Introductory Lectures:
Resurgence in Differential Equations, and
Effective Summation Methods

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Isaac Newton Institute Spring School: *Asymptotic Methods and Applications*, March 22-26, 2021

Isaac Newton Institute Programme: *Applicable Resurgent Asymptotics, 2021/2022*
Lecture 2
Basic Introduction to Resurgence: A Beginner’s Guide

1. Lecture 1: Resurgence & Linear Differential Equations
   ▶ Trans-series and Stokes Phenomenon in ODEs
   ▶ Borel Summation basics
   ▶ Recovering Non-perturbative Connection Formulas

2. Lecture 2: Resurgence & Nonlinear Differential Equations
   ▶ Nonlinear Stokes Phenomenon
   ▶ Painlevé Equation examples
   ▶ Parametric Resurgence & Phase Transitions

3. Lecture 3: Effective Summation Methods
   ▶ Probing the Borel Plane Numerically
   ▶ The Physics of Padé Approximation
   ▶ Optimal Summation and Extrapolation
Resurgence in Nonlinear Differential Equations

- trans-series for an $n^{\text{th}}$ order linear ODE has $n$ non-perturbative exponential terms ($n$ Borel singularities)
- new feature: trans-series for a nonlinear ODE has infinitely many non-perturbative exponential terms
- new feature: Borel singularities are repeated in (certain) integer multiples, due to the nonlinearity
- new feature: “resonance” of Borel singularities can lead to \( \log(x) \) terms, which also become iterated \( \Rightarrow \) sums over \( (\log(x))^n \) terms in the trans-series \( \Rightarrow \) more general trans-series
- new feature: nonlinear Stokes phenomenon
- Painlevé: "special functions of nonlinear ODE’s" (dlmf)
- many applications: fluids, gravity, statistical physics, optics, random matrices, matrix models, QM, QFT, strings, ...
- N.B. integrability is NOT required for resurgence
Resurgence in Nonlinear ODEs: e.g. Painlevé II = “nonlinear Airy”

Painlevé II:

\[ y'' = x y(x) + 2 y^3(x) \]

- Tracy-Widom law for statistics of maximum eigenvalue for Gaussian random matrices
- correlators in polynuclear growth; directed polymers (KPZ)
- double-scaling limit in unitary matrix models
- double-scaling limit in 2d Yang-Mills
- double-scaling limit in 2d supergravity
- non-intersecting Brownian motions
- longest increasing subsequence in random permutations
- ... universal!

• here: illustrate differences from the **linear** Airy equation
Resurgence in Nonlinear ODEs: Painlevé II = “nonlinear Airy”

\[ y''(x) = x y(x) + 2 y^3(x) \]

- \( x \to +\infty: y''(x) \approx x y(x) + \ldots \implies y(x) \sim \sigma_+ \text{Ai}(x) + \ldots \)

- \( x \to -\infty: x y(x) + 2 y^3(x) \approx 0 + \ldots \implies y(x) \sim \pm \sqrt{-\frac{x}{2}} + \ldots \)

- general solution is meromorphic (Painlevé integrability)

\[ y(x) = \frac{1}{x - x_0} - \frac{x_0}{6}(x - x_0) - \frac{1}{4}(x - x_0)^2 + h_0(x - x_0)^3 + \frac{x_0}{72}(x - x_0)^4 + \ldots \]

- numerical instability: separatrix
Resurgence in Nonlinear ODEs: Painlevé II = “nonlinear Airy”

\[ y''(x) = x y(x) + 2 y^3(x) \]

Exercise 2.1:

1. Show that the general Painlevé II solution has a meromorphic expansion with only poles for moveable singularities (those associated with boundary conditions):

\[
y(x) = \frac{1}{x - x_0} - \frac{x_0}{6} (x - x_0) - \frac{1}{4} (x - x_0)^2 + h_0 (x - x_0)^3 + \frac{x_0}{72} (x - x_0)^4 + \ldots
\]

2. Change the nonlinearity of the equation from \( y^3(x) \) to \( y^4(x) \) and show that this Painlevé integrability condition fails (comment: nevertheless, despite being nonintegrable, all the subsequent resurgent trans-series analysis still holds for such an equation)
• Hastings-McLeod: $\sigma_+ = 1$ unique real solution on $\mathbb{R}$ that matches $\sigma_+ \text{Ai}(x)$ asymptotics as $x \to +\infty$ with $\sqrt{-\frac{x}{2}}$ asymptotics as $x \to -\infty$
Painlevé II: Boutroux’s Asymptotic Weierstrass Form

\[ y''(x) = 2y^3 + xy \]

- define a new function, \( y(x) = \sqrt{x}w(x) \):

\[ w'' + \frac{1}{x} w' - \frac{1}{4x^2} w(x) = 2xw^3(x) + xw(x) \]

- change variable to "Écalle variable", \( z = \frac{2}{3}x^{3/2} \):

\[ w''(z) = [2w^3(z) + w(z)] + \left\{ \frac{1}{9z^2}w(z) - \frac{w'(z)}{z} \right\} \]

- approximate solution in terms of Weierstrass \( \mathcal{P} \) function

\[ y(x) \approx \sqrt{x} \sqrt{\mathcal{P} \left( \frac{2}{3}x^{3/2}; \left\{ g_2, \frac{4 - 9g_2}{27} \right\} \right)} - \frac{1}{3} \]

- delicate tuning removes groups of poles to infinity
Painlevé II: Boutroux’s Asymptotic Weierstrass Form

- asymptotic $\frac{2\pi}{3}$ structure
Painlevé II: trans-series analysis as $x \to +\infty$: $y''(x) = 2y^3 + xy$

- exact integral equation:

$$y(x) = \sigma_+ \text{Ai}(x) + 2\pi \int_x^\infty dz y^3(z) [\text{Ai}(x) \text{Bi}(z) - \text{Ai}(z) \text{Bi}(x)]$$

- iterate $\to$ trans-series (all odd powers of $\sigma_+$)

$$y_+(x) \sim \sum_{n=0}^\infty \sigma_+^{2n+1} Y_{2n+1}(x) \quad , \quad x \to +\infty$$

- $Y_1(x) = \text{Ai}(x)$

- $Y_3(x) = 2\pi \left( \text{Ai}(x) \int_x^\infty \text{Ai}^3(z) \text{Bi}(z) \, dz - \text{Bi}(x) \int_x^\infty \text{Ai}^4(z) \, dz \right)$

- $Y_5(x) = 6\pi \left( \text{Ai}(x) \int_x^\infty Y_3(z) \left( Y_1(z) \right)^2 \text{Bi}(z) \, dz - \text{Bi}(x) \int_x^\infty Y_3(z) \left( Y_1(z) \right)^2 \text{Ai}(z) \, dz \right)$
Painlevé II: trans-series analysis as $x \to +\infty$

- trans-series with all odd powers of Airy exponentials

$$y_+(x) \sim \sum_{n=0}^{\infty} \left( \frac{\sigma_+ e^{-\frac{2}{3} x^{3/2}}}{2 \sqrt{\pi} x^{1/4}} \right)^{2n+1} \mathcal{F}_{2n+1}(x)$$

- instanton fluctuation series of the form

$$\mathcal{F}_{2n+1}(x) \sim \frac{1}{x^n} \sum_{m=0}^{\infty} \frac{d_m^{2n+1}}{x^{3m/2}}$$

- evaluate $y(0)$ and $y'(0)$ for Hastings-McLeod $\sigma_+ = 1$:

$$y_+(0) \approx \text{Ai}(0) + 2\pi \left( \frac{1}{24\pi} \text{Ai}(0) - \frac{\ln 3}{24\pi^2} \text{Bi}(0) \right) = 0.366693782...$$

$$y'_+(0) \approx \text{Ai}'(0) + 2\pi \left( \frac{1}{24\pi} \text{Ai}'(0) - \frac{\ln 3}{24\pi^2} \text{Bi}'(0) \right) = -0.293451526...$$

note: $\int_0^{\infty} \text{Ai}^4(z) \, dz = \frac{\ln 3}{24\pi^2}$ and $\int_0^{\infty} \text{Bi}(z)\text{Ai}^3(z) \, dz = \frac{1}{24\pi}$
Painlevé II: trans-series analysis as $x \to +\infty$

\[ y_+(0) \approx \text{Ai}(0) + 2\pi \left(\frac{1}{24\pi} \text{Ai}(0) - \frac{\ln 3}{24\pi^2} \text{Bi}(0)\right) = 0.366693782... \]

\[ y'_+(0) \approx \text{Ai}'(0) + 2\pi \left(\frac{1}{24\pi} \text{Ai}'(0) - \frac{\ln 3}{24\pi^2} \text{Bi}'(0)\right) = -0.293451526... \]

- compare with numerics:
Painlevé II: trans-series analysis as $x \to -\infty$

- now consider the opposite direction: $x \to -\infty$

$$y''(x) = x y(x) + 2 y^3(x)$$

- separatrix tracking as $x \to -\infty \Rightarrow$

$$0 \approx x y(x) + 2 y^3(x), \quad x \to -\infty$$

- formal series solution

$$y_-(x) \sim \sqrt{-\frac{x}{2}} \left(1 - \frac{1}{8(-x)^3} - \frac{73}{128(-x)^6} - \frac{10567}{1024(-x)^9} - \cdots\right)$$

- no parameter! $\Rightarrow$ something is missing (non-perturbative corrections)

- non-alternating factorially divergent $\Rightarrow$ something is missing (non-perturbative corrections)
Trans-series Ansatz for Nonlinear ODEs

• finding the building blocks of the trans-series

\[ \mathcal{D}_{\text{linear}} y(x) = F_{\text{nonlinear}}(y(x)) \]

• formal perturbative series \( y[0](x) \) satisfies this equation term-by-term

• extend to include first non-pert. trans-series correction

\[ y(x) \approx y[0](x) + y[\text{np}](x) + \ldots \]

\[ \rightarrow \quad \mathcal{D}_{\text{linear}} y[0] + \mathcal{D}_{\text{linear}} y[\text{np}] = F_{\text{nonlinear}}(y[0]) + F'_{\text{nonlinear}}(y[0]) y[\text{np}] \]

• linear and homogeneous equation for \( y[\text{np}](x) \)

• solve by ansatz, where \( x_c = \acute{\text{E}}\text{calle critical variable} \)

\[ y[\text{np}](x) \sim \sigma x_c^\beta e^{-\gamma x_c} \sum_n \frac{d_n}{x_c^n} \]

• all higher trans-series terms solve linear equations
Painlevé II: trans-series analysis as $x \to -\infty$

- formal perturbative series solves $y'' = x y(x) + 2 y^3(x)$
  
  $$y[0](x) \sim \left(\frac{-x}{2}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{c_n}{(-x)^{3n}}$$

- trans-series ansatz for $x \to -\infty$
  
  $$y(x) \sim \left(\frac{-x}{2}\right)^{\frac{1}{2}} \sum_{k=0}^{\infty} \sigma_k^k y[k](x)$$

- $\Rightarrow$ tower of linear equations for $y[k]$

- $y[1]$: linear & homogeneous equation
  
  $$y''[1] = \left(6 y_0^2 + x\right) y[1]$$

  
  $$y''[2] = \left(6 y_0^2 + x\right) y[2] + 6 y[0] y_1^2$$
Painlevé II: trans-series analysis as $x \to -\infty$

- first non-perturbative correction “beyond all orders”:

$$y''_{[1]} = \left( 6y_0^2 + x \right) y_{[1]} \sim \left( -2x - \frac{3}{4x^2} + \ldots \right) y_{[1]}$$

- exponential ansatz (using Écalle critical variable):

$$y_{[1]}(x) \sim (-x)^\beta e^{-\gamma(-x)^{\frac{3}{2}}} (1 + \ldots)$$

- matching terms ⇒

$$y_{[1]}(x) \sim \frac{\sigma_-}{(-x)^{1/4}} e^{-\sqrt{2} \frac{2}{3} (-x)^{3/2}} \left( 1 - \frac{17}{72} \frac{2}{3} (-x)^{3/2} + \frac{1513}{10368} (\sqrt{2} \frac{2}{3} (-x)^{3/2})^2 - \ldots \right)$$

- the $\sqrt{2}$ factor is not a misprint!
Resurgence Relation for Painlevé II

• recall formal perturbative series as $x \to -\infty$

$$y[0](x) \sim \sqrt{-\frac{x}{2}} \left(1 - \frac{1}{8(-x)^3} - \frac{73}{128(-x)^6} - \frac{10567}{1024(-x)^9} - \cdots \right)$$

• large order growth of coefficients as $n \to \infty$

$$c_n \sim -(0.1466323\ldots) \frac{\Gamma \left(2n - \frac{1}{2}\right)}{\left(\frac{2\sqrt{2}}{3}\right)^{2n}} \left(1 - \frac{17}{72} \left(\frac{2n}{2n - \frac{3}{2}}\right) + \frac{1513}{10368} \left(\frac{2n}{2n - \frac{3}{2}}\right)^2 \right) - \cdots$$

• compare with fluctuations around the first exponential:

$$y[1](x) \sim \frac{\sigma_-}{(-x)^{1/4}} e^{-\sqrt{2}\frac{2}{3} (-x)^{\frac{3}{2}}} \left(1 - \frac{17}{72} \left(\frac{2}{\sqrt{2} \frac{2}{3} (-x)^{\frac{3}{2}}}\right) + \frac{1513}{10368} \left(\frac{2}{\sqrt{2} \frac{2}{3} (-x)^{\frac{3}{2}}}\right)^2 \right) - \cdots$$

• large-order/low-order resurgence relation (cf. Airy)

• continues to all orders of trans-series (bridge equations)
Exercise 2.2:

1. Generate many terms, and verify the large order behavior of the coefficients of the formal $x \to -\infty$ series for the Painlevé II Hastings-McLeod solution.

2. Explain why you should have been able to predict that the factorial growth involves $\Gamma (2n - ...)$ instead of $\Gamma (n - ...)$. 

3. Numerically identify the Stokes constant to high precision

$$0.1466323... = \frac{1}{\pi} \sqrt{\frac{2}{3\pi}}$$

4. Confirm the large-order/low-order resurgence relation.
Resurgence in Nonlinear ODEs: e.g. Painlevé II

- Hastings-McLeod solution
- **different** trans-series solutions for $x \to \pm \infty$
- $x \to +\infty$

\[
y_+(x) \sim \sum_{k=0}^{\infty} \left( \frac{\sigma_+ e^{-\frac{2}{3}x^{3/2}}}{2 \sqrt{\pi} x^{1/4}} \right)^{2k+1} \mathcal{F}[2k+1](x)
\]

- $x \to -\infty$

\[
y_-(x) \sim \sqrt{-x} \sum_{k=0}^{\infty} \left( \frac{\sigma_- e^{-\frac{2\sqrt{2}}{3}(-x)^{3/2}}}{2 \sqrt{\pi} (-x)^{1/4}} \right)^{k} \mathcal{Y}[k](x)
\]

- non-linear Stokes phenomenon

- "condensation of instantons" across the transition

- trans-asymptotic analysis describes the transition
Resurgence in Nonlinear ODEs: Painlevé II = “nonlinear Airy”

- recall the interesting structure in the complex plane ...
- Hastings-McLeod: $\text{Ai}(x)$ asymptotics as $x \to +\infty$, $\sqrt{-\frac{x}{2}}$ asymptotics as $x \to -\infty$

- there are other more general solutions ...
Gross-Witten-Wadia Matrix Model and Painlevé III

- partition function

\[ Z(t, N) = \int_{U(N)} DU \exp \left[ \frac{N}{t} \text{tr} \left( U + U^\dagger \right) \right] \]

- two variables: 't Hooft coupling \( t \), and matrix size \( N \)

- 3rd order phase transition at \( N = \infty, \, t = 1 \) (universal)

\begin{center}
\begin{figure}
\centering
\includegraphics{fig2}
\caption{The specific heat per degree of freedom, \( C/N^2 \), as a function of \( \lambda \) (temperature).}
\end{figure}
\end{center}

- random matrix theory: \( Z(t, N) = \det \left[ I_{j-k} \left( \frac{N}{t} \right) \right]_{j,k=1,...,N} \)

- large \( N \) asymptotics? Beyond the double-scaling region
Gross-Witten-Wadia Matrix Model and Painlevé III

• “order parameter” $\Delta(t, N) \equiv \langle \det U \rangle$ satisfies a Painlevé III eqn

$$t^2 \Delta'' + t \Delta' + \frac{N^2 \Delta}{t^2} (1 - \Delta^2) = \frac{\Delta}{1 - \Delta^2} \left( N^2 - t^2 (\Delta')^2 \right)$$

• $' \equiv \frac{\partial}{\partial t}$; $N$ appears only as a parameter

• direct relation to the partition function

$$\Delta^2(t, N) = 1 - \frac{Z(t, N - 1) Z(t, N + 1)}{Z^2(t, N)}$$

• non-perturbative large $N$ effects from the ODE

$$\Delta(t, N) = \sum_n \frac{c^{(0)}(t)}{N^n} + e^{-N S(t)} \sum_n \frac{c^{(1)}(t)}{N^n} + e^{-2N S(t)} \sum_n \frac{c^{(2)}(t)}{N^n} + \ldots$$

• all physical observables inherit this trans-series structure

• trans-series changes from strong to weak coupling: Stokes jump from real to complex saddles of eigenvalue integrals
Gross-Witten-Wadia Matrix Model and Painlevé III

\[ \Delta(t,N) \]
Exercise 2.3: Consider the Painlevé III (Okamoto form) for the GWWW model:

\[ t^2 \Delta'' + t \Delta' + \frac{N^2 \Delta}{t^2} (1 - \Delta^2) = \frac{\Delta}{1 - \Delta^2} \left( N^2 - t^2 (\Delta')^2 \right) \]

1. Show that in the \( t > 1 \) region this equation linearizes to

\[ t^2 \Delta'' + t \Delta' + \frac{N^2}{t^2} (1 - t^2) \Delta \approx 0 \]

and this is solved by the Bessel functions \( J_N \left( \frac{N}{t} \right), Y_N \left( \frac{N}{t} \right) \).

2. Hence show that a solution decreasing at large \( t \) can be written as an exact integral equation, which can be iterated to generate the \( t > 1 \) large \( N \) trans-series.

3. Show that for \( t < 1 \) the dominant large \( N \) solution is algebraic, \( \Delta(t) \sim \sqrt{1 - t} \), from which the formal large \( N \) series solution can be generated.
Resurgence: Large $N$ 't Hooft limit at Weak Coupling

- large $N$ trans-series at weak-coupling ($t < 1$)

$$\Delta(t, N) \sim \sqrt{1 - t} \sum_{n=0}^{\infty} \frac{d^{(0)}_n(t)}{N^{2n}} - \frac{\sigma_{\text{weak}}}{2\sqrt{2\pi N}} \frac{te^{-NS_{\text{weak}}(t)}}{(1 - t)^{1/4}} \sum_{n=0}^{\infty} \frac{d^{(1)}_n(t)}{N^n} + \ldots$$

- weak-coupling large $N$ instanton action

$$S_{\text{weak}}(t) = \frac{2\sqrt{1 - t}}{t} - 2 \text{arctanh} (\sqrt{1 - t})$$

- fluctuation around first exponential from the ODE

$$\sum_{n=0}^{\infty} \frac{d^{(1)}_n(t)}{N^n} = 1 + \frac{(3t^2 - 12t - 8)}{96(1 - t)^{3/2}} \frac{1}{N} + \ldots$$

- large-order growth of perturbative coefficients ($\forall t < 1$):

$$d^{(0)}_n(t) \sim \frac{-1}{\sqrt{2}(1 - t)^{3/4}\pi^{3/2}} \frac{\Gamma(2n - \frac{5}{2})}{(S_{\text{weak}}(t))^{2n-\frac{5}{2}}} \left[ 1 + \frac{(3t^2 - 12t - 8)}{96(1 - t)^{3/2}} \frac{S_{\text{weak}}(t)}{(2n - \frac{7}{2})} + \ldots \right]$$

- (parametric) resurgence relation, for all $t$
• large $N$ trans-series at strong-coupling ($t > 1$)

\[
\Delta(t, N) \approx J_N \left( \frac{N}{t} \right) \sim \frac{\sqrt{t} e^{-N S_{\text{strong}}(t)}}{\sqrt{2\pi N} (t^2 - 1)^{1/4}} \sum_{n=0}^{\infty} \frac{U_n(t)}{N^n} + \ldots
\]

• strong-coupling large $N$ instanton action

\[
S_{\text{strong}}(t) = \text{arccosh}(t) - \sqrt{1 - \frac{1}{t^2}}
\]

• complex saddles of eigenvalue integrals (cf. Stokes)

• more accurate uniform large $N$ instanton expansion

\[
\Delta(t, N) \sim \left( \frac{4 \left( \frac{3}{2} S_{\text{strong}}(t) \right)^{2/3}}{1 - 1/t^2} \right)^{1/4} \frac{\text{Ai} \left( N^{2/3} \left( \frac{3}{2} S_{\text{strong}}(t) \right)^{2/3} \right)}{N^{1/3}} + \ldots
\]

• analogous to uniform WKB

• double-scaling limit = Painlevé II = non-linear Airy eqn
Resurgence in GWW: double-scaling limit = Painlevé II

- reduction cascade of Painlevé equations
- "zoom in" on vicinity of phase transition:

\[ \kappa \equiv N^{2/3}(t - 1) \quad ; \quad \Delta(t, N) = \frac{t^{1/3}}{N^{1/3}} y(\kappa) \]

- \( N \to \infty \) with \( \kappa \) fixed:

\[ \Delta \quad \text{PIII equation} \quad \longrightarrow \quad \frac{d^2y}{d\kappa^2} = 2y^3(\kappa) + 2\kappa y(\kappa) \quad (\text{PII}) \]

- e.g. on strong-coupling side:

\[ \lim_{N \to \infty} J_N(N - N^{1/3} \kappa) = \left( \frac{2}{N} \right)^{1/3} \text{Ai} \left( 2^{1/3} \kappa \right) \]

- the immediate vicinity of the physical phase transition region is described by the Hastings-McLeod PII solution
Resurgence in GWW: double-scaling limit = Painlevé II

- "zoom in" on vicinity of phase transition:
- integral equation form of PII:

$$y(\chi) = \sigma \text{Ai}(\chi) + 2\pi \int_{\chi}^{\infty} \left[ \text{Ai}(\chi)\text{Bi}(\chi') - \text{Ai}(\chi')\text{Bi}(\chi) \right] y^3(\chi') d\chi'$$
Lee-Yang: complex zeros of $Z$ pinch the real axis at the phase transition point in the thermodynamic limit.
Resurgence for Painlevé VI

- Painlevé VI reduces to the other Painlevé equations by a well-defined cascade of coalescence limits, analogous to what happens for the familiar (linear) special functions.

- Remarkable recent result (Gamayun, Iorgov, Lisovyy, 2013): the Jimbo expansion of the Painlevé VI tau function has an all orders convergent conformal block expansion (cf. CFT)

\[
\tau(t) \sim \sum_{n=\infty}^{\infty} s^n C(\vec{\theta}, \sigma+n) B(\vec{\theta}, \sigma+n; t)
\]

\[
B(\vec{\theta}, \sigma; t) \propto t^{\sigma^2} \sum_{\lambda, \mu \in \mathcal{Y}} B_{\lambda, \mu}(\vec{\theta}, \sigma) t^{||\lambda||+||\mu||}
\]

$\vec{\theta}$: monodromy parameters; $C(\vec{\theta}, \sigma)$: known combinatorial functions; $s \& \sigma$ are b.c parameters; the $t$-dependent term $B(\vec{\theta}, \sigma; t)$ is written as a sum over partitions.

- Expression is manifestly resurgent: the sum over $n$ is an instanton sum, encoded in the zero-instanton sector.
A Few Selected References

- Digital Library of Mathematical Functions: https://dlmf.nist.gov/

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