Orbifold Conformal Field Theory and Cohomology of the Monster

Geoffrey Mason
University of California, Santa Cruz

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◇1: Introduction.

In this lecture, Conformal Field Theory (CFT) will be synonomous with *Vertex Operator Algebra* (VOA). These objects embody much of the formalism behind bosonic CFT. The use of VOAs in this context is close to the physicist's approach to CFT, and involves working directly with the quantum fields (or vertex operators) that define the theory. We will say nothing about theories involving fermions (super VOAs) and other variations such as the Ghost System, which plays a role in the bosonic string and the chiral de Rham complex, for example.

Physicists are particularly interested in *rational* CFT; there is a corresponding notion of rational VOA. One of the main problems of the subject is to understand the structure of the category of representations over a RCFT/RVOA, which is expected to be (at least) a braided tensor category. Roughly speaking, the passage from a RCFT to its module category is a forgetful functor

$$\{CFT\} \longrightarrow \{TQFT\}.$$

A complete understanding of this functor at the level of full mathematical rigor is still some way off. We will describe what is known in a special case (so-called *holomorphic orbifolds*) and what some of the cohomological implications are. We illustrate with the example of the Frenkel-Lepowsky-Meurman Moonshine Module $V^\natural$, which is probably the most interesting example of a VOA.

$\diamond$2: VOAs.

'Space' refers to a linear space over the complex numbers. Fix such a space V, sometimes called the *Fock* space. Elements of V are called *states*. A *(quantum) field* on V is an element

$$a(z) = \sum_{n \in Z} a(n) z^{-n-1} \in \mathrm{End(V)}[[z, z^{-1}]]$$

with the property that

> if $v \in V$ then $a(n).v = 0$ for all large enough integers n.

(z is a formal variable.) The space of all fields on V is denoted by $\mathcal{F}(V)$. The endomorphisms $a(n)$ are the *(Fourier) modes*.

(a) LOCALITY: We say that a pair of fields $a(z), b(z) \in \mathcal{F}(V)$ are *mutually local* in case the following holds:

> there is $k \geq 0$ such that $(y - z)^k [a(y), b(z)] = 0$.

Here, y and z are independent variables. We write $a(z) \sim b(z)$ if $a(z)$ and $b(z)$ are mutually local fields.

The relation $\sim$ is symmetric but generally neither reflexive nor transitive.

EXAMPLES:

(Derivative of a field): If $a(z) \in \mathcal{F}(V)$ then

$$a'(z) = \sum(-n-1)a(n)z^{-n-2} \in \mathcal{F}(V).$$

$$b(z) \in \mathcal{F}(V), a(z) \sim b(z) \Rightarrow a'(z) \sim b(z).$$

(Tensor Products): Given spaces V, W, there is a natural injection

$$\mathcal{F}(V) \otimes \mathcal{F}(W) \longrightarrow \mathcal{F}(V \otimes W).$$

(The map arises via $a(z) \otimes b(z) \mapsto \Sigma c(n) z^{-n-1}$ where $c(n) = \Sigma_{i \in Z}\, a(i) \otimes b(n-i-1)$. Although this is an infinite sum of operators on $V \otimes W$, it is well-defined because $a(z), b(z)$ are fields: a state in $V \otimes W$ is annihilated by all but a finite number of the summands. Operators defined in this manner occur frequently.)

(Virasoro Algebra): The *Virasoro algebra* is the Lie algebra

$$Vir = \bigoplus_{n \in Z} CL(n) \oplus Ck$$

with relations $[Vir, k] = 0$ and

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3 - m}{12}\delta_{m+n,0}k.$$

Set $\omega(z) = \Sigma_{n\in Z} L(n)z^{-n-2}$. Then

$$\omega(z) \sim \omega(z).$$

(c) VERTEX OPERATOR ALGEBRA of CFT-TYPE:
$(V, Y, \mathbf{1}, \omega)$
Ingredients:

1. N-graded Fock space $V = V_0 \oplus V_1 \oplus ..., dimV_n < \infty$
2. $C\mathbf{1} = V_0$, vacuum state $\mathbf{1}$.
3. Linear map $Y : V \longrightarrow \mathrm{EndV}[[z, z^{-1}]] \subseteq \mathcal{F}(V)$,

$$v \mapsto Y(v, z) = \sum_{n\in Z} v(n)z^{-n-1}$$

4. Distinguished state $\omega \in V_2$ with

$$Y(\omega, z) = \sum_{n\in Z} L(n)z^{-n-2}.$$

Axioms:

1. Locality: $Y(u, z) \sim Y(v, z)\ \forall u, v \in V$.
2. Creativity: $Y(u, z).\mathbf{1} = u + O(z)$.
3. Virasoro : modes of $Y(\omega, z)$ close on Virasoro algebra represented on V with central charge c;

$$L(0).v = nv,\ \ v \in V_n.$$
$$Y'(u, z) = Y(L(-1)u, z)$$

Let $\mathcal{L}$ denote the space of vertex operators $\{Y(v, z)\}$. The axioms amount to a linear bijection, the *State-Field Correspondence*,

$$V \longleftrightarrow \mathcal{L}$$

in which each state v corresponds to a field $Y(v, z)$ and each field creates the corresponding state from the vacuum. Moreover the fields are mutually local and closed under derivativation.

EXAMPLES:
1. Tensor product $V_1 \otimes V_2$ of VOAs; $c = c_1 + c_2$.
2. Given a positive-definite even lattice L, set $H = C \otimes_Z L$. Then

$$V_L = S(H_1 \oplus H_2 \oplus ...) \otimes C[L]$$

is the Fock space for a *lattice* VOA (FKS construction); $c = rkL$.
3. $V_{L_1} \otimes V_{L_2} = V_{L_1 \perp L_2}$.
4. Moonshine Module $V^\natural$; c=24.
5. Let M be a linear space with a Virasoro field $\omega(z) \in \mathcal{F}(M)$. Any maximal set of mutually local fields $\mathcal{L}$ in $\mathcal{F}(M)$ which contains $\omega(z)$ carries a canonical structure of VOA.

$\diamond$3: Modules over a VOA.

$V = (V, Y, \mathbf{1}, \omega)$ is a VOA. A *module* over V is a linear space M together with a morphism of vertex algebras $V \longrightarrow \mathcal{L}$ where $\mathcal{L} \subseteq F(M)$ is a VOA of local fields of the type discussed in example 5 above.

If M is an irrep then its grading has the shape

$$M = \sum_{n \geq 0} M_{n+\lambda}$$

for a constant λ, the *conformal weight* of M.

Modules over V form an abelian category $V - Mod$

V is called *rational* in case $V - Mod$ is a semi-simple category.

Theorem (Dong-Li-Mason): If V is rational then it has only finitely many (inequivalent) irreps.

EXAMPLES:
1. V is a module over itself (adjoint module).
2. Irreps of $V_1 \otimes V_2$ are $M_1 \otimes M_2$ for irreps M_1 of V_1 and M_2 of V_2.

3. Irreps of V_L have Fock spaces

$$S(H_1 \oplus H_2 \oplus ...) \otimes C[L+\alpha]$$

for $\alpha \in L^o/L$. Indeed, V_L is rational and V_L has precisely $|L^0 : L|$ inequivalent irreps (Dong-Li-Mason, Dong).

Call V *holomorphic* if V is rational and V has a *unique* irrep, namely the adjoint module (i.e., V is simple).

EXAMPLES:
1. V_L is holomorphic iff $L = L^o$ is *self-dual*.
2. $V^\natural$ is holomorphic (Dong, DLM)

$\diamond$4: Witt-Grothendieck Groups.

Tensor product gives (isomorphism classes of) VOAs the structure of a commutative monoid $\mathcal{M}$. It has lots of interesting submonoids e.g., $\mathcal{M}^{rat}$, $\mathcal{M}^{hol}$, $\mathcal{M}^{latt}$ corresponding to rational, holomorphic, and lattice VOAs respectively. Denote the corresponding Witt-Grothendieck groups by

$$\mathcal{W} \supseteq \mathcal{W}^{rat} \supseteq \mathcal{W}^{hol}, \mathcal{W}^{latt}.$$

I expect that $\mathcal{M}$ is cancellative, so that $\mathcal{M} \longrightarrow \mathcal{W}$ is injective. This is certainly true for $\mathcal{W}^{latt}$ which is isomorphic to the Witt-Grothendieck group of positive-definite, even quadratic forms.

$\diamond$5: Automorphism Groups.

$$Aut(V) = \{g \in GL(V) : gY(v,z)g^{-1} = Y(gv,z), g\omega = \omega\}$$

EXAMPLES:
1. If L is an ADE root lattice then $Aut(V_L)$ contains the corresponding complex Lie group.
2. In general $Aut(V_L)$ contains an extension of $Aut(L)$ by the torus R^c/L.
3. $Aut(V^\natural) = M$ (Griess, FLM, Tits).
4. $V^G = \{v \in V : g.v = v, g \in G\}$ is a VOA .

If $G_i \subseteq Aut(V_i)$ for $i = 1, 2$ then G_1 x $G_2 \subseteq Aut(V_1 \otimes V_2)$. This allows us to define *equivariant* monoids $\mathcal{M}_G$ and Witt-Grothendieck groups

$$\mathcal{W}_G \supseteq \mathcal{W}_G^{rat} \supseteq \mathcal{W}_G^{hol}$$

consisting of (isomorphism classes of) pairs (V, G) where $G \subseteq Aut(V)$ and $(V_1, G) = (V_2, G)$ iff V_1 and V_2 are isomorphic as G-module VOAs.

Basic Problem in RCFT: given $(V, G) \in \mathcal{M}_G^{rat}$, G finite, describe the structure of $V^G - Mod$.

Theorem(Schur-Weyl duality)(Dong-Mason): Let $G \subseteq Aut(V), |G| < \infty$. There is a decomposition of V as V^G-module:

$$V = \bigoplus_{\chi \in irrG} M_\chi \otimes V^\chi$$

where M_χ ranges over all irreps of G and V^χ ranges over *distinct, non-zero* irreps for V^G.

So there is a natural family of irreps for V^G indexed by irreps of G. In effect $G \otimes V^G$ acts like a dual pair in the sense of Howe. The group algebra CG is the commutant of the algebra of operators on V spanned by the Fourier modes of fields determined by $v \in V^G$.

These are not the only irreps for V^G.

$\diamond$ 6: Twisted Sectors.

For simplicity we now consider only pairs $(V, G) \in \mathcal{M}_G^{hol}$, $|G| < \infty$. For technical reasons we also assume that V is *C_2-cofinite*. This means that the linear subspace spanned by states $u(-2).v$ with u, v $\in V$ has finite codimension in V. It is expected that V rational $\Rightarrow$V C_2-cofinite, but this is not known. C_2-cofiniteness behaves well wrt tensor products and we can define additional Witt-Grothendieck groups to incorporate it.

Twisted representations of V are a kind of representation for which there is no classical analog. Fix $g \in G$ of order N. In essence, a *g-twisted module* is a C-graded space M together with a Y-map

$$Y_g : V \longrightarrow EndM[[z^{1/N}, z^{-1/N}]]$$

each $Y_g(v, z) \in \mathcal{F}(M)$, $Y_g(u, z) \sim Y_g(v, z)$, and

$$\begin{aligned} Y_g(v, z) &= \sum_{n \in r/N + Z} v(n) z^{-n-1} \; for \\ g.v &= e^{-2\pi i r/N} v, \end{aligned}$$

together with usual Virasoro axiom.

If M is an irr g-twisted module it has grading of the shape

$$M = \sum_{n \geq 0} M_{\lambda + n/N}$$

for some fixed constant λ (*conformal weight* of M), moreover dim $M_{\lambda+n/N} < \infty$.

EXAMPLES:
1. A V-module is precisely a 1-twisted module.
2. For a lattice theory V_L with L self-dual, let $\sigma \in \frac{1}{N} L/L \subseteq Aut(V_L)$. Then

$$S(H_1 + H_2 + \ldots) \otimes C[L + \sigma]$$

is an irreducible σ-twisted module.
3. The restriction of a g-twisted module to V^G ($g \in G$) is a (untwisted) V^G-module.

Theorem(Dong-Li-Mason): Let V be a C_2-cofinite, holomorphic VOA, and let g be an automorphism of finite order. Then there is a *unique* irreducible g-twisted module, call it $V(g)$.

This applies to the Moonshine Module $V^\natural$, for example. So for each element $g \in Monster$ there is a unique g-twisted module $V^\natural(g)$. This fact is the basis for understanding Monstrous Moonshine (see below).

Uniqueness of twisted sectors informs us that G acts on isomorphism classes of irreducible twisted modules :

$$g : (M, Y_h) \mapsto (M, Y_h og) \simeq (M, Y_{g^{-1}hg})$$

$$Y_h og\ (v, z) := Y_h(g.v, z)$$

Thus each $g \in G$ induces a linear map $\phi(g)$ on the *intertwining algebra*

$$V_G = \bigoplus_{g \in G} V(g).$$

However the operators $\phi(g)$ may not close on a group isomorphic to G; rather, if we let $\phi_h(g)$ denote the action of $\phi(g)$ on $V(h)$ then

$$\phi_g(hk) = \alpha_g(h, k)^{-1} \phi_{kgk^{-1}}(h)\phi_g(k).$$

Set

$$\alpha(h, k) = \sum_{g \in G} \alpha_g(h, k)e(g) \in U(C[G]^*)$$

(Unit group in the dual group algebra, where $e(g)e(h) = \delta_{g,h}e(g)$.)

Thus

$$[\alpha] \in H^2(G, U) = H^3(ZG)$$

(Hochschild cohomology.) Note that

$$H^3(ZG) = \bigoplus_{g^G} H^3(C_G(g), Z) = \bigoplus_{g^G} H^2(C_G(g), C^*).$$

Thus

$$\alpha \mapsto (..., \theta_g, ...)$$

where $\theta_g \in Z^2(C_G(g), C^*)$: the operators $\phi_g(h)$ for $h \in C_G(g)$ acting on the twisted sector $V(g)$ span not the group algebra $C[C_G(g)]$, but a *twisted* group algebra $C^{\theta_g}[C_G(g)]$.

In this way we have defined a morphism of groups

$$\mathcal{W}_G^{hol} \longrightarrow H^3(ZG).$$

This map is generally not close to being surjective, and it is a problem to give a good description of the image.

$\diamond$7: Loop Space Interpretation.

Let $(V, G) \in \mathcal{M}_G^{hol}, |G| < \infty$. Let BG = classifying space for G, LBG = free loop space on BG.

It is well-known that

$$LBG = \bigcup_{g^G} BC_G(g)$$

(disjoint union) and

$$H^3(ZG) = H^3(LBG, Z).$$

V determines a bundle over BG via the Borel construction

$$EG \times_G V \longrightarrow BG.$$

The theorem on Twisted Sectors shows that there is a unique way to extend this to a (projective) bundle of twisted sectors over loop space via

$$EC_G(g) \times_{C_G(g)} PV(g) \longrightarrow BC_G(g).$$

The cohomology class $[\alpha] = [(..., \theta_g, ...)] \in H^3(LBG)$ tells us how to linearize the projective action of $C_G(g)$.

$\diamond$8: Associativity.

We will define maps

$$\begin{array}{ccc} \mathcal{W}_G^{hol} & \longrightarrow & H^4(G,Z) \\ & \searrow & \downarrow \\ & & H^3(ZG) \end{array}$$

where the se arrow is the one defined before. Define

$$D^\alpha(G) = \bigoplus_{g\in G} C^{\theta_g}[C_G(g)] = C[G] \bowtie_\alpha C[G]^*.$$

(Smash product.) From what we have already said, it can be seen that $D^\alpha(G)$ lies in the commutant of the algebra generated by the Fourier modes of the (twisted) vertex operators $Y_g(v,z), v \in V^G, g \in G$, acting on the intertwining algebra V_G. Following the idea of the Schur-Weyl Duality Theorem, one expects that $D^\alpha(G)$ is the full commutant, so that roughly speaking

$$D^\alpha(G) \otimes V^G \subseteq End(V_G)$$

is a dual pair. This is known in only a few cases.

Much more is expected: the duality underlies a Morita equivalence of categories

$$D^\alpha(G) - Mod \simeq V^G - Mod$$

If this is true it tells us among other things that:

(a) All irreps of V^G lie in V_G i.e., in some twisted sector. They are indexed by pairs (θ_g, χ_g), where χ_g ranges over the irreps of the twisted group algebra $\mathrm{C}^{\theta_g}[C_G(g)]$ and g ranges over an element in each conjugacy class.

(b)V^G is rational.

Now $V^G - Mod$ admits a product

$$U \Box V$$

which is hard to define and harder to understand! One knows under certain conditions (Huang-Lepowsky, et al) and expects quite generally that it is associative. Since the irreps of V^G are indexed by group elements one expects that the associativity isomorphism for irreps is multiplication by a scalar

$$\omega(g, h, k) : (U_g \Box V_h) \Box W_k \xrightarrow{\sim} U_g \Box (V_h \Box W_k).$$

Similarly we expect

$$U_g \Box V_h \xrightarrow{\sim} V_h \Box U_g$$

In this way, $V^G - Mod$ will be a braided tensor category, in particular there are hexagonal and pentagonal consistency conditions, and the latter translates into the assertion that

$$[\omega] \in H^3(G, C^*) = H^4(G, Z).$$

This defines the group morphism

$$\mathcal{W}_G^{hol} \longrightarrow H^4(G, Z).$$

Via the Morita equivalence, $D^\alpha(G) - Mod$ acquires an associative product and $D^\alpha(G)$ is promoted to a quasi-triangular quasi-Hopf algebra $D^\omega(G)$, the *twisted quantum double* of G. This is the same associative algebra $D^\alpha(G)$ inflicted with a quasi-coassociative coproduct and (Drinfeld) associativity constraint

$$\Phi = \sum_{g,h,k \in G} \omega(g, h, k)^{-1} 1 \bowtie e(g) \otimes 1 \bowtie e(h) \otimes 1 \bowtie e(k)$$

Thus we have

$$(Id \otimes \Delta) o \Delta = \Phi^{-1}((\Delta \otimes Id) o \Delta)\Phi$$

These quasi-Hopf algebras were introduced by Dijkgraaf-Pasquier-Roche. Φ determines the structure of the twisted double, in particular ω

determines the cocycles θ_g via the remarkable formula (DPR)

$$\theta_g(x,y) = \frac{\omega(g,x,y)\omega(x,y,(xy)^{-1}gxy)}{\omega(x,x^{-1}gx,y)}.$$

This formula defines the group morphism

$$H^4(G,Z) \longrightarrow H^3(ZG).$$

Andy Baker explained to me how this map can be viewed at the level of loop spaces as arising from the evaluation map

$$S^1 \text{ x } LBG \xrightarrow{ev} BG.$$

Taking cohomology yields

$$H^4(G,Z) = H^4(BG,Z) \to H^4(S^1 \text{ x } LBG) \\ \to H^3(LBG,Z) = H^3(ZG).$$

$\diamond$9: Connections with algebraic K-Theory.

Let $(V, G) \in \mathcal{M}_G^{hol}, |G| < \infty$ as before. Assume for convenience that G is perfect (i.e., $G = [G, G]$) and let $\hat{G}$ be the universal central extension of G:

$$1 \longrightarrow H_2(G, Z) \longrightarrow \hat{G} \longrightarrow G \longrightarrow 1.$$

There are maps

$$\begin{array}{ccccccc} & & & & H_3(\hat{G}, Z) & \xrightarrow{B\iota^+} & K_3(ZG) \\ & & & & \downarrow & & \downarrow \\ \mathcal{W}_G^{hol} & \longrightarrow & H^4(G, Z) & = & H_3(G, Z) & \xrightarrow{assembly} & HC_3^-(ZG) \\ & \searrow & \downarrow & & \downarrow & & \\ & & H^3(ZG) & = & H_2(G, Z) & & \end{array}.$$

Apart from the left triangle discussed so far, the other maps are standard in algebraic K-Theory: it is well-known that $H_3(\hat{G}, Z) = \pi_3(BG^+)$ where $+$ refers to Quillen's plus construction wrt G. The top map then arises from

$$G \xrightarrow{\iota} GL(ZG) = \xrightarrow{Lim} GL_n(ZG)$$

The top left vertical arrow is the composite

$$\pi_3(BG^+) \rightarrow H_3(BG^+, Z) = H_3(BG, Z) = H_3(G, Z)$$

The top right vertical map is the Chern map.

In particular, if G has trivial Schur multiplier (e.g., $G = Monster$) then we have

$$\begin{array}{ccccc} \mathcal{W}_G^{hol} & \longrightarrow & H^4(G,Z) & \longrightarrow & K_3(ZG) \\ & \searrow & \downarrow & & \\ & & H^3(ZG) & & \end{array}$$

and the top right map is an *injection*.

We can detect elements in these groups by utilizing more ideas from orbifold CFT...

Remark: Witten, G.Moore and others have pointed out the relevance of algebraic K-theory to M-theory. It is likely that one can directly relate VOAs to algebraic K-theory.

$\diamond$10: Modular-Invariance.

Continue with $(V,G) \in \mathcal{M}_G^{hol}$, $|G| < \infty$ and C_2-cofiniteness. For a pair of commuting elements $g, h \in G$ we have introduced the operator $\phi_g(h)$ acting on the twisted sector $V(g)$. There is thus the possibility of considering the following formal traces (equivariant 1-point correlation function):

$$\begin{aligned} Z(g,h,\tau) &= tr_{V(g)}\ \phi_g(h)\ q^{L(0)-c/24} \\ &= q^{\lambda - c/24} \sum_{n \geq 0} tr_{V(g)_{n+\lambda}}\ \phi_g(h) q^{n/N} \end{aligned}$$

where $N =$ order of g. As usual, we are taking $\tau \in \mathcal{H}$ (complex upper half-plane), and $q = e^{2\pi i \tau}$.

Theorem (Modular-Invariance) (Dong-Li-Mason): With the above assumptions, we have for $\gamma \in SL(2,Z)$,

$$Z(g,h,\gamma\tau) = \epsilon Z((g,h)\gamma, \tau)$$

for a constant $\epsilon(g,h,\gamma)$, and where

$$(g,h).\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (g^a h^c, g^b h^d)$$

Furthermore c is a non-negative integer divisible by 8 (Zhu) and the conformal weight $\lambda \in Q$.

It is conjectured that each $Z(g,h,\tau)$ is a classical modular function. This is not known in general. The last theorem reduces the general question to showing simply that the constants are of absolute value 1.

The Conway-Norton Moonshine Conjecture = Borcherds' Theorem shows that the functions

$$Z(1,h,\tau) = q^{-1} \sum_{n\geq 0} tr_{V_n^\natural}\ h\ q^n$$

in the case of the Monster are indeed modular, indeed they are all *hauptmoduln*. This property is peculiar to the Monster VOA, whereas modular-invariance is generic.

The Modular-Invariance Theorem has powerful consequences. Taking $g = 1$ and $\gamma = S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ yields

$$\begin{aligned} Z(1,h,-1/\tau) &= Z(h,1,\tau) \\ &= \epsilon(h) q^{\lambda - c/24} \sum_{n\geq 0} \dim V(h)_{\lambda+n/N} q^{n/N} \end{aligned}$$

So if we 'know' the trace of h on V we can deduce the graded dimension of the h-twisted sector. (Up to a scalar $\epsilon(h)$ that is. Physicists always take the scalar to be 1, and they are probably right.)

For the Moonshine Module we know all the traces of elements in the Monster (Borcherds, Conway-Norton) so we can apply this. Here are 5 pertinent examples:

$$
\begin{aligned}
Z(1,2-,\tau) &= 1^{24}/2^{24}+24; Z(2-,1,\tau)=24+... \\
Z(1,4-,\tau) &= 1^{8}/4^{8}+8; \quad Z(4-,1,\tau)=8+... \\
Z(1,8-,\tau) &= 1^{4}4^{2}/2^{2}8^{4}+4; Z(8-,1,\tau)=4+... \\
Z(1,16-,\tau) &= 1^{2}8/2.16^{2}+2; Z(16-,1,\tau)=2+... \\
Z(1,3-,\tau) &= 1^{12}/3^{12}+12; \quad Z(3-,1,\tau)=12+...
\end{aligned}
\tag{2}
$$

where

$$
\begin{aligned}
1^{24}/2^{24}+24 &= \eta(\tau)^{24}/\eta(2\tau)^{24}+24 \\
\eta(\tau) &= q^{1/24}\prod_{n\geq 1}(1-q^{n})
\end{aligned}
$$

etc.

$\diamond 11 : H^4(M, Z)$ and $K_3(ZM)$.

M denotes the Monster simple group. One knows that

$$H^2(M, Z) = H_1(M, Z) = H^3(M, Z) = H_2(M, Z) = 0.$$

Little is known beyond this, in particular it is not known if $H^4(M, Z)$ is non-zero. There is some information about the p-parts $H^4(M, Z)_p$, which for large enough p are zero (Charles Thomas, others?)

We've already established the diagram

$$\begin{array}{ccccc} \mathcal{W}^{hol}_M & \longrightarrow & H^4(M, Z) & \hookrightarrow & K_3(ZM) \\ & \searrow & \downarrow & & \\ & & H^3(ZM) & & \end{array}$$

We will explain how the element $(V^\natural, M) \in \mathcal{W}^{hol}_M$ detects a non-zero element in $H^4(M, Z)$.

It is very difficult to do this directly, but we can detect a non-zero image in $H^3(ZM)$ by looking at the twisted sectors and determining whether the representation of a centralizer $C_M(g)$ on the twisted sector $V^\natural(g), g \in M$, is truly projective i.e., whether θ_g is *not* a coboundary.

One can do this by using group theory and the Theorem on Modular-Invariance, which carries information about the traces of elements acting on twisted sectors. Without giving details, it can be shown that the representation of $C_M(g)$ on $V^\natural(g)$ is truly projective if g is of type $2-, 3-, 4-$ by showing that the representation on the constant homogeneous piece is already projective. It is expected that this holds also for type $8-$ and $16-$, but these two cases seem quite hard. Let's assume the $16-$ element is properly projective.

The following is true in general:

$$[Res_{C(g)}^{C(g^2)} \theta_{g^2}] = [\theta_g]^2.$$

This result is really a consequence of the fact that the coproduct in $D^\omega(G)$ is an algebra morphism.

It follows that $[\theta_{16-}]$ has order 16. Also, $[\theta_{3-}] \neq [0]$. The conclusion is that the image of $(V^\natural, M)$ under the map

$$\mathcal{W}_M^{hol} \longrightarrow H^3(ZG)$$

has order divisible by 48.

Meta-Theorem:

(a) The Moonshine Module/Monster pair $(V^\natural, M) \in \mathcal{W}_M^{hol}$ detects elements of order (at least) 48 in $H^4(M, Z)$ and $K_3(ZM)$.

(b) The 2-torsion part of the element in $H^4(M, Z)$ lies in the nil radical $J(H^*(M, Z))$ of the cohomology ring.

It turns out that we can detect when the 2-part of ω lies in $J(H^*(M, Z))$ using orbifold theory, a theorem of Quillen in group cohomology, and the Verlinde formula. (b) follows along these lines.

This will have to wait.