Cauchy-type integrals in Multivariable Complex Analysis

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Credits

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Overview of this talk

Let $D \subset \mathbb{C}$ be a domain in the complex plane, and suppose that $f : D \mapsto \mathbb{C}$ is holomorphic in D and continuous on $\overline{D}$, that is

$$\overline{\partial} f(z) = 0, \quad z \in D \quad \text{and} \quad f \in C^0(\overline{D})$$

The classical

- Cauchy theorem:

$$\int_{w \in bD} f(w)\, dw = 0$$

- Cauchy integral formula:

$$f(z) = \frac{1}{2\pi i} \int_{w \in bD} \frac{f(w)}{w - z}\, dw, \quad z \in D$$

have many applications in Physics and Engineering, and as such are essential components of the Complex Analysis Toolbox.

Overview of this talk

Two main assets:

- *holomorphicity*: Cauchy kernel

$$\frac{1}{w - z}$$

  is **holomorphic** as a function of the output variable $z \in D$ (recall that $w \in bD$, so $z \neq w$).
- *Universality*: Cauchy integral is **meaningful** for most contour shapes.

One major drawback:

- Cauchy integral *lacks a good transformation law under conformal maps* (with a few exceptions).

Overview of this talk

This brings up three questions, which are the focus of this talk:

- Is there another integration kernel that retains the main assets of the Cauchy kernel but also has a good transformation law under conformal maps?
- Is there an analog of the Cauchy kernel for holomorphic functions of two (or more) complex variables that retains the main assets of the 1-dimensional Cauchy kernel?
- Do the answers to questions above depend on holomorphic &/or geometric properties of the underlying domain D?

Cauchy Integral: a closer look

- The Cauchy Integral along the boundary of a (simply connected) domain $D \subset \mathbb{C}$:

$$\mathcal{C}f(z) \;=\; \frac{1}{2\pi i}\int\limits_{w\in bD} \frac{f(w)}{w-z}\,dw\,, \quad z\in D$$

More precisely, we regard $\mathcal{C}$ as a Singular Integral Operator (SIO) and in this context it is often referred to as the Cauchy Transform ("transform" as opposed to "integral" – but I will keep calling it "Integral"):

- $\mathcal{C}f(z) \;=\; p.v.\, \frac{1}{2\pi i}\int\limits_{w\in bD} \frac{f(w)}{w-z}\,dw\,,\; z\in bD$ ("principal value")

  Alternate approach: exploit Cauchy Formula:

- $\mathcal{C}f(z_\epsilon) \;=\; f(z_\epsilon) + \frac{1}{2\pi i}\int\limits_{w\in bD} \frac{f(w)-f(z_\epsilon)}{w-z_\epsilon}\,dw\,,\; z\in bD,\; z_\epsilon\to z \quad,$

  $f \in C^\alpha(\overline{D})$ and holomorphic in D.

Landmark Results

- Theorem [Calderòn (1977); Coifman-McIntosh-Meyer (1982)]:

  Suppose $D \subset \mathbb{C}$ is a Lipschitz domain, i.e.

  $$bD = \left\{ w = t + i\,A(t)\,, \quad |A(t) - A(s)| \le M|s - t|,\ s, t \in \mathbb{R} \right\}$$

  Then, the Cauchy Integral

  $$f \mapsto \mathcal{C}(f)$$

  is bounded: $L^p(bD, \sigma) \to L^p(bD, \sigma), \quad 1 < p < \infty$

  with respect to $\sigma =$ arc-length measure for bD, that is

  $$\|\mathcal{C}(f)\|_{L^p(bD,\sigma)} \le C\|f\|_{L^p(bD,\sigma)}$$

  with $C = C(p, bD)$.
  Here: $L^p(bD, \sigma) := \{f \mid \int_{bD} |f(w)|^p d\sigma(w) < \infty\},\ p > 1$

A remark about the Cauchy Integral

The Cauchy integral for a Lipschitz curve is the prototypical **Calderòn-Zygmund operator**:

A crucial feature of CZ operators, T:

$$T \text{ is bounded} : L^2(bD, \sigma) \to L^2(bD, \sigma)$$

$$\Longleftrightarrow$$

$$T \text{ is bounded} : L^p(bD, \sigma) \to L^p(bD, \sigma) \quad \text{all } 1 < p < \infty$$

(Something to keep in mind for future reference)

Impact of Landmark results

- Elliptic Linear PDEs: *BVPs on non-smooth domains*
- Harmonic Analysis: *New Techniques for SIOs* (*T(1)-Theorem*)
- Geometric Function Theory: *holomorphic Capacity* (Vitushkin conjecture on removable sets for holomorphic functions)

- **One & Several complex variables**:
  - Solution of $\overline{\partial}$-problem **with estimates**.
  - **Orthogonal projections** of L^2 onto spaces of holomorphic functions associated with

$$D \Subset \mathbb{C}^n, \quad n \geq 1$$

Specifically, the Szegő projection and Bergman projection, which map L^2 onto, respectively, the holomorphic Hardy space, and onto the Bergman space

Orthogonal projections: a case study

- holomorphic Hardy Space for $D \subset \mathbb{C}$:

$$H^p(bD, \sigma) := \left\{ F \,\middle|\, \overline{\partial} F(z) = 0, z \in D;\ \sup_{\epsilon > 0} \int_{z \in bD_\epsilon} |F(z)|^p d\sigma_\epsilon(z) < +\infty \right\}$$

(A closed subspace of $L^p(bD, \sigma)$, $1 \leq p < \infty$).

- Pick $p = 2$: Orthogonal Projection $\mathbf{S}_D : L^2(bD) \mapsto H^2(bD)$:
  - "Projection": $\mathbf{S}_D \circ \mathbf{S}_D = \mathbf{S}_D$.
  - $\mathbf{S}_D$ "Orthogonal" $\iff$ $\mathbf{S}_D = \mathbf{S}_D^*$ $\iff$ $\|\mathbf{S}_D\|_{L^2 \to L^2} = 1$

    $\mathbf{S}_D$ = *Szegő Projection*
  - $\mathbf{S}_D^*$ denotes the $L^2(bD, \sigma)$-*adjoint* of $\mathbf{S}_D$:

$$\int_{bD} (\mathbf{S}_D^* f)\,\overline{g}\, d\sigma = \int_{bD} f\, \overline{\mathbf{S}_D g}\, d\sigma, \quad \text{(here } \overline{A + iB} := A - iB\text{)}$$

Szegő projection: similarities with Cauchy Integral

- Szegő projection is an integral operator over bD:

$$\mathbf{S}_D f(z) = \int\limits_{w\in bD} f(w) S_D(w,z)\, d\sigma(w), \quad z \in D \subset \mathbb{C}$$

Here $d\sigma$ is arc-length for bD.

- Szegő kernel $S_D(w,z)$ is holomorphic as a function of $z \in D$ and becomes singular as $z \to w \in bD$ (interpret above as "p.v.").

- "Cauchy-type" formula:

$$f(z) = \int\limits_{w\in bD} f(w) S_D(w,z)\, d\sigma(w), \quad z \in D \subset \mathbb{C}$$

for any f holomorphic in D and continuous on $\overline{D}$.

- Two Szegő kernel retains two main assets of Cauchy kernel

Szegő projection: dissimilarities with Cauchy Integral

Szegő kernel is not universal (it is domain-specific)

On the other hand:

- If $D = \mathbb{D}_1(0)$ then $S_D(w,z) \equiv$ Cauchy kernel. In fact

$$S_D(w,z)\, d\sigma(w) = \frac{1}{2\pi i} \frac{dw}{(w-z)} \iff D = \mathbb{D}_1(0)$$

- $S_D(w,z)$ enjoys a good transformation law under conformal maps (somewhat making up for lack of universality):
- $\varphi : D_1 \to D_2$ conformal
- $S_{D_1}(w,z)\, d\sigma_1(w) =$

$$= \sqrt{\varphi'(z)}\, S_{D_2}(\varphi(w), \varphi(z))\, \overline{\sqrt{\varphi'(w)}}\, d\sigma_2(\varphi(w))$$

Szegő projection: dissimilarities with Cauchy Integral

Recall that: $\mathbf{S}_D = \mathbf{S}_D^* \iff \|\mathbf{S}_D\|_{L^2\to L^2} = 1$, that is:

$\mathbf{S}_D$ is trivially bounded on $L^2(bD, \sigma)$. On the other hand

Regularity on $L^p(bD, \sigma)$ for $p \neq 2$ is non-trivial:

- ***L^p-Regularity problem for Szegő projection:***

under minimal assumptions on D, find $P = P(D) \in [2, +\infty]$ so that

$\mathbf{S}_D : L^p(bD, \sigma) \to L^p(bD, \sigma)$ *is bounded for all $P' < p < P$*

$$(\textit{Note:} \quad \mathbf{S}_D = \mathbf{S}_D^* \Rightarrow \frac{1}{P} + \frac{1}{P'} = 1)$$

L^p-regularity of Szegő projection: History.

Size of P is related to geometry and regularity of D e.g.,

S. Semmes (1983):

- If $D \Subset \mathbb{C}$ is Vanishing Chord-Arc (e.g., D of class C^1), then $P = +\infty$.

L. - Stein (2004):

- If $D \Subset \mathbb{C}$ is Lipschitz with constant M, then

$$P = 2\left(1 + \frac{\pi}{2 \arctan M}\right) > \ 4$$

- If $D \Subset \mathbb{C}$ is a rectifiable local graph, then $P = 4$.

- If D is "Ahlfors Regular", then $P = 2 + \epsilon$, $\epsilon = \epsilon(D)$.

(*This class includes shapes shown in Martine Ben Amar's talk*).

D. Bekolle' (1990):

- There is rectifiable $D \Subset \mathbb{C}$ with $P = 2$.

Connection with Cauchy Integral

- Let T be another projection: $L^2 \mapsto H^2$ i.e.,
  - T produces holomorphic functions from, say, C^1-smooth boundary data (*T has "holomorphic kernel"*)

    and

  - T reproduces holomorphic functions from their boundary values (*"Cauchy formula"*).

- Compare action of T on $L^2(bD, \sigma)$ with orthogonal proj. $\mathbf{S}_D$:

$$\mathbf{S}_D T = T; \qquad T\mathbf{S}_D = \mathbf{S}_D \quad \Rightarrow \quad \mathbf{S}_D T^* = \mathbf{S}_D$$

$$\mathbf{S}_D(T^* - T) = \mathbf{S}_D - T$$

$$T = \mathbf{S}_D\,[I - (T^* - T)] \quad \text{on} \quad L^2 \qquad (I = \text{Identity op.})$$

The basic idea, after Kerzman & Stein

$$T \;=\; \mathbf{S}_D\,[I - (T^* - T)] \quad \text{on} \quad L^2 \tag{0.1}$$

- Basic idea: *if $T^* - T$ is "better" than T ("cancellation of singularities") then may use* (0.1) *to draw information: from $\mathbf{S}_D$ to T and vice-versa, from T to $\mathbf{S}_D$.*
  - From $\mathbf{S}_D$ to T: another proof of $T : L^2 \to L^2$ (regularity of T).
  - From T to $\mathbf{S}_D$: Suppose T bounded in L^2: can we solve (0.1) for $\mathbf{S}_D$?

$$(T^* - T)^* \;=\; -(T^* - T)$$

$$\Longrightarrow$$

$$\mathbf{S}_D \;=\; T\,[I - (T^* - T)]^{-1} \quad \textbf{in} \quad L^2$$

??? What about L^p, $p \neq 2$???

From T to $\mathbf{S}_D$ via: $\mathbf{S}_D = T\,[I - (T^* - T)]^{-1}$

Settings where we can deal with $p \neq 2$:

- $D \subset \mathbb{C}$ and T = Cauchy integral:
  - D of class C^2: $1 < p < \infty$ (Kerzman-Stein 1978):

    via: $T^* - T$ smoothing, which implies

    $[I - (T^* - T)]^{-1} : L^p \to L^p\,, \quad 1 < p < \infty$

  - D vanishing-constant chord-arc: $1 < p < \infty$ (Semmes, 1983):

    via $T^* - T$ compact in L^p for $1 < p < \infty$, which implies

    $[I - (T^* - T)]^{-1} : L^p \to L^p\,, \quad 1 < p < \infty$

Thus $\mathbf{S}_D : L^p \to L^p \quad 1 < p < \infty$

The situation in higher dimensions

$$D \subset \mathbb{C}^n, \quad n \geq 2$$

- The notion of Hardy space $H^p(bD, \sigma)$ is taken verbatim from $n = 1$. (Now σ stands for "Induced Lebesgue measure".)
- In particular $H^2(bD, \sigma)$ is (again) a closed subsp. of $L^2(bD, \sigma)$
- So, the existence of $\mathbf{S}_D$ is guaranteed
- Example: The Szegő kernel for $D :=$ the unit ball in $\mathbb{C}^2$:
  - $\mathbb{B} = \{z = (z_1, z_2) \in \mathbb{C}^2,\ |z_1|^2 + |z_2|^2 < 1\}$
  - $S_{\mathbb{B}}(z, w) = \dfrac{(n-1)!}{2\pi^n} \dfrac{1}{[1 - (z_1\overline{w}_1 + z_1\overline{w}_2)]^2}$
- However for $D \subset \mathbb{C}^n$ with $n \geq 2$ there is **no** good transformation law under bi-holomorphic maps $\varphi : D_1 \to D_2$
- This makes the study of $\mathbf{S}_D$ for $D \subset \mathbb{C}^n$ with $n \geq 2$ **much harder** than the 1-dimensional situation.

The Szegő projection for $D \subset \mathbb{C}^n$ when $n \geq 2$

On the other hand, the "comparison with Cauchy Integral" approach to $\mathbf{S}_D$ described earlier is, in principle, applicable in any dimension. This approach has been pursued extensively over the past 40 years.

Some of the key ingredients for successful implementation of comparison with Cauchy Integral:

(i.) One has to have a meaningful notion of "Cauchy Kernel" $H(w, z)$ for multi-variable w and z.

(ii.) $H(w, z)$ must be holomorphic as a function of $z \in D \subset \mathbb{C}^2$, $n \geq 2$.

(iii.) The corresponding "Cauchy Integral" operator T must be bounded on $L^2(bD, \sigma)$.

(iv.) The difference $H(w, z) - H^*(w, z)$ must be "better" than each of $H(w, z)$ and $H^*(w, z)$ taken alone.

(v.) Need all of the above to work for $D \subset \mathbb{C}^n$ with "minimal" boundary regularity.

The Szegő projection for $D \subset \mathbb{C}^n$ when $n \geq 2$

- Steps `(i.)-(iv.)` have been carried out in the situation when $D \subset \mathbb{C}^n$ is **smooth** (class C^3 or better) and satisfies certain (natural) geometric requirements: the theory for smooth D is by now classical (Kerzman-Stein 1978)
- The threshold for the classical theory is the **class** C^2.
- In the sequel, I will relinquish step `(iv.)` (Szegő projection) and focus on steps `(i.)-(iii.)` in the setting when $D \subset \mathbb{C}^n$ has minimal regularity (**below the class** C^2).

Objectives

Extend the 1-dimensional theory for the Cauchy Integral to

$$D \subset \mathbb{C}^n, \quad n \geq 2$$

- *Find* a higher-dimensional analog of the Cauchy kernel:

$$H(w, z) = \frac{1}{2\pi i} \frac{dw}{w - z}, \quad z \in D \subset \mathbb{C}, \quad w \in bD \subset \mathbb{C}$$

  which is

  - **meaningful** when $z \in D \subset \mathbb{C}^n$, $w \in bD \subset \mathbb{C}^n$, $n \geq 1$
  - **holomorphic** as a function of $z \in D$
  - and, **valid for D with "minimal" regularity**

- Show that the operator defined via this new kernel is **bounded**: $L^p(bD, \sigma) \to L^p(bD, \sigma)$, $\quad 1 < p < \infty$

Effects of the requirement that the kernel be holomorphic

Domain must be "pseudoconvex":

- We seek a kernel $H(w, z)$ that is holomorphic in $z \in D$ and **singular** at $z = w$ for any $w \in bD$.
- This means that D must be the **maximal domain of holomorphicity** of the function $z \mapsto H(w, z)$
- .... which is equivalent to the geometric requirement that D be "pseudoconvex" (**Levi Problem**).
- Any $D \Subset \mathbb{C}$ (i.e. $n = 1$) is pseudoconvex.
- Hartogs (1906): existence of non-pseudoconvex domains in $\mathbb{C}^2$.

Settings where things are known to work ($D \subset \mathbb{C}^n$, $n \geq 2$)

Classical setting:

Henkin; Ramirez; Kerzman-Stein (1978):

$D \in C^k$, $k \geq 3$ and strongly Levi-pseudoconvex

- Kernel: via an algebraic construct (*Cauchy-Fantappié theory*)
- Proof of $L^p(bD, \sigma) \to L^p(bD, \sigma)$-regularity:

  by way of "*osculation by model domain*":

$$\{z \mid \operatorname{Im} z_n > |z_1|^2 + \cdots + |z_{n-1}|^2\}$$

the Siegel Upper Half Space

Settings where things are known to work:

New technology:

L. - Stein (2016)

$D \in C^k$, $\mathbf{k = 2}$ and strongly Levi-pseudoconvex

- Kernel(s): a family of Cauchy-Fantappiè terms
- Threshold for original method ("osculation by model domain"): class C^2.
- Proof of $L^p \to L^p$-regularity: $T(1)$ **theorem**.

Setting of current interest:

L. - Stein (2014):

$D \Subset \mathbb{C}^n$, $D \in C^{1,1}$ and strongly $\mathbb{C}$-linearly convex, i.e.

- D has a defining function of class $C^{1,1}$, i.e.
  - $D = \{\rho < 0\}$ and $bD = \{w \mid \rho(w) = 0\}$, with
  - $\rho : \mathbb{C}^n \to \mathbb{R}$, $\rho \in C^1(\mathbb{C}^n)$
  - $\nabla\rho(w) \neq 0$, $w \in bD$, and $\nabla\rho \in \text{Lip}(\mathbb{C}^n)$

and

- $d^E(z, w + T_w^{\mathbb{C}}) \geq c|w - z|^2$ if $z \in \overline{D}$ and $w \in bD$

Example: Siegel upper half space: $D = \{z \in \mathbb{C}^2 \mid \text{Im}\, z_2 > |z_1|^2\}$ is strongly $\mathbb{C}$-linearly convex, but not strongly convex (because $\ell = \{(0 + i0, x_2 + i0) \mid x_2 \in \mathbb{R}\} \subset bD$)

Kernel

Cauchy-Leray kernel:

$$H(w,z) = \frac{1}{(2\pi i)^n}\frac{\partial\rho(w)\wedge(\overline{\partial}\partial\rho(w))^{n-1}}{\langle\partial\rho(w), w-z\rangle^n}, \quad w\in bD, \quad z\in D$$

- $\rho : \mathbb{C}^n \to \mathbb{R}$ is (*any*) defining function for D.
- $\partial\rho(w) = \sum_{j=1}^{n}\frac{\partial\rho}{\partial\zeta_j}(w)dw_j\,; \quad \overline{\partial}\partial\rho = \sum_{j,k=1}^{n}\frac{\partial^2\rho}{\partial\zeta_j\partial\overline{\zeta}_k}(w)dw_j\wedge d\overline{w}_k$
- $\langle\zeta,\eta\rangle := \sum\zeta_j\,\eta_j, \quad \zeta,\eta\in\mathbb{C}^n$
- first introduced by J. Leray (1950s) in the setting of C^2-smooth, strongly convex domains D.
  N. Stanton (1979): connection with Monge-Ampére eqn.
  Revisited by T. Hansson (1999) in the context of a family of C^∞-smooth, weakly convex ellipsoids.

Cauchy-Leray kernel: good news

Suppose for the moment that D is of class C^2:

$$H(w,z) = \frac{1}{(2\pi i)^n} \frac{\partial\rho(w) \wedge (\bar{\partial}\partial\rho(w))^{n-1}}{\langle \partial\rho(w), w - z\rangle^n} \tag{0.2}$$

- If $n = 1$ then Cauchy-Leray is the one-dim Cauchy kernel:

$$\frac{\partial\rho(w)}{\langle \partial\rho(w), w - z\rangle} = \frac{\rho'(w)dw}{\rho'(w)(w-z)} = \frac{dw}{w-z}$$

- For any $n \geq 1$, $H(w,z)$ is holomorphic wrt $z \in D$ because denominator does not vanish **by strong $\mathbb{C}$-linear convexity**:

$$|\langle \partial\rho(w), w - z\rangle| \approx d^E(z, w + T_w^{\mathbb{C}}) \;\geq\; c|w-z|^2 > 0$$

Cauchy-Leray kernel: caveats

Suppose now D that is only of class $C^{1,1}$:

$$H(w,z) = \frac{1}{(2\pi i)^n} \frac{\partial\rho(w) \wedge (\bar\partial\partial\rho(w))^{n-1}}{\langle \partial\rho(w), w-z\rangle^n} \qquad (0.3)$$

- By Rademacher Theorem:

$$\rho \in C^{1,1}(\mathbb{C}^n) \quad \Rightarrow \quad \nabla^2\rho \in L^\infty(\mathbb{C}^n)$$

- in particular, $\nabla^2\rho(w)$ is defined only a.e. $w \in \mathbb{C}^n$
- but bD has measure 0 in $\mathbb{C}^n$
- so, $\nabla^2\rho$ may be undefined on bD. In particular

$$\bar\partial\partial\rho, \text{ and thus } H(w,z), \text{ may be undefined}$$

An example

For

$$D := \{x + iy \,|\, x < 0\} \subset \mathbb{C}$$

and

$$F : \mathbb{C} \to \mathbb{R} \quad \text{given by} \quad F(x + iy) := |x|$$

we have

- D is a smooth domain in $\mathbb{C}$;
- $F \in Lip(\mathbb{C})$ and so $\nabla F \in L^\infty(\mathbb{C})$ by Rademacher

However, ∇F and hence dF are undefined on $bD = \{x + iy \,|\, x = 0\}$

On the other hand, *$j^* dF$ is well-defined on bD. In fact, $j^*(dF) = d(j^* F) \equiv 0$*

Here $\quad j : bD \hookrightarrow \mathbb{C} \quad$ is the inclusion map

Cauchy-Leray kernel: the role of tangential components

Proposition (L. – Stein)

Suppose $F \in C^{1,1}(\mathbb{C}^n)$ (with $n \geq 2$) and $D \subset \mathbb{C}^n$ is of class $C^{1,1}$. Then there exists a (unique) 2-form on bD, which we write as $j^*(\bar\partial\partial F)$, whose coefficients are in $L^\infty(bD)$ and satisfies

$$\int_{bD} j^*(\bar\partial\partial F) \wedge \psi = \int_{bD} j^*(\partial F) \wedge d(\psi)$$

for all $(2n-3)$-forms ψ on bD that are of class C^1.

Outcomes for $F := \rho$ (defining function of D):

- Cauchy-Leray kernel is well-defined (meaningful):
- $H(w, z)$ reproduces holomorphic functions (*"Cauchy formula"*) and is "canonical" (*independent of choice of ρ).*

Cauchy-Leray integral: main result

Theorem (L. – Stein)

Suppose $D \subset \mathbb{C}^n$ is strongly $\mathbb{C}$-linearly convex and of class $C^{1,1}$. Then, the Cauchy-Leray integral:

$$f \mapsto \mathcal{C}(f)(z) := \frac{1}{(2\pi i)^n} \int\limits_{w \in bD} f(w)\, j^* \left(\frac{\partial \rho(w) \wedge (\overline{\partial} \partial \rho(w))^{n-1}}{\langle \partial \rho(w), w - z \rangle^n} \right)$$

initially defined for functions in $C^1(bD)$, extends to a bounded linear operator:

$$L^p(bD, \lambda) \to L^p(bD, \lambda), \quad 1 < p < \infty$$

where λ is the Leray-Levi measure

$$d\lambda(w) := \frac{1}{(2\pi i)^n} j^* (\partial \rho(w) \wedge (\overline{\partial} \partial \rho(w))^{n-1})$$

Cauchy-Leray integral: L^p-regularity

Proof of $L^p \to L^p$-regularity: goes by way of

- $T(1)$-Theorem in the special case:

$$T(1) = 0; \quad T^*(1) = 0$$

- for a space of homogeneous type $\{X, \mathtt{d}, \lambda\}$ informed by the geometry and regularity of the ambient domain D:
    - $X := bD$
    - $\mathtt{d}(w, z) := |\langle \partial\rho(w), w - z\rangle|^{1/2}$
    - $d\lambda := j^*(\partial\rho \wedge (\overline{\partial}\partial\rho)^{n-1})$ (The Leray-Levi measure)

$T(1)$ theorem for spaces of homogeneous type

Theorem (Coifman)

- *Let $(X, \mathtt{d}, \lambda)$ be a space of homogeneous type.*
- *Let $K(w,z)$ be a standard kernel, i.e.:*
  - *Size estimates for any $w, z \in X$:*
    $$|K(w,z)| \lesssim \big(\lambda(\mathcal{B}_r(w))\big)^{-1}, \quad r := \mathtt{d}(w,z)$$
  - *Regularity estimates for any $w, z, \zeta \in X$:*
    $$|K(w,\zeta) - K(z,\zeta)| + |K(\zeta,w) - K(\zeta,z)| \lesssim$$
    $$\lesssim \frac{\mathtt{d}(w,z)}{\mathtt{d}(w,\zeta)} \left(\lambda(\mathcal{B}_r(w))\right)^{-1}, \quad r := \mathtt{d}(w,z)$$
- *Suppose that T with kernel $K(w,z)$ is weakly bounded: $|(T\varphi,\psi)| \lesssim \lambda(\mathcal{B}_r(w))$ for any φ and ψ normalized bump functions supported in $\mathcal{B}_r(w)$ (any $w \in X$, $r > 0$)*
- *Cancellation conditions: Suppose $T(1),\ T^*(1) \in BMO(X,\lambda)$.*

Then, T is bounded: $L^2(X,\lambda) \to L^2(X,\lambda)$.

$T(1)$-theorem for the Cauchy-Leray integral

Theorem (L. – Stein)

Suppose $D \Subset \mathbb{C}^n$ is strongly $\mathbb{C}$-linearly convex and of class $C^{1,1}$, and let:

$$K(w,z) := \langle \partial\rho(w), w - z \rangle^{-n}.$$

Then, for our choice of space of homogeneous type $(bD, \mathtt{d}, \lambda)$ we have that

- *$K(w,z)$ is a standard kernel (size and regularity estimates hold).*
- *The Cauchy-Leray integral, $\mathcal{C}$, is weakly bounded.*
- *Also, $\mathcal{C}(1) = 1$ (Cauchy Formula), and in fact we may reduce to the simpler situation*

$$T(1) = 0; \quad T^*(1) = 0$$

for a suitable modification of $\mathcal{C}$ (which we denote T).

A few words about the proof

A key ingredient in the proof of the weak-boundedness property and of the cancellation conditions:

$$T(1) = 0; \quad T^*(1) = 0$$

are two basic identities that in effect express the Cauchy-Leray kernel and its dual as *appropriate derivatives*. Namely:

$$H(w,z) = d_w\omega(w,z) + \mathfrak{r}(w,z), \quad \text{and}$$

$$\overline{H(z,w)} = d_w\widetilde{\omega}(w,z) + \widetilde{\mathfrak{r}}(w,z), \quad \text{where}$$

- the coefficients of ω (resp. $\widetilde{\omega}$) are absolutely integrable and have better homogeneity than $H(z,w)$ (resp. $\overline{H(w,z)}$), i.e.

  they have $\quad \langle\partial\rho(w), w-z\rangle^{-n+1} \quad$ vs. $\quad \langle\partial\rho(w), w-z\rangle^{-n}$
- the remainders $\mathfrak{r}$ and $\widetilde{\mathfrak{r}}$ are harmless: $C(bD) \mapsto C(\overline{D})$
- From these it follows that $\mathcal{C}$ is weakly bounded

Comparison with proof for 1-dimensional setting

Remarkably the "basic identities" are meaningful only for $n > 1$, because a one-dimensional analogue would necessarily involve a logarithmic term, invalidating their use: i.e., for $n = 1$ one has:

$$H(w,z) = \frac{dw}{w-z} = d_w\omega(w,z) + \mathfrak{r}(w,z)$$

with

- $\omega(w,z) := \log(w-z)$
- $\mathfrak{r}(w,z) = 0$

but $\log(w-z)$ does not have the appropriate homogeneity that would automatically ensure the weak boundedness property.

Hypotheses are optimal

- L.-Stein (2018): Strong $\mathbb{C}$-lin. convexity is optimal:

$$D := \{x_1^2 + y_1^4 + x_2^2 + (y_2 - 1)^2 < 1\}$$

  - D is smooth and strictly (but not strongly) $\mathbb{C}$-linearly convex
  - $\mathcal{C}$ unbounded in L^p for all $1 < p < \infty$

- L.-Stein (2018): $C^{1,1}$ category also optimal:

$$D_\alpha := \{|x_1|^{1+\alpha} + y_1^2 + x_2^2 + (y_2 - 1)^2 < 1\}, \; 0 < \alpha < 1.$$

  - D_α strongly convex and of class $C^{1,\alpha}$ (but not $C^{1,1}$)
  - $\mathcal{C}$ unbounded in L^p for all $1 < p < \infty$.

Thank You!