Isaac Newton Institute for Mathematical Sciences

Noncommutative Geometry

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It is very natural to think that space and time are primary notions and that physics should be based on them. It would, however, be more reasonable to say that our notions of geometry emerged in the 19th century from and together with the classical physics of the day, a process that led ultimately to Einstein's formulation of gravity as curvature. Geometry now also plays a role all branches of mathematics, indeed wherever one finds a continuum space of interest.

This continuum assumption of classical mechanics was, however, already shattered in the 1920's with the discovery of the quantum nature of the phase space of the microscopic mechanical system describing an atom. Such a system manifests itself through discrete spectral lines (see fig. 1) and its basic laws, such as the Ritz–Rydberg law of spectroscopy, are in direct contradiction with a continuum picture of the phase space. Heisenberg was the first to understand that for a microscopic mechanical system the coordinates, namely the real numbers x^1, $p^1, \ldots,$ such as the positions and momenta that one would like to use to parameterise points of the phase space, actually do not commute. More recent developments such as string theory similarly imply that our classical geometrical framework is too narrow to describe in a faithful manner the physical spaces of great interest when one deals with microscopic systems: one needs some form of 'noncommutative' geometry in which the "algebra of coordinates" is no longer commutative.

Although the first examples of noncommutative spaces came from quantum mechanics, there turn out to be a great many others, such as the leaf spaces of foliations, duals of nonabelian discrete groups, the space of Penrose tilings, the noncommutative torus $\mathbb{T}^2_\theta$ which plays a role in the quantum Hall effect and in M-theory compactification and the space of $\mathbb{Q}$-lattices. The latter has an action of the scaling group providing a spectral interpretation of the zeros of the L-functions of number theory and an interpretation of the Riemann–Weil explicit formulae as trace formulae. Another rich class of examples arises from deformation theory, such as deformation of Poisson manifolds, quantum groups and their homogeneous spaces.

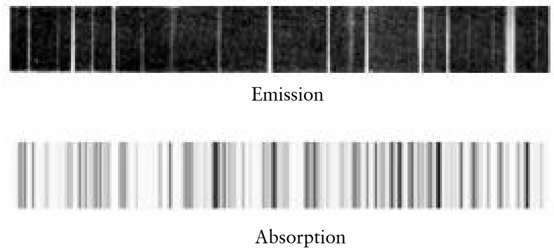

Fig. 1. Emission and absorption spectra

The new theory adapts all classical geometric concepts and tools to this new class of geometric spaces, leading to far reaching generalisations such as cyclic cohomology and K-homology but also to great surprises. One of them is that non-commutative spaces even at the coarsest level of "measure theory" generate their own proper time evolution, whereas classical spaces are in this deep sense static.

Another deep feature is that the concept of "metric" in noncommutative geometry is precisely based on spectral data for operators of a quantum mechanical nature. The development of the ideas of curvature and characteristic classes in this context is then made possible by a far reaching extension of the calculus and of the Atiyah–Singer index theorem to such operators.

Noncommutative geometry is also tied in via quantum groups to q-series (and hence to number theory), to knot theory and to invariants of 3-manifolds. Quantum groups arose as generalised or 'quantum' symmetries and now play a role in various contexts in mathematics and physics, for example in the book-keeping of divergences in quantum field theories. This led in particular to a surprising relation between the motivic Galois theory of Grothendieck and the fine features of the renormalisation procedure in quantum field theory.

The ubiquity of quantum groups is not surprising given the diverse roles that ordinary symmetry groups already play. Their reconciliation, however, as important examples of noncommutative geometry is not at all obvious and is just one of the several current directions to be looked at during the programme.