FLAT CONNECTIONS AND COMODULES

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ABSTRACT. This is the text of a talk given at the Isaac Newton Institute in Cambridge on 4th August, 2006.

This talk is motivated by a recent paper [A Kaygun and M Khalkhali, Hopf modules and noncommutative differential geometry, Lett. Math. Phys. 76 (2006), 77–91] in which Hopf modules appearing as coefficients in Hopf-cyclic cohomology are interpreted as modules with flat connections.

We start by describing how all the algebraic structure involved in a universal differential calculus fits in a natural way into the notion of a coring (or a coalgebra in the category of bimodules). We recall the theorem of Roiter [A.V. Roiter, Matrix problems and representations of BOCS's. [in:] Lecture Notes in Mathematics, vol. 831, Springer-Verlag, Berlin and New York, 1980, pp. 288-324] in which a bijective correspondence is established between semi-free differential graded algebras and corings with a grouplike element. A brief introduction to the theory of comodules is given and the theorem establishing a bijective correspondence between comodules of a coring with a grouplike element and flat connections (with respect to the associated differential graded algebra) is given [T Brzeziński, Corings with a grouplike element, Banach Center Publ., 61 (2003), 21-35].

Finally we specialise to corings which are built on a tensor product of algebra and a coalgebra. Such corings are in one-to-one correspondence with so-called entwining structures, and their comodules are entwined modules. The latter include all known examples of Hopf-type modules such as Hopf modules, relative Hopf modules, Long dimodules, Doi-Koppinen and alternative Doi-Koppinen modules. In particular they include Yetter-Drinfeld and anti-Yetter-Drinfeld modules and their generalisations, hence all the modules of interest to Hopf-cyclic cohomology. In this way the interpretation of the latter as modules with flat connections is obtained as a corollary of a more general theory.

1. INTRODUCTION AND MOTIVATION

The motivation for this talk comes from a recent paper [21] in which Kaygun and Khalkhali prove that anti-Yetter-Drinfeld (or aYD) modules introduced in [20], [18], [19], as well as their generalisation from [28] can be understood as modules with a flat connection. The aim of this talk is to give an explanation of this identification in terms of corings and comodules.

Throughout this talk A denotes an associative unital algebra over a commutative ring k. Product in A is denoted by $\mu : A \otimes A \to A$. The identity morphism for an object, say, V is denoted by V.

In back of our minds we should keep an example provided by the universal differential envelope of A which should be very familiar to any expert in non-commutative geometry. Recall that this is a differential graded algebra $\Omega A = \oplus_{n=0}^{\infty} \Omega^n A$ over A (i.e.

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$A = \Omega^0 A$) defined as follows. The bimodule of one-forms is given by

$$\Omega^1 A := \ker\mu = \{\sum_i a_i \otimes b_i \in A\otimes A \mid \sum_i a_i b_i = 0\}. \tag{1.1}$$

$\Omega^1 A$ has the obvious A-bimodule structure. The differential $d : A \to \Omega^1 A$ is defined as

$$d : a \mapsto 1\otimes a - a\otimes 1 = [1\otimes 1, a]. \tag{1.2}$$

One defines higher differential forms by iteration

$$\Omega^{n+1} A := \Omega^1 A \otimes_A \Omega^n A, \tag{1.3}$$

i.e. (or, more precisely) ΩA is the tensor algebra of the A-bimodule $\Omega^1 A$, $\Omega A = T_A(\Omega^1 A)$. The differential d is extended to the whole of Ω by requiring the graded Leibniz rule (and that $d\circ d = 0$). This amounts to inserting the unit of the algebra A in all possible places in $\Omega^n A \subset A^{\otimes n+1}$ with alternating signs.

2. Sweedler's example and definition of corings

The universal differential envelope of an algebra A uses all the structure that is encoded in the notion of an algebra, i.e. the product (in the definition of $\Omega^1 A$), the unit (in the definition of d) and the tensor product over A (in the definition of $\Omega^n A$). In [30], Sweedler proposed a different point of view on algebras. He suggested to look at the A-bimodule $\mathfrak{C} = A\otimes A$ (with the obvious A-actions) and consider two A-bilinear maps

$$\Delta_{\mathfrak{C}} : \mathfrak{C} \to \mathfrak{C}\otimes_A\mathfrak{C} \simeq A\otimes A\otimes A, \qquad a\otimes a' \mapsto a\otimes 1\otimes a', \tag{2.1}$$

$$\varepsilon_{\mathfrak{C}} : \mathfrak{C} \to A, \qquad \varepsilon_{\mathfrak{C}} = \mu : a\otimes a' \mapsto aa'. \tag{2.2}$$

The algebra structure of A is fully encoded in the maps (2.1), (2.2). It is an elementary exercise to check that the maps $\Delta_{\mathfrak{C}}$ and $\varepsilon_{\mathfrak{C}}$ make the following diagrams commute

$$\begin{array}{ccc} \mathfrak{C} & \xrightarrow{\Delta_{\mathfrak{C}}} & \mathfrak{C}\otimes_A\mathfrak{C} \\ {\scriptstyle\Delta_{\mathfrak{C}}}\downarrow & & \downarrow{\scriptstyle\mathfrak{C}\otimes\Delta_{\mathfrak{C}}} \\ \mathfrak{C}\otimes_A\mathfrak{C} & \xrightarrow{\Delta_{\mathfrak{C}}\otimes\mathfrak{C}} & \mathfrak{C}\otimes_A\mathfrak{C}\otimes_A\mathfrak{C}, \end{array} \tag{2.3}$$

$$\begin{array}{ccc} \mathfrak{C} & \xrightarrow{\Delta_{\mathfrak{C}}} & \mathfrak{C}\otimes_A\mathfrak{C} \\ & \searrow{\scriptstyle\simeq} & \downarrow{\scriptstyle\varepsilon_{\mathfrak{C}}\otimes\mathfrak{C}} \\ & & A\otimes_A\mathfrak{C}\,, \end{array} \qquad \begin{array}{ccc} \mathfrak{C} & & \\ {\scriptstyle\Delta_{\mathfrak{C}}}\downarrow & \searrow{\scriptstyle\simeq} & \\ \mathfrak{C}\otimes_A\mathfrak{C} & \xrightarrow{\mathfrak{C}\otimes\varepsilon_{\mathfrak{C}}} & \mathfrak{C}\otimes_A A\,. \end{array} \tag{2.4}$$

Note that the diagrams (2.4) simply express that 1 is the unit in the algebra A. Also, note that the diagrams (2.3) state that $\Delta_{\mathfrak{C}}$ is a *coassociative map*, while (2.4) state the *counitality axiom*. In other words, these diagrams mean that $A\otimes A$ is a *coalgebra over a non-commutative ring* A. These observations lead to the following general definition (no relation of $\mathfrak{C}$ to $A\otimes A$).

Definition 2.1. An A-bicomodule $\mathfrak{C}$ is called an *A-coring* iff there are A-bimodule maps $\Delta_{\mathfrak{C}} : \mathfrak{C} \to \mathfrak{C}\otimes_A\mathfrak{C}$, $\varepsilon_{\mathfrak{C}} : \mathfrak{C} \to A$ rendering diagrams (2.3), (2.4) commutative.

As for coalgebras, $\Delta_{\mathfrak{C}}$ is called a *coproduct* and $\varepsilon_{\mathfrak{C}}$ is called a *counit*. The coring $\mathfrak{C} = A\otimes A$ is known as the *Sweedler* or *canonical* coring associated to the ring extension $k \to A$. Note in passing that A itself is an A-corings. Thus the notion of a coring includes that of a ring.

3. Roiter's theorem

Going back to the universal differential envelope and realising that $\mathfrak{C} = A\otimes A$ is a coring, we can identify $\Omega^1 A$ with the kernel of the counit $\varepsilon_{\mathfrak{C}}$. A question thus arises: is this a coincidence, or are there other corings, for which the kernel of the counit gives rise to a differential graded algebra? Before this question can be answered, one needs to tackle the problem of defining a differential. By equation (1.2), the universal differential is given as a commutator with $1\otimes 1 \in \mathfrak{C} = A\otimes A$. Note that

$$\Delta_{\mathfrak{C}}(1\otimes 1) = (1\otimes 1)\otimes_A(1\otimes 1), \qquad \varepsilon_{\mathfrak{C}}(1\otimes 1) = 1.$$

In the case of a general A-coring $\mathfrak{C}$ we can distinguish elements which have above properties and thus arrive at the following

Definition 3.1. An element g of an A-coring $\mathfrak{C}$ is called a *group-like element* provided that

$$\Delta_{\mathfrak{C}}(g) = g\otimes_A g, \qquad \varepsilon_{\mathfrak{C}}(g) = 1.$$

The following remarkable result of Roiter [27] states that in fact any differential graded algebra of certain kind comes from a coring with a group-like element.

Theorem 3.2. *(1) Any A-coring $\mathfrak{C}$ with a group-like element g gives rise to a differential graded algebra ΩA defined as follows: $\Omega^1 A = \ker\varepsilon_{\mathfrak{C}}$, $\Omega^{n+1}A = \Omega^1 A\otimes_A\Omega^n A$ (i.e., $\Omega A = T_A(\ker\varepsilon_{\mathfrak{C}})$). The differential is defined by $d(a) = ga - ag$, for all $a\in A$, and, for all $c^1\otimes_A\cdots\otimes_A c^n \in (\ker\varepsilon_{\mathfrak{C}})^{\otimes_A n}$,*

$$\begin{aligned} d(c^1\otimes_A\cdots\otimes_A c^n) &= g\otimes_A c^1\otimes_A\cdots\otimes_A c^n + (-1)^{n+1}c^1\otimes_A\cdots\otimes_A c^n\otimes_A g \\ &\quad + \sum_{i=1}^{n}(-1)^i c^1\otimes_A\cdots\otimes_A c^{i-1}\otimes_A\Delta_{\mathfrak{C}}(c^i)\otimes_A c^{i+1}\otimes_A\cdots\otimes_A c^n. \end{aligned}$$

(2) A differential graded algebra ΩA over A such that $\Omega A = T_A(\Omega^1 A)$ (i.e. $\Omega^{n+1}A = \Omega^1 A\otimes_A\Omega^n A$; following [26]*, a differential graded algebra with this property is said to be* semi-free*), defines a coring with a grouplike element.*

(3) The operations described in items (1) and (2) are mutual inverses.

Proof. (1) and (3) are proven by straightforward calculations, so we only indicate how to construct a coring from a differential graded algebra (i.e. sketch the proof of (2)). Starting with ΩA, define

$$\mathfrak{C} = Ag \oplus \Omega^1 A,$$

where g is an indeterminate. In other words we define $\mathfrak{C}$ to be a direct sum of A and $\Omega^1 A$ as a left A-module. We now need to specify a compatible right A-module structure. This is defined by

$$(ag + \omega)a' := aa'g + ada' + \omega a'.$$

The coproduct is specified by

$$\Delta_{\mathfrak{C}}(ag) = ag\otimes_A g, \qquad \Delta_{\mathfrak{C}}(\omega) = g\otimes_A\omega + \omega\otimes_A g - d(\omega),$$

and the counit

$$\varepsilon_{\mathfrak{C}}(ag+\omega) := a,$$

for all $a \in A$ and $\omega \in \Omega^1 A$. Note that this structure is chosen in such a way that g becomes the required group-like element. □

The Roiter theorem teaches us that:

Semi-free differential graded algebras are in bijective correspondence with corings with a group-like element.

The canonical coring construction can be performed for any algebra map $B \to A$ (i.e. it is not necessary that $B = k$) – this is the original Sweedler's example from [30]. In this case $\mathfrak{C} = A \otimes_B A$ and the resulting differential graded algebra (defined with respect to the group-like element $1_A \otimes_B 1_A$ corresponds to the relative universal differential forms as studied, for example, in [15].

4. Comodules and flat connections

An A-coring is an algebraic structure and we would like to study its (co)representations. These are given in terms of *comodules*.

Definition 4.1. A right A-module M is called a *right* $\mathfrak{C}$*-comodule* iff there exists a right A-linear map $\varrho^M : M \to M \otimes_A \mathfrak{C}$ rendering the following diagrams

$$\begin{array}{ccc} M & \xrightarrow{\varrho^M} & M\otimes_A\mathfrak{C} \\ {\scriptstyle \varrho^M}\downarrow & & \downarrow{\scriptstyle M\otimes\Delta_{\mathfrak{C}}} \\ M\otimes_A\mathfrak{C} & \xrightarrow{\varrho^M\otimes\mathfrak{C}} & M\otimes_A\mathfrak{C}\otimes_A\mathfrak{C} \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\varrho^M} & M\otimes_A\mathfrak{C} \\ & {\scriptstyle \simeq}\searrow & \downarrow{\scriptstyle M\otimes\varepsilon_{\mathfrak{C}}} \\ & & M\otimes_A A \end{array}$$

commutative.

As for coalgebras, the map ϱ^M is called a *coaction*. Comodules of the Sweedler coring $\mathfrak{C} = A \otimes_B A$ associated to a ring extension $B \to A$ correspond bijectively to *descent data* for the extension $B \to A$ [14], [22]. Thus corings are nowadays effectively used to describe a (generalised) noncommutative descent theory, cf. [12] (on the other hand, it is my personal belief that the proper framework for noncommutative descent theory should be provided by bicategories).

The existence of a group-like element in an A-coring $\mathfrak{C}$ has a very natural explanation in terms of comodules [6]: $\mathfrak{C}$ has a group-like element if and only if A is a right (or, equivalently, left) $\mathfrak{C}$-comodule.

The noncommutative differential geometric interpretation of comodules of a coring with a group-like element is provided by the following theorem taken from [7]. First recall that a *connection* in a right A-module M (with respect to a differential graded algebra ΩA over A) is a k-linear map $\nabla : M \otimes_A \Omega^\bullet A \to M \otimes_A \Omega^{\bullet+1} A$ such that, for all $\omega \in M \otimes_A \Omega^k A$ and $\omega' \in \Omega A$,

$$\nabla(\omega\omega') = \nabla(\omega)\omega' + (-1)^k \omega d(\omega').$$

A *curvature* of a connection ∇ is a (right A-linear) map

$$F_\nabla : M \to M \otimes_A \Omega^2 A,$$

defined as a restriction of $\nabla \circ \nabla$ to M, that is, $F_\nabla = \nabla \circ \nabla\mid_M$. A connection is said to be *flat* if its curvature is identically equal to 0.

Theorem 4.2. *Assume that $\mathfrak{C}$ is an A-coring with a group-like element g, and write ΩA for the associated differential graded algebra.*

(1) If (M, ϱ^M) is a right $\mathfrak{C}$-comodule, then the map

$$\nabla : M \to M \otimes_A \Omega^1 A, \qquad m \mapsto \varrho^M(m) - m \otimes_A g,$$

is a flat connection.

(2) If M is a right A-module with a flat connection $\nabla : M \to M \otimes_A \Omega^1 A$, then M is a right $\mathfrak{C}$-comodule with the coaction

$$\varrho^M : M \to M \otimes_A \mathfrak{C}, \qquad m \mapsto \nabla(m) + m \otimes_A g.$$

(3) The operations described in items (1) and (2) are mutual inverses.

This theorem is proven by a straightforward calculation and, combined with the Roiter theorem, teaches us that:

Flat connections with respect to a semi-free differential graded algebra are in bijective correspondence with comodules of a coring with a group-like element.

Combined with the identification of right comodules of the Sweedler A-coring $A \otimes_B A$ with descent data, the above observation explains the appearance of flat connections in the descent theory cf. [25].

5. Entwined modules

Typically, Hopf-type modules involve data consiting of an algebra and a coalgebra, and objects which are at the same type modules and comodules with some compatibility condition. It is quite natural, therefore, to ask the following question.

Suppose that, given an algebra A and a coalgebra C (with coproduct Δ and counit ε), we would like to construct an A-coring structure on $\mathfrak{C} = A \otimes C$. As written, $\mathfrak{C}$ has an obvious left A-multiplication

$$a(a' \otimes c) := aa' \otimes c, \tag{5.1}$$

it has also an obvious candidate for a counit,

$$\varepsilon_{\mathfrak{C}} := A \otimes \varepsilon. \tag{5.2}$$

In view of the identification $\mathfrak{C} \otimes_A \mathfrak{C} = (A \otimes C) \otimes_A (A \otimes C) \simeq A \otimes C \otimes C$, the map

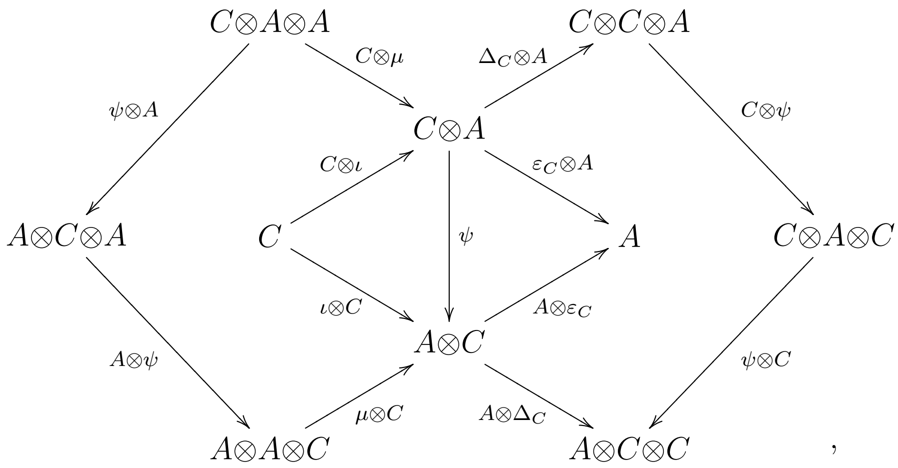
$$\Delta_{\mathfrak{C}} := A \otimes \Delta, \tag{5.3}$$

is an obvious candidate for a coproduct for $\mathfrak{C}$. To make $A \otimes C$ into an A-coring with already specified structures (5.1)–(5.3) we need to introduce a suitable right A-multiplication. Obviously since $A \otimes C$ must be an A-bimodule, in view of (5.1) any such a right A-multiplication is determined by a map $\psi : C \otimes A \to A \otimes C$,

$$\psi(c \otimes a) := (1 \otimes c)a. \tag{5.4}$$

The map ψ must satisfy (four) conditions corresponding to unitality and associativity of the right A-multiplication and to the facts that both $\Delta_{\mathfrak{C}}$ and $\varepsilon_{\mathfrak{C}}$ are right A-linear maps. As observed in [6] (following a comment by M. Takeuchi), these four conditions are equivalent to the commutativity of the following *bow-tie diagram*

(5.5)



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where μ is the product in A and $\iota : k \to A$ is the unit map. The map ψ satisfying the conditions (5.5) is known as an *entwining map*, C and A are said to be *entwined* by ψ, and the triple (A, C, ψ) is called an *entwining structure*. These are notions introduced originally in [9] (with no reference to corings, but with an aim to recapture missing Hopf algebra symmetry needed for the construction of principal bundles over quantum homogeneous spaces). The corresponding coring $\mathfrak{C} = A \otimes C$ is often referred to as the coring associated to an entwining structure (A, C, ψ) (of course, it depends on the point of view, whether we want to see a coring as being determined by the map ψ or the map ψ as being determined by a coring).

One easily checks that right comodules of the A-coring $\mathfrak{C} = A \otimes C$ associated to an entwining structure are simply k-modules M which are both right A-modules and right C-comodules such that both structures satisfy the following compatibility condition, for all $m \in M$ and $a \in A$,

$$\varrho^M(ma) = m_{(0)}\psi(m_{(1)} \otimes a), \tag{5.6}$$

where $\varrho^M(m) = m_{(0)} \otimes m_{(1)}$ is the C-coaction on M. Such k-modules are known as *entwined modules* and were introduced in [5].

Although entwining structures in this form were introduced in [9], and, at least on the first sight, the conditions expressed by the bow-tie diagram (5.5) might seem a bit complicated, in fact they are a special case of the structure which appeared in category theory some forty years ago and is known as a *(mixed) distributive law* [1], [33].

6. Anti-Yetter-Drinfeld and other Hopf-type modules

Since the end of the sixties, Hopf algebraists studied intensively objects with both an action and a coaction of a Hopf algebra or, more generally, with an action of an algebra and a coaction of a coalgebra which are compatible one with the other through an action/coaction of a Hopf algebra. Such objects are known as *Hopf-type modules*, and examples include Hopf modules originally introduced by Sweedler, relative Hopf-modules of Doi and Takuechi [16], [32], Doi-Koppinen Hopf modules [17], [23] or (as a special case of the latter) Yetter-Drinfeld modules [29], [34]. Essentially, compatibility conditions for all known Hopf-type modules can be recast in the form of an entwining structure and are of the form of equation (5.6). For more information about entwining structures and their connection with Hopf-type modules we refer to [13] or to [10, Section 33].

The qualification *essentially* appears here, since there are also variants of Hopf-type modules for *weak Hopf algebras* [4] (such as weak Doi-Hopf modules [2]) and for *bialgebroids* [31], [24] (such as Doi-Koppinen modules for quantum groupoids [8]). To describe the former one needs to study corings built not on $A\otimes C$ but on a (left A-module) direct summand of $A\otimes C$. Such corings are equivalently described in terms of *weak entwining structures* [11]. To describe the latter, on works over a non-commutative ring R from the onset, and studies A-corings on $A\otimes_R C$ (to make sense of these, C has to be an R-coring and A must be an R-ring, i.e. there must be a ring map $R\to A$). These lead to *entwining structures over non-commutative rings* [3]. In any case, to the best of my knowledge, every known Hopf-type module (whether weak or over a non-commutative ring) is a comodule of an associated coring. This, in particular, implies to the newest additions to the family of Hopf-type modules, i.e. anti-Yetter-Drinfeld modules which play the role of coefficients in Hopf-cyclic cohomology [20], [18], [19], and to their generalisations termed (α,β)-equivariant C-modules [28].

We illustrate the general theory of the previous sections on the example of anti-Yetter-Drinfeld modules. To this end, take $A=C=H$, where H is a Hopf algebra with a bijective antipode S. Then one can define an entwining map $\psi: H\otimes H\to H\otimes H$ by

$$\psi(c\otimes a) = a_{(2)}\otimes S^{-1}(a_{(1)})ca_{(3)}, \tag{6.1}$$

for all $a,c\in H$. Here $a_{(1)}\otimes a_{(2)}\otimes a_{(3)} := (\Delta\otimes H)\circ\Delta(a)$. That ψ is an entwining map indeed can be easily checked by a routine calculation. While doing this exercise, the reader should notice that the only significant property (apart from multiplicativity and unitality of the coproduct) is the fact that the antipode is an anti-algebra and anti-colagebra map. Consequently, there is an H-coring $\mathfrak{C}=H\otimes H$ with the right H-multiplication

$$(b\otimes c)a = ba_{(2)}\otimes S^{-1}(a_{(1)})ca_{(3)}. \tag{6.2}$$

The compatiblity (5.6) for right H-module and H-comodule M comes out as, for all $a\in H$,

$$\varrho^M(ma) = m_{(0)}a_{(2)}\otimes S^{-1}(a_{(1)})m_{(1)}a_{(3)}, \tag{6.3}$$

i.e. entwined modules for (6.1) coincide with (right-right) anti-Yetter-Drinfeld modules. Since $C=H$ is a Hopf algebra 1_H is a group-like element in H, and hence $1_H\otimes 1_H$ is a group-like element in the H-coring $\mathfrak{C}$. By the Roiter theorem there is the associated differential graded algebra and by Theorem 4.2 anti-Yetter-Drinfeld modules are modules with a flat connection with respect to this differential graded structure. Explicitly,

$$\Omega^1 H = \{\sum_i a_i\otimes c_i \in H\otimes H \mid \sum_i a_i\varepsilon(c_i) = 0\}.$$

Thus, in particular $\Omega^1 H = H\otimes H^+$, where $H^+ := \ker\varepsilon$, provided H is a flat k-module. The right H-action on $\Omega^1 H$ is given by the formula (6.2). Thus the differential comes out as

$$d(a) = (1\otimes 1)a - a(1\otimes 1) = a_{(2)}\otimes S^{-1}(a_{(1)})a_{(3)} - a\otimes 1.$$

Anti-Yetter-Drinfeld modules are an example of *(α,β)-equivariant C-modules* introduced in [28]. In this case A is a bialgebra, C is an A-bimodule colagebra, $\alpha : A \to A$ is a bialgebra map and $\beta : A \to A$ is an anti-bialgebra map (i.e. β is both anti-algebra and anti-coalgebra map). All these data give rise to an entwining map $\psi : C{\otimes}A \to A{\otimes}C$ defined by

$$\psi(c{\otimes}a) = a_{(2)}{\otimes}\beta(a_{(1)})c\alpha(a_{(3)}).$$

I leave it as an exercise to work out explicitly the form of the corresponding coring $\mathfrak{C} = A{\otimes}C$ and of the compatibility condition (5.6). If, in addition, C has a group-like element e, then $1{\otimes}e$ is a group-like element in $\mathfrak{C}$. Again, the derivation of the explicit form of the associated differential graded algebra is left as an exercise.

7. A few comments on conventions.

Throughout this talk we used the *right-right conventions*, i.e. we studied right actions and right coactions. Obviously, one can study left comodules over an A-coring (these will correspond to left A-modules with a flat connection). In the case of entwining structures, there are four possible conventions (right-right, right-left, left-right, left-left); thus, for example, there are four types of entwining structures corresponding to four types of anti-Yetter-Drinfeld modules. One can move freely between these conventions by using opposite/co-opposite algebras and/or coalgebras. Obviously, although this requires some care, but does not introduce any new mathematical features.

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