{N\}}u_2^{\{N\}} + \ldots + \zeta_{m-1}^{\{N\}}u_{m-1}^{\{N\}} + f_m^+\overline{u_m^{\{N\}}} = \phi_m^+, \end{cases}$$

The following Hankel operators are compact:

$$\mathcal{K}_f : L_\infty^- \to L_2^+ \qquad u \to \mathbb{P}^+(fu)$$
$$\mathcal{K}_\zeta : L_\infty^+ \to L_2^- \qquad u \to \mathbb{P}^-(\zeta u)$$

where $f, \zeta \in L_2$ and

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Stability against errors

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L. Ephremidze, G. Janashia, and E. Lagvilava, *On approximate spectral factorization of matrix functions*, J. Fourier Anal. Appl. 17 (2011), no. 5, 976–990.

Even in the scalar case the spectral factorization is not stable if S is singular, i.e. zero appears on the boundary

$$S^+(z) = \exp\left(\frac{1}{4\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log S(e^{i\theta})\,d\theta\right).$$

A number of deterministic singular matrices (useful for wavlet constructions) has been factorised by the proposed method

L. Ephremidze, F. Saied, and I. M. Spitkovsky, *On the algorithmization of Janashia-Lagvilava matrix spectral factorization method*, IEEE Trans. Inform. Theory 64 (2018), no. 2, 728–737.

What makes the procedures stable?

Singularity of the power spectral density matrix S is reflected on diagonal terms when we do a lower-upper triangular factorization, i.e. f_m^+ can be singular in the matrix

$$F(t) = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ \zeta_1(t) & \zeta_2(t) & \cdots & \zeta_{m-1}(t) & f_m^+(t) \end{bmatrix}$$

We approximate F by $\hat{F}$ and prove that the convergence of $\hat{F}U_{\hat{F}}$.

We can develop the same construction procedure of $U_{\hat{F}}$ and prove the convergence of $\hat{F}U_{\hat{F}}$ under the mild condition $f_m^+(0) \neq 0$ instead of $f_m^+ \in H_2^O$.

L. Ephremidze and I. Spitkovsky, *Matrix Spectral Factorization with Perturbed Data*, Mem. Differential Equations Math. Phys. 66 (2015), 65-82.

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L. Ephremidze and I. Spitkovsky, *Matrix Spectral Factorization with Perturbed Data*, Mem. Differential Equations Math. Phys. 66 (2015), 65-82.

Applications of the algorithm in Neuroscience

PHYSICAL REVIEW E **99**, 042404 (2019)

Determination of effective brain connectivity from activity correlations

J. N. MacLaurin and P. A. Robinson
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(Received 12 May 2017; revised manuscript received 12 February 2019; published 15 April 2019)

Effective connectivity embodied in transfer functions is derived from symmetric-network activity correlations under task-free conditions via a recent causal spectral factorization method. This generalizes previous covariance-based analyses to include frequency dependencies and time delays. Results are verified against analytic solutions of equations that have reproduced many aspects of experimental brain dynamics and against cases of more complex connectivity. Robustness to noise is also demonstrated.

DOI: 10.1103/PhysRevE.99.042404

$$\mathbf{C}(\omega) = \mathbf{T}(\omega)\mathbf{T}^{\dagger}(\omega), \qquad (4)$$

The problem of finding a causal solution of Eq. (4) was first addressed over 60 years ago [36], but a recent highly efficient method is used here [28]. This finds a succession of unitary

[36] N. Wiener and P. Masani, Acta Math. **98**, 111 (1957).
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pendencies, using a recent causal spectral factorization that we demonstrate to be accurate and robust. These results

Applications of the algorithm for real-time computations

US009318232B2

(12) **United States Patent**
Ephremides et al.

(10) **Patent No.:** **US 9,318,232 B2**
(45) **Date of Patent:** **Apr. 19, 2016**

(54) **MATRIX SPECTRAL FACTORIZATION FOR DATA COMPRESSION, FILTERING, WIRELESS COMMUNICATIONS, AND RADAR SYSTEMS**

(75) Inventors: **Anthony Ephremides**, Bethesda, MD (US); **Gigla Janashia**, Tbilisi (GE); **Levan Janashia**, legal representative, Tbilisi (GE); **Lasha Ephremidze**, Tbilisi (GE); **Edem Lagvilava**, Tbilisi (GE)

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CPC *H01B 1/10* (2013.01); *G06F 17/16* (2013.01); *H04N 19/635* (2014.11)

(58) **Field of Classification Search**
USPC 702/191
See application file for complete search history.

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Written Opinion of the International Searching Authority application No. PCT/US09/02719 dated Nov. 2, 2010.

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Primary Examiner — Tung S Lau
(74) *Attorney, Agent, or Firm* — Squire Patton Boggs (US) LLP

(57) **ABSTRACT**

A new apparatus, system, and technique for performing Matrix Spectral Factorization has been developed, which can factorize in real time high-dimensional matrices with high-order polynomial or non-rational matrices. The method can be

Thanks for your attention