On the classification and topology of complex map-germs of corank one and $A_e$-codimension one

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Abstract
Corank one map-germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, $n < p$, of $A_e$-codimension one are classified and their vanishing topology is shown to be homotopically equivalent to a sphere.
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1 Introduction

In his classic paper [10] Mather classified the $A$-stable map-germs. The next target for classification, the $A_e$-codimension one germs, appears to be considerably more difficult, as one does not have an equivalent of Mather’s result that $K$-equivalent $A$-stable maps are $A$-equivalent. For example, the two real maps, $(x, y) \to (x, y^3 \pm x^2y)$, have $A_e$-codimension one, are $K$-equivalent but not $A$-equivalent, see [11]. However, this problem does not occur in the complex situation for this example.

In his Ph.D. thesis, [1], Cooper classified corank 1 $A_e$-codimension 1 map-germs $\mathbb{C}^n$ to $\mathbb{C}^{n+1}$ by using explicit changes in source and target to reduce the map to a normal form. A more elementary proof of the classification is given in [2]. Surprisingly, just as in the stable case the situation comes down to dealing with the $K$-equivalence class of the germ mainly because if the map is not an augmentation then the $A$-orbit is open in the $K$-orbit.

In this paper we generalise to the case of corank 1 $A_e$-codimension 1 map-germs $\mathbb{C}^n$ to $\mathbb{C}^p$, $n < p$, i.e. the dimension of the target space is increased.

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2 The results

The main theorem is the following.

Theorem 2.1 Suppose that $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, $n < p$, is a corank 1 $A_e$-codimension 1 map-germ, then the following are true.
1. $f$ is $A$-equivalent to a map of the form,

$$
(u_1, \ldots, u_{l-1}, v_1, \ldots, v_{l-1}, w_1, w_{l+1}, \ldots, w_{l+1}, x_1, \ldots, x_{n-l-(r+2)+1}, y)
$$

$$
\mapsto (u_1, \ldots, u_{l-1}, v_1, \ldots, v_{l-1}, w_1, w_{l+1}, \ldots, w_{l+1}, x_1, \ldots, x_{n-l-(r+2)+1},
$$

$$
y^{l+1} + \sum_{i=1}^{l-1} v_i y^i, y^{l+2} + \sum_{i=1}^{l-1} v_i y^i + y^l \sum_{i=1}^{l-1} v_i y^i, \ldots, y^{n-l-(r+2)+1} + \sum_{i=1}^{l-1} w_i y^i),
$$

where $r = p - n - 1$ and $l + 1$ is the multiplicity of the germ. Conversely, any such germ has $A_l$-codimension $1$.

2. The germ is precisely $l + 2$-determined.

3. An $A_l$-versal unfolding is given by unfolding with the addition of the term $\lambda y^l$ to the $(p - rl - 1)$th component function.

One immediately deduces the following.

**Corollary 2.2** Corank 1 $A_l$-codimension 1 map-germs from $C^n$ to $C^p$ which are $K$-equivalent are $A$-equivalent.

To every finitely $A$-determined corank 1 map-germ there exists a unique stabilisation, see [7]. The image of this stabilisation is called the disentanglement of $f$. One can also investigate the multiple points in this image.

**Definition 2.3** Let $h : X \rightarrow Y$ be a continuous map. The $k$th image multiple point space of $h$, denoted $M_k(h)$, is defined to be,

$$
M_k(h) := \text{closure}\{y \in Y|\#h^{-1}(y) \geq k\}.
$$

**Definition 2.4** We define the $k$th disentanglement of $f$, denoted $\text{Dis}_k(f)$, to be the $k$th multiple point space of the stabilisation of $f$.

Suppose that $f_R : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$ is a finitely $A$-determined map-germ, with a real stabilisation $f_{R,t}$ and that the complexification of $f_R$, denoted $f_C$ has stabilisation arising from complexifying $f_{R,t}$. We can denote the $k$th image multiple point spaces of these maps by $\text{Dis}(f_R)$ and $\text{Dis}(f_C)$.

**Definition 2.5** The map $f_{R,t}$ is a good real perturbation if $\dim H_i(\text{Dis}_i(f_R); \mathbb{Z}) = \dim H_i(\text{Dis}_i(f_C); \mathbb{Z})$ for all $i = p - (p - n - 1)k - 1$, with $2 \leq k \leq p/(p - n)$.

This is a generalisation of the notion given in [12] and [9]. The idea is that the complex topology is visible over the reals.

**Theorem 2.6** Suppose that $f : (C^n, 0) \rightarrow (C^p, 0)$, $n < p$, is a corank 1 $A_l$-codimension 1 map-germ.

1. The disentanglement $\text{Dis}_1(f)$ is homotopically equivalent to a $(n-(p-n-1))$-sphere. The higher disentanglements are empty or contractible.

2. It is obvious that $f$ is the complexification of a real map-germ. This map has a good real perturbation and in fact the natural inclusion for this perturbation $\text{Dis}_k(f_R) \rightarrow \text{Dis}_k(f_C)$ is a homotopy equivalence for all $k \geq 1$.

These results are analogous to the case of a quadratic isolated complete intersection singularity. For then the Milnor fibre is homotopically equivalent to a single sphere and it is possible to define a real Milnor fibre with the same topology. (In fact the above theorem is a consequence of these results).

When an isolated complete intersection singularity has Milnor number equal to one then it is $K$-equivalent to a quadratic singularity. One may ask for corank 1 maps in the range $n < p$, if the disentanglement is homotopically a sphere, then is the map $A_l$-codimension 1?
3 Classification

3.1 Proof of Theorem 2.1 part 1

Firstly we define the augmentation of a map-germ.

Definition 3.1 Let \( f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^p, 0) \) be a map with a 1-parameter stable unfolding \( F : (\mathbb{C}^n \times \mathbb{C}, 0) \rightarrow (\mathbb{C}^p \times \mathbb{C}, 0) \), where \( F(x, \lambda) = (f(x), \lambda) \). Then the augmentation of \( f \) by \( F \) is the map \( A_F(f) := (f(x), \lambda) \).

If \( f \) has \( A_e \)-codimension 1 then \( A_F(f) \) has \( A_e \)-codimension 1 and the equivalence class of \( A_F(f) \) is independent of the choice of miniversal unfolding of \( f \). See Proposition 2.1 and Theorem 2.4 of [2]. Thus we can produce new codimension 1 maps from old codimension 1 maps. If \( f \) is not the augmentation of another germ then \( f \) is called primitive.

One can also generalise this definition so that the unfolding parameter is replaced by a function, see [4].

To prove part 1 of Theorem 2.1 we use results from the classification in the \( p = n + 1 \) case given in [2]. Let us follow them and begin by defining a map \( f^l : (\mathbb{C}^{l-1}, 0) \rightarrow (\mathbb{C}^l, 0) \) by

\[
f^l(u, v, y) = (u, v, y^{l+1} + \sum_{i=1}^{l-1} u_i y^i, y^{l+2} + \sum_{i=1}^{l-1} v_i y^i).
\]

By Lemma 4.1 of [2] the \( A_e \)-codimension is 1. If we label the last two coordinates of \( \mathbb{C}^l \) \( Y_1 \) and \( Y_2 \) then the \( A_e \)-tangent space is

\[
T A_e f^l = \theta(f^l) \left\{ y^l \partial / \partial Y_2, y^{l-1} \partial / \partial v_1, \ldots, y \partial / \partial v_{l-1} \right\} + y^{l-1} \partial / \partial v_1 + y^l \partial / \partial Y_2, \ldots, y \partial / \partial v_{l-1} + y^l \partial / \partial Y_2 \right\}.
\]

Let us now define an extension of this map, \( f^{l,r} : (\mathbb{C}^{l-1+r}, 0) \rightarrow (\mathbb{C}^{l+r+1}, 0) \):

\[
f^{l,r}(u, v, y, w) = (u, v, y^{l+1} + \sum_{i=1}^{l-1} u_i y^i, y^{l+2} + \sum_{i=1}^{l-1} v_i y^i, w, \sum_{i=1}^{l-1} w_i y^i, \ldots, \sum_{i=1}^{l-1} w_{r-1} y^i).
\]

Through augmentation we get a map of the form in Theorem 2.1. By the proof of Proposition 3.7 of [6] it is known that \( f^{l,r} \) is finitely determined. However we can do better than this as the following shows.

**Theorem 3.2** The map \( f^{l,r} \) has \( A_e \)-tangent space equal to

\[
T A_e f^{l,r} = \theta(f^{l,r}) \left\{ y^l \partial / \partial Y_2, y^{l-1} \partial / \partial v_1, \ldots, y \partial / \partial v_{l-1} \right\} + y^{l-1} \partial / \partial v_1 + y^l \partial / \partial Y_2, \ldots, y \partial / \partial v_{l-1} + y^l \partial / \partial Y_2 \right\}.
\]

Hence \( f^{l,r} \) has \( A_e \)-codimension equal to 1. To prove the above theorem let us investigate what the effect of extension is.

Suppose we have a finitely determined map \( h : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^p, 0) \) such that

\[
h(w_1, \ldots, w_l, y, u_1, \ldots, u_{l-1}, x) = (w_1, \ldots, w_l, \sum_{i=1}^{l-1} u_i y^i, y^{l+1} + \sum_{i=1}^{l-1} u_i y^i, u_1, \ldots, u_{l-1}, x, f_1(u, x, y), \ldots, f_l(u, x, y)).
\]

Let \( O_X \) denote the ring of function germs at 0 for the germ \( (X, 0) \). The tangent space \( T A_e \) is a \( h^*(O_{C^n}) \) submodule of \( (O_{C^n})^l \). Let \( e_i \) denote the standard basis vector for the \( i \)th copy of \( O_{C^n} \).
Lemma 3.3 \( \sigma \), \( r_i \in \mathcal{T} \mathcal{A}_e \) for all \( 1 \leq i \leq l + 1 \).

**Proof.** It is evident that we can reduce the requirement to \( y^k e_i \in \mathcal{T} \mathcal{A}_e \) for all \( i = 1, \ldots, l + 1 \).

Note that

\[
y^k e_{i+1} \in \mathcal{T} \mathcal{A}_e \iff y^{k-1} e_i \in \mathcal{T} \mathcal{A}_e, \quad k - i \geq 0. (+)
\]

This follows from the fact that \( y^i e_i + y^i e_{i+1} \in \mathcal{T} \mathcal{R}_e \) and it implies that it suffices to show that \( y^k e_{i+1} \in \mathcal{T} \mathcal{A}_e \) for all \( k \).

For \( 1 \leq s \leq l \), \( e_s + y^s e_{i+1} \in \mathcal{T} \mathcal{R}_e \) and \( \mathcal{E}_s \in \mathcal{T} \mathcal{L}_e \) so \( y^s e_{i+1} \in \mathcal{T} \mathcal{A}_e \). We will now use induction: Suppose \( y^s e_{i+1} \in \mathcal{T} \mathcal{A}_e \) for all \( s < k \) then \( y^k e_{i+1} \in \mathcal{T} \mathcal{A}_e \).

The number \( k \) will be of the form \( k = r(l+1) + i \) with \( r \geq 1 \) (assuming \( k < l + 1 \) already dealt with as above) and \( 0 \leq i \leq l \).

Case \( i = 0 \): Clearly \( y^{l+1} + \sum_{j=1}^{l-1} u_j y^j e_{i+1} \in \mathcal{T} \mathcal{L}_e \) so \( y^{r(l+1)} e_{i+1} \in \mathcal{T} \mathcal{A}_e \) as the other terms in \( y \) in the expansion have order less than \( r(l+1) \).

Case \( i > 0 \): The assumption \( y^s e_{i+1} \in \mathcal{T} \mathcal{A}_e \) for all \( s < r(l+1) + i \) implies that \( y^s e_i \in \mathcal{T} \mathcal{A}_e \) for all \( i \leq s < r(l+1) + i \) by (.), i.e.

\[
y^s e_i \in \mathcal{T} \mathcal{A}_e \quad \text{for all} \quad s < r(l+1). (++)
\]

After applying this lemma to the map \( f^{i,r} \) and all that is required is to check that if \( g \) is a function in variables \( w_1 \) to \( w_l \) then \( g y^t \partial / \partial Y_2 \) is in the target space. This is easy to check.

The maps \( f^{i,r} \) have a very interesting property which will be very useful.

**Lemma 3.4** The \( A \)-orbit of \( f^{i,r} \) is open in its \( K \)-orbit.

**Proof.** Let the dimension of the source be \( n \) and that of the target be \( p \). and denote the normal space of the \( G \)-orbit by \( NG_e \). It is easy to calculate that \( \dim NG_e(f^{i,r}) = p + 1 \) (It should be noted that this is not true for augmentations of \( f^{i,r} \) as then we have \( e_i \in \mathcal{T} \mathcal{L}_e \) for at least one \( i \)). Thus we find that \( \dim \mathcal{R}_e = \dim \mathcal{N}_e - p \). But \( \dim \mathcal{N}_e = \dim \mathcal{N}A - n \) (as \( f^{i,r} \) is not \( A \)-stable, see [14] p.110) and \( \dim \mathcal{N}_e = \dim \mathcal{N}C + (p-n) \) ([14] p.509). So \( \dim \mathcal{N}_e = \dim \mathcal{N}_e \), implying that the \( A \)-orbit is open in the \( K \)-orbit.

**Proof (of Theorem 2.1).** We now generalise the proof of Proposition 4.3 of [2]. Suppose that \( f : (C^p, 0) \to (C^p, 0) \) is a corank 1 \( A \)-codimension 1 map-germ, \( n < p \) with multiplicity \( l + 1 \). The versal unfolding \( G : (C^n \times C^0, 0) \to (C^n \times C^0, 0) \) is a \( n - l(p - n + 1) + 1 \)-fold prism on a minimal stable map-germ of multiplicity \( l + 1 \). Thus by Theorem 2.7 of [2] \( f \) is the \( n - l(p - n + 1) + 1 \)-fold augmentation of an \( A \)-codimension 1, corank 1, multiplicity \( l + 1 \) map-germ \( f' : (C^{l+1+p-n-1}, 0) \to (C^{l+1+p-n-1}, 0) \). Such a map is obviously \( K \)-equivalent to the map \( f^{i,p-n-1} \). The \( A \)-orbit of \( f^{i,p-n-1} \) is open in its \( K \)-orbit by Lemma 3.4 and by Lemma 3.12 of [2] there is at most one \( A \)-orbit in a given complex contact class, thus we conclude that \( f' \) and \( f^{i,p-n-1} \) are \( A \)-equivalent.

The \( n - l(p - n + 1) + 1 \)-fold augmentation of \( f^{i,p-n-1} \) is \( A \)-equivalent to \( f \) as the \( A \)-equivalence class of the augmentation of codimension 1 map-germ \( g \) depends only on the \( A \)-equivalence class of \( g \).
3.2 Order of determinacy

To find the order of determinacy we use the techniques of [13], in particular his Proposition 3.8, which we summarise as the following. Denote the maximal ideal of \( \mathcal{O}_C \) by \( m_d \) and use the standard \( tf \) and \( w_f \) notation of Singularity Theory, see [14].

**Proposition 3.5** Let \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}^r, 0) \) be a map-germ. Let

\[
D \subseteq tf(\theta_{\mathbb{C}^n}) + w_f(\theta_{\mathbb{C}^r}) + m_n^s \theta_f
\]

be an \( \mathcal{O}_{\mathbb{C}^n} \)-module such that

\[
m_n^s \theta_f \subseteq tf(m_n \theta_{\mathbb{C}^n}) + f^*(m_p) \cdot D + m_n^{s+1} \theta_f.
\]

Then \( f \) is \( s \)-determined.

Let \( f \) be as in Theorem 2.1. Then by calculation one can see that \( T_{\mathcal{A}_c}f \) has the same type of structure as \( T_{\mathcal{A}_c}(f^{l,r}) \): Let \( m = p - r - 1 \), then \( y^i e_m \) and \( y^{l-i} e_{l+i+1} \), \( i = 1, \ldots, l-1 \) are missing from \( T_{\mathcal{A}_c}f \), but \( y^i e_m + y^{l-i} e_{l+i+1} \) is included. Thus if we let \( m_{n-1} \) denote the ideal generated by the variables other than \( y \) and

\[
D = \langle \mathcal{O}_n, \ldots, \mathcal{O}_n, m_{n-1} \mathcal{O}_n + \langle y^i \rangle \mathcal{O}_n, \ldots, m_{n-1} \mathcal{O}_n + \langle y^l \rangle \mathcal{O}_n, \mathcal{O}_n, m_{n-1} \mathcal{O}_n + \langle y^{l+1} \rangle \mathcal{O}_n, \mathcal{O}_n, \ldots, \mathcal{O}_n \rangle
\]

where the \( m_{n-1} \mathcal{O}_n + \langle y^i \rangle \mathcal{O}_n \) terms begin at position \( i \), then \( D \) is an \( \mathcal{O}_n \)-module contained in \( T_{\mathcal{A}_c}f \).

The non-trivial problem is to show that, for all \( i \), \( y^{l+2} e_i \) is in the right hand side of the second inclusion in the proposition. For the positions corresponding to the functions \( u_1, \ldots, u_{l-1}, v_1, \ldots, v_{l-1} \) and \( w_{11}, \ldots, w_{l1} \) we can use elements of \( tf(m_n \theta_{\mathbb{C}^n}) \) modulo \( m_n^{l+3} \). For the \( r \) extension terms and position \( 2l - 1 \) we use \( y^{l+2} + \sum_{i=1}^{l-1} v_i y^i \), elements of \( tf \) and \( f^*(m_p) \cdot D \). For the remaining position we use \( y \theta_f / \partial y \) and terms in \( tf \) and \( f^*(m_p) \cdot D \).

So \( f \) is at least \( (l+2) \)-determined. This is in fact exact. The \( (l+1) \)-jet is not finitely \( \mathcal{A} \)-determined as can be seen by showing (using the method of [8]) that \( (l+1) \)th multiple point space has dimension greater than that of a finitely determined map-germ.

4 Topology

Theorem 2.6 part 1 on the topology of the \( k \)th disentanglement has been proved for the \( p = n + 1 \) in Corollary 5.3 of [5], though note that this was first proved in this case for \( k = 1 \) in [1], see [2]. For more general \( p \) that \( \text{Dis}_1(f) \) is homotopically equivalent to a sphere can be deduced from the proof of Proposition 3.7 of [6] and Theorem 4.24 of [3] but the following, which investigates higher disentanglements, also shows it.

We begin with noting from Theorem 3.2 of [5] that for an augmentation \( \text{Dis}_n(\mathcal{A}_F f) \) is homotopically equivalent to the suspension of \( \text{Dis}_n(f) \). Thus we can assume our map is primitive.

Define \( f_t : \mathbb{C}^{d-l} \times \mathbb{C}^l \to \mathbb{C}^d \times \mathbb{C}^{(l+1)} \) by

\[
f_t(u, v, w) = (u, v, y^{l+1} + \sum_{i=1}^{l-1} u_i y^i, y^{l+2} + \sum_{i=1}^{l-1} v_i y^i + t v^l, w, \sum_{i=1}^l w_{1i} y^i, \ldots, \sum_{i=1}^l w_{ri} y^i).
\]

Then, for \( t \neq 0 \) we can produce the disentanglement map for \( f_0 \).
Define \( g_t : \mathbb{C}^{2l-1} \to \mathbb{C}^l \) by \( g_t := f_t | f_t^{-1}(\mathbb{C}^l \times \{0\}) \), then \( g_t \) for \( t \neq 0 \) gives the disentanglement map for \( g_0 \), a corank 1 map-germ of \( A_\nu \)-codimension 1. The space \( \text{Dis}_m(g) \) is homotopically equivalent to a \( 2l-1 \) sphere if \( m = 1 \), and contractible or empty for \( m > 1 \) by Corollary 5.3 of [5]. We shall show that \( \text{Dis}_m(f_0) \) is homotopically equivalent to this space. In the following we assume \( t \neq 0 \) defines the disentanglement maps.

For a continuous map \( h : X \to Y \) of topological spaces let \( D^k(h) \) denote the \( k \)th multiple point space as defined in [8].

From the natural inclusion of \( \mathbb{C}^l \) into \( \mathbb{C}^{l+r(i+1)} \) we induce a natural map \( \phi^k : D^k(g_t) \to D^k(f_t) \).

It is shown in the proof of Proposition 3.7 of [6] that \( D^k(g_t) \) and \( D^k(f_t) \) are non-singular for \( k < l+1 \), and from the description there we can deduce that \( D^k(f_t) \) contracts equivariantly onto \( D^k(g_t) \). The only other non-trivial spaces are \( D^{l+1}(f_t) \) and \( D^{l+1}(g_t) \) and from the description in [6] it follows that these are \( S_k \)-equivariantly homeomorphic Milnor fibres of what is effectively the same isolated complete intersection singularity.

To conclude that the natural map \( \text{Dis}_m(g_0) \to \text{Dis}_m(f_0) \) induces an isomorphism on integer homology for all \( m \) we use Theorem 3.2 of [6]:

**Lemma 4.1** Suppose that \( h_i : X_i \to Y_i \), \( i = 1, 2 \), are finite and proper continuous maps for which the image computing spectral sequence exists (this is a technical condition which is true for the maps under consideration here) and that there exist continuous maps \( \phi \) and \( \psi \) such that the diagram

\[
\begin{array}{ccc}
\ h_1 : & X_1 & \to & Y_1 \\
\phi & \downarrow & \downarrow & \psi \\
\ h_2 : & X_2 & \to & Y_2 \\
\end{array}
\]

commutes. Then if the map \( \phi^k : D^k(h_1) \to D^k(h_2) \) is an \( S_k \)-homotopy equivalence for all \( k \geq 1 \), then \( \psi_{|M_m(h_1)} : M_m(h_1) \to M_m(h_2) \) induces an isomorphism on integer homology groups for all \( m \geq 1 \).

We turn our attention to the fundamental groups of the image multiple point spaces and to this end we prove the following.

**Lemma 4.2** Suppose that \( f : X \to Y \) is a finite and proper continuous map, \( D^m(f) \) is path connected and that there exists a point \( (x_1, \ldots, x_m) \in D^m(f) \) such that \( x_c = x_d \) for \( c \neq d \).

1. If \( D^{m-1}(f) \) is path connected then the natural map of fundamental groups

\[ \pi_1(D^{m-1}(f)) \to \pi_1(M_{m-1}(f)) \]

is surjective.

2. If \( D^{m+1}(f) \) is empty then

\[ \pi_1(D^m(f)) \to \pi_1(M_m(f)) \]

is surjective.

**Proof.** (i) For a continuous map \( h \) we can define \( \varepsilon^k : D^k(h) \to D^{k-1}(h) \) by projecting through omission of the last copy of the source of \( h \). Let \( D^k_r(h) \) be the image of \( D^k(h) \) in \( D^l(h) \) (through composition of maps \( \varepsilon^i \)). Then \( M_r(h) \) is the image of \( h_r := h|D^r_r \). We have

\[
D^j(f_r) = \begin{cases} 
D^j_r(h), & \text{for } j < r, \\
D^j(h), & \text{for } j \geq r.
\end{cases}
\]

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As $D^{m-1}(f)$ is path connected, $D^j(f_{m-1})$ is path connected for $j < m - 1$ as it is the image of $D^{m-1}(f)$ in $D^j(f)$. As $D^m(f)$ has a point with $x_e = x_d$, $c \neq d$, then so does $D^j(f_{m-1})$ for all $2 \leq j < m$. These two facts imply that every point in $D^j(f_m)$ is connected by a path to a point with $x_e = x_d$, $c \neq d$.

Now, for any continuous map $h$, $D^{i+1}(h) = D^2(e^i : D^i(h) \to D^{i-1}(h))$. From this and Theorem 4.18 of [3] we deduce that for $j \leq m + 1$ that

$$\pi_1(D^j(f_{m-1})) \to \pi_1\left(e^i(D^j(f_{m-1}))\right) = \pi_1(D^{i-1}(f_{m-1}))$$

is surjective and produce a chain of maps to get

$$\pi_1(D^{m-1}(f)) = \pi_1(D^{m-1}(f_{m-1})) \to \pi_1(M_{m-1}(f))$$

surjective.

(ii) One can follow a similar argument to show that $\pi_1(D^{m-1}(f_m)) \to \pi_1(M_m(f))$ is surjective. As $D^{m+1}(f)$ is empty then $e^m : D^m(f) \to e^m(D^m(f)) = D^{m-1}(f_m)$ is a bijective and proper map so this is a homomorphism. \qed

**Proposition 4.3** The inclusion $\text{Dis}_m(g_0) \to \text{Dis}_m(f_0)$ is a homotopy equivalence for all $m \geq 1$ and hence Theorem 2.6 part 1 is proved.

**Proof.** Note that $M_m(f_l)$ and $M_m(g_l)$ are Stein spaces and so are homotopy equivalent to CW-complexes of dimension equal to their complex dimension.

If $\dim_C M_m(f_l) \leq 1$ then the statement is elementary to prove. If $\dim_C M_m(f_l) > 1$ then it is enough to show that $M_m(g_l)$ and $M_m(f_l)$ are simply connected because a map between simply connected CW-complexes that induces an isomorphism on integer homology is a homotopy equivalence by Whitehead's theorem, [15], p.220. In our given range we know that $M_m(g_l)$ is simply connected.

Note that $D^j(f_l)$ is contractible for $j < l + 1$ and $D^{l+1}(f_l)$ is the Milnor fibre of an isolated complete intersection singularity and so is homotopically equivalent to a wedge of spheres. Higher multiple point spaces are empty.

Case $\dim D^{l+1}(f_l) > 0$: Here $D^{l+1}(f_l)$ is connected and since the restriction to a reflecting hyperplane in the ambient space is the Milnor fibre of an isolated complete intersection singularity, see [8] Theorem 2.14, there exists a point $(x_1, \ldots, x_{l+1})$ such that $x_c = x_d$ for some $c \neq d$. From Lemma 4.2 we deduce that $\pi_1(D^{m}(f_l)) \to \pi_1(M_{m-1}(f_l))$ is surjective for all $m \leq l + 1$. For $m < l + 1$ the result is then true. For the $l + 1$ case we note that we have are only concerned with $\dim C M_{l+1}(f_l) \geq 2$, i.e. $D^{l+1}(f_l)$ is simply connected.

Case $\dim D^{l+1}(f_l) = 0$: As $\dim D^{l+1}(f_l) = l - 1$ the only situations to check are for $M_{l}(f_l)$, which is simple, it is homotopically a circle, and for $M_{2}(f_l)$ which has dimension 0. \qed

**Proof (of Theorem 2.6 part 2).** From Proposition 3.7 of [6] we see that a good real perturbation exists, (use $t < 0$ in $f_l$) and that the natural map $\text{Dis}_m(f_k) \to \text{Dis}_m(f_c)$ induces an isomorphism of integer homology groups.

If $\dim M_m(f_c) \leq 1$ then the statement is trivial. For the other situations we must show that $\text{Dis}_m(f_k)$ is simply connected. Calculations show that $D^k(f_{k,t})$ and $D^k(f_{c,t})$ are connected, non-singular and contract onto the diagonal for $k < l + 1$. The space $D^{l+1}(f_{c,t})$ is simply connected when its dimension is greater than 1, and $D^{l+1}(f_{k,t})$ is $S_{l+1}$-homotopically equivalent to it. Thus by Lemma 4.2 the image multiple point sets for $f_{k,t}$ are simply connected.

Again using Whitehead's theorem we conclude that the spaces are homotopically equivalent. \qed

We finish with a theorem on augmentations.
Theorem 4.4 Suppose that \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0) \) is the augmentation by the isolated hypersurface singularity \( g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) of the corank 1 \( \mathcal{A}_e \)-codimension 1, multiplicity \( l + 1 \) map-germ. Let \( g \) have Milnor number \( \mu(g) \).

Then \( \text{Dis}_1(\mathcal{A}_{F,g}(f)) \) is homotopically equivalent to a wedge of \( \mu(g) n - 1(p - n - 1) + q \)-spheres. Higher disentanglements are contractible or empty. Furthermore,

\[
\mu(g) \leq \mathcal{A}_e - \text{cod}(\mathcal{A}_{F,g}(f)),
\]

with equality if \( g \) is quasihomogeneous.

**Proof.** The result on homotopy follows from Theorem 3.2 of [5].

Note that \( f \) is quasihomogeneous and hence so is the unfolding \( F \). Then, (denoting Tyurina number of \( g \) by \( \tau(g) \) and Milnor number by \( \mu(g) \)),

\[
\mathcal{A}_e - \text{cod}(\mathcal{A}_{F,g}(f)) = \tau(g)\mathcal{A}_e - \text{cod}(f), \quad \text{by Theorem 3.3 of [4]},
\]

\[
= \tau(g), \quad \text{by equality if } g \text{ quasihomogeneous.}
\]

\( \square \)

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