Spectral Triples

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is given by an involutive unital algebra $\mathcal{A}$ represented as operators in a Hilbert space $\mathcal{H}$ and a self-adjoint operator D with compact resolvent such that all commutators $[D, a]$ are bounded for $a \in \mathcal{A}$.

A spectral triple is *even* if the Hilbert space $\mathcal{H}$ is endowed with a $\mathbb{Z}/2$- grading γ which commutes with any $a \in \mathcal{A}$ and anticommutes with D.

Real Structure

A real structure of KO-dimension $n \in \mathbb{Z}/8$ on a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is an antilinear isometry $J : \mathcal{H} \to \mathcal{H}$, with the property that

$$J^2 = \varepsilon, \qquad JD = \varepsilon' DJ, \quad \text{and} \quad J\gamma = \varepsilon'' \gamma J \tag{1}$$

The numbers $\varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\}$ are a function of n mod 8 given by

n	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

Moreover, the action of $\mathcal{A}$ satisfies the commutation rule

$$[a, b^0] = 0 \quad \forall\, a, b \in \mathcal{A}, \tag{2}$$

where

$$b^0 = J b^* J^{-1} \qquad \forall b \in \mathcal{A}, \tag{3}$$

and the operator D satisfies the order one condition :

$$[[D, a], b^0] = 0 \qquad \forall\, a, b \in \mathcal{A}\,. \tag{4}$$

One defines a right $\mathcal{A}$-module structure on $\mathcal{H}$ by

$$\xi\, b = b^0\, \xi\,, \quad \forall\, \xi \in \mathcal{H}\,, \quad b \in \mathcal{A} \tag{5}$$

The unitary group of the algebra $\mathcal{A}$ then acts by the "adjoint representation" in $\mathcal{H}$ in the form

$$\xi \in \mathcal{H} \to \mathrm{Ad}(u)\, \xi = u\, \xi\, u^*\,, \quad \forall\, \xi \in \mathcal{H} \tag{6}$$

The inner fluctuations of the metric are given by

$$D \to D_A = D + A + \varepsilon'\, J\, A\, J^{-1} \tag{7}$$

where A is a self-adjoint operator of the form

$$A = \sum a_j [D, b_j]\,, \quad a_j, b_j \in \mathcal{A}. \tag{8}$$

Proposition

Let $(\mathcal{A}, \mathcal{H}, D)$ be a real spectral triple with antilinear isometry J fulfilling (3) and (4). Then for any gauge potential $A \in \Omega^1_D$, $A = A^*$ and any unitary $u \in \mathcal{A}$, one has

$$\mathrm{Ad}(u)(D + A + \varepsilon' \, J \, A \, J^{-1})\mathrm{Ad}(u^*) =$$

$$D + \gamma_u(A) + \varepsilon' \, J \, \gamma_u(A) \, J^{-1}$$

where

$$\gamma_u(A) = \, u \, [D, u^*] + \, u \, A \, u^*$$

Proposition

1) Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple and $D' = D + A$ for some $A \in \Omega^1_D$, $A = A^*$. Then for any $B \in \Omega^1_{D'}$, $B = B^*$ one has

$$D' + B = D + A' \,, \quad A' = A + B \in \Omega^1_D$$

2) Let $(\mathcal{A}, \mathcal{H}, D)$ be a real spectral triple with antilinear isometry J fulfilling (3) and (4). Let $A \in \Omega^1_D$, $A = A^*$ and $D' = D + A + \varepsilon' \, J \, A \, J^{-1}$. Then for any $B \in \Omega^1_{D'}$, $B = B^*$ one has

$$D' + B + \varepsilon' \, J \, B \, J^{-1} = D + A' + \varepsilon' \, J \, A' \, J^{-1} \,,$$

$$A' = A + B \in \Omega^1_D$$

Spectral Action

The starting point is the discussion of observables in gravity. By the principle of gauge invariance the only quantities which have a chance to be observable in gravity are those which are invariant under the gauge group *i.e.* the group of diffeomorphisms of the space-time M. Assuming first that we deal with a classical manifold, one can form a number of such invariants (under suitable convergence conditions) as the integrals of the form

$$\int_M F(K)\,\sqrt{g}\,d^4x \tag{9}$$

where $F(K)$ is a scalar invariant function* of the Riemann curvature K. Such invariants, of the form (9) appear as the *single integral* observables *i.e.* those which add up when evaluated on the direct sum of geometric spaces.

*the scalar curvature is one example of such a function but there are many others

Now while in theory a quantity like (9) is observable it is almost impossible to evaluate since it involves the knowledge of the entire spacetime and is in that way highly non localized. On the other hand, spectral data[†] are available in localized form anywhere, and are (asymptotically) of the form (9) when they are of the additive form

$$\mathrm{Trace}\,(f(D/\Lambda)), \tag{10}$$

where D is the Dirac operator and f is a positive even function of the real variable while the parameter Λ fixes the mass scale.

[†] the data of spectral lines are intimately related to the Dirac Hamiltonian, hence to the geometry of "space"

The spectral action principle asserts that the fundamental action functional S that makes it possible to compare different geometric spaces at the classical level and is used in the functional integration (after Wick rotation to euclidean signature) to go to the quantum level, is itself of the form (10). The detailed form of the even function f is largely irrelevant since, assuming* that the dimension spectrum is *simple*, the spectral action (10) can be expanded in decreasing powers of the scale Λ in the form

$$\mathrm{Trace}\,(f(D/\Lambda)) \sim \sum_{k \in \Pi^+} f_k\,\Lambda^k \int\!\!\!\!\!- |D|^{-k} + \quad (11)$$

$$+\, f(0)\,\zeta_D(0) + o(1),$$

where the function f only appears through the scalars

$$f_k = \int_0^\infty f(v)\,v^{k-1}\,dv. \qquad (12)$$

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*this hypothesis fails for conical singularities

The relation between the asymptotic expansion,

$$\text{Trace}\,(e^{-t\Delta}) \sim \sum a_\alpha\, t^\alpha \qquad (t \to 0) \tag{13}$$

and the ζ function,

$$\zeta_D(s) = \text{Trace}\,(\Delta^{-s/2}) \tag{14}$$

is given by,

- A non-zero term a_α with $\alpha < 0$ gives a *pole* of ζ_D at -2α with

$$\text{Res}_{s=-2\alpha}\, \zeta_D(s) = \frac{2\, a_\alpha}{\Gamma(-\alpha)} \tag{15}$$

- The absence of $\log t$ terms gives regularity at 0 for ζ_D with

$$\zeta_D(0) = a_0\,. \tag{16}$$

For the positive operator $\Delta = D^2$ one has,

$$|D|^{-s} = \Delta^{-s/2} = \frac{1}{\Gamma\left(\frac{s}{2}\right)} \int_0^\infty e^{-t\Delta}\, t^{s/2-1}\, dt \tag{17}$$

using

$$\int_0^1 t^{\alpha+s/2-1}\, dt = (\alpha + s/2)^{-1}$$

one gets the first statement. The second follows from the equivalence

$$\frac{1}{\Gamma\left(\frac{s}{2}\right)} \sim \frac{s}{2}, \quad s \to 0$$

so that only the pole part at $s = 0$ of

$$\int_0^\infty \mathrm{Tr}(e^{-t\Delta})\, t^{s/2-1}\, dt$$

contributes to the value $\zeta_D(0)$. But this pole part is given by

$$a_0 \int_0^1 t^{s/2-1}\, dt = a_0 \frac{2}{s}$$

and thus one gets (16).

Riemannian geometry and spectral triples

A spin Riemannian manifold M gives rise in a canonical manner to a spectral triple. The Hilbert space $\mathcal{H}$ is the Hilbert space $L^2(M,S)$ of square integrable spinors on M and the algebra $\mathcal{A}=C^\infty(M)$ of smooth functions on M acts in $\mathcal{H}$ by multiplication operators :

$$(f\,\xi)(x)=\,f(x)\,\xi(x)\,,\quad \forall x\in M \tag{18}$$

The operator D is the Dirac operator,

$$\partial\!\!\!/_M=\sqrt{-1}\,\gamma^\mu\,\nabla^s_\mu \tag{19}$$

where ∇^s is the spin connection which we express in a vierbein e so that

$$\begin{aligned}\gamma^\mu &= \gamma^a e^\mu_a\\ \nabla^s_\mu &= \partial_\mu+\frac{1}{4}\omega_\mu^{ab}\,(e)\,\gamma_{ab}\end{aligned} \tag{20}$$

The grading γ is given by the chirality operator which we denote by γ_5 in the 4-dimensional case. The operator J is the charge conjugation operator.

The finite geometry

Our only input is the following algebra :

$$\mathcal{A}_{LR} = \mathbb{C} \oplus \mathbb{H}_L \oplus \mathbb{H}_R \oplus M_3(\mathbb{C}) \qquad (21)$$

which is the direct sum of the matrix algebras $M_N(\mathbb{C})$ for $N = 1, 3$ with two copies of the algebra $\mathbb{H}$ of quaternions. The indices L, R are just there for book-keeping. By construction $\mathcal{A}_{LR}$ is an involutive algebra, with involution

$$(\lambda, q_L, q_R, m)^* = (\bar{\lambda}, \bar{q}_L, \bar{q}_R, m^*) \qquad (22)$$

using the involution $q \to \bar{q}$ of the algebra of quaternions. It admits a natural subalgebra $\mathbb{C} \oplus M_3(\mathbb{C})$ corresponding to integer spin, which is an algebra over $\mathbb{C}$. The subalgebra $\mathbb{H}_L \oplus \mathbb{H}_R$ corresponding to half-integer spin is an algebra over $\mathbb{R}$.

The bimodule $\mathcal{M}_F$

Let $\mathcal{M}$ be a bimodule over an involutive algebra $\mathcal{A}$. For $u \in \mathcal{A}$ unitary, *i.e.* such that $uu^* = u^*u = 1$, one defines Ad(u) by Ad$(u)\xi = u\xi u^* \, \forall \xi \in \mathcal{M}$.

Let $\mathcal{M}$ be an $\mathcal{A}_{LR}$-bimodule. Then $\mathcal{M}$ is *odd* iff the adjoint action of $s = (1,-1,-1,1)$ fulfills Ad$(s) = -1$.

Such a bimodule is a representation of the reduction $\mathcal{B} = (\mathcal{A}_{LR} \otimes_{\mathbb{R}} \mathcal{A}^0_{LR})_p$ of $\mathcal{A}_{LR} \otimes_{\mathbb{R}} \mathcal{A}^0_{LR}$ by the projection $p = \frac{1}{2}(1 + s \otimes s^0)$. This subalgebra is an algebra over $\mathbb{C}$ and we restrict to complex representations.

One defines the contragredient bimodule of a bimodule $\mathcal{M}$ as the complex conjugate space

$$\mathcal{M}^0 = \{\bar{\xi} \, ; \, \xi \in \mathcal{M}\}, \quad a\,\bar{\xi}\,b = \overline{b^* \xi \, a^*} \qquad (23)$$

Let $\mathcal{M}_F$ be the direct sum of all inequivalent irreducible odd $\mathcal{A}_{LR}$-bimodules.

- The dimension of the complex vector space $\mathcal{M}_F$ is 32.
- The $\mathcal{A}_{LR}$-bimodule $\mathcal{M}_F$ is the direct sum of the following bimodule and its contragredient

$$\mathcal{E} = \mathbf{2}_L \otimes \mathbf{1}^0 \oplus \mathbf{2}_R \otimes \mathbf{1}^0 \oplus \mathbf{2}_L \otimes \mathbf{3}^0 \oplus \mathbf{2}_R \otimes \mathbf{3}^0 \tag{24}$$

- The $\mathcal{A}_{LR}$-bimodule $\mathcal{M}_F$ is isomorphic with the contragredient bimodule $\mathcal{M}_F^0$ by the antilinear isometry J_F,

$$J_F(\xi, \bar{\eta}) = (\eta, \bar{\xi}) \,, \quad \forall \xi \,,\ \eta \in \mathcal{E} \tag{25}$$

- One has

$$J^2 = 1 \,, \quad \xi \, b = J b^* J \, \xi \,, \quad \forall \xi \in \mathcal{M}_F \,,\ b \in \mathcal{A}_{LR} \tag{26}$$

The algebra $\mathcal{B} = (\mathcal{A}_{LR} \otimes_{\mathbb{R}} \mathcal{A}^0_{LR})_p$ is the direct sum of 4 copies of the algebra $M_2(\mathbb{C}) \oplus M_6(\mathbb{C})$.

The sum of irreducible representations of $\mathcal{B}$ has dimension 32 and is given by

$$\mathbf{2}_L \otimes \mathbf{1}^0 \oplus \mathbf{2}_R \otimes \mathbf{1}^0 \oplus \mathbf{2}_L \otimes \mathbf{3}^0 \oplus \mathbf{2}_R \otimes \mathbf{3}^0$$

$$\oplus \mathbf{1} \otimes \mathbf{2}^0_L \oplus \mathbf{1} \otimes \mathbf{2}^0_R \oplus \mathbf{3} \otimes \mathbf{2}^0_L \oplus \mathbf{3} \otimes \mathbf{2}^0_R$$

The subalgebra and the order one condition

We let $\mathcal{H}_F$ be the sum of $N = 3$ copies of the $\mathcal{A}_{LR}$-bimodule $\mathcal{M}_F$. The multiplicity $N = 3$ is an input. We define the $\mathbb{Z}/2$-grading γ_F by

$$\gamma_F = c - J_F\, c\, J_F\,, \quad c = (0, 1, -1, 0) \in \mathcal{A}_{LR}$$

One then checks that the following holds

$$J_F^2 = 1\,, \quad J_F\, \gamma_F = -\, \gamma_F\, J_F \tag{27}$$

which together with the commutation of J_F with the Dirac operators, is characteristic of KO-dimension equal to 6 modulo 8.

The left action of $\mathcal{A}_{LR}$ splits as the sum of two representations π (in $\mathcal{H}_f$) and π' (in $\mathcal{H}_{\bar{f}}$). These representations of $\mathcal{A}_{LR}$ are disjoint (*i.e.* they have no equivalent subrepresentations) and this precludes the existence of operators D in $\mathcal{H}_F$ which fulfill the order one condition (4) and mix the subspaces $\mathcal{H}_f$ and $\mathcal{H}_{\bar{f}}$.

We shall now show that such mixing is obtained by passing to a unique subalgebra of maximal dimension.

Proposition

There exists a unique (up to an automorphism of $\mathcal{A}_{LR}$) subalgebra $\mathcal{A}_F \subset \mathcal{A}_{LR}$ of maximal dimension admitting an off diagonal Dirac. It is given by

$$\mathcal{A}_F = \{(\lambda, q_L, \lambda, m) \mid \lambda \in \mathbb{C}\,,\ q_L \in \mathbb{H}\,,\ m \in M_3(\mathbb{C})\}$$

$$\sim \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}).$$

For any operator $T : \mathcal{H}_f \to \mathcal{H}_{\bar{f}}$ we let

$$\mathcal{A}(T) = \{b \in \mathcal{A}_{LR} \mid \pi'(b)T = T\pi(b)\,,$$

$$\pi'(b^*)T = T\pi(b^*)\}$$

It is by construction an involutive unital subalgebra of $\mathcal{A}_{LR}$.

Let $\mathcal{A} \subset \mathcal{A}_{LR}$ be an involutive unital subalgebra of $\mathcal{A}_{LR}$.

1. If the restriction of π and π' to $\mathcal{A}$ are disjoint, there is no off diagonal Dirac for $\mathcal{A}$.
2. If there exists an off diagonal Dirac for $\mathcal{A}$, there exists a pair e, e' of minimal projections in the commutants of $\pi(\mathcal{A}_{LR})$ and $\pi'(\mathcal{A}_{LR})$ and an operator T such that $e'Te = T \neq 0$ and $\mathcal{A} \subset \mathcal{A}(T)$.

Unimodular unitary group of $\mathcal{A}_F$

The unitary group of an involutive algebra $\mathcal{A}$ is given by

$$\mathsf{U}(\mathcal{A}) = \{u \in \mathcal{A} \mid uu^* = u^*u = 1\}$$

In our context we define the special unitary group $\mathsf{SU}(\mathcal{A}) \subset \mathsf{U}(\mathcal{A})$ by :

$$\mathsf{SU}(\mathcal{A}_F) = \{u \in \mathsf{U}(\mathcal{A}_F) \ : \ \mathrm{Det}(u) = 1\}$$

where $\mathrm{Det}(u)$ is the determinant of the action of u in $\mathcal{H}_F$.

1. The group SU($\mathcal{A}_F$) is, up to an abelian finite group,

$$\text{SU}(\mathcal{A}_F) \sim \text{U}(1) \times \text{SU}(2) \times \text{SU}(3)$$

2. The adjoint action of the U(1) factor is given by multiplication of the basis vectors in $\mathcal{H}_f$ by the following powers of $\lambda \in \text{U}(1)$:

	$\uparrow \otimes \mathbf{1}^0$	$\downarrow \otimes \mathbf{1}^0$	$\uparrow \otimes \mathbf{3}^0$	$\downarrow \otimes \mathbf{3}^0$
$\mathbf{2}_L$	-1	-1	$\frac{1}{3}$	$\frac{1}{3}$
$\mathbf{2}_R$	0	-2	$\frac{4}{3}$	$-\frac{2}{3}$

The classification of Dirac operators

We now characterize all operators D_F which qualify as Dirac operators and moreover commute with the subalgebra

$$\mathbb{C}_F \subset \mathcal{A}_F\,, \quad \mathbb{C}_F = \{(\lambda,\lambda,0)\,,\lambda\in\mathbb{C}\} \qquad (28)$$

A Dirac operator is a self-adjoint operator D in $\mathcal{H}_F$ commuting with J_F, $\mathbb{C}_F$, anticommuting with γ_F and fulfilling the order one condition $[[D,a],b^0]=0$ for any $a,b\in\mathcal{A}_F$.

In order to state the classification of Dirac operators we introduce the following notation : let $Y_{(\downarrow 1)}$, $Y_{(\uparrow 1)}$, $Y_{(\downarrow 3)}$, $Y_{(\uparrow 3)}$ and Y_R be three by three matrices, we then let $D(Y)$ be the operator in $\mathcal{H}_F$ given by

$$D(Y)=\begin{bmatrix} S & T^* \\ T & \bar{S} \end{bmatrix} \tag{29}$$

where

$$S=S_1\oplus(S_3\otimes 1_3) \tag{30}$$

and in the decomposition $(\uparrow_R,\downarrow_R,\uparrow_L,\downarrow_L)$

$$S_1=\begin{bmatrix} 0 & 0 & Y^*_{(\uparrow 1)} & 0 \\ 0 & 0 & 0 & Y^*_{(\downarrow 1)} \\ Y_{(\uparrow 1)} & 0 & 0 & 0 \\ 0 & Y_{(\downarrow 1)} & 0 & 0 \end{bmatrix}$$

$$S_3 = \begin{bmatrix} 0 & 0 & Y^*_{(\uparrow 3)} & 0 \\ 0 & 0 & 0 & Y^*_{(\downarrow 3)} \\ Y_{(\uparrow 3)} & 0 & 0 & 0 \\ 0 & Y_{(\downarrow 3)} & 0 & 0 \end{bmatrix}$$

while the operator T is 0 except on the subspace $E_R = \uparrow_R \otimes \mathbf{1}^0 \subset \mathcal{H}_F$ which it maps, using the matrix Y_R, to the conjugate $J_F \, E_R$.

Theorem

1. Let D be a Dirac operator. There exists 3×3 matrices $Y_{(\downarrow 1)}$, $Y_{(\uparrow 1)}$, $Y_{(\downarrow 3)}$, $Y_{(\uparrow 3)}$ and Y_R, with Y_R symmetric, such that $D = D(Y)$.
2. All operators $D(Y)$ (with Y_R symmetric) are Dirac operators.
3. The operators $D(Y)$ and $D(Y')$ are conjugate by a unitary operator commuting with $\mathcal{A}_F$, γ_F and J_F iff there exists unitary matrices V_j and W_j such that

$$Y'_{(\downarrow 1)} = V_1\, Y_{(\downarrow 1)} V_3^* \,,\; Y'_{(\uparrow 1)} = V_2\, Y_{(\uparrow 1)}\, V_3^* \,,$$

$$Y'_{(\downarrow 3)} = W_1\, Y_{(\downarrow 3)}\, W_3^* \,,\; Y'_{(\uparrow 3)} = W_2\, Y_{(\uparrow 3)}\, W_3^* \,,$$

$$Y'_R = V_2\, Y_R\, \bar{V}_2^*$$

Lemma

Let $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$ be an operator in $\mathcal{H}_F = \mathcal{H}_f \oplus \mathcal{H}_{\bar{f}}$. Then $P \in \mathcal{A}'_F$ iff the following holds

- P_{11} is block diagonal with three blocks in $M_{12}(\mathbb{C})$, $M_{12}(\mathbb{C})$, and $1_2 \otimes M_{12}(\mathbb{C})$ corresponding to the subspaces where the action of (λ, q, m) is by λ, $\bar{\lambda}$ and q.
- P_{12} has support in $\mathbf{1} \otimes \mathbf{2}_L^0 \oplus \mathbf{1} \otimes \mathbf{2}_R^0$ and range in $\uparrow_R \otimes \mathbf{1}^0 \oplus \uparrow_R \otimes \mathbf{3}^0$.
- P_{21} has support in $\uparrow_R \otimes \mathbf{1}^0 \oplus \uparrow_R \otimes \mathbf{3}^0$ and range in $\mathbf{1} \otimes \mathbf{2}_L^0 \oplus \mathbf{1} \otimes \mathbf{2}_R^0$.
- P_{22} is of the form

$$P_{22} = T_1 \oplus (T_2 \otimes 1_3)$$

The moduli space

Let us start by the moduli space $\mathcal{C}_3$ of pairs of three by three matrices $(Y_{(\downarrow 3)}, Y_{(\uparrow 3)})$ modulo the equivalence relation :

$$Y'_{(\downarrow 3)} = W_1\, Y_{(\downarrow 3)}\, W_3^*\,,\; Y'_{(\uparrow 3)} = W_2\, Y_{(\uparrow 3)}\, W_3^*$$

where the W_j are unitary matrices. Each equivalence class contains a pair $(Y_{(\downarrow 3)}, Y_{(\uparrow 3)})$ where $Y_{(\uparrow 3)}$ is diagonal (in the given basis) and with positive entries, while $Y_{(\downarrow 3)}$ is positive. Indeed the freedom to chose W_2 and W_3 allows to make $Y_{(\uparrow 3)}$ positive and diagonal and the freedom in W_1 then allows to make $Y_{(\downarrow 3)}$ positive. The eigenvalues are the charateristic values of $Y_{(\uparrow 3)}$ and $Y_{(\downarrow 3)}$ and are invariants of the pair. We can thus find diagonal matrices $\delta_\uparrow$ and $\delta_\downarrow$ and a unitary matrix C such that

$$Y_{(\uparrow 3)} = \delta_\uparrow\,, \quad Y_{(\downarrow 3)} = C\,\delta_\downarrow\, C^*$$

Lemma

Two pairs of the form $(\delta_\uparrow, C\,\delta_\downarrow\,C^*)$ are equivalent iff there exists diagonal unitary matrices $A, B \in N$ such that

$$A\,C = C'\,B$$

Proof

If $A\,C = C'\,B$ one has

$$A\,Y_{(\uparrow 3)}\,A^* = Y'_{(\uparrow 3)}\,, \quad A\,Y_{(\downarrow 3)}\,A^* = Y'_{(\downarrow 3)}$$

Conversely one gets $W_1 = W_3$ and $W_2 = W_3$ from the uniqueness of the polar decomposition, and thus $W_3 = W$ is diagonal (provided we ordered the eigenvalues in the same order) and we get

$$W\,C\,\delta_\downarrow\,C^*\,W^* = \,C'\,\delta_\downarrow\,C'^*$$

so that $W\,C = C'\,B$ for some diagonal matrix B. Since W and B have the same determinant one can assume that they both belong to N.

The dimension of the moduli space is thus $3+3+4=10$ where the $3+3$ comes from the eigenvalues and the $4=8-4$ from the above double coset space of C's. One way to parameterize the representatives of the double cosets of the matrix C is by means of three angles θ_i and a phase δ,

$$V = \begin{pmatrix} c_1 & -s_1c_3 & -s_1s_3 \\ s_1c_2 & c_1c_2c_3 - s_2s_3e_\delta & c_1c_2s_3 + s_2c_3e_\delta \\ s_1s_2 & c_1s_2c_3 + c_2s_3e_\delta & c_1s_2s_3 - c_2c_3e_\delta \end{pmatrix},$$

for $c_i = \cos\theta_i$, $s_i = \sin\theta_i$, and $e_\delta = \exp(i\delta)$.

Let us now consider the moduli space $\mathcal{C}_1$ of triplets $(Y_{(\downarrow 1)}, Y_{(\uparrow 1)}, Y_R)$, with Y_R symmetric, modulo the equivalence relation

$$Y'_{(\downarrow 1)} = V_1 \, Y_{(\downarrow 1)} V_3^* \, , \; Y'_{(\uparrow 1)} = V_2 \, Y_{(\uparrow 1)} \, V_3^* \, ,$$

$$Y'_R = V_2 \, Y_R \, \bar{V}_2^*$$

By construction one has a natural surjective map

$$\pi \; : \; \mathcal{C}_1 \mapsto \mathcal{C}_3$$

just forgetting about Y_R. The generic fiber of π is the space of symmetric complex there by three matrices modulo the action of a complex scalar λ of modulus one by

$$Y_R \to \lambda^2 \, Y_R$$

The (real) dimension of the fiber is $12 - 1 = 11$. The total dimension of the moduli space is thus 31.

Dimension, KO-theory, and Poincaré duality

The notion of manifold in noncommutative geometry is discussed in terms of Poincaré duality in KO-homology. We now have to find out how the new finite noncommutative geometry F behaves with respect to this duality. We first note that now the dimension being equal to 6 modulo 8 the intersection pairing is *skew symmetric*. It is given explicitly as follows :

The following defines an antisymmetric bilinear pairing on $K_0 \times K_0$:

$$\langle e, f \rangle = \mathrm{Tr}(\gamma\, e\, J f J^{-1}) \tag{31}$$

The group $K_0(\mathcal{A}_F)$ is the free abelian group generated by the classes of $e = (1, 0, 0)$, $e_L = (0, 1, 0)$ and $f_3 = (0, 0, f)$.

1. The representation of the algebra generated by $(\mathcal{A}_F, D_F, J_F, \gamma_F)$ in $\mathcal{H}_F$ splits as a direct sum of two subrepresentations,

$$\mathcal{H}_F = \mathcal{H}_F^{(1)} \oplus \mathcal{H}_F^{(3)}$$

2. In the generic case each of these subrepresentations is irreducible.
3. In the basis (e, e_L, f_3) the pairing (31) is up to an overall multiplicity three (the number of generations) given by,

$$\mathcal{H}_F^{(1)} : \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathcal{H}_F^{(3)} : \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & 0 \end{bmatrix}$$

The product geometry

We now consider a 4-dimensional smooth compact Riemannian manifold M with a fixed spin structure and consider its product with a geometry of dimension 6 modulo 8. With $(\mathcal{A}_j, \mathcal{H}_j, \gamma_j)$ of KO-dimensions 4 for $j = 1$ and 6 for $j = 2$, the product geometry is given by the rules,

$$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2 \ , \ \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \ ,$$

$$D = D_1 \otimes 1 + \gamma_1 \otimes D_2 \, , \ \gamma = \gamma_1 \otimes \gamma_2 \, , \ J = J_1 \otimes J_2$$

Notice that it matters that J_1 commutes with γ_1 to check that J commutes with D. One checks that the order one condition is fulfilled by D if it is fulfilled by the D_j.

For the product of the manifold M by the finite geometry F we thus have $\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_F = C^\infty(M, \mathcal{A}_F)$, $\mathcal{H} = L^2(M,S) \otimes \mathcal{H}_F = L^2(M, S \otimes \mathcal{H}_F)$ and $D = \partial\!\!\!/_M \otimes 1 + \gamma_5 \otimes D_F$ where $\partial\!\!\!/_M$ is the Dirac operator on M. It is given by equations (19) and (20).