# Growth of thin fingers in Laplacian and Poisson fields

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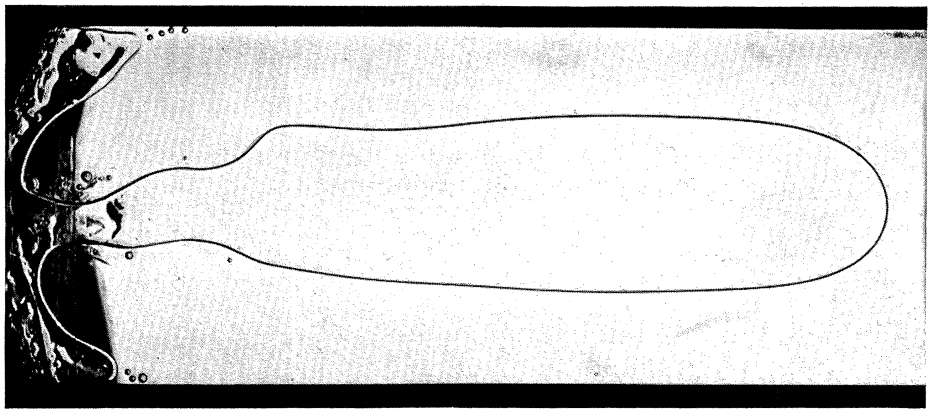
Part I: The Schwarz-Christoffel map and the Laplacian growth of needles and fingers (acknowledgement: Callum Millar)

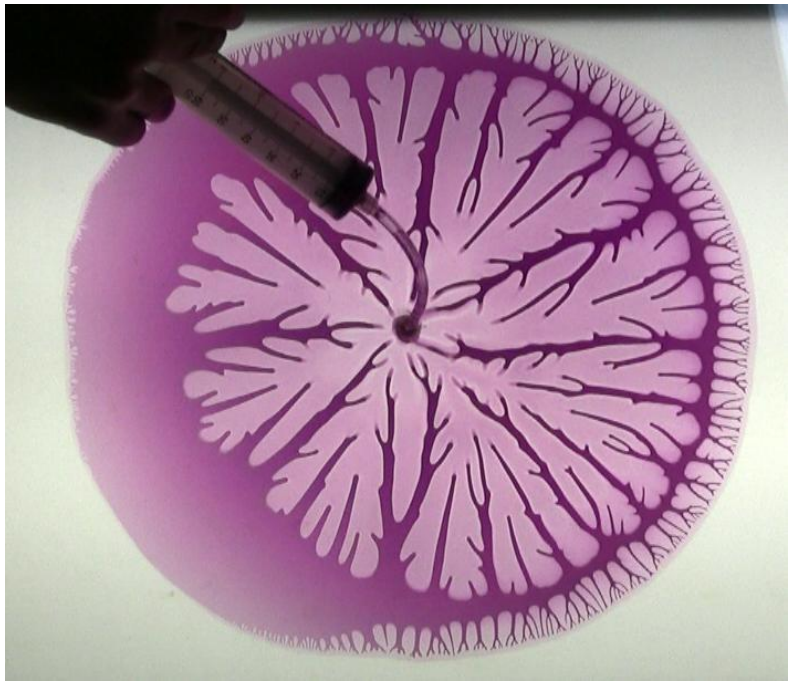
Part II: Finger growth and selection in a Poisson field

# Laplacian growth examples

## Hele-Shaw flow



Saffman & Taylor, *The penetration of a fluid into a medium or Hele-Shaw cell containing more viscous fluid*, Proc. Roy. Soc. 1958



Hele-Shaw Cell (from n-e-r-v-o-u-s.com)

# Laplacian growth examples (cont.)

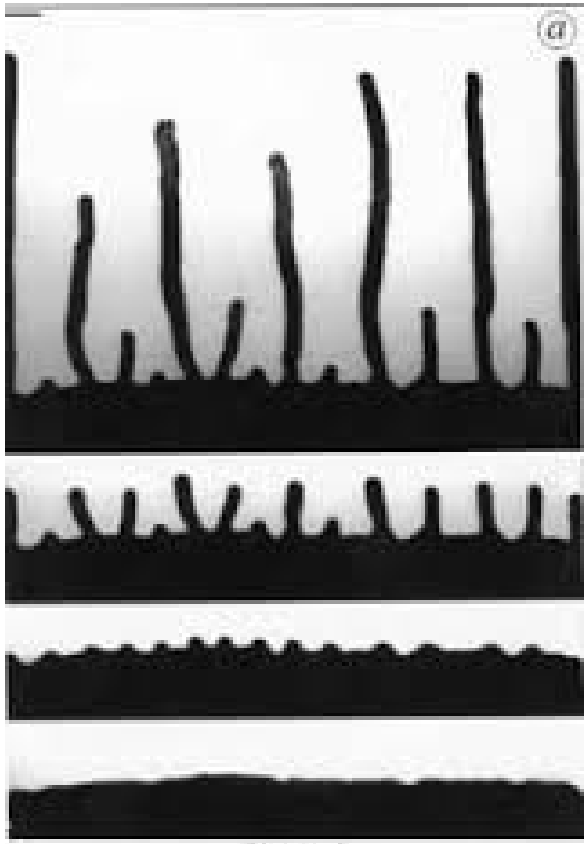


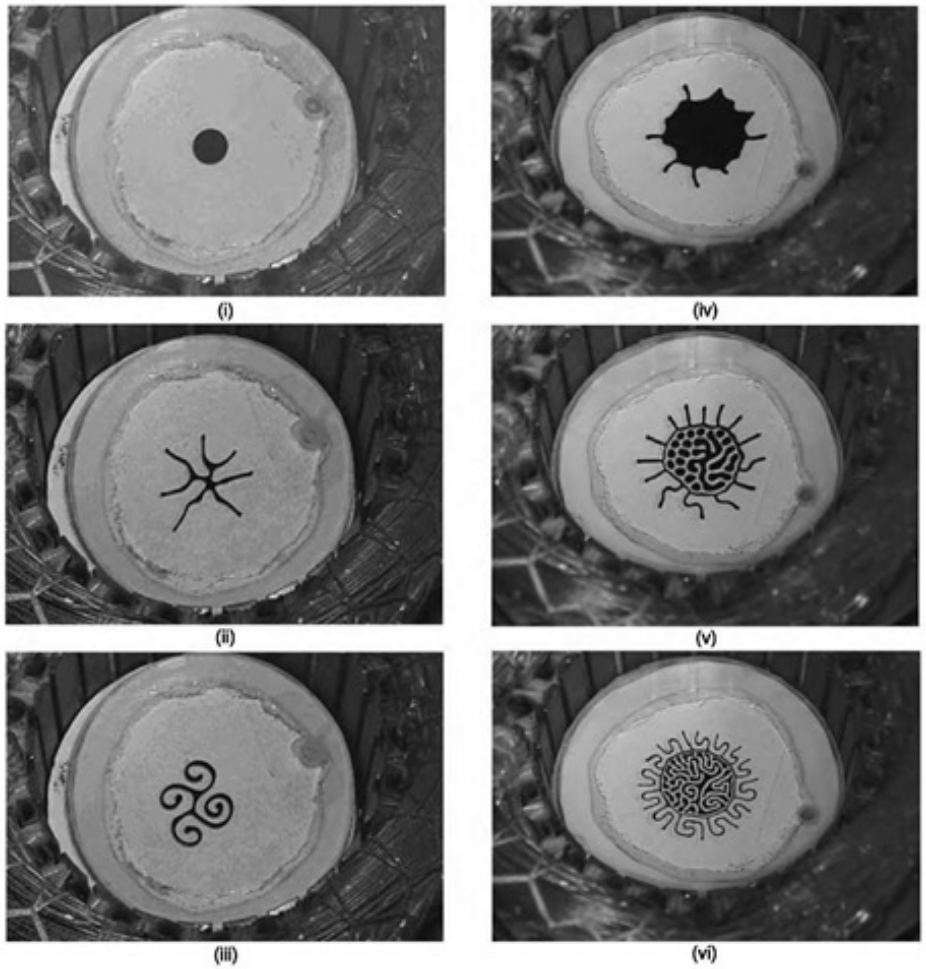

Figure 3

beach rills

Zik & Moses, *Fingering instability in combustion: An extended view*. Phys. Rev E. 1999

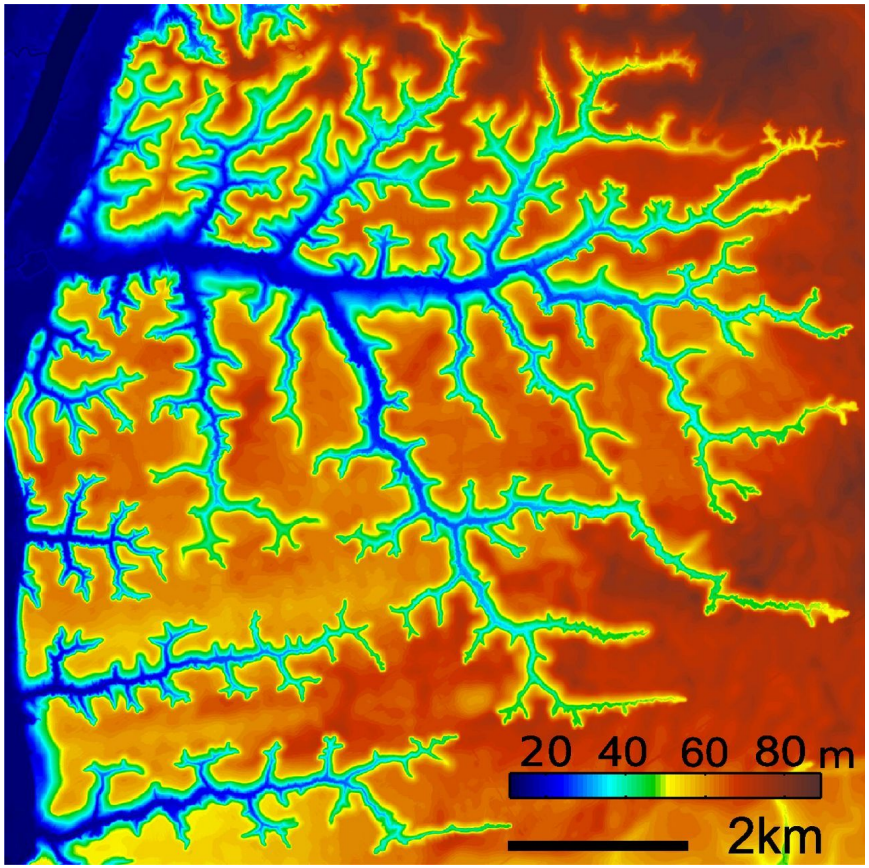
# ‘Laplacian’ growth examples (cont.)

(see e.g. Rhodes *et al.*, *J. Magnetism and Mag. Materials* 2005)



# Groundwater-fed streams

Groundwater sculpted landscape, Bristol, Florida ('Rothman group', EAPS, MIT)



# Valley network dynamics

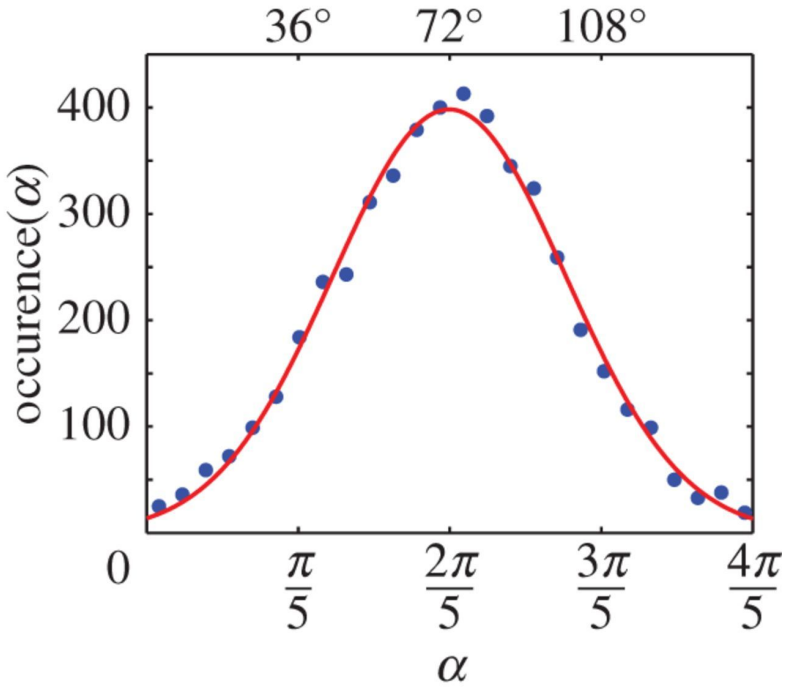
A flow in porous media problem

- $h(x, y)$ elevation of the water table above impermeable layer
- groundwater flux (Darcy's law) $q = -kh\nabla h$ ($k \equiv$ homogeneous sand conductivity).
- $\nabla . q = P$, where $P$ is the constant precipitation rate
- $\nabla^2 h^2 = -\frac{2P}{k}$

Drainage network growth: 2D interface problem in a Poisson or harmonic field with $\phi(x, y) = 0$ on the interface.

e.g. Petroff *et al. Bifurcation dynamics of natural drainage networks*, Phil. Trans. 2013

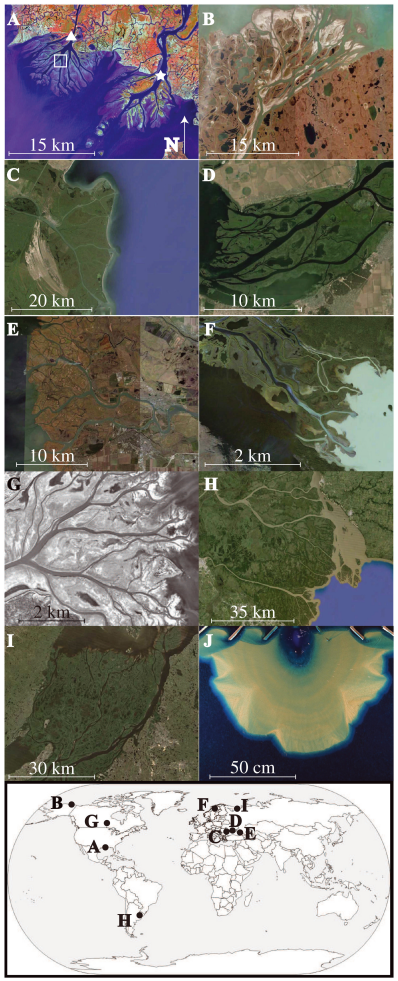
# Bifurcation of streams




The measured distribution of bifurcations ($N = 4966$). The measured probability density function (blue points) is well approximated as a normal distribution (red curve).
[From Petroff *et al.* 2013].

# Branching in river deltas

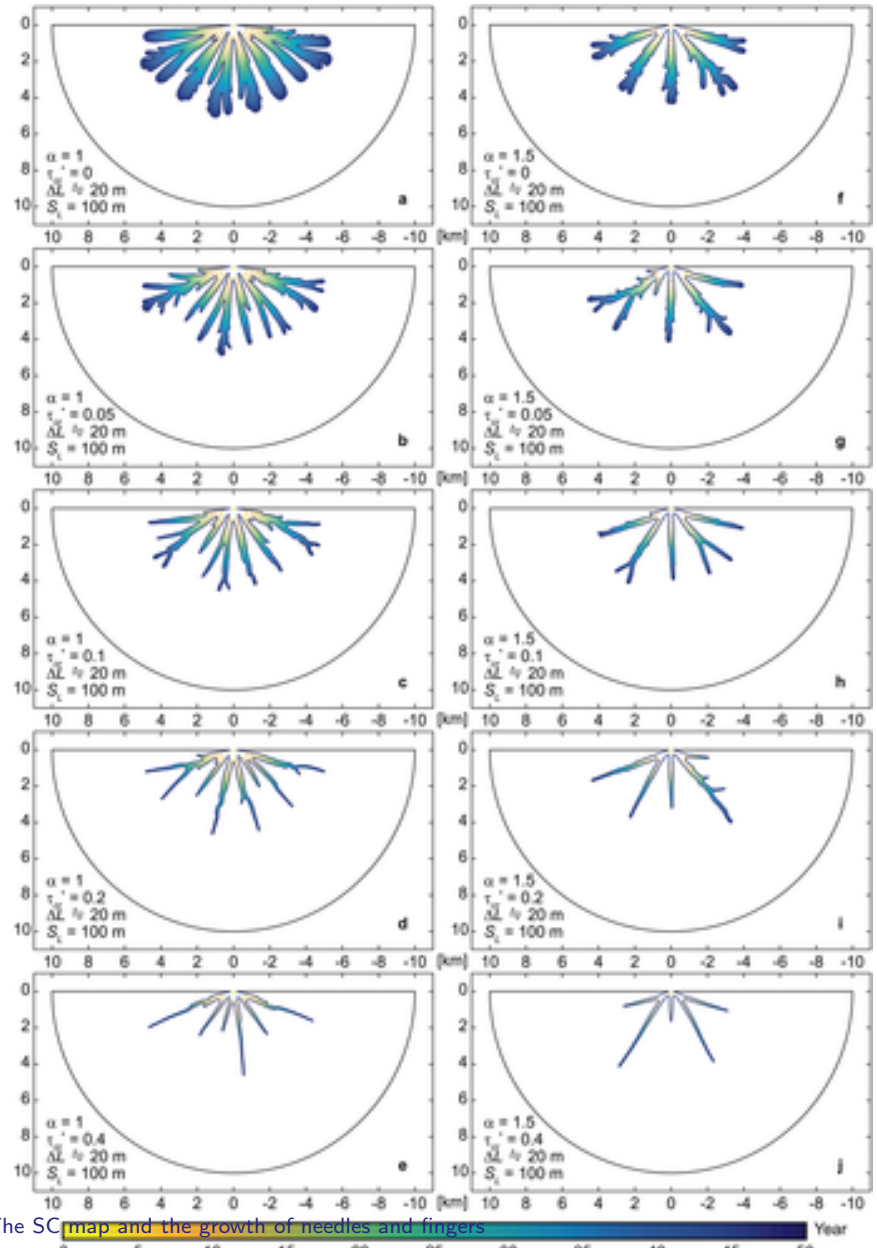
Coffey and Shaw: *Congruent bifurcation angles in river delta and tributary channel networks*, Geophys. Res. Lett. 2017




Distributary networks: shallow, friction dominated limit $\Longrightarrow$ Laplacian dynamics.

Typical branching angle:
Field (10 sites): $70.4^\circ \pm 2.6^\circ$
Experiment: $68.3^\circ \pm 8.7^\circ$

# Numerical computation of distributary deltas



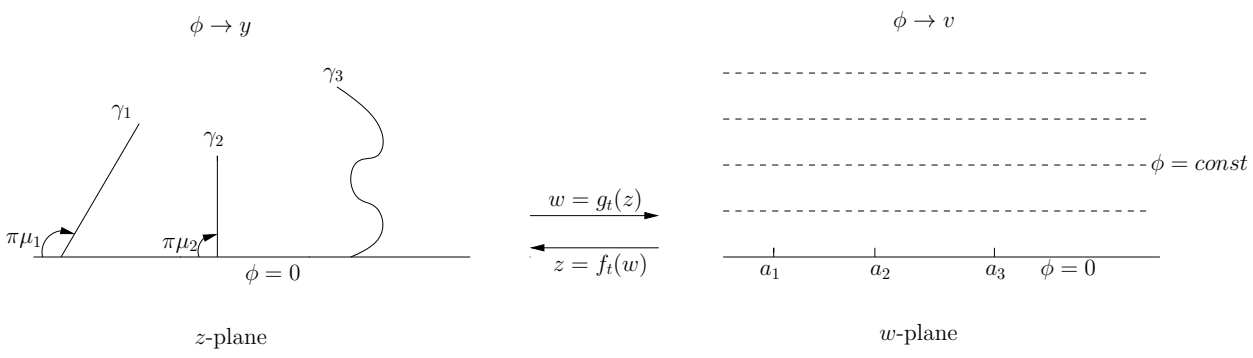

Ke, Shaw, Mahon & Cathcart: *Distributary channel networks as moving boundaries: causes and morphodynamic effects*, JGR 2019

Finite element method solving Laplace's equation; sediment transport coupled to hydrodynamics determines boundary condition. Simulates Wax Lake Delta, Louisiana.

# Martian valleys?

Relatively narrow branching angle $\implies$ formed primarily by overland flow erosion.
e.g. Seybold, Kite & Kirchner: *Branching geometry of valley networks on Mars and Earth and its implications for early Martian climate.* Sci. Advances (2018)

# Laplacian growth of fingers and needles in a half-plane




$\Delta\phi = 0$ in the slit half-plane, with $\phi = 0$ on fingers/needles and $\mathrm{Im}(z) = 0$, $\phi \to y$ as $z \to \infty$.

$g_t$ maps from the slit $z$-plane to the upper half of the $w = u + iv$plane, normalised by $g_t(z) = z + K(t)/z + \mathcal{O}\left(z^{-2}\right)$ as $z \to \infty$.

$f_t(w) = g_t^{-1}$

# Growth mechanism and speed

At the tip $z = \gamma$ of the finger or needle, the gradient of $\phi$ has the usual inverse square root singularity:

$$|\nabla\phi(\gamma + r, t)| \sim \frac{C(t)}{\sqrt{r}},$$

where $r$ is a local radius about $z = \gamma$.

Integrating the flux of $\phi \implies$ tip speed $v$ is given by $v = |C(t)| = |f_t''(a(t))|^{-1/2}$ e.g. Derrida & Hakim (1992), Carleson & Makarov (2002), Gubiec & Szymczak (2008).

More generally $v = |C(t)|^\eta = |f''(a(t))|^{-\eta/2}$ where $\eta$ is constant. Value of $\eta$ influences dynamics e.g. screening.

# Loewner's equation

For a single slit (curved finger), $g_t$ satisfies Loewner's equation:

$$\dot{g}_t(z) = \frac{d(t)}{g_t(z) - a(t)},$$

where $a(t)$ is the (real) driving function, and $d(t)$ is related to the growth speed of the fingers. Usually, $g_t|_{t=0}(z) = g_0(z) = z$.

Note $\phi = \mathrm{Im}(w) = \mathrm{Im}(g_t)$.

$N$ fingers:

$$\dot{g}_t(z) = \sum_{k=1}^{N} \frac{d_k(t)}{g_t(z) - a_k(t)}.$$

# Loewner's equation (cont.)

## Growth speed

Recall $v = |f_t''(a(t))|^{-\eta/2}$. Can show

$$|d_k(t)| = |f_t''(a_k(t))|^{-1-\eta/2}.$$

If $\eta = -2$ then $d_k = const$...but this is a special case.

For the 'natural choice' of $\eta = 1$ the Loewner growth rates $d_k(t)$ vary in time $\implies$ screening (see later).

# Loewner's equation (cont.)

## Example

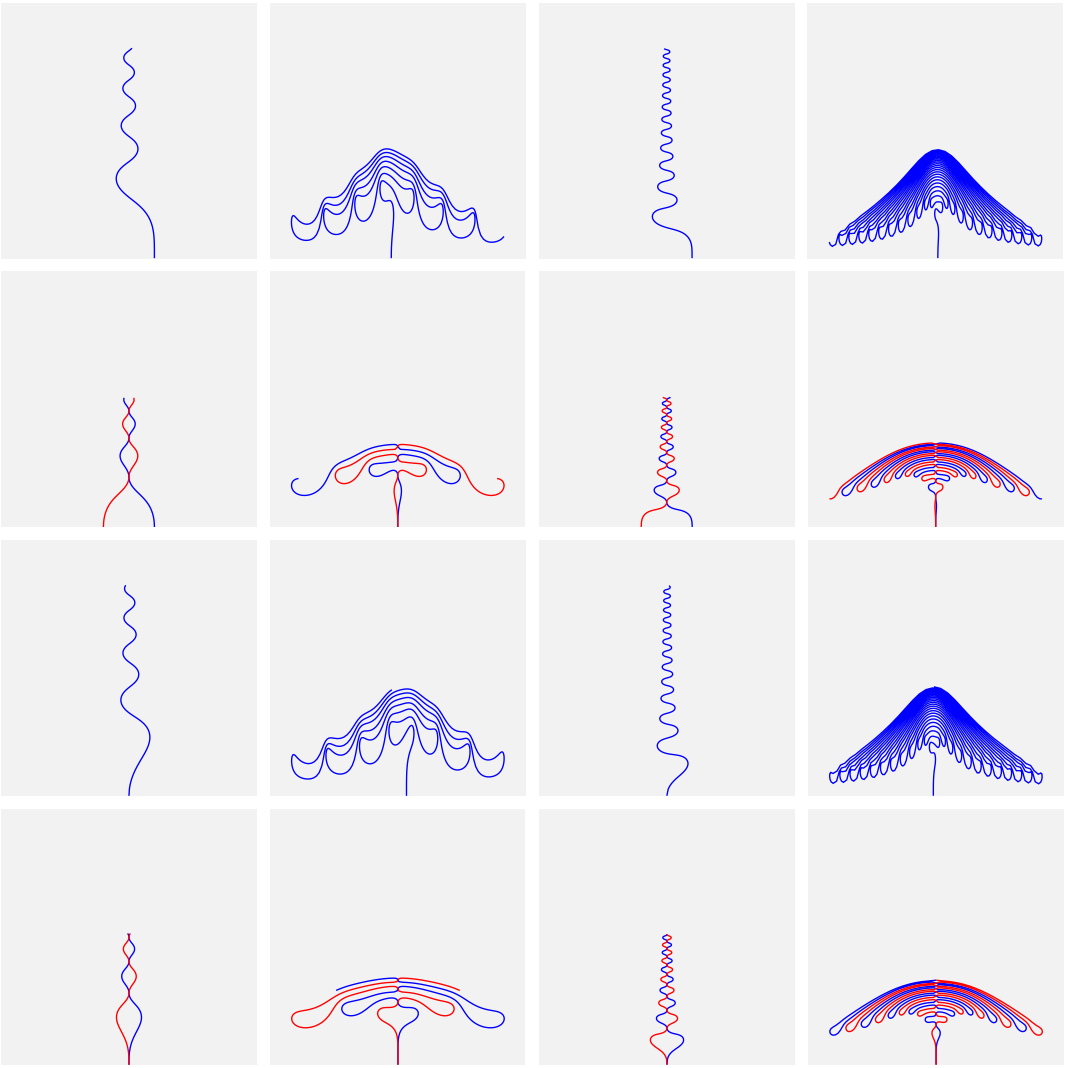
A single finger with $a(t) = 0$:

$$\dot{g}_t(z) = \frac{d(t)}{g_t(z)},$$

Let (wlog) $d(t) = const = 2$; solving Loewner gives $g_t^2 = 4t + C$. For $g_0(z) = z$, $g_t^2 = 4t + z^2$. The curve $z = \gamma(t)$ satisfies $g_t(\gamma) = 0$ and so $\boxed{\gamma(t) = 2i\sqrt{t}}$.

Other choices of $a(t)$ give more exotic curves e.g. $a(t)$ is a Brownian motion (Schramm-Loewner evolution).

Examples traces for oscillatory driving functions (Dolica Akello-Egwel):



# Loewner's equation (cont.)

In Laplacian 'geodesic' growth, how is $a(t)$ chosen? We demand the $a_j(t)$ satisfy

$$\frac{da_j}{dt} = \sum_{k \neq j} \frac{d_k(t)}{a_j(t) - a_k(t)}.$$

(see e.g. Carleson & Makarov, 2002)

Equivalent to 'principle of local symmetry' (PLS) and maximising flux into finger tips (see Devauchelle et al., 2017).

# Loewner dynamics (cont.)

## Example 2: two symmetric fingers

Growth of two symmetric fingers (see e.g. Gubiec & Szymczak 2008)
Let $d_1 = d_2 = 1$ (wlog) and $a_1(t) = -a_2(t) = a(t)$ which gives

$$\frac{dg_t}{dt} = \frac{2g_t}{g_t^2 - a(t)^2},$$

and for geodesic growth $da/dt = 1/2a$.
Solution: $5g_t^{1/2}a^2 - g_t^{5/2} = g_0^{5/2} - 5g_0^{1/2}a_0^2$, where $a = \sqrt{t + a_0^2}$.
Since $g_0 = z$, the tips satisfy

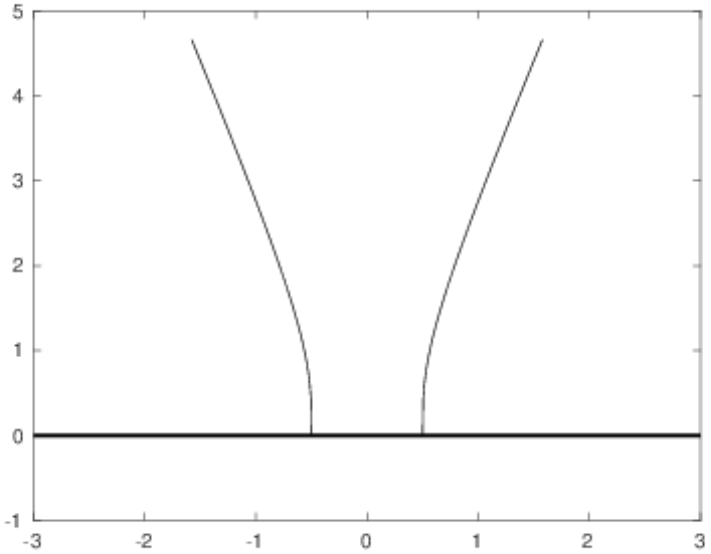
$$\gamma_\pm(\gamma_\pm^2 - 5a_0^2)^2 = \pm 16(a_0^2 + t)^{5/2}.$$

# Loewner dynamics (cont.)

## Example 2: two symmetric fingers (cont.)

$$\gamma_\pm(\gamma_\pm^2 - 5a_0^2)^2 = \pm 16(a_0^2 + t)^{5/2}$$

As $t \to \infty$, fingers tend to straight lines with opening angle $\pi/5$.
E.g. $a_0 = 0.5$



# Needle growth

Needles grow in straight lines with speed $v = |f_t''(a(t))|^{-1/2}$ where $f_t$ is the SC map from the $w$- to $z$-plane e.g. for $N$ needles

$$z = f_t(w) = A \int_{w_0}^{w} \prod_{i=1}^{N} \frac{(s - a_i)}{(s - a_{iL})^{\mu_i}(s - a_{iR})^{1-\mu_i}} ds + f(w_0),$$

where $\pi\mu_i$ is the angle the $i$th needle makes with the negative real axis. The parameters $a_{iL}$ and $a_{iR}$ lie on the real $w$-axis to the left and right of $a_i$ respectively s.t. ordering
$a_{iL} < a_i < a_{iR} < a_{(i+1)L} < a_{i+1} < a_{(i+1)R}$, $i = 1, \cdots, N-1$.

Far-field condition requires $A(t) = 1$.

Use Driscoll's SC Toolbox to compute $f_t(w)$.

# Needle growth (cont.)

## Alternative analytical approach

J. Tsai (2009): consider needles to be straight fingers evolving according to Loewner's equation. What choice of driving function $a_k(t)$ do straight paths result?

Leads to coupled ODEs for SC parameters e.g. two fingers there are 6 coupled ODEs for the SC parameters. For $j = 1, 2$:

$$\dot{a}_j = (-1)^j \frac{d_1 + d_2}{a_2 - a_1} + \frac{\mu_1 d_j}{a_{1L} - a_j} + \frac{(1-\mu_1) d_j}{a_{1R} - a_j} + \frac{\mu_2 d_j}{a_{2L} - a_j} + \frac{(1-\mu_2) d_j}{a_{2R} - a_j},$$

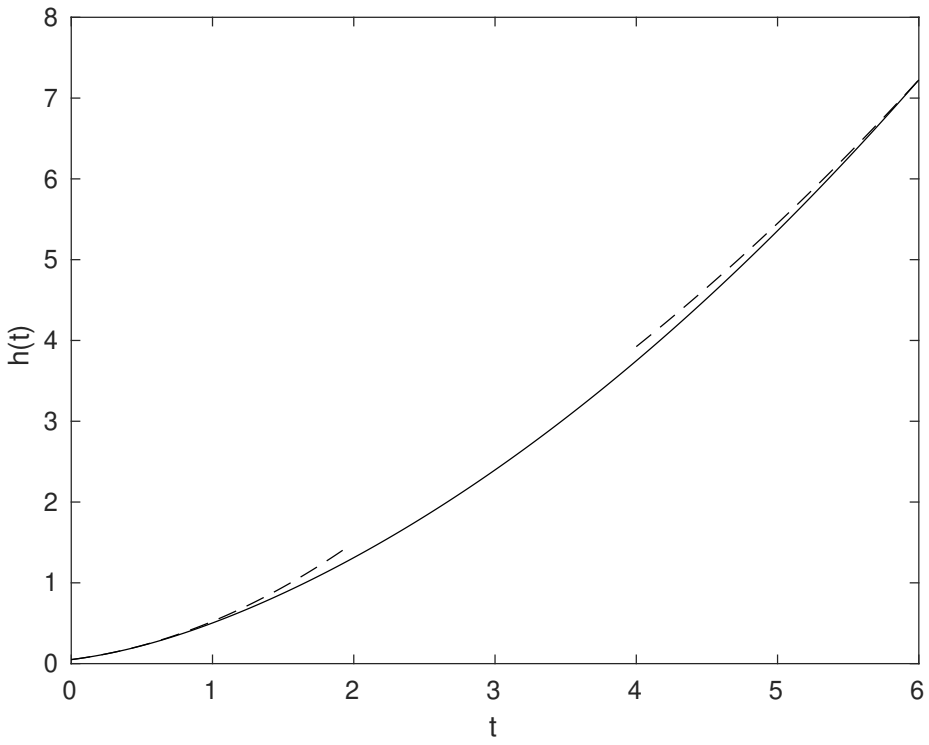
$$\dot{a}_{jL} = \sum_{i=1}^{2} \frac{d_i}{a_{jL} - a_i},$$

$$\dot{a}_{jR} = \sum_{i=1}^{2} \frac{d_i}{a_{jR} - a_i}.$$

# Examples of two needle growth $\eta = 1$

## Equal length needles

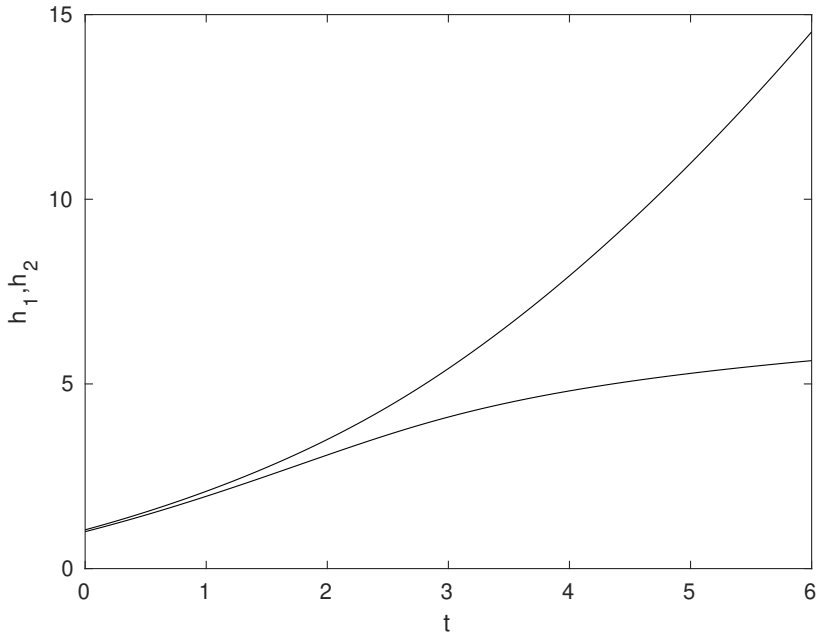
small time: $h \sim t^2/4$
large time: $h \sim t^2/4\sqrt{2}$



# Examples of two needle growth

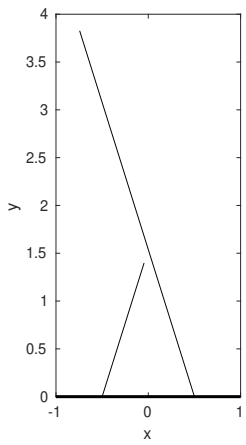
## Two initially nearly-equal length needles

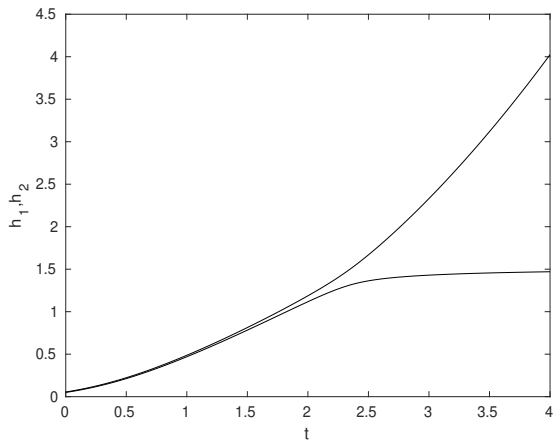
Initial lengths 1.0 and 1.05:




Screening

# Two inclined needles

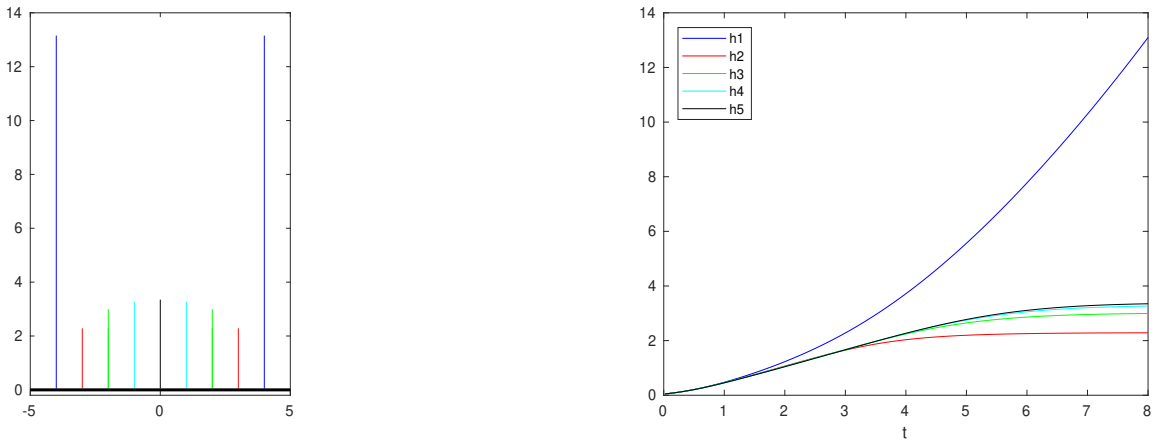



(a) Snapshot of a pair of needles at $t = 4$ with initial lengths 0.05 (left) and 0.055 (right). The left needle has $\mu_1\pi = 3\pi/5$ and the right needle $\mu_2\pi = 2\pi/5$. (b) Needle lengths as functions of time.

# $N$ needles

## Nine evenly spaced needles in a half plane

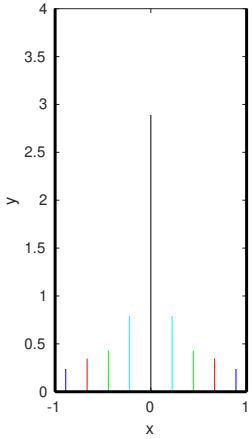


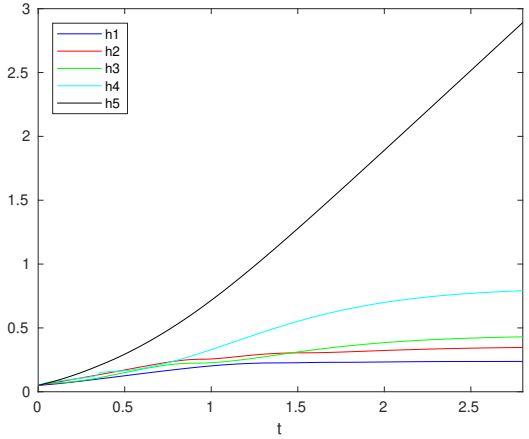

(a) Nine evenly spaced needles growing vertically in a half-plane, each having initial length 0.05. (b) Needle lengths as functions of time.

# $N$ needles

## Nine evenly spaced needles in a semi-infinite strip with zero flux sidewalls

Note: need to map from $w$-plane to empty semi-infinite strip in $\zeta$-plane c.f. Gubiec & Szymczak (2008).

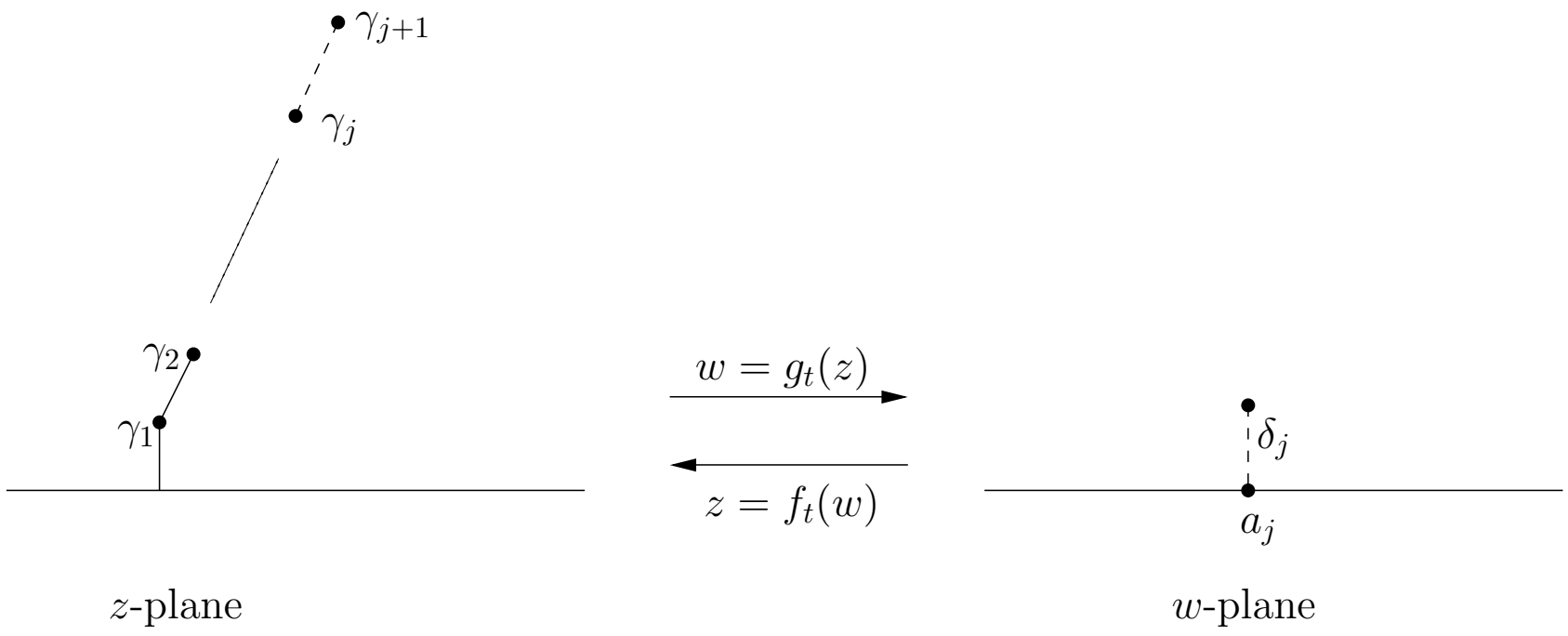



Nine evenly spaced needles growing vertically in a semi-infinite strip with zero flux conditions on the vertical sidewalls. The initial lengths of the needles is 0.05. (b) Lengths of the needles as function of time.

# Using the SC Toolbox to compute curved fingers

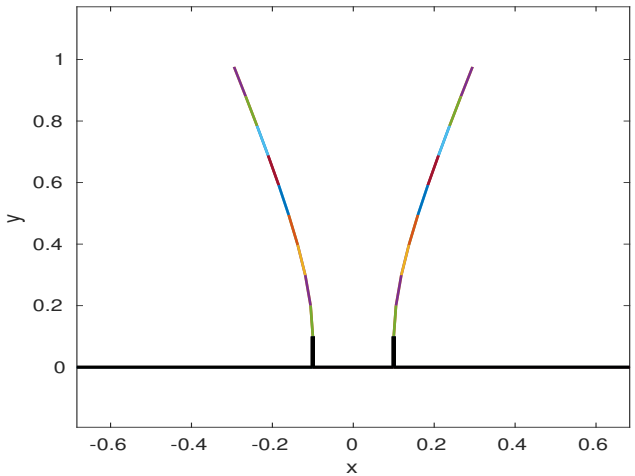
Idea: discretize curved fingers into straightline segments and then map to upper half of $w$-plane using SC Toolbox.

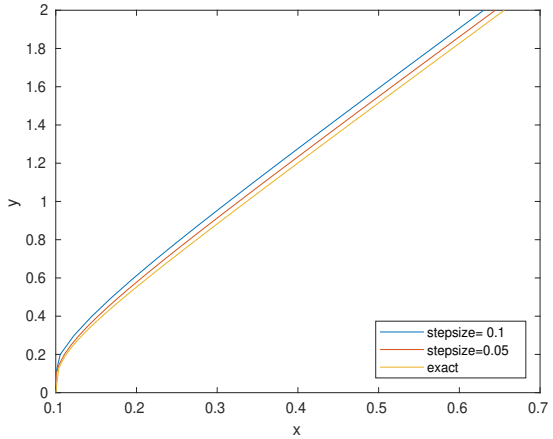



Map discretized finger to the upper half of the $w$-plane: $\gamma_j$ maps to $a_j$ on the $\mathrm{Re}(w)$-axis. The point $w = a_j + i\delta_j$ (i.e. growth along a 'streamline', PLS) maps to $\gamma_{j+1}$.

# Using the SC Toolbox to compute curved fingers (cont.)

## Test problem: two symmetric fingers

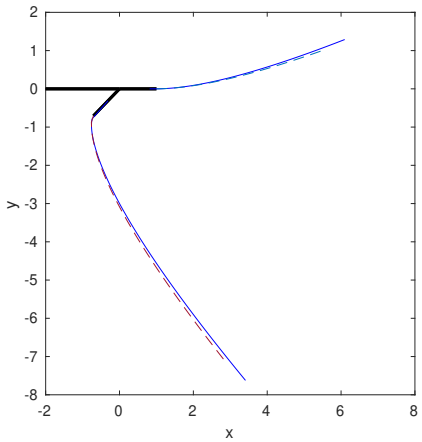


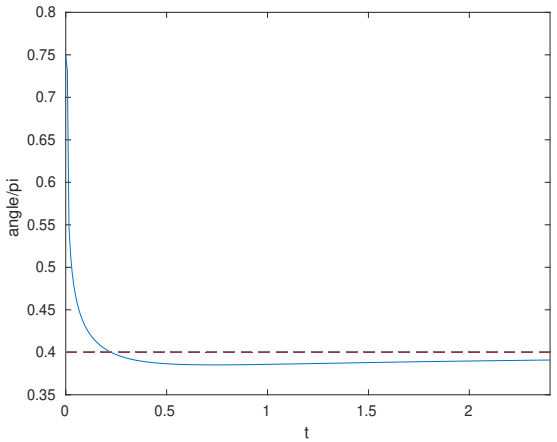




(a) Nine iterations are shown with each step of length 0.1. The dark line shows the initial starting configuration with fingers 0.2 apart. (b) Shape of the computed right finger for different step sizes $\delta l = 0.05, 0.1$ compared with the exact two finger solution.

# Using the SC Toolbox to compute bifurcating fingers

Problem: Asymmetric bifurcation from semi-infinite 'primary' needle: how do 'secondary' fingers behave e.g. do they evolve toward a bifurcation angle of $2\pi/5$? (see Devauchelle et al. 2017 for symmetric bifurcation case).
Choose $\eta = -2$ so $d_1 = d_2$.

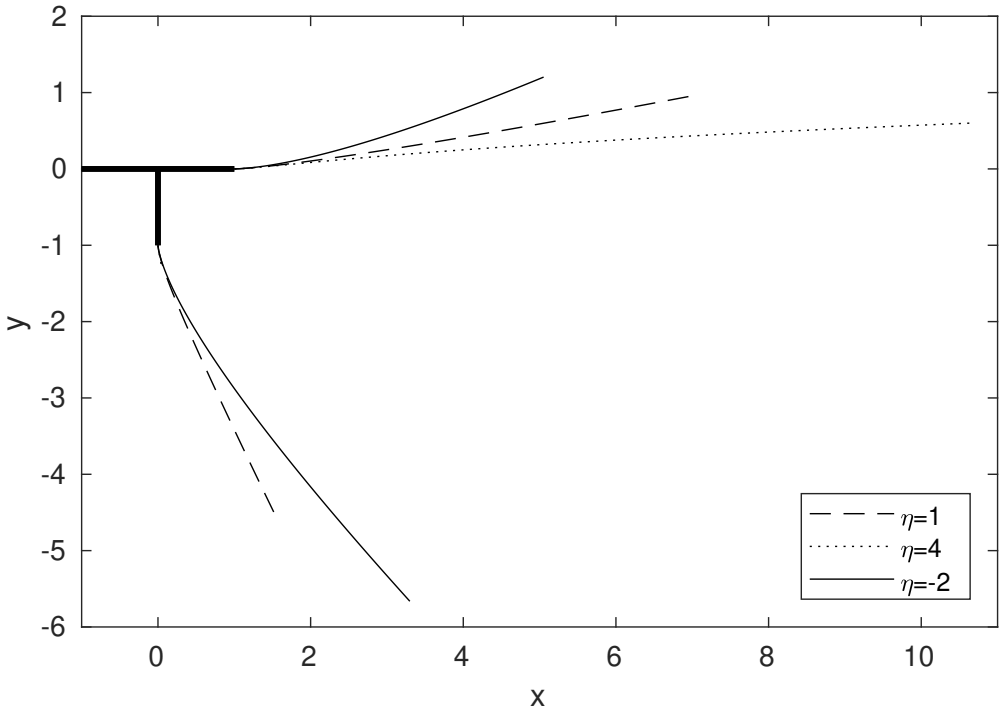



(a) Exact finger trajectories (solid line); numerical (dashed lines).
(b) Evolving angle between the tangents of the finger tips.

# Using the SC Toolbox to compute bifurcating fingers

## The effect of screening parameter $\eta$



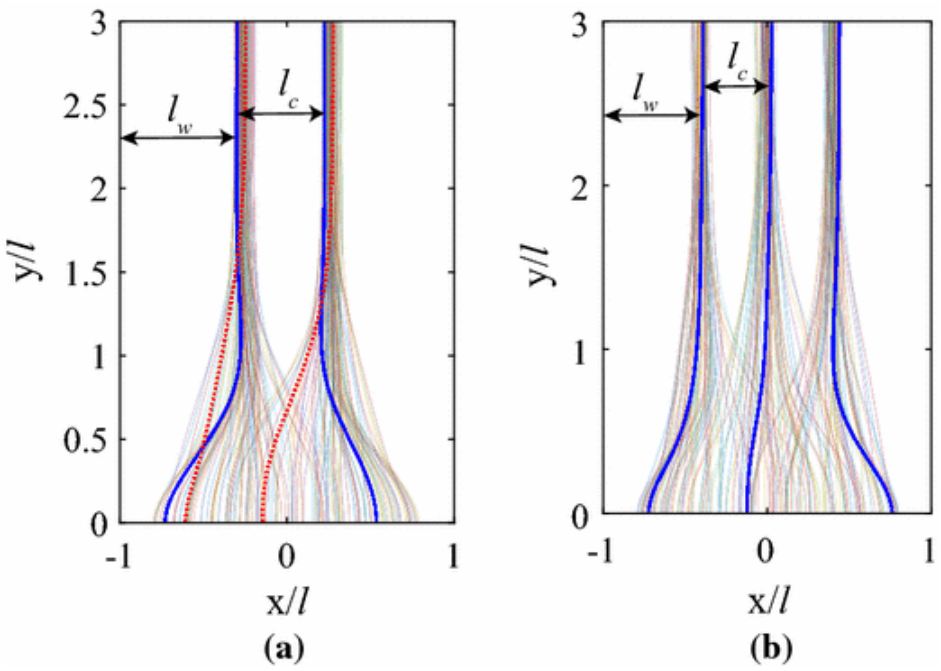

Screening of fingers from an initial right-angled bifurcation (thick dark line) from a semi-infinite needle for $\eta = -2, 1, 4$.

# Part I: Concluding remarks

- Needle growth in half-space is usefully studied using the SC map e.g. asymptotics, numerical computation using SC Toolbox, reduction to coupled system of ODEs for SC parameters.
- Strong screening effects in needle growth when $\eta = 1$.
- Curved fingers can discretized and their evolution computed using SC Toolbox and PLS to give direction of growth.
- Extend approach to other (i) geometries and (ii) PDEs?
- Applications e.g. river deltas

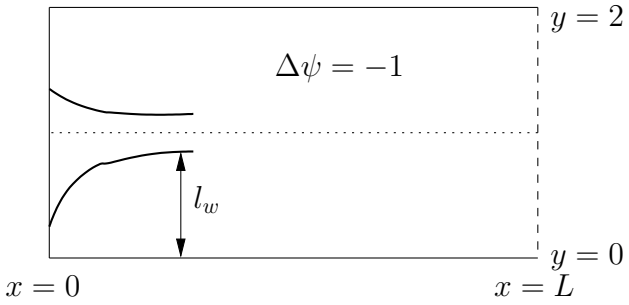
# Part II: Finger growth and selection in a Poisson field

Cohen and Rothman, *J. Stat. Phys.* vol. 167, 2017




- $\Delta\phi = -1$ solved in a truncated strip using finite elements
- $\phi = 0$ on boundaries and fingers; zero flux at far boundary
- Fingers advanced according to PLS
- Two or three fingers cases considered
- Fingers follow selected paths

# Analytical derivation of selected paths: two fingers



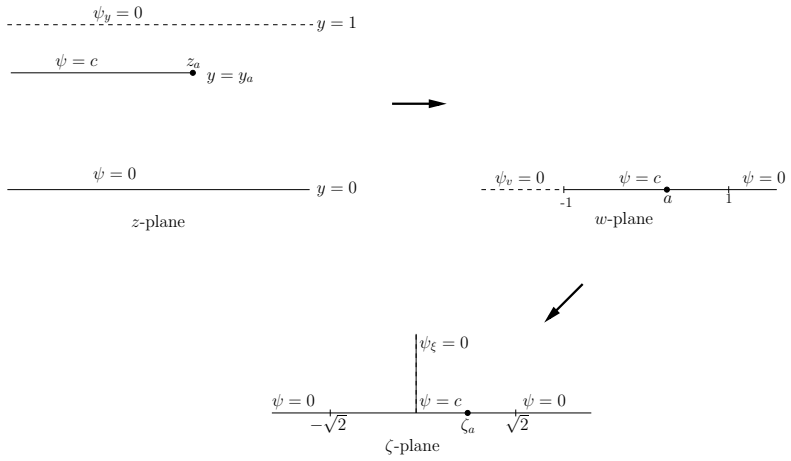

Cohen & Rothman solve $\Delta\phi = -1$ with $\phi = 0$ on boundaries/fingers and $\phi_x = 0$ on $x = L$.

Can $l_w$ be found explictly?

Step 1: Consider steady problem in the infinite strip $0 \leq y \leq 1$.
Step 2: Let $\psi = \phi + y^2/2 - y$, so $\Delta\psi = 0$ and (i) $\psi = 0$ on $y = 0$, (ii) $\psi = c = y_a^2/2 - y_a$ on the finger, $\psi_y = 0$ on $y = 1$ and $\psi_x \to 0$ as $|x| \to \infty$.

# Analytical derivation of selected paths: two fingers (cont.)

Task: find $y = y_a$ s.t. PLS holds at the finger tip $z = z_a$.
Solve for $\psi$ in the cut strip in the $z$-plane:




Map 1: $z = f(w) = \frac{1-a}{2\pi}\log(w-1) + \frac{1+a}{2\pi}\log(w+1)$
Map 2: $\zeta = (1+w)^{1/2}$
Now, $\psi = \mathrm{Im}F(\zeta)$ where

$$F(\zeta) = \frac{c}{\pi}\log\left(\frac{\zeta-\sqrt{2}}{\zeta+\sqrt{2}}\right).$$

# Analytical derivation of selected paths: two fingers (cont.)

## Principle of local symmetry (PLS)

Behaviour of $\psi$ near $z_a$: let $\delta = z - z_a$ and $\epsilon = \zeta - \zeta_a$; and $z = h(\zeta)$ the composite mapping from the $\zeta$ to $z$-plane. Then $\epsilon = \alpha_1 \delta^{1/2} + \alpha_2 \delta + \cdots$ where

$$\alpha_1^2 = \frac{2}{h''(\zeta_a)}, \quad \alpha_2 = -\frac{\alpha_1^2}{6} \frac{h'''(\zeta_a)}{h''(\zeta_a)}.$$

Also, from the complex potential $F(z)$

$$F(z) - F(z_a) = \alpha_1 \beta_1 \delta^{1/2} + (\beta_1 \alpha_2 + \beta_2 \alpha_1^2)\delta + \mathcal{O}(\delta^{3/2}),$$

where

$$\beta_1 = \frac{2c\sqrt{2}}{\pi(\zeta_a^2 - 2)}, \quad \beta_2 = -\frac{2c\sqrt{2}\zeta_a}{\pi(\zeta_a^2 - 2)^2}.$$

# Analytical derivation of selected paths: two fingers (cont.)

## Principle of local symmetry (cont.)

Laplacian case: PLS states that the coefficient of $\delta$ vanishes.

Poisson case: modification needed owing to the contribution of the term $y - y^2/2$. PLS becomes

$$\beta_1 \alpha_2 + \beta_2 \alpha_1^2 + V_a = 0,$$

where $V_a = 1 - y_a$.

# Analytical derivation of selected paths: two fingers (cont.)

## Principle of local symmetry (cont.)

Poisson PLS $\implies$ algebraic equation for $a$:

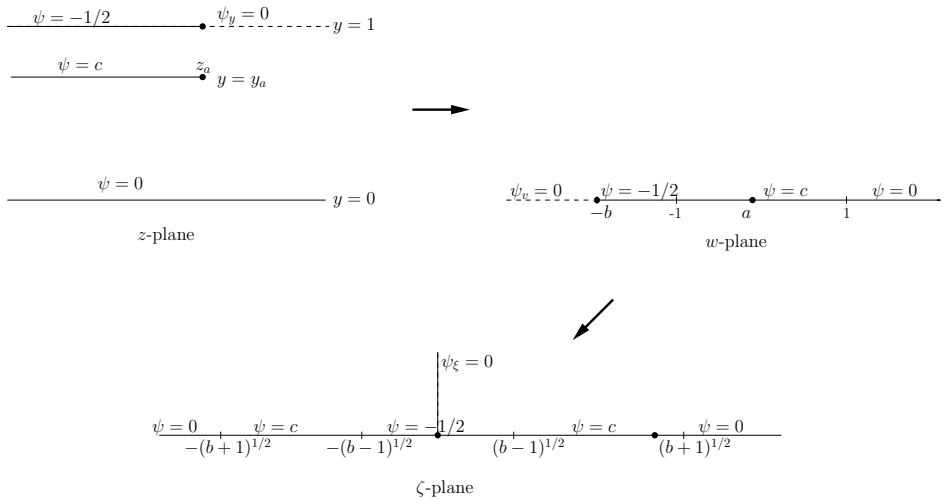
$$(a+3)^2 = \frac{24}{\sqrt{2}}(1+a)^{3/2},$$

with solution $a \approx -0.48099$ on the permitted interval $|a| \leq 1$. This in turn gives $y_a = l_w \approx 0.74$

(Cohen & Rothman: $l_w = 0.74 \pm 0.027$)

# Analytical derivation of selected paths: three fingers)

Mapping sequence:




$$F(\zeta) = \frac{c}{\pi} \log\left(\frac{\zeta - (b+1)^{1/2}}{\zeta + (b+1)^{1/2}}\right) - \frac{(1/2 + c)}{\pi} \log\left(\frac{\zeta - (b-1)^{1/2}}{\zeta + (b-1)^{1/2}}\right)$$

$$z = z_a \to w = a \to \zeta = \zeta_a = (a+b)^{1/2}.$$

# Analytical derivation of selected paths: three fingers (cont.))

Two parameters: $a$ and $b$.
(i) PLS gives

$$\pi(5a^2 + 8ab + 3)\beta_1 + 6\pi(a^2 - 1)\zeta_a\beta_2 = -12\zeta_a^3 V_a,$$

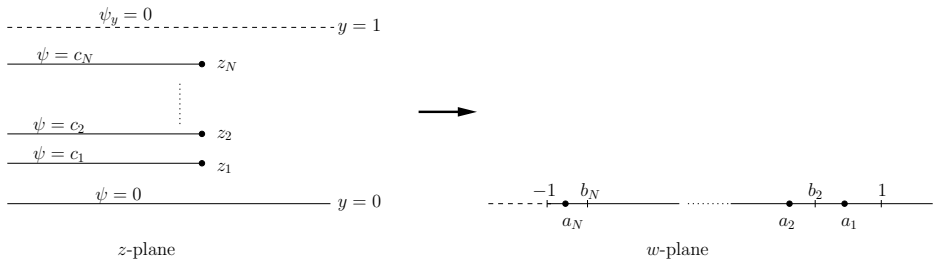
where $\beta_i = \beta_i(a, b)$ $i = 1, 2$ are known functions.

(ii) Assume finger lengths are the same:

$$(1-a)\log(1-a)+(1+a)\log(1+a) = (1-a)\log(b+1)+(1+a)\log(1-b).$$

Solving numerically gives $a \approx -0.2213$ and $b \approx 1.2904 \implies$
$y_a = l_w \approx 0.61$ c.f. Cohen & Rothman who find $l_w \approx 0.60 \pm 0.031$.

# Analytical derivation of selected paths: $2N$ fingers)




i.e. a $N$-cut strip. Map has $2N - 1$ parameters determined by $N$ PLS conditions and $N - 1$ equal length conditions.

e.g. $N = 4$ has solution $y_1 \approx 0.53$ and $y_2 \approx 0.85$.

# Connection with Laplacian growth of fingers in a (semi)strip

Consider $\Delta\phi = 0$ with $\phi = 0$ on all boundaries/fingers and $\phi \to x$ as $x \to \infty$. Can show e.g. by conformal mapping of the canonical 2-finger solution of Loewner's equation in a half-space, the asymptotic behaviour of fingers is

$$x + iy = \frac{2}{\pi}\cosh^{-1}\left(re^{2\pi i/5}\right), \quad \text{as} \quad r \to \infty.$$

and so $y \to 4/5$ i.e. distance 0.2 from centreline $y = 1$.

Alternatively, using the present technique of mapping the cut strip, the PLS immediately gives $a = -3/5$ or $y_a = l_w = 4/5$ i.e distance 0.2 from the centreline (as above).

# Part II: concluding remarks

- Exact solution for thin Poisson fingers have been obtained
- Other geometries/boundary conditions?
- Can selected paths be used to infer forcing? e.g. $P(x,t)$ in $\Delta\phi = -P(x,t)$