

A Markov chain for certain triple systems

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Latin squares

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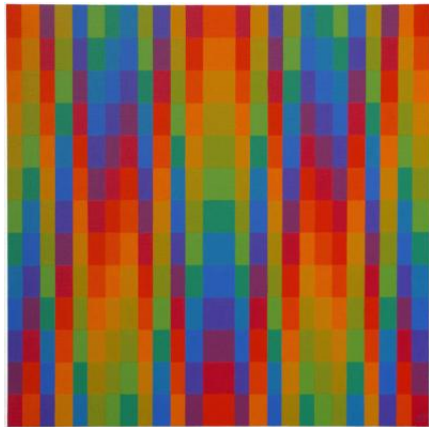
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R. A. Fisher memorial window, Gonville & Caius College

Commercial break



Raymond Brownell
Great Expectations Gallery, London

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We don't know. There are upper and lower bounds of the form $(n/c)^{n^2}$ (with different values of c).

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The exact numbers are known for $n \leq 11$ (McKay and Wanless). The number of Latin squares of order 11 is

776 966 836 171 770 144 107 444 346 734 230 682 311 065 600 000.

A typical Latin square

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3. **Are the parities of the rows approximately independent** (so that the number of rows which are odd permutations is approximately binomial $\text{Bin}(n, \frac{1}{2})$)?

Transversals, orthogonal mates, intercalates

1	2	3	4
2	1	4	3
3	4	1	2
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- ▶ **Transversal:** set of cells, one for each row, column, symbol.

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$1a$	$2b$	$3c$	$4d$
$2c$	$1d$	$4a$	$3b$
$3d$	$4c$	$1b$	$2a$
$4b$	$3a$	$2d$	$1c$

- ▶ **Orthogonal mate:** Latin square whose symbols are transversals to the original.

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- **Intercalate:** subsquare of order 2 (two rows, two columns, two symbols).

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Such a method was found by Jacobson and Matthews.

Improper Latin squares

The method is a random walk around an enlarged space containing both Latin squares and “improper” Latin squares. We need to know that not too infrequently we are at a proper Latin square, and that in the limit all proper Latin squares are equally likely to occur.

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Suppose we have a Latin square in which the entry in row i and column j is k , and we want to change this entry to k' . The entry k' already occurs in row i (say in column j'), and in column j (say in row i'), and these entries must be changed to k . We need to remove k from position (i', j') and replace it with k' .

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This is OK if the entry there is k , otherwise we create a “fault”.

Improper Latin squares

	j	j'	
i	k	k'	
i'	k'	k''	

→

	j	j'	
i	k'	k	
i'	k	$k' + k'' - k$	

The random walk

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The random walk is reversible and connected; so in the limiting distribution, any two Latin squares have the same probability (as do any two improper Latin squares).

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From any improper Latin square, we expect to reach a Latin square in a polynomial number of steps. Jacobson and Matthews also gave a polynomial upper bound for the diameter of the graph around which we are walking. The most important open question about this method is how rapidly we converge to the uniform distribution (what is the mixing time).

Some exploration

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The conjecture about parity of rows is more interesting. It seems to be correct to a reasonable approximation. But Thomas Prellberg, using a technique to explore the tails of a distribution, found evidence that the extreme cases are a little more likely than expected.

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Hägglkvist and Janssen showed that the probability that all rows have the same parity is exponentially small, but couldn't get the right constant. An interesting problem remains here!

First generalisation: Other kinds of triple system

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Triple systems

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Take a composition (k_1, \dots, k_m) of 3 (an ordered sum of positive integers summing to 3, that is, $(1, 1, 1)$, $(2, 1)$ or (3)). Take sets X_1, \dots, X_m , one for each part of the composition. Our design will be a subset

$$\mathcal{B} \subseteq \binom{X_1}{k_1} \times \dots \times \binom{X_m}{k_m}$$

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with the following property:

We require that, for any tuple of integers of the same length as the given composition and having sum 2, say (t_1, \dots) , and any choice of subsets T_i of X_i with $|T_i| = t_i$, there is a **unique** $(K_1, \dots, K_m) \in \mathcal{B}$ with $T_i \subseteq K_i$ for $i = 1, \dots, m$.

The case $(1, 1, 1)$

We have three sets X_1, X_2, X_3 , which we regard as indexing the rows, columns and symbols. Choose one element in any two of these sets; there should be a unique triple in \mathcal{B} containing these elements.

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This is just a Latin square looked at a different way!

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We have a single set X_1 , and \mathcal{B} is a collection of 3-element subsets of X_1 with the property that any 2-element subset of X_1 is contained in a unique member of \mathcal{B} .

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In other words, a **Steiner triple system!**

The case $(2, 1)$

We have two sets X_1, X_2 which we think of as the set of vertices of a complete graph and the set of colours in an edge-colouring. The design \mathcal{B} consists of pairs $(\{x_1, x_2\}, y)$ – a 2-subset of X_1 and an element of Y . We require:

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- ▶ any pair of vertices lies in a unique such object (that is, each edge gets a unique colour);
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So we have a **1-factorisation** of a complete graph K_n , i.e. a proper edge-colouring with the minimum number $n - 1$ of colours (so each colour is a 1-factor).

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But in the other two cases, we have no proof of connectedness.

Second generalisation: Larger values of λ

In the definition of triple systems, we can replace the condition that any pair is in a **unique** triple by the condition that any pair is in a constant number λ of triples. The resulting generalisation of Steiner triple systems consists of 2-designs, or “balanced incomplete-block designs” with block size 3; the generalisation of Latin squares consists of orthogonal arrays of strength 2 and degree 3. For 1-factorisations, there seems to be no previous account of these structures (which might be called λ -factorisations of the λ -fold complete multigraph).

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Some things we can't do

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- ▶ Choose a random filled Sudoku grid (of arbitrary size).
- ▶ Choose a random BIBD with larger block size.