The Generalized Polarization Tensors (GPTs) contain significant information on the shape of the domain and its material parameter. The aim of our efforts is to provide a method of constructing GPT-vanishing structures using shape derivative for two-dimensional conductivity or anti-plane elasticity problem. We design an optimization approach which solves the problem by minimizing a cost functional. In order to compute the shape derivative of this functional, we rigorously derive an asymptotic expansion of the perturbations of the GPTs that are due to small deformations of the boundaries of the coatings. We also present some numerical examples of GPT-vanishing structures for several different shaped inclusions.

Neutral inclusion

- Conductivity distribution
  \[ \sigma = \sigma_c + \sigma_a(x) \]
  \( (D) \) is the core and \( \Omega \) is the shell

- Conductivity (anti-plane elasticity) problem:
  \[ \nabla \cdot (\sigma \nabla u) = 0 \quad \text{in} \ R^2 \]
  \( \Omega \) is said to be neutral to the field \( \tau \) if the solution \( u \) satisfies
  \[ u(x) - H(x) = 0 \quad \text{in} \ R^2 \Omega \]
  The field \( \tau \) can not probe the neutral inclusion.

Why consider GPT-vanishing structure?

For radial structures, neutral inclusion is equivalent to PT-vanishing structure, but not for general shapes.

It was proved that it is only confocal ellipses which is neutral to multiple fields. Milton-Serkov (01) and Kang-Lee (14)

We can make the inclusion vaguely seen by making a first few terms of its GPTs vanish, Milton-Serkov (01) and Kang-Lee (14)

Consider the problem (1) with inclusion \( \sigma = \sigma_c + \sigma_a(x) \)

Conductivity distribution:

\[ \sigma = \sigma_c + \sigma_a(x) \quad D \]

A GPT-vanishing structure of order \( N \) is an inclusion \( \Omega \) such that

\[ \sum_{\alpha \beta} a_{\alpha \beta} M_{\alpha \beta}(\sigma, \Omega) = 0 \quad \text{for all} \ |\alpha|, |\beta| \leq N. \]

Construction of GPT-vanishing structures

The multipolar expansion:

\[ (u - H)(x) = \sum_{|\alpha| = 1}^{\infty} (-1)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x^{|\alpha|}} \mathcal{H}(x) \partial^{|\alpha|} \frac{M_{\alpha \beta}(\sigma, \Omega) \partial^{|\alpha|} M_{\alpha \beta}(\sigma, \Omega)}{|x|} \]

\( \mathcal{H} \) : a given entire harmonic function.

\( M_{\alpha \beta} \) : Generalized Polarization Tensors (GPTs).

Conductivity distribution:

\[ \sigma = \sigma_c + \sigma_a(x) \quad D \]

Aim: To construct GPT-vanishing structure of order \( N \) for the inclusion of general shape by using shape derivative.

Optimization Problem

Minimize a cost functional over \( \Omega_N \)

\[ J = \sum_{n \in \mathbb{N}} \sum_{i \in \mathbb{N}} |\alpha| < N \sum_{|\beta| < N} a_{\alpha \beta} M_{\alpha \beta}(\sigma, \Omega) \]

Here \( n \) is a integer with \( 2 \leq n \leq 2N \)

The GPT-vanishing structure of higher order

The GPT-vanishing structure of the inclusion can be constructed by minimizing the simulated GPTs. (HEM)

Algorithm

Gradient Descent Method:

\[ \sigma_{k+1} = \sigma_k - \frac{\partial J}{\partial \sigma_k} \]

where \( \sigma_k \) is the outward unit normal to \( \partial \Omega_{k+1} \)

Calculate \( [M_{\alpha \beta}] \) using modified boundaries

Conclusion

- Either the value of the conductivity \( \sigma_c \) of the core or the ratio \( \rho \) is higher, the coating of PT-vanishing structure is thinner.

- Numerical results show that the conductivities of GPT-vanishing structure from inside layer to the outmost layer should behave oscillated.

- A good initial guess will dramatically improve the speed for construction of GPT-vanishing structure.