GREATER GENERALITY BRINGS SIMPLICITY

Leonid I. Slepyan

School of Mechanical Engineering, Tel Aviv University
P.O. Box 39040, Ramat Aviv 69978 Tel Aviv, Israel
email: slepyanl@tauex.tau.ac.il

Following the title, I discuss four topics developed in

Slepyan, L.I., 2016. Mechanical wave momentum from the first principles. Wave Motion 68, 283-290.
Slepyan, L.I., 2015. On the energy partition in oscillations and waves. Proc. R. Soc. A 471: 20140838.
Slepyan, L.I., 1979. Betty Theorem and Orthogonality Relations for Eigenfunctions. Mechanics of Solids, 14, 74-77.
Slepyan, L.I., 1963. On a displacement of a deformable body in acoustic medium. J. Appl. Math. Mech., 27, 1402-1411.

I also mention some other studies related to the Title.

1 Wave momentum

Along with energy, momentum plays a defining role in wave actions, and different related questions were debated since Lord Rayleigh's works on the theory (Rayleigh, 1902, 1905). Brillouin (1925), McIntyre (1981), Ostrovsky and Potapov (1988), Peskin (2010), Falkovich (2011) and Maugin and Rousseau (2015) are among others who considered various aspects of this topic (a comprehensive list of references can be found in the book by Maugin and Rousseau, 2015).

1.1 Linearisation complicates

The mechanical momentum density is defined as the product of the material density and particle velocity $\boldsymbol{p} = \varrho\boldsymbol{v}$. In a linearised formulation

$$\varrho = \varrho_0 \quad \Longrightarrow \quad \langle \boldsymbol{p} \rangle = 0 \quad \text{if} \quad \langle \boldsymbol{v} \rangle = 0\,. \tag{1}$$

However, the last condition $\langle \boldsymbol{v} \rangle = 0$ can actually concern the velocity of fixed particles (as the wave excited by the *Lagrangian emitter*) or the particle velocity detected at a fixed spatial point (the wave excited by the *Eulerian emitter*). The results corresponding to these different formulations appears different (Falkovich, 2011. Fluid Mechanics (A short course for physicists). Cambridge University Press. ISBN 978-1-107-00575-4, pp. 75-76).

Another doubtful point. Along with the classical definition of momentum, the so-called "wave momentum" $-\varrho_0 u'v$ was considered as the product of the initial density, 1D strain and particle velocity. What is it?

It is "... a much debated notion" (Maugin and Rousseau, 2015. Wave Momentum and Quasi-Particles in Physical Acoustics. World Scientific Series on Nonlinear Science Series A: Volume 88., p. 2.)

"Of course, considering a "definition" of wave momentum ... would be quite reasonable from a strict mathematical viewpoint, but the physical meaning would be doubtful." (ibid, p.14).

So, the theory was not completed, and the task was to find an exact expression which also could be useful for the determination of the wave momentum based on the linearised formulation.

1.2 The starting points

We consider the 1D axial momentum

$$\boldsymbol{p = mv}\,, \tag{2}$$

where $m = m(x,t)$ is the waveguide mass and $v = v(x,t)$ is the particle velocity. The considerations are based on the following two statements (the **Basic conditions**)

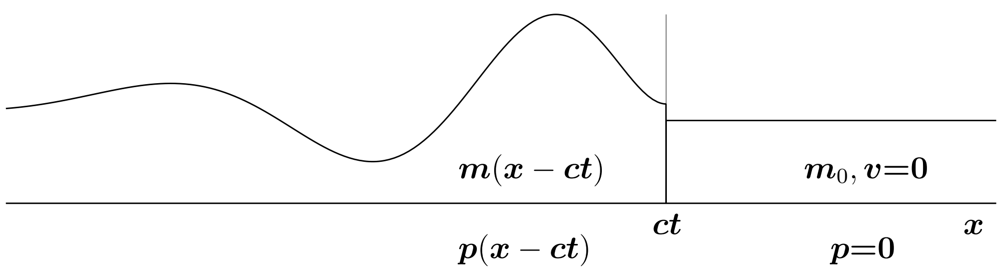
The mass conservation

$$\boldsymbol{\partial m/\partial t = -\partial p/\partial x} \tag{3}$$

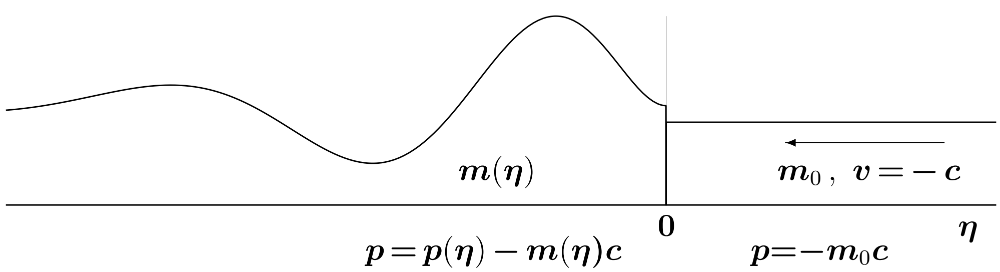
and a kind of the causality principle: **ahead of the wave the uniform waveguide is in the initial static state**

$$\boldsymbol{m = m_0 = \mathrm{const}\,, \quad v = 0}\,. \tag{4}$$

1.3 Steady-state wave



The wave from a moving observer point of view, where $\eta = x - ct$ is the space coordinate:



It directly follows from the Starting points Sect. 1.2 and the fact that the momentum p_c is independent of time that

$$p_c = p(\eta) - m(\eta)c = -m_0 c\,. \tag{5}$$

Thus, the **wave momentum** is the product of the wave mass and the wave speed

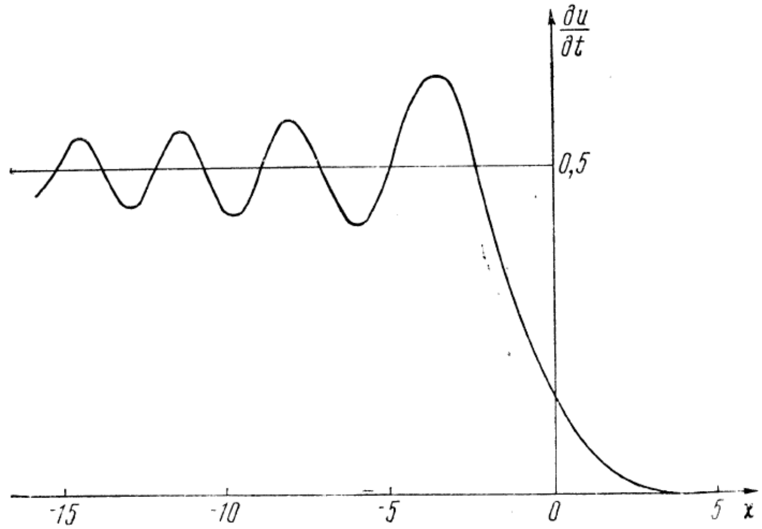
$$\boldsymbol{p = m_w(\eta)c}\,, \tag{6}$$

where the **wave mass**

$$\boldsymbol{m_w(\eta) = m(\eta) - m_0}\,. \tag{7}$$

1.3.1 Non-steady waves

Note that some waves have no front at a moving point but a quasi-front in an area. Such a quasi-step wave forms in a discrete chain under suddenly applied constant force:



The quasi-front area expands relatively slowly (usually $\sim t^{1/3}$), which allows us to ignore it considering the wave momentum in an a main, established area of the wave.The quasi-front area expands relatively slowly (usually $\sim t^{1/3}$), which allows us to ignore it considering the wave momentum in a main, established area of the wave. Indeed, consider the momenta in front and behind the quasi-front area (a time-averaged momentum in the latter). These momenta detected in the moving frame must be equal (that is equal to $-m_0 c$). Otherwise, there will be an unbounded growth (or decrease) mass in the area.

Besides, in the non-steady wave, some motions may persist being considered in the moving frame too. It occurs in dispersive sinusoidal waves where the quasi-front velocity (equal to the wave group velocity v_g) can differ from the phase velocity v_p. For example, $v_p = 2v_g$ in waves on deep water, whereas the opposite relation corresponds to a flexural wave in an elastic beam.

So, the wave momentum of a non-steady wave observed in the moving frame is

$$p_c = p(\eta, t) - m(\eta, t)c\,, \tag{8}$$

where $c = c_g$ for a sinusoidal wave.

In order not to conflict with the title, we assume the established part of the wave to be periodic. In this case, we get time-independent values averaged over the period (in the frame moving with speed c).

$$\langle p_c(\eta)\rangle = \langle p(\eta)\rangle - \langle m(\eta)\rangle c\,. \tag{9}$$

Now it follows from the Starting points that $p_c(\eta)\rangle = -m_0 c$. As above for the steady-state wave, we obtain

$$\boldsymbol{\langle p(\eta)\rangle = \langle m_w(\eta)\rangle c\,, \quad \langle m_w(\eta)\rangle = \langle m(\eta)\rangle - m_0\,.} \tag{10}$$

1.3.2 The group velocity

For a sinusoidal wave, the last relation also follows from the remarkable theorem on the group velocity, the best proof of which was presented by M. Hayes:

Hayes M. (1977) A Note on Group Velocity. Proc R Soc Lond A 354: 533-535.

$$u = \exp[\mathrm{i}(\omega(k)t - kx)]\,, \quad k \to k + \mathrm{i}\varepsilon\,, \quad \omega \to \omega + \omega'\mathrm{i}\varepsilon\,,$$

$$u = \boldsymbol{\exp[-\varepsilon(\omega'(k)t - x)]}\exp[\mathrm{i}(\omega(k)t - kx)]\,. \tag{11}$$

Not only the energy but any value for which the conservation law is valid propagates with the group velocity. Thus, it concerns the mass and, consequently, the sinusoidal wave momentum.

1.4 Comments

(1) The expression for the steady-state mechanical wave momentum now coincides with that for light (and other electromagnetic waves) where the rest mass $m_0 = 0$. Indeed, the light's mass follows from Einstein's $\mathcal{E} = mc^2$ as

$$m = m_w = \frac{\mathcal{E}}{c^2} \quad \Longrightarrow \quad p = m_w c = \frac{\mathcal{E}}{c}\,, \tag{12}$$

as it is.

(2) The mechanical wave momentum for a low amplitude wave can now be found based on the linear solution. Indeed, it depends on a single variable, m_w, which can be calculated based on strains obtained in the solution.

(3) The physical meaning of the "wave momentum". The so-called "wave momentum," $-\varrho_0 u'v$, can be treated as the momentum without a zero-mean term.

Let $\varrho = \varrho_0 + \varrho_w$. Noting that in the 1D **Eulerian formulation**, $\varrho_w = -\varrho_0 u'$ we represent the momentum density with a linear part separated

$$p = \varrho v = p_0 + p_w\,, \quad p_0 = \varrho_0 v\,, \quad p_w = \varrho_w v = -\varrho_0 u'v\,, \tag{13}$$

and

$$\langle p\rangle = \langle \varrho_w v\rangle = -\varrho_0\langle u'v\rangle \quad \text{if} \quad \tau\langle v\rangle \equiv \int_\tau v\,\mathrm{d}t = 0\,, \tag{14}$$

where τ is the period.

(4) Binary wave. Consider a semi-infinite string or an elastic beam, $0 < x < \infty$, loaded by a transverse periodic force applied at $x = 0$, Fig. 1. A note can be met that "Obviously, no axial momentum is excited since there is no longitudinal force." However, it is true for the total momentum only. A binary wave arises, such that its parts possess oppositely directed momentum of the same value.

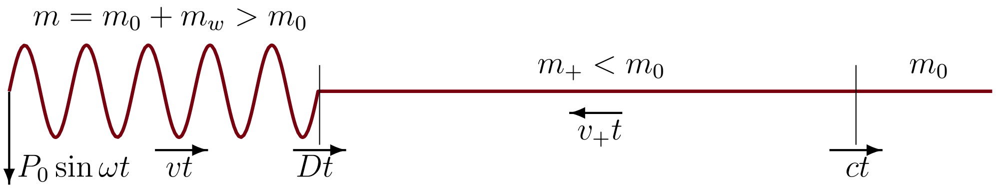
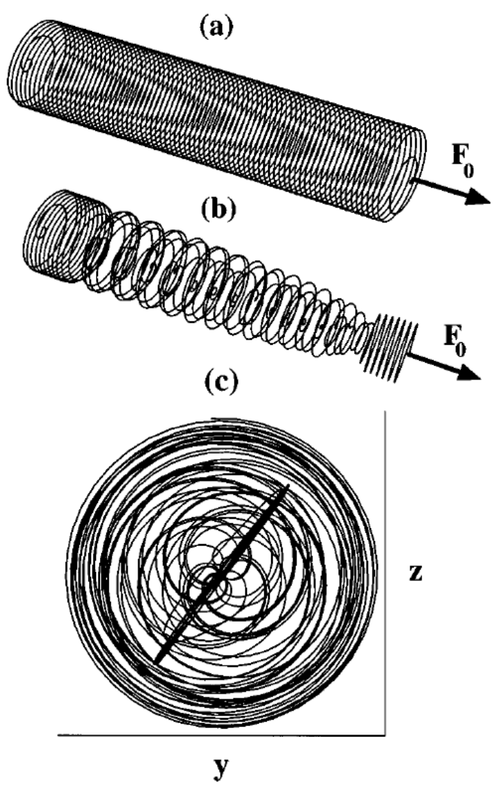


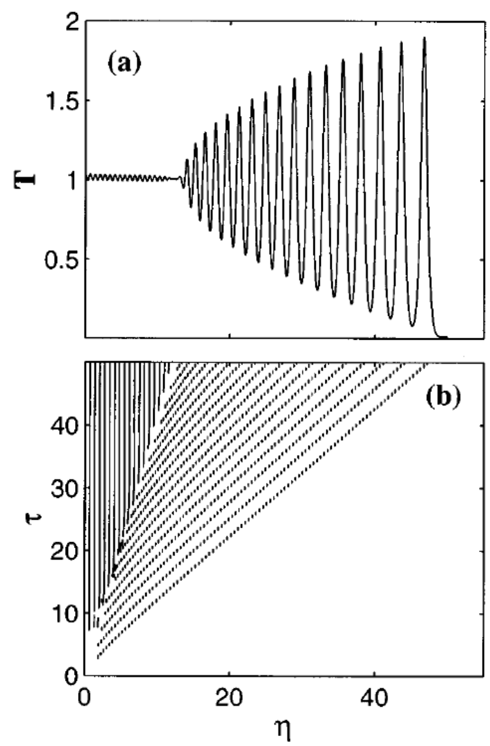
Figure 1: Binary wave: a forerunner (the front part of the wave, $Dt < x < ct$) and the successive part (the wave excited transversely, $0 \leq x < Dt$), which have oppositely directed momenta of the same value. There are two moving discontinuities, $x = ct$ and $x = Dt$.

Note that it is the next type of the binary wave first revealed in a helical fiber under axial tension, where the angular momenta are self-compensated. The forerunner corresponds to the free solitary wave, while the successive part rotates as a rigid body.

Krylov, V. and Slepyan, L., 1997. Binary Wave in a Helical Fiber. Physical Review B 55(21) (June 1),14067-14070.

Slepyan, L., Krylov, V. and Parnes, R., 1995. Solitary Waves in an Inextensible, Flexible, Helicoidal Fiber. Physical Review Letters, 74, No. 14, 2725-2728.





The key point: The causality principle

2 Energy partition

Shortly after my above-titled paper was published, I found that **the energy partition problem was set, in fact, by Lord Rayleigh in 1877**. He wrote:

"It has often been noticed, in particular cases of progressive waves, that the potential and kinetic energies are equal; but I do not call to mind any general treatment of the question."

Rayleigh, M.A., F.R.S., 1877. On Progressive Waves. Proceedings of the London Mathematical Society, 1-9 (1), 21-26.

For free linear oscillations and sinusoidal waves, it is long recognised that kinetic and potential energies averaged over the period are equal. In some partial cases, it is observed directly. This fact is also confirmed by Whitham for a non-specified linear sinusoidal wave.

Along with this, the energy partition in different areas of nonlinear dynamics is also of interest. However, even the linear case was not fully explored. For instance, the cases of systems with time-dependent parameters and forced motions are not discussed. The topic was also discussed, in particular, in

Mask, L.R., and Jay, B.E., 1967. The Partition of Energy in Standing Gravity Waves of Finite'Amplitude. J. of Geophys. Research 72, 573-581.

Whitham, G.B., 1974. Linear and Nonlinear Waves. John Wiley & Sons, NY.

Lighthill, J., 1978. Waves in Fluids. Cambridge University Press, Cambridge.

Antenucci, J.P., and Imburger, J., 2001. Energetics of long internal gravity waves in large lakes. Limnol. Oceanogr., 46(7), 1760-1773.

Falnes, J., 2007. A review of wave-energy extraction. Marine Structures 20, 185-201.

Thiago Messias Cardozo and Marco Antonio Chaer Nascimento, 2009. Energy partitioning for generalized product functions: The interference contribution to the energy of generalized valence bond and spin coupled wave functions. The Journal of Chemical Physics 130, 104102-1-8.

Korycansky, D.G., 2013. Energy conservation and partition in CTH impact simulations. Lunar and Planetary Science Conference, p.1370.

2.1 The main relations

Consider the Euler-Lagrange equations

$$\frac{\partial L}{\partial u_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{u}_i} = 0\,, \quad i = 1, 2, \ldots\,, \tag{15}$$

Multiplying the equations by u_i and integrating over an arbitrary segment, $t_1 \le t \le t_2$, we obtain (with summation on repeated indices)

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial u_i} u_i + \frac{\partial L}{\partial \dot{u}_i} \dot{u}_i \right) \mathrm{d}t + B_1 - B_2 = 0\,, \tag{16}$$

where

$$B_{1,2} = \frac{\partial L}{\partial \dot{u}_i} u_i \quad (t = t_{1,2})\,. \tag{17}$$

Thus

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial u_i} u_i + \frac{\partial L}{\partial \dot{u}_i} \dot{u}_i \right) \mathrm{d}t = 0 \quad \text{if} \quad B_1 = B_2\,. \tag{18}$$

Regarding this formulation, we note that the relation (18) follows from Hamilton's principle of least action with the variation $\delta \boldsymbol{u}$ replaced by $\boldsymbol{u}$. This, however, entails a change in the additional conditions. Namely, in the variational formulation, the integration limits, $t_{1,2}$, are arbitrary under the condition $\delta \boldsymbol{u} = 0$ at $t = t_{1,2}$, while in the modified formulation they are not arbitrary but must meet the condition in (18).

We now suppose that the Lagrangian is a sum of homogeneous functions of $\boldsymbol{u}, \dot{\boldsymbol{u}}$

$$L(\lambda \boldsymbol{u}, \lambda \dot{\boldsymbol{u}}, t) = \lambda^{\nu_n} L_n(\boldsymbol{u}, \dot{\boldsymbol{u}}, t)\,, \tag{19}$$

where ν_n is the homogeneity order.

With refer to Euler's theorem on homogeneous functions

$$\frac{\partial f(\boldsymbol{x})}{\partial x_i} x_i = \nu f(\boldsymbol{x}) \quad \text{if} \quad f(\lambda \boldsymbol{x}) = \lambda^{\nu} f(\boldsymbol{x}) \tag{20}$$

we rewrite the relation (18) in the form

$$\int_{t_1}^{t_2} \nu_n L_n \,\mathrm{d}t = 0 \quad (B_1 = B_2)\,. \tag{21}$$

Let L be the difference between the kinetic and potential energies, $L = \mathcal{K} - \mathcal{P}$, where $\mathcal{K}$ and $\mathcal{P}$, are homogeneous functions of the orders μ (K) and ν ($\mathcal{P}$). In this case, the relation between the averaged energies

$$\langle \mathcal{K} \rangle = \frac{1}{\tau} \int_{t_1}^{t_1+\tau} \mathcal{K} \,\mathrm{d}t\,, \quad \langle \mathcal{P} \rangle = \frac{1}{\tau} \int_{t_1}^{t_1+\tau} \mathcal{P} \,\mathrm{d}t \quad (\tau = t_2 - t_1) \tag{22}$$

is

$$\frac{\langle\mathcal{K}\rangle}{\langle\mathcal{P}\rangle} = \frac{\nu}{\mu} \quad \Longrightarrow \quad \langle\mathcal{K}\rangle = \frac{\nu}{\mu+\nu}\langle\mathcal{E}\rangle\,, \quad \langle\mathcal{P}\rangle = \frac{\mu}{\mu+\nu}\langle\mathcal{E}\rangle\,, \tag{23}$$

where $\mathcal{E} = \mathcal{K} + \mathcal{P}$ is the total energy.

Thus, the equipartition does hold in a general case but for the energies multiplied by the corresponding homogeneity orders

$$\boldsymbol{\mu}\langle\boldsymbol{\mathcal{K}}\rangle = \boldsymbol{\nu}\langle\boldsymbol{\mathcal{P}}\rangle\,. \tag{24}$$

In particular, in the linear case with $\nu = \mu = 2$, the averaged energies are equal.

Remarkable that the energy ratio in (23) is equal to the homogeneity order ratio regardless of the other parameters of the system and the dynamic process and whether the system is linear or nonlinear.

2.1.1 Waves

In the case where the total energy of a wave is infinite, we have to choose a finite segment of the waveguide (x_1, x_2), similar to (t_1, t_2) in time, to avoid boundary terms in integration by parts over this segment. Namely, in the conversions

$$\int_{x_1}^{x_2} u_i \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial u_i'}\,\mathrm{d}x = -\int_{x_1}^{x_2} \frac{\partial L}{\partial u_i'}u_i'\,\mathrm{d}x + \frac{\partial L}{\partial u_i'}u_i|_{x_1}^{x_2}\,,$$
$$\int_{x_1}^{x_2} u_i \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial L}{\partial u_i''}\,\mathrm{d}x = \int_{x_1}^{x_2} \frac{\partial L}{\partial u_i''}u_i''\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial u_i''}u_i|_{x_1}^{x_2} - \frac{\partial L}{\partial u_i''}u_i'|_{x_1}^{x_2} \tag{25}$$

and so on, the segment should be chosen such that the boundary terms vanish. In this case, the energy partition relation (21) remains valid with respect to the energies averaged over the space-time region $(x_1 < x < x_2, t_1 < t < t_2)$.

However, in some classes of waves, which are considered below, the averaging over one variable, t or x, appears to be sufficient.

2.1.2 Derrick-Pohozaev identity

In the case of solitary waves (at least of some of the known solitary waves), the potential energy consists of two terms differ by the homogeneity orders. So there are three unknown energy terms, and the single equality (21) is not sufficient to obtain the $\mathcal{K} - \mathcal{P}$ relation. Fortunately, for such a solitary wave the Derrick-Pohozaev identity presents an additional energy relation, which allowed the complete separation to be done.

I thankful to the member of the Proc. R. Soc. A. editorial board, who drew my attention to the identity.

2.2 Examples

The general results are illustrated by examples of various types of oscillations and waves, linear and nonlinear, homogeneous and forced, steady-state and transient, periodic and non-periodic, parametric and resonant, regular and solitary.

The key point: The Euler theorem on homogeneous functions

3 Orthogonality of waves

Consider a free linear wave propagating in an elastic layer (a 2D problem)

$$\begin{aligned} &\boldsymbol{u}\mathrm{e}^{\mathrm{i}(\omega t-kx)}\,, \quad \boldsymbol{u}=\boldsymbol{u}(u_x(y),u_y(y))\,,\\ &\boldsymbol{\sigma}\mathrm{e}^{\mathrm{i}(\omega t-kx)}\,, \quad \boldsymbol{\sigma}=\boldsymbol{\sigma}(\sigma_{xx}(y),\sigma_{xy}(y))\,, \end{aligned} \tag{26}$$

where $\boldsymbol{u}$ is the displacement and $\boldsymbol{\sigma}$ is the stress vector acting on a cross-section from the right.

A. S. Silbergleit and B. M. Nuller proved the orthogonality relation for two such waves (of the same frequency but different in the wavenumber) propagating in an elastic layer

$$\int_0^h (\sigma'_{xx}u''_x-\sigma''_{xy}u'_y)\,\mathrm{d}y=0 \quad ((k')^2\neq(k'')^2)\,, \tag{27}$$

where the prime and double prime are used to distinguish the waves.

The authors brought the paper to V.V. Novizhilov to consider communicating it to the Sov. Phys. Dokl. He sent the paper to me for a review, and I was reckless, saying that such a relation could correspond to a more general case. However, after some discussion, the manuscript was submitted and published:

A. S. Silbergleit, B. M. Nuller, 1977. Generalized orthogonality of homogeneous solutions in dynamic problems of the theory of elasticity, Dokl. Akad. Nauk SSSR, 234, 2, 333-335.

Soon after, Novozhilov reminded me of my remark in such a way that I had no choice but to find a more general solution. It looked like I should have done it on a bet.

I new that Betti's theorem is just what I need for such a task, but it was not sufficient, and I spent the whole week thinking on how an additional condition could be. Finally, I found that symmetry is the crucial point. Namely,

The existence of a wave $\boldsymbol{u}'$ propagating in a direction entails the existence of the same wave $\boldsymbol{u}'''$ propagating oppositely.

The generalized theorem followed directly from these two statements. I found that the orthogonality relation is valid (with a tiny correction) not only for a straight elastic layer but for a straight or circular waveguide of an arbitrary cross-section. And (**Greater generality brings simplicity!**) the result was obtained without using the dynamic equations and boundary conditions, neither the waveguide cross-section geometry nor the constitutive equation of its material was assumed, and no wave mode data were required.

Slepyan, L.I., 1979. Betty Theorem and Orthogonality Relations for Eigenfunctions. Mechanics of Solids, 14, 74-77.

The displacements and stress components of the oppositely propagating waves are

$$\boldsymbol{u}'_0(u_1\boldsymbol{\nu}'_1,\boldsymbol{u}'_2)\,, \quad \boldsymbol{\sigma}'_0(\sigma'_1\boldsymbol{\nu}_1,\boldsymbol{\sigma}'_2)\,, \quad k' \tag{28}$$

and

$$\boldsymbol{u}_0'''(-u_1\boldsymbol{\nu}_1', \boldsymbol{u}_2'), \quad \boldsymbol{\sigma}_0'''(\sigma_1'\boldsymbol{\nu}_1, -\boldsymbol{\sigma}_2'), \qquad -k', \tag{29}$$

where index 1 corresponds to the axial displacement and normal (to the cross-section) stress, index 2 corresponds to the tangent vectors in the cross-section and $\boldsymbol{\nu}_1$ is the axial unit vector. The coordinate x is the axial coordinate.

Now consider a part of the waveguide, $a < x < b$, and express the Betti theorem twice, with respect to the first and second waves, and also for the second and the third ones. We have

$$\begin{aligned} M_+ \int_S (\sigma_1' u_1'' + \boldsymbol{\sigma}_2' \boldsymbol{u}_2'' - \sigma_1'' u_1' - \boldsymbol{\sigma}_2'' \boldsymbol{u}_2') \, \mathrm{d}S = 0\,, \\ M_- \int_S (-\sigma_1' u_1'' + \boldsymbol{\sigma}_2' \boldsymbol{u}_2'' - \sigma_1'' u_1' + \boldsymbol{\sigma}_2'' \boldsymbol{u}_2') \, \mathrm{d}S = 0\,, \\ M_\pm = \exp(-\mathrm{i}(k'' \pm k')b) - \exp(-\mathrm{i}(k'' \pm k')a)\,. \end{aligned} \tag{30}$$

The condition $(k')^2 \neq (k'')^2$ allows the multipliers $M_\pm$ to be omitted, after which subtracting the second expression from the first one we obtain the generalised orthogonality relation

$$\int_S (\sigma_1' u_1'' - \boldsymbol{\sigma}_2'' \boldsymbol{u}_2') \, \mathrm{d}S = 0\,. \tag{31}$$

The key point: the symmetry condition.

4 Dynamics of a submerged body under a pressure wave

4.1 Final displacement

V.V. Novozilov obtained an elegant relation between the displacements of an arbitrary shape rigid body and the water particles (calculated for the free wave).

$$u = \frac{M_0 + m}{M + m} u_0 \,, \tag{32}$$

where M and M_0 are masses of the body and the water in the body volume, u and u_0 are the respective final displacements under a plane wave, and m is the added mass (which depends on the body volume and form). The derivation was based on the wave equation for a perfect liquid:

Novozhilov, V.V., 1959. On the displacement of an absolutely rigid body under the action of an acoustic pressure wave. J. Appl. Math. Mech. 23, 1138-1142.

In the following, using a different approach I had shown that Novozhilov's formula is still valid for any elastically deformable body under an arbitrary wave, and that $u(t) \to= u_0$ $(t \to \infty)$ independently of the mass ratio, - in the case of a linearly viscose liquid:

Slepyan, L.I., 1963. On a displacement of a deformable body in acoustic medium. J. Appl. Math. Mech., 27, 1402-1411.

We now base on the the parallel analysis of the dynamics of the body and the *liquid body* (if the body were consist of the surrounding water), - without using the wave equation (**Greater generality brings simplicity**):

$$\begin{gathered} M\ddot{u} + Q = P \,, \quad M^0\ddot{u}^0 + Q^0 = P \,, \\ Q = \sum_i F_i(t) * \ddot{u}_i(t) \,, \quad Q^0 = \sum_i F_i(t) * \ddot{u}_i^0(t) \,, \end{gathered} \tag{33}$$

where P is the force, which would act on the body surface if the latter were rigid and unmoving, Q and Q^0 are forces of the same direction at which the motion and deformation of the surface act on the surrounding water, the set of $u_i(t)$, $(u_i^0(t)), i = 1, 2, ...$ defines the displacement of the moving and deforming body (liquid body). The first six functions correspond to the displacements and rotations of the rigid surface, whereas the other reflects its deformation. Lastly, $F_i(t)$ is the corresponding force acting on the surrounding water at $\dot{u}_i = H(t)$.

Using the Laplace transform on time we have

$$P^L(s) = M^0 s^2 (u^0)^L(s) + \sum_i F_i^L(s) s^2 (u_i^0)^L(s) \,. \tag{34}$$

This equality allows us to express the body dynamics through the corresponding values for the liquid body. So we rewrite the first equation in (33) in the form as

$$Ms(u)^L(s) + \sum_i F_i^L(s)s(u_i)^L(s) = M^0 s(u^0)^L(s) + \sum_i F_i^L(s)s(u_i^0)^L(s)\,. \tag{35}$$

Noting that

$$\lim_{t\to\infty} u(t) = \lim_{s\to+0} s u_i^L(s) \tag{36}$$

(if the left limit exists as assumed) and that the elastic deformations vanished as $t \to \infty$, we obtain an equation for the final displacements

$$Mu + \sum_{i=1}^{6} F_i^L(0)u_i = M^0 u^0 + \sum_{i=1}^{6} F_i^L(0)u_i^0\,, \tag{37}$$

where

$$F_i^L(0) = \int_0^\infty F_i(t)\,\mathrm{d}t\,, \quad u_1 = u\,, \quad u_1^0 = u^0 \tag{38}$$

with

$$\int_0^\infty F_i(t)\,\mathrm{d}t = m_i \text{ (perfect liquid)}, \quad \int_0^t F_i(t)\,\mathrm{d}t \to \infty, \quad t \to \infty, \quad \text{(viscous liquid)}, \tag{39}$$

where m_i are the added masses.

In a general case, similar equations should be written for all the displacements and rotations, that is for $u_i, i = 1, ..., 6$. Here, however, we consider the case where due to symmetry only a single displacement exists, $u_1 = u$. In this case, Novozhilov's formula

$$u = \frac{M^0 + m}{M + m} u^0 \tag{40}$$

follows from Eqs. (38) and (39) for perfect fluid, whereas $u = u^0$ for viscous one.

This problem in more detail, including both the body displacement and rotation, is discussed in the book

Slepyan, L.I., 1972. Non-stationary elastic waves. Sudostroenie.

4.2 Time-dependent displacement

It was also shown that not only the final displacements coincide, $u = u^0$ (if the masses are equal), but the motions of the body and liquid body are rather close to each other:

The liquid body approach was then used for the evaluation of shaking of more complex structures under different wave orientations.

The key point: the liquid body approach.

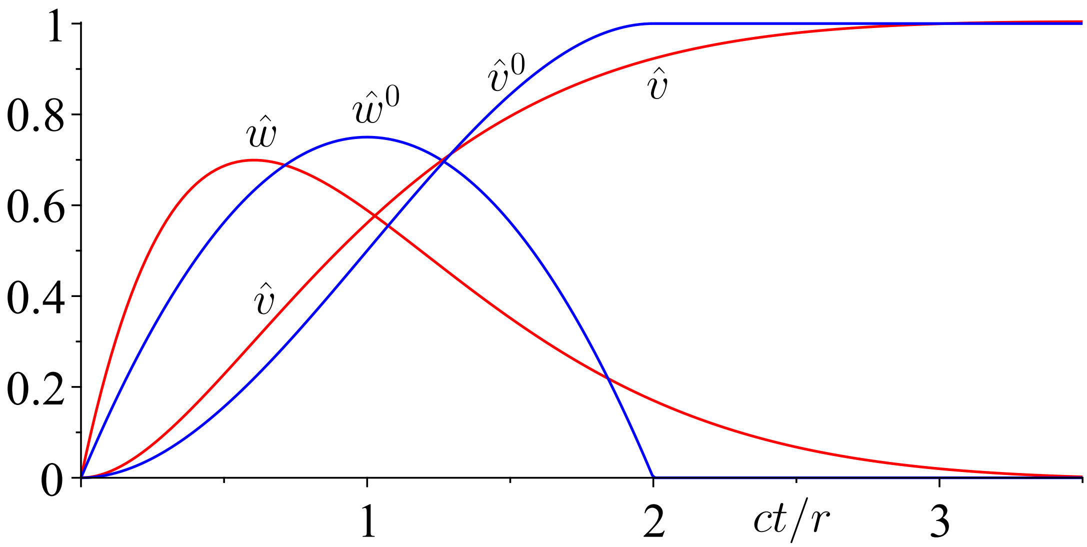


Figure 2: The non-dimensional acceleration $\hat{w} = w\,r/(cv_0)$ and speed $\hat{v} = v/v_0$ of the rigid sphere, and of the inertia center of the liquid sphere of the same mass, $\hat{w}^0$ and $\hat{v}^0$ (normalized similarly), under a step wave.

5 J-integral

Rice J.R. (1968) A Path Independent Integral and the Approximate Analysis of Strain Concentration by Notches and Cracks. J Appl Mech 35: 379-386.

6 Driving forces in dynamic elasticity

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