# The Fredholm Factorization of matrix Generalized Wiener-Hopf equations

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# INTRODUCTION

- The problem to factorize a matrix kernel constitutes an interesting/beautiful mathematical problem that in the past has become a ‘cult’ activity for many scientists.

- The state of the art is now that in the ninety years since the solution of the scalar equation was found, the WH vector equation has been solved in closed forms only in particular cases [Book] V. Daniele, R. Zich, The Wiener-Hopf method in electromagnetics, SciTech Pub, 2014

- Growing importance of the formulations on applied mathematical, physics and engineering

- Equation forms assumed by the WH equations are more numerous,
  see for instance GWHEs for angular regions (Daniele, 2001)

- Closed form WH factorization of an arbitrary matrix remains a challenging and fascinating problem

- Strong need of the introduction of a general and efficient method of solutions of matrix equations

# AIM

- Aim of this presentation is to illustrate:

  the reduction of the WH equations to Fredholm equation of second kind,

  inspired by Vekua's book (1967)

- The <<Fredholm factorization>> is available to a wide spectrum of applications

- The <<Fredholm factorization>> is a

  powerful, flexible and general tool

- Focus on diffraction problems with plane wave source that produces single pole (say $\eta_o$)

  Note:
  in this case, without any loss of generality the WH equations can be assumed homogeneous.

# Wiener Hopf (WH) equations in spectral domain

**Notation**

$$G(\eta)F_+(\eta)=F_-(\eta)$$

Engineering assumption in complex variable: j

$$F_+(\eta)=\int_0^{\infty} f(x)e^{j\eta x}dx \qquad F_-(\eta)=\int_{-\infty}^{0} f(x)e^{j\eta x}dx$$

$$F_{+,-}(\eta)\to 0 \qquad \frac{G(\eta)}{\eta}\to 0\,, \quad \frac{[G(\eta)]^{-1}}{\eta}\to 0 \quad \text{as } \eta\to\infty$$

**Non standard parts of the Laplace Transforms**

$$F_{+,-}(\eta)=F_{+,-}^{s}(\eta)+F_{+,-}^{n.s}(\eta)$$

$F_+^{s}(\eta)$ (standard part regular in the upper half-plane $\mathrm{Im}[\eta]\geq 0$)

$F_-^{s.}(\eta)$ (standard part regular in the lower half-plane $\mathrm{Im}[\eta]\leq 0$)

$F_+^{n.s.}(\eta)$ (not-standard part of plus functions. It is constitued by the characteristic part of the poles located in the upper half-plane $\mathrm{Im}[\eta]>0$)

$F_-^{n.s.}(\eta)$ (not-standard part of minus functions. It is constitued by the characteristic part of the poles located in the lower half-plane $\mathrm{Im}[\eta]<0$

**Properties**

$$F_{+,-}(\eta) = F^{s}_{+,-}(\eta) + F^{n.s}_{+.-}(\eta)$$

$F^{n.s.}_{+}(\eta)$ (not-vanishing if there are poles in the upper half-plane Im[$\eta$]>0)

$F^{n.s.}_{-}(\eta)$ (not-vanishing if there are poles in the lower half-plane Im[$\eta$]<0)

Focus on diffraction problems :
We define geometrical optical contribution (GO) the contribution obtained by the geometrical optical solution.
GO is known without the necessity to solve the WH equations

**REMARK**

The not standard part of the WH unknowns are coincident with the not standard part of the geometrical optical contribution. These terms are known without the necessity to solve the WH equation

$$F^{n.s}_{+,-}(\eta) = \left[ F^{G.O}_{+,-}(\eta) \right]^{n.s.}$$

## The classical factorization method to solve the WH equation

A very ingenious idea: Factorization of the Matrix kernel

$$G(\eta) = G_-(\eta) G_+(\eta)$$

$G_-(\eta)$ and $G_+(\eta)$ with their inverses $[G_-(\eta)]^{-1} = G_-^{-1}(\eta)$ and $[G_+(\eta)]^{-1} = G_+^{-1}(\eta)$ regular in the half-planes $\mathrm{Im}[\eta] \le 0$ and $\mathrm{Im}[\eta] \ge 0$ respectively

$\frac{G_\pm(\eta)}{\eta} \to 0, \quad \frac{[G(\eta)]^{-1}}{\eta} \to 0$ as $\eta \to \infty$. The equation $G(\eta) F_+(\eta) = F_-(\eta)$ can be rewritten:

$$G_+(\eta) F_+(\eta) - G_+(\eta_o) F_+^{n.s.}(\eta) - G_-^{-1}(\eta_o') F_-^{n.s.}(\eta) =$$

$$G_-^{-1}(\eta) F_-(\eta) - G_+(\eta_o) F_+^{n.s.}(\eta) - G_-^{-1}(\eta_o') F_-^{n.s.}(\eta) = w(\eta)$$

*simple pole $\eta_o$*
*simple pole $\eta_o'$*

$w(\eta) = 0$ because is entire and vanishing at $\infty$ (Liouville)

HP: Considering the unknown as Laplace transforms

# Solution of th WH equation by factorization

$$F_+(\eta) = \left[G_+(\eta)\right]^{-1}\left[G_+(\eta_o)F_+^{n.s.}(\eta) + G_-^{-1}(\eta_o')F_-^{n.s.}(\eta)\right]$$

$$F_-(\eta) = \left[G(\eta)\right]^{-1} F_+(\eta)$$

Greater convenience:

The obtained exact solution provides the values of the WH unknowns for <u>every value of $\eta$</u> even for points not located in the principal sheet i.e. the sheet where is defined the branch of the multivalued functions present in the kernel .

PROBLEM:

<u>Exact factorization can be obtained only for particular class of matrix kernels.</u>
<u>Furthermore their expressions are often very complicated to be obtained.</u>

# The Fredholm factorization to solve the WH equations

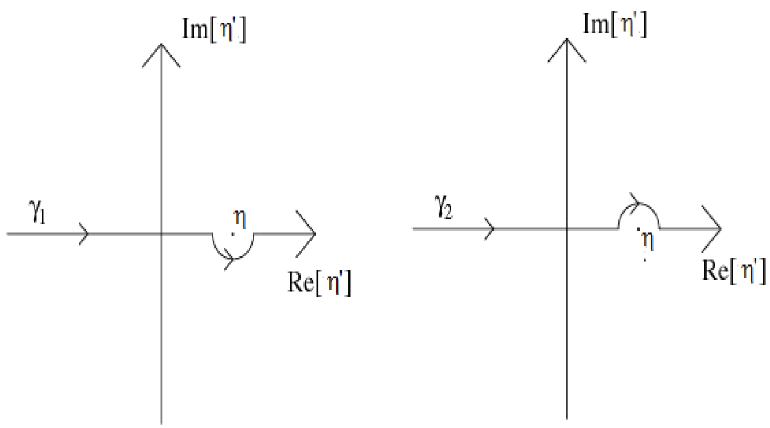
**IDEA**

Eliminate the unknown minus (plus) function using the Cauchy decomposition.
This elimination produces integral representations.

We use the word «factorization» ( instead of the more correct word «decomposition») to try to create more attention to this new technique of solution of the Wiener Hopf equations.

Also used to obtain the factorization of kernel (see Lecture 1 of short course)

In decomposing a given functions in plus and minus contributions, the "smile" and the "frown" integration lines have important properties




$$\frac{1}{2\pi j}\int_{\gamma_1}\frac{F_+(\eta')}{\eta'-\eta}d\eta' = F_+(\eta) - F_+^{n.s.}(\eta) \qquad \eta\in\mathbb{R}$$

$$\frac{1}{2\pi j}\int_{\gamma_2}\frac{F_+(\eta')}{\eta'-\eta}d\eta' = -F_+^{n.s.}(\eta) \qquad \eta\in\mathbb{R}$$

$$\frac{1}{2\pi j}\int_{\gamma_2}\frac{F_-(\eta')}{\eta'-\eta}d\eta' = -F_-(\eta) + F_-^{n.s.}(\eta) \qquad \eta\in\mathbb{R}$$

$$\frac{1}{2\pi j}\int_{\gamma_1}\frac{F_-(\eta')}{\eta'-\eta}d\eta' = F_-^{n.s.}(\eta) \qquad \eta\in\mathbb{R}$$

**Remark**

The above decomposition equations hold also in presence of plus (minus) functions diverging in the lower (upper) half planes $\mathrm{Im}[\eta]<0$ ( $\mathrm{Im}[\eta]>0$)

**Properties**

$$\boxed{G(\eta)F_+(\eta) = F_-(\eta)}$$

The «smile» decomposition eliminates the unknown standard part of the minus function $F_-^s(\eta')$
(recall that the **not standard** part of the minus and plus function is **known**):

$$\frac{1}{2\pi j}\int_{\gamma_1}\frac{G(\eta')F_+(\eta')}{\eta'-\eta}d\eta' = \frac{1}{2}G(\eta)F_+(\eta) + \frac{P.V.}{2\pi j}\int_{-\infty}^{\infty}\frac{G(\eta')F_+(\eta')}{\eta'-\eta}d\eta' = \frac{1}{2\pi j}\int_{\gamma_1}\frac{F_-(\eta')}{\eta'-\eta}d\eta' = F_-^{n.s.}(\eta)$$

From the other side the «frown» decomposition yields:

$$\frac{1}{2\pi j}\int_{\gamma_2}\frac{G(\eta)F_+(\eta')}{\eta'-\eta}d\eta' = -\frac{1}{2}G(\eta)F_+(\eta) + \frac{P.V.}{2\pi j}\int_{-\infty}^{\infty}\frac{G(\eta)F_+(\eta')}{\eta'-\eta}d\eta' = -G(\eta)F_+^{n.s.}(\eta)$$

The difference yields the integral equation of second kind in terms of plus WH unknown

$$\boxed{G(\eta)F_+(\eta) + \frac{1}{2\pi j}\int_{-\infty}^{\infty}\frac{G(\eta')-G(\eta)}{\eta'-\eta}F_+(\eta')d\eta' = F_-^{n.s.}(\eta) + G(\eta)F_+^{n.s.}(\eta), \quad \eta \in \mathbb{R}}$$

**Remark:**

The kernel of the integral is quadratically integrable in the space $L_2[R,p(\eta)]$ (Daniele, 2003, sez-6.2.5)

**Comparison**

$$F_+(\eta)+\frac{[G(\eta)]^{-1}}{2\pi j}\int_{-\infty}^{\infty}\frac{G(\eta')-G(\eta)}{\eta'-\eta}F_+(\eta')d\eta'=[G(\eta)]^{-1}F_-^{n.s.}(\eta)+F_+^{n.s.}(\eta),\quad \eta\in\mathbb{R}\qquad \text{(FIE)}$$

$$F_+(\eta)=\left[G_+(\eta)\right]^{-1}\left[G_+(\eta_o)F_+^{n.s.}(\eta)+G_-^{-1}(\eta_o')F_-^{n.s.}(\eta)\right]\qquad \text{(WHF)}$$

**Benefits of FIE:**

- Fredholm factorization provides approximate solutions with the correct singularities of the WH unknowns as the classical WH factorization (Take into account that the singularities of F(η’) do not produce singularities in the integral of the FIE)
- The obtained integral equation always is Fredholm integral equation of second kind (FIE).
- The equation is valid for arbitrary WH matrix G(η).
- The Fredholm factorization does not require a WH equation written in normal form

**Defect of FIE**:

Even though the numerical solution of the FIE is obtained very easily, to evaluate the plus function for values of η far from the integration line requires an analytic continuation.

# Fredholm factorizations of WH equations not presenting the normal form

The classical factorization applies to a complete system of WH equations presenting the «normal» form

$$G(\eta)F_+(\eta) = F_-(\eta)$$

However many problems are formulated in terms of WH equations that do not present the "normal" form. For instance let's consider the WH equation

$$A(\eta)F_{1+}(\eta) + F_-(\eta) + B(\eta)\,\mathrm{e}^{j\eta s}\,F_{2+}(-\eta) = 0$$

Appling the "smile" decomposition the first two terms yield:

$$A(\eta)F_{1+}(\eta) + \frac{1}{2\pi j}\int_{-\infty}^{\infty}\frac{A(\eta') - A(\eta)}{\eta' - \eta}F_{1+}(\eta')d\eta' + A(\eta)F_{1+}^{n.s.}(\eta) + F_-^{n.s.}(\eta), \quad \eta \in \mathbb{R}$$

The third term requires η' –> –η' in the integral of the «smile» integration. Taking into account that:

$$\frac{B(\eta)\,\mathrm{e}^{j\eta s}}{2\pi j}\int_{\gamma_{1\eta}}\frac{F_{2+}(-\eta')}{\eta' - \eta}d\eta' = B(\eta)\,\mathrm{e}^{j\eta s}\,F_{2+}^{n.s.}(-\eta),$$

$$\frac{1}{2\pi j}\int_{\gamma_{1\eta}}\frac{B(\eta')\,\mathrm{e}^{j\eta' s}}{\eta' - \eta}F_{2+}(-\eta')d\eta' - \frac{1}{2\pi j}\int_{\gamma_{1\eta}}\frac{B(\eta)\,\mathrm{e}^{j\eta s}}{\eta' - \eta}F_{2+}(-\eta')d\eta' = \frac{1}{2\pi j}\int_{-\infty}^{\infty}\frac{-B(-\eta')\,\mathrm{e}^{-j\eta' s} + B(\eta)\,\mathrm{e}^{j\eta s}}{\eta' + \eta}F_{2+}(\eta')d\eta'$$

we get:

$$\frac{1}{2\pi j}\int_{\gamma_{1\eta}}\frac{B(\eta')\,\mathrm{e}^{j\eta' s}\,F_{2+}(-\eta')}{\eta' - \eta}d\eta' = \frac{1}{2\pi j}\int_{-\infty}^{\infty}\frac{-B(-\eta')\,\mathrm{e}^{-j\eta' s} + B(\eta)\,\mathrm{e}^{j\eta s}}{\eta' + \eta}F_{2+}(\eta')d\eta' + B(\eta)\,\mathrm{e}^{j\eta s}\,F_{2+}^{n.s.}(-\eta), \quad \eta \in \mathbb{R}$$

# Fredholm factorizations of WH equations not presenting the normal form

**Some important forms are:**

a) Transversally modified WH equations (modified of 2nd kind)
b) Longitudinal modified WH equations (modified of 1st kind)
c) Generalized WH equations.

*As it is well known, the modified WH equations have been introduced and solved by Jones by using an efficient approximate method*

The application of the Fredholm Factorization to these equations (possibly vectorial) illustrates the flexibility of this technique.

Without any loss of generality, we will consider closed modified WH equations .
We call "closed" the WH equations that do not require other independent additional functional equations to get the involved WH unknowns.

As it is well known, Jones ideated two ingenious method to solve closed modified equations.

To this author opinion the Fredholm factorization is more advantageous because it does not require preliminar factorizations.

# Transversally modified WH equation

$$G(\eta)F_+(\eta)+H(\eta)F_+(-\eta)=F_-(\eta)$$

## Associated Fredholm Integral equation

Again the «smile» decomposition eliminates the unknown standard part of the minus function in the second member (recall that the **not standard** parts of the minus and plus functions are **known**):

$$G(\eta)F_+(\eta)+\frac{1}{2\pi j}\int_{-\infty}^{\infty}\frac{G(\eta')-G(\eta)}{\eta'-\eta}F_+(\eta')d\eta'+\frac{1}{2\pi j}\int_{\gamma_1}\frac{H(\eta')F_+(-\eta')}{\eta'-\eta}d\eta'=F_-^{n.s.}(\eta)+G(\eta)F_+^{n.s.}(\eta),\quad \eta\in\mathbb{R}$$

Taking into account that (see previous example)

$$\frac{1}{2\pi j}\int_{\gamma_{1\eta}}\frac{H(\eta')F_+(-\eta')}{\eta'-\eta}d\eta'=\frac{1}{2\pi j}\int_{-\infty}^{\infty}\frac{(H(-\eta')-H(\eta))F_+(\eta')}{\eta'+\eta}d\eta'+H(\eta)F_+^{n.s.}(-\eta)$$

# Transversally modified WH equation

we get the FIE associated to the transversally modified WH equation

$$G(\eta)F_+(\eta)+\frac{1}{2\pi j}\int_{-\infty}^{\infty}\frac{G(\eta')-G(\eta)}{\eta'-\eta}F_+(\eta')d\eta'+$$

$$\frac{1}{2\pi j}\int_{-\infty}^{\infty}\frac{(-H(-\eta')+H(\eta))F_+(\eta')}{\eta'+\eta}d\eta'=-H(\eta)F_+^{n.s.}(-\eta)+F_-^{n.s.}(\eta)+G(\eta)F_+^{n.s.}(\eta),\quad \eta\in\mathbb{R}$$

**Remark:**

<u>The Fredholm factorization yields to FIE that does not require any factorization of G(η)</u>

# Longitudinally modified WH equation

$$G(\eta)F_o(\eta)+e^{j\eta s}F_+(\eta)=F_-(\eta)=F_{\pi+}(-\eta) \qquad \text{(L1)}$$

**Remark**

The plus unknown Fo($\eta$) is an entire function regular in the whole plane $\eta$.

Fo($\eta$) is vanishing in the upper half plane Im[$\eta$]>0, and not bounded in the lower half plane Im[$\eta$]<0

However It is possible to see that the auxiliary function:

$$F_{\pi o}(\eta)=e^{j\eta s}F_o(-\eta)$$

has the same characteristics of Fo($\eta$).

By resorting to this analytical property we will show that (L1) is a closed WH equation

**Fredholm integral equation associated the (L1)**

Taking into account that $e^{j\eta s}F_+(\eta)$ is a plus funtion we have: $\frac{1}{2\pi j}\int_{\gamma_1}\frac{e^{j\eta' s}F_+(\eta')}{\eta'-\eta}d\eta'=e^{j\eta s}F_+(\eta)-e^{j\eta_o s}F_+^{n.s.}(\eta) \qquad \eta\in\mathbb{R}$

$$-e^{j\eta s}F_+(\eta)+e^{j\eta_o s}F_+^{n.s.}(\eta)+F_{\pi+}^{n.s}(-\eta)=G(\eta)F_o(\eta)+\frac{1}{2\pi j}\int_{-\infty}^{\infty}\frac{[G(\eta')-G(\eta)]F_o(\eta')}{\eta'-\eta}d\eta', \quad \eta\in\mathbb{R}$$

# Longitudinally modified WH equation

Even though we have eliminated the minus $F_{\pi+}(-\eta)$, we have only one equation that involves two unknowns: $F_+(\eta)$ and $F_o(\eta)$.

The application of the Fredholm Factorization on the following three independent WH equations obtained by L1 changing $\eta$ with $-\eta$ and by multiplying for *exp(jηs),* is enough to get a closed mathematical problem

$$G(-\eta)F_o(-\eta) + e^{-j\eta s}F_+(-\eta) = F_{\pi+}(\eta) \qquad \text{(L2)}$$

$$G(-\eta)F_{\pi o}(\eta) + F_+(-\eta) = e^{j\eta s}F_{\pi+}(\eta) \qquad \text{(L3)}$$

$$G(\eta)F_{\pi o}(-\eta) + F_+(\eta) = e^{-j\eta s}F_{\pi+}(-\eta) \qquad \text{(L4)}$$

Applying the Fredholm factorization procedure on (L2)-(L4) and taking into account the FIE obtained on the previous slide we get a system of four FIEs having the four unknowns $F_o$, $F_{\pi o}$, $F_+$, $F_{\pi+}$.

Algebraic manipulations reduce the system of order 4 to the system of order 2 having as unknowns only the two functions $F_o$, $F_{\pi o}$.

# Longitudinally modified WH equation

The FIE associated to the longitudinal modified WH equations are

$$G(\eta)F_o(\eta)+\frac{1}{2\pi j}\int_{-\infty}^{\infty}\frac{[G(\eta')-G(\eta)]F_o(\eta')]}{\eta'-\eta}d\eta'+\frac{e^{j\eta s}}{2\pi j}\int_{-\infty}^{\infty}\frac{[G(\eta)-G(-\eta')]F_{\pi o}(\eta')]}{\eta'+\eta}d\eta' = \mathrm{N}_o(\eta) \quad \eta\in\mathbb{R}$$

$$G(-\eta)F_{\pi o}(\eta)+\frac{1}{2\pi j}\int_{-\infty}^{\infty}\frac{[G(-\eta')-G(-\eta)]F_{\pi o}(\eta')]}{\eta'-\eta}d\eta'+\frac{e^{j\eta s}}{2\pi j}\int_{-\infty}^{\infty}\frac{[G(-\eta)-G(\eta')]F_o(\eta')]}{\eta'+\eta}d\eta' = \mathrm{N}_{\pi o}(\eta) \quad \eta\in\mathbb{R}$$

$$F_+(\eta)=F_+^{n.s.}(\eta)-e^{-j\eta_o s}F_{\pi+}^{n.s}(-\eta)+\frac{1}{2\pi j}\int_{-\infty}^{\infty}\frac{[G(-\eta')-G(\eta)]F_{\pi o}(\eta')}{\eta'+\eta}d\eta'$$

$$F_{\pi+}(\eta)=e^{j\eta_o s}F_+^{n.s.}(-\eta)+F_{\pi+}^{n.s}(-\eta)+\frac{1}{2\pi j}\int_{-\infty}^{\infty}\frac{[G(\eta')-G(-\eta)]F_o(\eta')}{\eta'+\eta}d\eta'$$

Where the source terms are:

$$N_o^{\cdot}(\eta)=F_{\pi+}^{n.s}(-\eta)+e^{j(\eta-\eta_o)s}F_{\pi+}^{n.s}(-\eta)+(e^{j\eta_o s}-e^{j\eta s})F_+^{n.s}(\eta)$$

$$N_{\pi o}^{\cdot}(\eta)=-F_+^{n.s}(-\eta)+e^{j(\eta+\eta_o)s}F_+^{n.s}(-\eta)-(e^{j\eta_o s}-e^{j\eta s})F_{\pi+}^{n.s}(\eta)$$

Remark

The first two equations provide the solution of $\mathrm{F}_o^{\cdot}(\eta)$ and $\mathrm{F}_{\pi o}^{\cdot}(\eta)$

The last two equations provide $\mathrm{F}_+^{\cdot}(\eta)$ and $\mathrm{F}_{\pi+}^{\cdot}(\eta)$ in terms of $\mathrm{F}_o^{\cdot}(\eta)$ and $\mathrm{F}_{\pi o}^{\cdot}(\eta)$

**Remarks:** Again the Fredholm factorization yields to FIE that does not require any factorization of G(η)

**Remarks [probably best route]:**

From L1-L4 we can deduce the two WH equations

$$F_o(-\eta) + G^{-1}(-\eta)e^{-j\eta s}F_+(\eta) = G^{-1}(-\eta)F_{\pi+}(\eta)$$
$$F_{\pi o}(-\eta) + G^{-1}(\eta)F_+(\eta) = G^{-1}(\eta)e^{-j\eta s}F_{\pi+}(-\eta)$$

The Fredholm factorization of these two equations eliminate the minus functions $F_o(-\eta)$ and $F_{\pi o}(-\eta)$ yielding directly two coupled equations in $F_+(\eta)$ and $F_{\pi+}(\eta)$

# The Fredholm factorization of generalized WH equations

Inspired to the generalized Hilbert-Riemann problems [Vekua, 1967] we introduce the GWHE

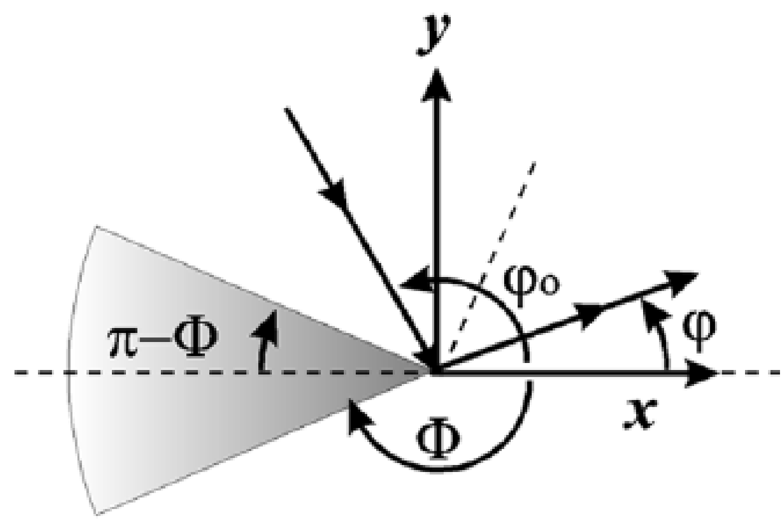
$$G(\eta)\,F_+(\eta) = F_-(m_\Phi(\eta))$$

In diffraction theory the GWHE arisen in presence of wedge with arbitrary angular aperture [Noble, Ex. 5.15]

For instance the diffraction by a PEC wedge yields the above GWHE where:

$$m_\Phi(\eta) = -\eta\cos\Phi + \sqrt{k^2-\eta^2}\,\sin\Phi$$

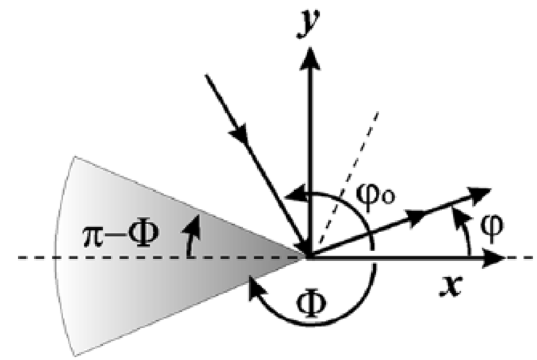
$$G(\eta) = \sqrt{k^2-\eta^2}$$



# The Fredholm factorization of generalized WH equations

**Remark:** the Generalized WH equations can be reduced to a classical WH equations using the mapping (Daniele,2001)

$$\eta = \eta(\alpha) = -k\cos\left[\frac{\Phi}{\pi}\arccos(-\frac{\alpha}{k})\right]$$




$$G(\eta)\,F_+(\eta) = F_-(m_\Phi(\eta))$$ Generalized WH equation

$$\Downarrow$$

$$\bar{G}(\alpha)\,\bar{F}_+(\alpha) = \bar{F}_-(\alpha)$$ Classical WH equation

For the PEC wedge we have the exact factorization $\bar{G}(\alpha) = \sqrt{k^2 - \eta(\alpha)^2}$

$$\bar{G}_-(\alpha) = \sqrt{\frac{k+\alpha}{2}} \qquad \bar{G}_+(\alpha) = \frac{\bar{G}(\alpha)}{\bar{G}_-(\alpha)}$$

Solution: $$\bar{F}_+(\alpha) = \frac{\pi\,\alpha_o}{\Phi\,\eta_o}\left[\bar{G}_+(\alpha)\right]^{-1}\bar{G}_+(\alpha_o)\frac{A_o}{\alpha-\alpha_o}$$ $\mathrm{Im}[\alpha_o] > 0$

with $\eta_o = -k\cos\varphi_o \to \alpha_o = -k\cos\frac{\pi}{\Phi}\varphi_o$

$$F_+^{n.s}(\eta) = \frac{A_o}{\eta-\eta_o}u(\frac{\pi}{2}-\varphi_o) \to \bar{F}_+^{n.s.}(\alpha) = \frac{\pi\,\alpha_o}{\Phi\,\eta_o}\frac{A_o}{\alpha-\alpha_o}u(\frac{\Phi}{2}-\varphi_o)$$

# The Fredholm factorization of generalized WH equations

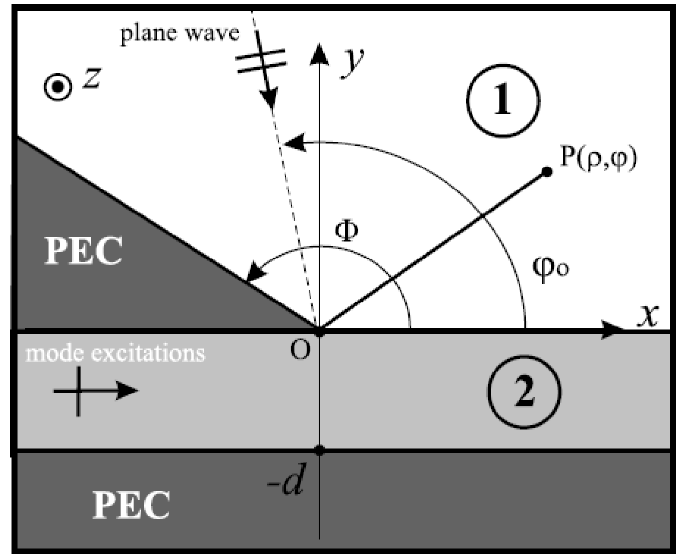
Let's consider a complex diffraction problem where several sub-regions are present
Each sub-region is described by WH equations. For example the sub-region 1 is described by the GWHE

$$Y_c(\eta)V_+(\eta) - I_+(\eta) = -I_{a+}(-m_a(\eta)) \qquad \text{G1}$$

$$m_a(\eta) = -\eta\cos\Phi_a + \sqrt{k^2-\eta^2}\sin\Phi_a \qquad Y_c(\eta) = \frac{\sqrt{k^2-\eta^2}}{k Z_o}$$

$$V_+(\eta) = \int_0^\infty E_z(x) e^{j\eta x} dx \qquad I_+(\eta) = \int_0^\infty H_x(x) e^{j\eta x} dx$$

k and $Z_o$ are the propagation constant and the impedance of the medium filling the sub-space 1




To eliminate the minus generalized function $I_{a+}(-m_a(h))$, again we can resort to the Fredholm factorization.

However the direct application in the η plane is very difficult.

Again we reduced to a classical WH equations using the mapping (Daniele, 2001)

$$\eta = \eta(\alpha) = -k\cos\left[\frac{\Phi_a}{\pi}\arccos\left(-\frac{\alpha}{k}\right)\right]$$

$$\bar{Y}_c(\alpha)\bar{V}_{1+}(\alpha) - \bar{I}_{1+}(\alpha) = -\bar{I}_{a+}(-\alpha)$$

Properties of the unknowns

# The Fredholm factorization of generalized WH equations

$$\bar{Y}_c(\alpha)\bar{V}_{1+}(\alpha) - \bar{I}_{1+}(\alpha) = -\bar{I}_{a+}(-\alpha)$$

where: $$Y_c(\eta) = \bar{Y}_c(\alpha), V_{1+}(\eta) = \bar{V}_{1+}(\alpha), I_{1+}(\eta) = \bar{I}_{1+}(\alpha), I_{a+}(-m_a) = \bar{I}_{a+}(-\alpha)$$

Fredholm factorization of Classical WH equation on the $\alpha$-plane
Subsequent use of the $\eta$-plane yields to the Fredholm factorization of the generalized WH equation

Back to $\eta$ $$\boxed{I_+(\eta) = Y_c(\eta)V_+(\eta) + \mathcal{Y}_a[V_+(\eta')] - S(\eta), \quad \eta \in \mathbb{R}}$$

$$\mathcal{Y}_a[...] = \frac{1}{2\pi j}\int_{-\infty}^{\infty} y_a(\eta,\eta')[...]d\eta \qquad y_a(\eta,\eta') = \frac{Y_c(\eta')}{\alpha(\eta')-\alpha(\eta)}\frac{d\alpha}{d\eta'} - \frac{Y_c(\eta)}{\eta'-\eta} + \sum_{n=1}^{n_{Max}} \frac{q_n^{\Phi_a}(\eta)}{\eta'-\mathrm{p}_n^{\Phi_a}(\eta)} u(\frac{\pi}{2} - n\Phi_a)$$

$$q_n^{\Phi_a}[\eta] = \frac{1}{\mathrm{k}\,\mathrm{Z_o}}(\eta\sin 2n\Phi_a + \sqrt{k^2-\eta^2}\cos 2n\Phi_a) \qquad p_n^{\Phi}[\eta] = \eta\cos(2\Phi_a) - \sqrt{k^2-\eta^2}\sin 2\Phi_a,$$

For space reason, the known function S($\eta$) that depends on not-standard GO contributions is not reported

Two completely different alternative deductions have been used to verify the correctness of the above equation.

The finite sum is extended to the values of n that makes positive the argument of the step function u.
This contribution is vanishing for obtuse angular regions.

# The Fredholm factorization of generalized WH equations

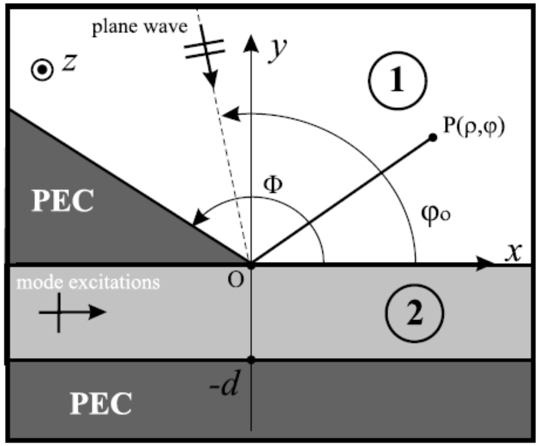
$$I_+(\eta) = Y_c(\eta)V_+(\eta) + \mathcal{Y}_a[V_+(\eta')] - S(\eta), \quad \eta \in \mathbb{R}$$

$$\mathcal{Y}_a[...] = \frac{1}{2\pi j}\int_{-\infty}^{\infty} y_a(\eta,\eta'][...]d\eta \qquad y_a(\eta,\eta'] = \frac{Y_c(\eta')}{\alpha(\eta')-\alpha(\eta)}\frac{d\alpha}{d\eta'} - \frac{Y_c(\eta)}{\eta'-\eta} + \sum_{n=1}^{n_{Max}} \frac{q_n^{\Phi_a}(\eta)}{\eta'-p_n^{\Phi_a}(\eta)} u\left(\frac{\pi}{2} - n\Phi_a\right)$$

**Remarks**

- The FIE of a sub region is independent of the other sub regions that constitute the whole diffraction problem. Thus the study of sub-region can be done once and for all.
- Only the source of the FIEs is related to the GO of the entire problem
- In presence of isolated wedges it can be more simple to work only in the $\alpha$-plane that involves classical WH equations. In this plane, also the analytic continuations are considerably more simple to get.
- Available solution of the diffraction by an arbitrary impenetrable wedge (Daniele and Lombardi (2006) or by a dielectric wedge (Daniele (2010,2011), Daniele and Lombardi 2011) have been accomplished

**Fredholm factorization  to solve novel complex canonical diffraction problems**




**Coupled planar and angular regions are present**

**Characteristics of the method:**

We can take advantage of this by breaking down the complexity of diffraction geometry in different sub homogeneous regions where the WH functional equations are known

Fredholm factorization  works on single functional WH equations without the necessity of normal form

The technique allows to introduce equivalent simple network modelling

An example of application is provided by the next presentation.

The validity of the solution has ascertained by comparing  with  exact solutions (when available) or numerical simulations done with the FEM.

We successfully  applied the above technique for a large variety of diffraction problems where coupled planar and angular regions are present (see next presentation).

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