# Design of Coherent Quantum Observers for Linear Quantum Systems

Shanon L. Vuglar and Hadis Amini


## Abstract

Quantum versions of control problems are often more difficult than their classical counterparts because of the additional constraints imposed by quantum dynamics. For example, the quantum LQG and quantum $H^\infty$ optimal control problems remain open. To make further progress, new, systematic and tractable methods need to be developed. This paper gives three algorithms for designing coherent quantum observers, i.e., quantum systems that are connected to a quantum plant and their outputs provide information about the internal state of the plant. Importantly, coherent quantum observers avoid measurements of the plant outputs. We compare our coherent quantum observers with a classical (measurement-based) observer by way of an example involving an optical cavity with thermal and vacuum noises as inputs.


## Quantum System Models

**Plant Model**

$$\begin{aligned} dx(t) &= Ax(t)\,dt + B\,dw(t); \\ dy(t) &= Cx(t)\,dt + D\,dw(t); \end{aligned} \qquad (1)$$

**Observer Model**

$$\begin{aligned} d\xi(t) &= \hat{A}\xi(t)\,dt + \hat{B}\,dy(t) + B_{v_1}\,dv_1(t) + B_{v_2}\,dv_2(t), \\ d\eta(t) &= \hat{C}\xi(t)\,dt + dv_1(t) \end{aligned} \qquad (2)$$

| | |
|---|---|
| $x(t)$ | plant state, |
| $dy(t)$ | plant output, |
| $\xi(t)$ | observer state, |
| $d\eta$ | observer output, |
| $dw(t)$ | quantum wiener processes, |
| $dv_1(t), dv_2(t)$ | quantum vacuum wiener processes. |

## Physical Realizability Result

*Physical Realizability* is the property of a set of equations of the form (1) or (2), that they correspond to a *quantum harmonic oscillator*, i.e. a meaningful quantum system.

**Theorem 1** *Consider an LTI system of the form (2) where $\hat{A}, \hat{B}$ and $\hat{C}$ are given. There exists $B_{v_1}$ and $B_{v_2}$ such that the system is physically realizable with the dimension of $dv_1$ equal to that of $dy$, and the dimension of $dv_2$ equal to the rank of the matrix*

$$\left(\Theta_\xi \hat{B}\Theta_y \hat{B}^T \Theta_\xi - \Theta_\xi \hat{A} - \hat{A}^T \Theta_\xi - \hat{C}^T \Theta_\eta \hat{C}\right).$$

*It is not possible to choose $B_{v_1}$ and $B_{v_2}$ such that the system is physically realizable and the sum of the dimensions of $dv_1(t)$ and $dv_2(t)$ is less than that given above.*

Here the various $\Theta$ are canonical commutaion matrices with dimensions corresponding to the dimensions of their subscripts.

## Coherent Observers

A coherent quantum observer is a quantum system with an internal state $\xi$ that is designed to estimate the internal variable of the plant's dynamics (1) and that has the following properties:

(i) it is designed such that the tracking error estimation $\langle x - \xi \rangle_\rho$ of the plant dynamics (1) exponentially converges to zero in the sense of expected values

(ii) it has second order dynamics such that the following limit exists,

$$\bar{J} = \lim_{t\to\infty} \frac{1}{2}\left[\left\langle (x-\xi)(x-\xi)^T \right\rangle_\rho + \left\langle \left((x-\xi)(x-\xi^T)\right)^T \right\rangle_\rho\right]. \qquad (3)$$

Here $\bar{J}$ is a performance metric which corresponds to the steady-state quantum expectation of the symmetrized error covariance matrix;

(iii) it is physically realizable.

## Algorithm 1

- Obtain a standard Kalman filter for the plant (1) by treating $dw$ as a classical Wiener processes with intensity $S_w$ where $dw\,dw^T = F_w\,dt$; $S_w = \mathfrak{Re}[F_w]$:

$$\begin{aligned} d\hat{x} &= (A - KC)\hat{x}\,dt + K\,dy; \\ d\hat{y} &= \hat{x}\,dt. \end{aligned}$$

Here, $K$ is the Kalman gain calculated in the usual way.

- Find $B_{v_1}$ and $B_{v_2}$ such that the following is physically realizable and hence a coherent observer.

$$\begin{aligned} d\xi &= (A - KC)\,\xi\,dt + K\,dy + B_{v_1}\,dv_1 + B_{v_2}\,dv_2; \\ d\eta &= \xi\,dt + dv_1. \end{aligned}$$

The coherent observer properties properties (i) and (ii) follow from the properties of the classical Kalman filter.
$\bar{J}$ is the unique symmetric positive definite solution to:

$$\begin{aligned} 0 &= \mathcal{A}_e \bar{J} + \bar{J}\mathcal{A}_e^T + \mathcal{B}_e S_{w,v} \mathcal{B}_e{}^T, \\ \mathcal{A}_e &= (A - KC), \\ \mathcal{B}_e &= \begin{bmatrix} (B - KD) & -B_{v_1} & -B_{v_2} \end{bmatrix}. \end{aligned}$$

$$\begin{bmatrix} dw \\ dv_1 \\ dv_2 \end{bmatrix}\begin{bmatrix} dw^T & dv_1^T & dv_2^T \end{bmatrix} = F_{w,v}\,dt; \quad S_{w,v} = \mathfrak{Re}[F_{w,v}].$$

## Algorithm 2

We introduce a free parameter $\rho > 0$ over which we optimize when designing the Kalman filter. This is to offset the impact of the as yet undetermined noise terms $B_{v_1}\,dv_1(t)$ and $B_{v_2}\,dv_2(t)$.

**Modified Plant Model**

$$\begin{aligned} dx &= Ax\,dt + B\,dw, \\ dy &= Cx\,dt + D\,dw + \rho\,d\tilde{w}. \end{aligned}$$

Here, $d\tilde{w}$ is a vacuum noise source with Ito product

$$d\tilde{w}\,d\tilde{w}^T = F_{\tilde{w}}\,dt,$$

where $F_{\tilde{w}}$ is a block diagonal matrix with each block equal to

$$\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}.$$

Take $S_{\tilde{w}}$ as the real part of $F_{\tilde{w}}$.
Repeat for different $\rho > 0$:

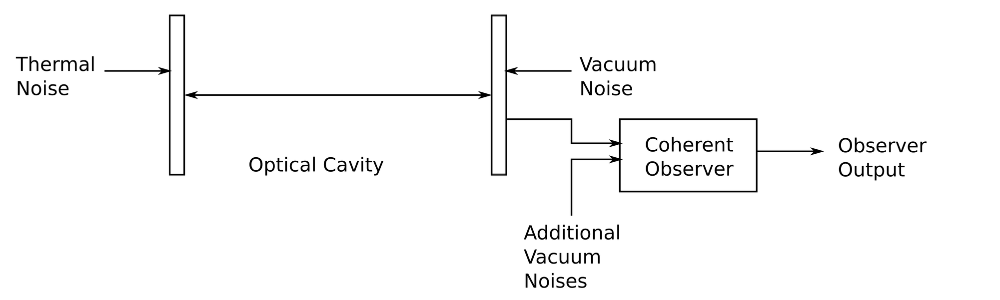
- Obtain the Kalman filter for the modified plant.
- Find $B_{v_1}$ and $B_{v_2}$ to obtain a coherent observer (2).
- Calculate $\bar{J}$ as in Algorithm 1. $\bar{J}$ is calculated for the actual plant and not for the modified plant.

Choose the coherent observer which gives the least value of $\bar{J}$.

## Example

We consider a quantum plant (right) consisting of an optical cavity with thermal and vacuum noise inputs. Its dynamics are described by the following QSDEs:

$$\begin{aligned} dx &= -\frac{1}{2}(\kappa_1 + \kappa_2)\,x\,dt - \sqrt{\kappa_1}\,dw_1 - \sqrt{\kappa_2}\,dw_2, \\ dy &= \sqrt{\kappa_1}x\,dt + dw_1. \end{aligned}$$

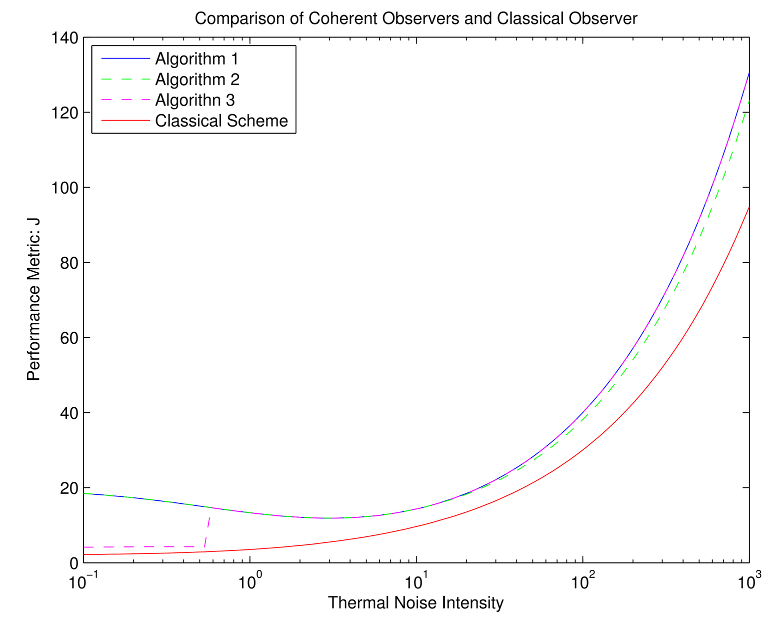



Here, $\kappa_1, \kappa_2$ are related to the mirror reflectances, $dw_1$ is vacuum noise and $dw_2$ is thermal noise of intensity $k_n$,

$$S_{w_1} = I_{2\times 2} \quad \text{and} \quad S_{w_2} = (1 + 2k_n)I_{2\times 2}.$$

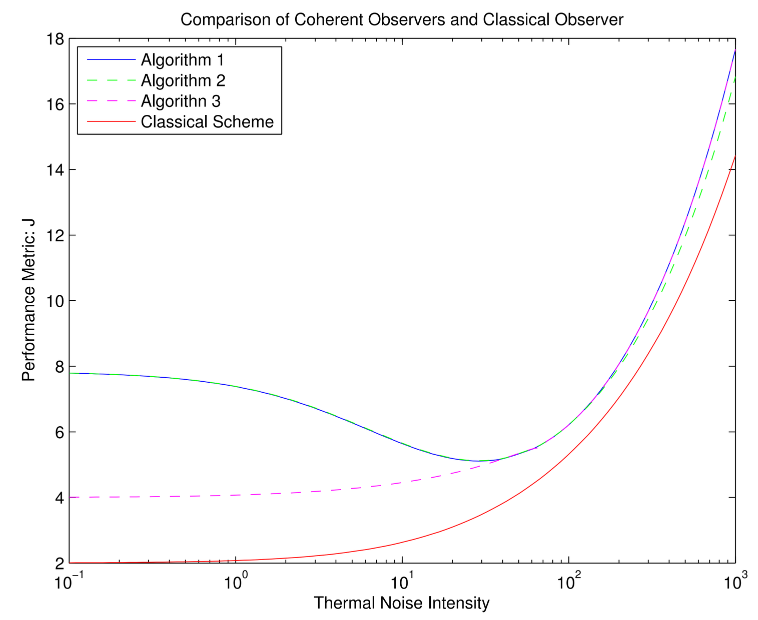
We compare the performance of our algorithms with a classical observer incorporating heterodyne measurement.

**Scenario 1**: $\kappa_1 = \kappa_2 = 0.1$



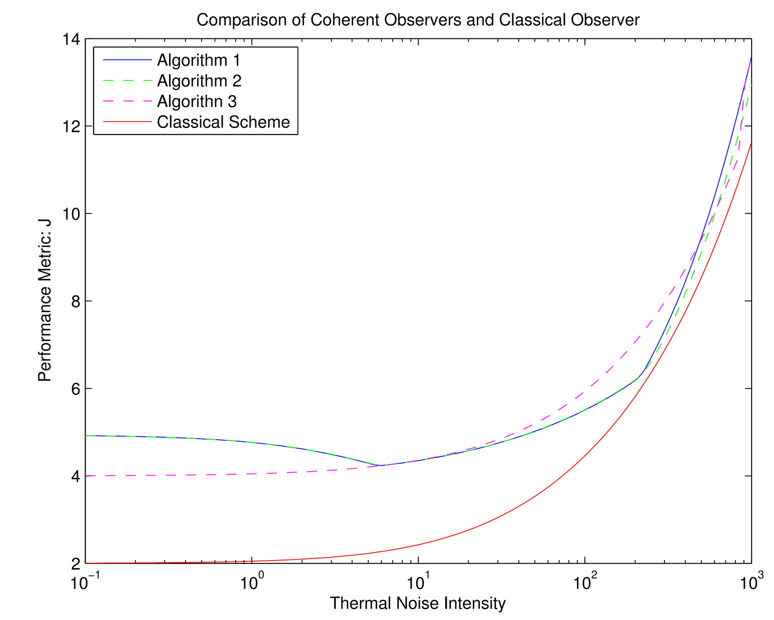

- The classical observer performs best. The coherent observers introduce at least as much additional quantum noise as does heterodyne measurement. The classical observer is optimal with respect to the measurement output whereas the coherent observers are suboptimal.
- As $k_n$ increases, the matrices $B_{v_1}$ and $B_{v_2}$ become more significant and there is greater scope for Algorithm 2 to outperform Algorithm 1.
- For small values of $k_n$, a suitable transformation matrix $T$ was found in Algorithm 3. In this regime, Algorithm 3 approaches the performance of the classical observer. The discontinuity in Algorithm 3's performance corresponds to the point above which, no suitable $T$ was found.
- For $k_n = 0$, we calculate $K = 0$; when $dw_2$ is a vacuum noise, the output of the plant gives no useful information about the internal state of the plant.

**Scenario 2**: $\kappa_1 = 0.5$; $\kappa_2 = 0.01$




- Compared to Scenario 1, mirror 1 is more lossy, mirror 2 is less lossy.
- Algorithm 3 outperforms Algorithm 1 for greater noise intensities $k_n$.
- For Algorithm 3, $T$ was found for $k_n < 69$.

**Scenario 3**: $\kappa_1 = 0.8$; $\kappa_2 = 0.01$




- Mirror 1 is even more lossy.
- For Algorithm 3, $T$ was found for $k_n < 910$.
- There is a region where Algorithm 2 performs better than Algorithm 3 despite introducing more quantum noises. In this region the impact of the $B_{v_1}$ term for Algorithm 3 is more significant than the combined impact of both the $B_{v_1}$ and $B_{v_2}$ terms for Algorithm 2.
- This scenario suggests that the performance metric $J$ obtained for Algorithms 1 and 2 is not necessarily smooth with respect to $k_n$. An explanation of this remains the subject of future research.

## Algorithm 3

We attempt to improve performance by using a state transformation of the Kalman filter obtained in Algorithm 1 that results in a coherent observer (2) where the $B_{v_2}\,dv_2(t)$ term vanishes.

- Obtain a Kalman filter as in Algorithm 1.
- Attempt to find a transformation $T$,

$$\tilde{\xi} = T\hat{x},$$

such that the system

$$\begin{aligned} d\tilde{\xi} &= T(A - KC)T^{-1}\tilde{\xi}\,dt + TK\,dy + \tilde{B}_{v_1}\,dv_1, \\ d\eta &= T^{-1}\tilde{\xi}\,dt + dv_1, \end{aligned}$$

is physically realizable for some $\tilde{B}_{v_1}$. This system is a coherent observer.

From previous results, if the Riccati equation

$$X\hat{B}\Theta\hat{B}^T X - \hat{A}^T X - X\hat{A} - \hat{C}^T\Theta\hat{C} = 0$$

has a non-singular, real, skew-symmetric solution $X$, then such a $T$ exists. Otherwise, we revert to Algorithm 1.

## Conclusion

Like the celebrated Kalman filter in the context of classical feedback control problems, it is envisaged that coherent quantum observers will play a pivotal role in solving coherent quantum feedback control problems. Here, we have proposed three algorithms for the design of coherent quantum observers. The key idea behind each of our algorithms was to first treat the quantum plants classically to obtain a Kalman filter. We then made use of previous results to obtain a physically realizable system by taking the Kalman filter obtained and allowing additional vacuum noise sources in its quantum implementation. Algorithm's 2 and 3 incorporate refinements to Algorithm 1 in an attempt to improve performance.
We compare the performance of the coherent quantum observers obtained with a measurement-based (classical) observer by way of an example involving an optical cavity with thermal and vacuum noise inputs. For each of the scenarios considered, the classical observer performs best. Algorithm 2 always performs at least as well as Algorithm 1. Algorithm 3 can potentially give a coherent quantum observer with a smaller number of quantum vacuum noise inputs than the other algorithms, however this does not guarantee better performance.


## Acknowledgments

The authors gratefully acknowledge Professors Hideo Mabuchi, Ian Petersen and Hendra Nurdin for helpful discussions. Shanon L. Vuglar is supported by the Australian Research Council Centre of Excellence for Quantum Computation and Communication Technology (Project No. CE110001027) and the Air Force Office of Scientific Research (Grant Nos. FA2386-09-1-4089 and FA2386-12-1-4075). Hadis Amini has a Math+X postdoctoral fellowship from the Simons Foundation.