Class forcing in a generalized context

We consider class forcing in a generalized context which allows more second-order objects than just the definable ones.

Definition 1. A pair $M = (\mathcal{M}, \mathcal{C})$ is a ground model if the following statements hold:

1. $\mathcal{C}$ is a countable subset of $\mathcal{M}$.
2. $\mathcal{M}$ is a countable transitive model of $\mathsf{ZF}^\omega$ in the language $\mathcal{L}_c$ enriched with additional predicates for every $A \subseteq \mathcal{C}$, i.e., $\mathcal{M}$ satisfies Separation and Replacement for $\mathcal{L}_c$-formulas in this extended language.
3. If $A_1, \ldots, A_n \subseteq \mathcal{C}$, then $\mathcal{M}$ contains all subsets of $M$ that are definable over $(\mathcal{M}, A_1, \ldots, A_n)$.

Example 1. Let $M$ be a countable transitive model of $\mathsf{ZFC}$ and let $\mathcal{M} = (\mathcal{M}, \mathcal{C})$ be the set of all subsets of $M$ that are definable over $(\mathcal{M}, \mathcal{C}_1, \mathcal{C}_2)$. Then $\mathcal{C} = \mathcal{M}$ is a ground model.

Sketch of the proof. Let for the Gödel code $\mathcal{G}$ be the formula in the forcing language of $\mathcal{M}$, then $\mathcal{M}$ is a ground model.

Let $\mathcal{M} = (\mathcal{M}, \mathcal{C})$ be a ground model.

In set forcing, every partial order $\mathcal{P}$ is countable and transitive, and $\mathcal{M} = (\mathcal{M}, \mathcal{C})$ is a model of Kelley-Morse class theory KM, then $\mathcal{M}$ is a ground model.

In class forcing, however, it is possible to have non-ground models. For example, consider a class forcing $\mathcal{P}$ in a generalized context which allows more second-order objects than just the definable ones. In this case, $\mathcal{M}$ may not be a ground model.

Boolean completions and the forcing theorem

Theorem 8. Assume that $\mathcal{M}$ satisfies either global choice or power set and let $\mathcal{P} = (\mathcal{P}, \mathcal{L})$ be a separative class forcing. Then the following statements are equivalent:

1. $\mathcal{P}$ satisfies the definability lemma for $\mathcal{M}$.
2. $\mathcal{P}$ satisfies the forcing theorem for all $\mathcal{L}$-formulas.
3. $\mathcal{P}$ satisfies the uniform forcing theorem for all $\mathcal{L}$-formulas.

Theorem 5. If $\mathcal{P}$ satisfies the definability lemma for $\mathcal{M}$, then $\mathcal{P}$ satisfies the forcing theorem for every $\mathcal{L}$-formula over $\mathcal{M}$.

Example. Let $\mathcal{P} = \mathcal{Col}(\omega_1, \mathcal{M})$ denote the class of all Cohen forcing conditions $\mathcal{C}$ such that $\mathcal{C} = \mathcal{M}$ is a ground model.

Theorem 10. If $\mathcal{M}$ is a countable transitive model of $\mathsf{ZFC}$ and $\mathcal{C} = \mathcal{M}(\mathcal{M})$, then $\mathcal{P}$ does not satisfy the forcing theorem over $\mathcal{M}$.

In particular, $\mathcal{P}$ does not have a Boolean completion.