CHRISTOFFEL-MINKOWSKI PROBLEM I: HESSIAN EQUATIONS ON S^n

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1. Introduction

Surface area measures are local versions of quermassintegrals in the theory of convex bodies. If the boundary of the convex body is smooth, the corresponding surface area function is a symmetric function of principal radii of its boundary. The general problem of finding a convex hypersurface with kth symmetric function of principal radii prescribed on its outer normals is often called Christoffel-Minkowski problem. It corresponds to find *convex* solutions of the nonlinear elliptic Hessian equation (see next section):

$$S_k(\{u_{ij} + u\delta_{ij}\}) = \varphi \quad \text{on} \quad S^n, \tag{1.1}$$

where S_k is the k-th elementary symmetric function.

In this paper, we are concerned with the existence, regularity and *convexity* for the equation (1.1).

Alexandrov-Fenchel-Jessen Theorem ([2] and [11]) asserts the uniqueness for the solutions of equation (1.1). In the case $k = 1$, this is the equation for Christoffel problem. The early treatments were given in Christoffel [9], Hurwitz [20], Hilbert [18], Suss [31] and others, the final solution was obtained in Firey [12], [13] and Berg [4]. The other extremal case is $k = n$, which corresponds to Minkowski problem. This case has also been settled due to the works of Minkowski [26], Alexandrov [1], Lewy [25], Nirenberg [27], Pogorelov [29], Cheng-Yau [8]. The intermediate problems still remain open, very little is known though there is an extensive literature devoted to it. We refer [5] and [30] for the references.

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It is known that for (1.1) to be solvable, the function $\varphi(x)$ has to satisfy

$$\int_{S^n} x_i\varphi(x)\,dx = 0, \qquad i = 1, ..., n+1. \tag{1.2}$$

For Minkowski problem, (1.2) is also sufficient. But is is not sufficient for the cases $1 \leq k < n$ [2]. For both Minkowski problem and Christoffel problem ($k = n$ and $k = 1$), the summation of the corresponding k-surfaces area functions of two convex bodies is also a k-surface area function of some other convex body. They are related to Blaschke and Minkowski sums of convex bodies respectively. For the intermediate cases $2 \leq k \leq n-1$, this is no longer true in general. There exist two strictly convex bodies with analytic boundaries, the sum of their k-surface area functions is not a k-surface area function of any convex body (see [10] and [16]). This suggests that the intermediate problems are much more complicated.

The intermediate Christoffel-Minkowski problems raise the following fundamental question in PDE:

Question: for what function φ on the right hand side of equation (1.1), there is a regular *convex* solution?

The structure of Hessian equations have been investigated in [6], [21], [32], [33], [24]. The natural solution class for this of type equations is k-convex functions (see Definition 2.1). If $k = n$, it is a Monge-Ampère equation, the convexity is built into the solution class. In general, k-convex function is not convex for $k < n$. Hence, the major issue here is to find conditions for the existence of *convex* solution of (1.1). In the case of Christoffel problem, equation (1.1) is linear. The necessary and sufficient conditions in [12] was derived from the linear representation formula of Green's function. For the intermediate cases ($2 \leq k \leq n-1$), (1.1) is a fully nonlinear equation. We have to take a different approach. We deal with the problem using continuity method as a deformation process together with strong minimum principle to force the *convexity*. This approach has been successfully used previously by Caffarelli-Friedman [7] and Korevaar-Lewis [23] (it appears that Yau also suggested similar approach, see [22]) for the semilinear equations in domains of $\mathbb{R}^n$. The crucial deformation lemma (Lemma 4.1) will be established in this paper for the fully nonlinear Hessian equation (1.1).

We introduce some notations.

Definition 1.1. Let f be a positive $C^{1,1}$ function on S^n satisfies (1.2), $\forall s \in \mathbb{R}$, we say f is in C_s if $(f^s_{ij} + \delta_{ij} f^s)$ is semi-positive definite almost everywhere in S^n. We say f is connected to g in $\mathcal{C}_s$ if there is a continuous path $h(t,.) \in \mathcal{C}_s$, such that $h(0,x) = f(x)$ and $h(1,x) = g(x)$, $\forall x \in S^n$.

We note that the definition is independent of the choice of orthonormal frame. A positive $C^{1,1}$ function f in $\mathcal{C}_s$ if and only if f satisfies (1.2) and f^s is a convex function in $\mathbb{R}^{n+1}$ as a homogeneous function of order 1.

We now state our main results.

Theorem 1.2. **(Full Rank Theorem)** *Suppose u is an admissible solution (Definition 2.1) of equation (1.1) with semi-positive definite spherical hessian $W = \{u_{ij} + u\delta_{ij}\}$ on S^n. If $\varphi \in \mathcal{C}_{-\frac{1}{k}}$, then W is positive definite on S^n.*

The following is the existence theorem.

Theorem 1.3. **(Existence Theorem)** *Let $\varphi(x) \in C^{1,1}(S^n)$ be a positive function, suppose φ is connected to 1 in $\mathcal{C}_{-\frac{1}{k}}$, then Christoffel-Minkowski problem (1.1) has a unique solution upto translations. More precisely, there exists a $C^{3,\alpha}$ ($\forall 0 < \alpha < 1$) closed strictly convex hypersurface M in $\mathbb{R}^{n+1}$ whose principal radii of curvature function of order k is $\varphi(x)$. M is unique upto translations. Furthermore, if $\varphi(x) \in C^{l,\gamma}(S^n)$ ($l \geq 2, \gamma > 0$), then M is $C^{2+l,\gamma}$. If φ is analytic, M is analytic.*

In a sequent paper [3] jointly with B. Andrews, we will study the curvature flow equation associated to Christoffel-Minkowski problems. The condition on φ in Theorem 1.3 will be replaced by simpler condition $\varphi \in \mathcal{C}_{-\frac{1}{k}}$ alone via curvature flow approach with the assistance of the Full Rank Theorem (theorem 1.2).

The organization of the paper is as follows. We derive the equation (1.1) together with some basic facts of elementary symmetric functions in the next section. In section 3, we establish C^2 a priori estimates for general k-convex solutions. The key deformation lemma will be proved in section 4 for the Hessian equations on S^n. In section 5, we prove Theorem 1.2 and Theorem 1.3.

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2. Preliminaries

We recall the definition of k-symmetric function: $\forall 1 \leq k \leq n$, $\forall \lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{R}^n$,

$$S_k(\lambda) = \sum \lambda_{i_1} ... \lambda_{i_k}, \tag{2.1}$$

where the sum is taking over for all increasing sequences $i_1, ..., i_k$ of the indices chosen from the set $\{1, ..., n\}$. The definition can be extended to symmetric matrices by letting $S_k(W) = S_k(\lambda(W))$, where $\lambda(W) = (\lambda_1(W), ..., \lambda_n(W))$ are the eigenvalues of symmetric matrix W. We also set $S_0 = 1$ and $S_k = 0$ for $k > n$.

For a strictly convex body K in $\mathbb{R}^{n+1}$ with smooth boundary M, the Gauss map $\vec{n}$ is a diffeomorphism from M to S^n. For $x \in S^n$, let $\lambda(x) = (\lambda_1(x), ..., \lambda_n(x))$ be the principal radii of curvature of M at the point $\vec{n}^{-1}(x)$. Then

$$S_k(x) = S_k(\lambda(x)) \tag{2.2}$$

is the k-surface area function over the unit sphere S^n at the point x. Let $u(x) = x\vec{n}^{-1}(x)$ be the support function. Let $e_1, ..., e_n$ be any orthonormal frame on S^n, and let u_{ij} to be the covariant derivative with respect to the frame. The eigenvalues $\lambda\{W(x)\} = (\lambda_1(x), ..., \lambda_n(x))$ of the Hessian matrix $W(x) = \{u_{ij}(x) + u(x)\delta_{ij}\}$ are the principal radii of M at $\vec{n}^{-1}(x)$ (see [8] and [29]). Hence, u satisfies the equation (1.1).

On the other hand, if u is a solution of (1.1) with the property that W is semi-positive definite on S^n, u is a support function of a convex body. If W is positive definite everywhere in S^n and $l \geq 2$, the corresponding hypersurface M is strictly convex. In this case, $\forall 0 \leq \alpha < 1$, M is $C^{l,\alpha}$ if and only if u is in $C^{l,\alpha}$ (e.g., see [29] and [30]).

Definition 2.1. Let $\mathcal{S}$ be the space consisting all $n \times n$ symmetric matrice. For $1 \leq k \leq n$, let Γ_k is the positive connected cone in $\mathcal{S}$ containing the identity matrix determined by

$$\Gamma_k = \{W \in \mathcal{S} : \quad S_1(W) > 0, ..., S_k(W) > 0\}$$

Suppose $u \in C^2(S^n)$, we say u is k-convex, if $W(x) = \{u_{ij}(x) + u(x)\delta_{ij}\}$ is in Γ_k for each $x \in S^n$. u is convex on S^n if W is n-convex. Furthermore, u is called an admissible solution of (1.1), if u is k-convex and satisfies (1.1).

It will become clear that algebraic properties of the elementary symmetric functions are crucial in our proofs. We also refer the recent work [19] for the important role of elementary symmetric functions in other contents. The following are some basic results of elementary symmetric functions.

Proposition 2.2. *If $\{W_{ij}\} = W$ is a $n \times n$ symmetric matrix, let*

$$F(W) = S_k(W)$$

for $1 \le k \le n$, then the following relations hold.

$$S_k(W) = \frac{1}{k!} \sum_{\substack{i_1,...,i_k=1\\ j_1,...,j_k=1}}^{n} \delta(i_1,...,i_k; j_1,...,j_k) W_{i_1 j_1} \cdots W_{i_k j_k},$$

$$F^{ij} := \frac{\partial F}{\partial W_{ij}}(W)$$
$$= \frac{1}{(k-1)!} \sum_{\substack{i_1,...,i_{k-1}=1\\ j_1,...,j_{k-1}=1}}^{n} \delta(i, i_1,...,i_{k-1}; j, j_1,...,j_{k-1}) W_{i_1 j_1} \cdots W_{i_{k-1} j_{k-1}}$$

$$F^{ij,rs} := \frac{\partial^2 F}{\partial W_{ij} \partial W_{rs}}(W)$$
$$= \frac{1}{(k-2)!} \sum_{\substack{i_1,...,i_{k-2}=1\\ j_1,...,j_{k-2}=1}}^{n} \delta(i, r, i_1,...,i_{k-2}; j, s, j_1,...,j_{k-2}) W_{i_1 j_1} \cdots W_{i_{k-2} j_{k-2}},$$

where $\delta(i_1,...,i_k; j_1,...,j_k)$ is the Kronecker symbol.

We will need the next two lemmas in the later sections.

Lemma 2.3. *For $1 \le k \le l$, $G = (\lambda_1, ..., \lambda_l)$, $\forall 1 \le i, j \le n, i \ne j$, we denote $S_k(G|i)$ to be the symmetric function with $\lambda_i = 0$ and $S_k(G|ij)$ to be the symmetric function with $\lambda_i = \lambda_j = 0$. Then, the followings are true,*

$$S_k(G)S_{l-1}(G|\alpha)S_{k-1}(G|\alpha) - S_l(G)S_{k-1}^2(G|\alpha)$$
$$= S_k(G|\alpha)S_{l-1}(G|\alpha)S_{k-1}(G|\alpha).$$

If $1 \le k \le l$, for $\alpha \ne \beta$,

$$S_k(G)S_{k-2}(G|\alpha\beta) - S_{k-1}(G|\alpha)S_{k-1}(G|\beta)$$
$$= S_k(G|\alpha\beta)S_{k-2}(G|\alpha\beta) - S_{k-1}^2(G|\alpha\beta).$$

Proof: We note that $S_l(G)$ is a nomial, now for any $\alpha \in G$,

$$S_k(G)S_{l-1}(G|\alpha)S_{k-1}(G|\alpha) - S_l(G)S_{k-1}^2(G|\alpha)$$
$$= [\lambda_\alpha S_{k-1}(G|\alpha) + S_k(G|\alpha)]S_{l-1}(G|\alpha)S_{k-1}(G|\alpha) - S_l(G)S_{k-1}^2(G|\alpha)$$
$$= S_l(G)S_{k-1}(G|\alpha)^2 + S_k(G|\alpha)S_{l-1}(G|\alpha)S_{k-1}(G|\alpha) - S_l(G)S_{k-1}^2(G|\alpha)$$
$$= S_k(G|\alpha)S_{l-1}(G|\alpha)S_{k-1}(G|\alpha).$$

The second identity in the lemma follows directly from the identities, $S_k(\lambda) = S_k(\lambda|i) + \lambda_i S_{k-1}(\lambda|i)$, $S_k(\lambda) = S_k(\lambda|ij) + \lambda_i S_{k-1}(\lambda|ij) + \lambda_j S_{k-1}(\lambda|ij)$. □

Lemma 2.4. *For $1 \le k \le l$, $G = (\lambda_1, ..., \lambda_l)$, $\forall \alpha \neq \beta$ and for all real numbers $\gamma_1, ..., \gamma_l$,*

$$\begin{aligned} &\sum_{\alpha \in G} S_k(G|\alpha) S_{l-1}(G|\alpha) S_{k-1}(G|\alpha)\gamma_\alpha^2 \\ \geq \;& S_l(G) \sum_{\alpha\neq\beta} \{S_{k-1}^2(G|\alpha\beta) - S_k(G|\alpha\beta)S_{k-2}(G|\alpha\beta)\}\gamma_\alpha\gamma_\beta. \end{aligned} \tag{2.3}$$

Proof: We first prove the following equality: $\forall 1 \le \alpha \le l$,

$$\sum_{\beta\in G, \beta\neq\alpha} \lambda_\beta [S_{k-1}^2(G|\alpha\beta) - S_k(G|\alpha\beta)S_{k-2}(G|\alpha\beta)] = S_k(G|\alpha)S_{k-1}(G|\alpha). \tag{2.4}$$

We note that,

$$\sum_{\beta\in G, \beta\neq\alpha} S_{k-1}(G|\alpha\beta)S_k(G|\alpha) = (l-k)S_k(G|\alpha)S_{k-1}(G|\alpha),$$

and

$$\sum_{\beta\in G, \beta\neq\alpha} S_k(G|\alpha\beta)S_{k-1}(G|\alpha) = (l-k-1)S_{k-1}(G|\alpha)S_k(G|\alpha).$$

As S_l is a nomial, we get

$$\begin{aligned} &\sum_{\substack{\beta\in G\\ \beta\neq\alpha}} \{\lambda_\beta S_{k-1}^2(G|\alpha\beta) - \lambda_\beta S_k(G|\alpha\beta)S_{k-2}(G|\alpha\beta)\} \\ &= \sum_{\substack{\beta\in G\\ \beta\neq\alpha}} [S_{k-1}(G|\alpha\beta)S_k(G|\alpha) - S_k(G|\alpha\beta)(S_{k-1}(G|\alpha\beta) + \lambda_\beta S_{k-2}(G|\alpha\beta))] \\ &= \sum_{\substack{\beta\in G\\ \beta\neq\alpha}} [S_{k-1}(G|\alpha\beta)S_k(G|\alpha) - S_k(G|\alpha\beta)S_{k-1}(G|\alpha)] \\ &= S_k(G|\alpha)[(l-k)S_{k-1}(G|\alpha) - (l-k-1)S_{k-1}(G|\alpha)] = S_k(G|\alpha)S_{k-1}(G|\alpha) \end{aligned}$$

Identity (2.4) is verified. Now we use Cauchy inequality and (2.4) to prove (2.3). For any $\alpha \neq \beta$, by nomiality of $S_l(G)$, $S_l(G) = \lambda_\alpha\lambda_\beta S_{l-2}(G|\alpha\beta)$.

$$\begin{aligned}
&S_l(G) \sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}} [S^2_{k-1}(G|\alpha\beta) - S_k(G|\alpha\beta)S_{k-2}(G|\alpha\beta)]\gamma_\alpha\gamma_\beta \\
&= \sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}} S_{l-2}(G|\alpha\beta)[S^2_{k-1}(G|\alpha\beta) - S_k(G|\alpha\beta)S_{k-2}(G|\alpha\beta)](\lambda_\beta\gamma_\alpha)(\lambda_\alpha\gamma_\beta) \\
&\le \sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}} S_{l-2}(G|\alpha\beta)[S^2_{k-1}(G|\alpha\beta) - S_k(G|\alpha\beta)S_{k-2}(G|\alpha\beta)]\frac{\lambda^2_\beta\gamma^2_\alpha + \lambda^2_\alpha\gamma^2_\beta}{2} \\
&= \sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}} S_{l-2}(G|\alpha\beta)\lambda_\beta[S^2_{k-1}(G|\alpha\beta) - S_k(G|\alpha\beta)S_{k-2}(G|\alpha\beta)]\lambda_\beta\gamma^2_\alpha \\
&= \sum_{\alpha\in G} S_{l-1}(G|\alpha) \sum_{\beta\in G,\beta\neq\alpha} \lambda_\beta[S^2_{k-1}(G|\alpha\beta) - S_k(G|\alpha\beta)S_{k-2}(G|\alpha\beta)]\gamma^2_\alpha \\
&= \sum_{\alpha\in G} S_k(G|\alpha)S_{l-1}(G|\alpha)S_{k-1}(G|\alpha)\gamma^2_\alpha.
\end{aligned}$$

This completes the proof of (2.3). □

3. A PRIORI ESTIMATES

As the first step in the proof of the theorem, we establish the a priori estimates for the solutions of equation (1.1) in this section .

We note that for any solution $u(x)$ of (1.1), $u(x) + l(x)$ is also a solution of the equation for any linear function $l(x) = \sum_{i=1}^{n+1} a_i x_i$. We will confine ourselves to solutions satisfying the following orthogonality condition:

$$\int_{S^n} x_i u\, dx = 0, \qquad \forall i = 1, 2, ..., n+1. \tag{3.1}$$

When u is convex, it is a support function of some convex body Ω. Condition (3.1) implies that the Steiner point of Ω coincides with the origin.

In the case of $k = 1$, the equation (1.1) is a linear elliptic equation on sphere. A priori estimates for solution u satisfies (3.1) in this case follows from standard linear elliptic theory. Therefore, we will restrict ourselves to the case $k \geq 2$. The equation (1.1) will be a uniformly elliptic once C^2 estimates are established for u (see [6]). By Evans-Krylov theorem and Schauder theory, one may obtain higher derivative estimates for u. Therefore, we only need to get C^2 estimates for u.

Proposition 3.1. *Let*

$$H := trace(u_{ij} + \delta_{ij}u) = \Delta u + nu. \tag{3.2}$$

If u is k-convex, then

$$0 < H \leq \max_{x\in S^n}(n\tilde{\varphi}(x) - \Delta\tilde{\varphi}(x)), \tag{3.3}$$

where $\tilde{\varphi} := (\frac{\varphi}{C_n^k})^{\frac{1}{k}}$.

Proof. The positivity of H follows from Newton-Maclaurin inequality. Assume the maximum value of H is attained at a point $x_0 \in S^n$. We choose an orthonormal local frame $e_1, e_2, ..., e_n$ near x_0 such that $u_{ij}(x_0)$ is diagonal. Let $W = \{u_{ij} + \delta_{ij}u\}$, we define

$$G(W_{ij}) := (\frac{S_k}{C_n^k})^{\frac{1}{k}}(W_{ij}).$$

Then the equation (1.1) becomes

$$G(W_{ij}) = \tilde{\varphi}. \tag{3.4}$$

For the standard metric on S^n, we have,

$$H_{ii} = \Delta W_{ii} - nW_{ii} + H.$$

By the assumption the matrix $W \in \Gamma_k$, so $\{G^{ij}\}$ is positive definite. Since $\{H_{ij}\} \leq 0$, and $\{G^{ij}\}$ is diagnol, it follows that at x_0,

$$0 \geq G^{ij}H_{ij} = G^{ii}(\Delta W_{ii}) - nG^{ii}W_{ii} + H\sum_i^n G^{ii}. \tag{3.5}$$

As G is homogeneous of degree one, we have

$$G^{ii}W_{ii} = \tilde{\varphi}. \tag{3.6}$$

Next we apply the Laplace operator to equation (3.4), we obtain

$$G^{ij}W_{ijk} = \nabla_k\tilde{\varphi},$$
$$G^{ij,rs}W_{ijk}W_{rsk} + G^{ij}\Delta W_{ij} = \Delta\tilde{\varphi}.$$

By the concavity of G, at x_o we have

$$G^{ii}\Delta(W_{ii}) \geq \Delta\tilde{\varphi}. \tag{3.7}$$

Combining (3.6), (3.7) and (3.5), we see that

$$0 \geq \Delta\tilde{\varphi} - n\tilde{\varphi} + H\sum_{i=1}^n G^{ii}.$$

As $\sum_{i=1}^n \frac{\partial G}{\partial W_{ii}} \geq 1$ (e.g, [32]), it follows that $H \leq n\tilde{\varphi} - \Delta\tilde{\varphi}$. □

In order to obtain a C^2 bound, we need a C^0 bound for u. In the case of Minkowski problem, such crucial C^0 bound was established by Cheng-Yau in [8] (see also [29]). Their estimates assume the convexity. Here, we use the a priori bounds in Proposition 3.1 to get a C^0 bound for k-convex solutions.

Lemma 3.2. *For any C^2 k-convex function v, there is a constant C depending only on n and $\max_{S^n}(n(\Delta v + nv) + |v|)$ such that,*

$$||v||_{C^2} \leq C. \tag{3.8}$$

Proof. At any point $x \in S^n$, we may assume the matrix $(v_{ij} + \delta_{ij}v)$ is diagonal. Let λ_i is a eigenvalue of that matrix, as $k \geq 2$, we have

$$v_{ii} + v \leq max_i\lambda_i \leq \Delta v + nv.$$

In turn,

$$v_{ii} \leq (\Delta v + nv) - v, \quad \forall i. \tag{3.9}$$

It follows from (3.9) that, for any $i = 1, ..., n$,

$$v_{ii} = (\Delta v + nv) - nv - \sum_{k\neq i} v_{kk} \geq -(n-2)(\Delta v + nv) - v.$$

Thus at x, as $\Delta v + nv \geq 0$,

$$|v_{ii}|_{C^0} \leq (n(\Delta v + nv) + |v|).$$

we obtain

$$|\nabla^2 v|_{C^0} \leq C(\max_{x\in S^n}(n(\Delta v + nv) + |v|)). \tag{3.10}$$

By interpolation, $|\nabla v|_{C^0}$ can be bounded by $|v|_{C^0}$ and $|\nabla^2 v|_{C^0}$. The lemma is proved. □

Now we establish the C^0-estimate. The proof is based on a rescaling argument.

Proposition 3.3. *If u is a admissible solution of equation (1.1) and satisfies (3.1), then there exist a positive constant C depending only on $n, k, \min_{S^n}\varphi, ||\varphi||_{C^{1,1}}$, such that,*

$$||u||_{C^0} \leq C. \tag{3.11}$$

Proof. Suppose (3.11) is false, then $\exists u^l (l = 1, 2, ...)$ satisfying (3.1), there is a constant $\tilde{C}$ independent of l, and $F(\{u^l_{ij} + \delta_{ij} u^l\}) = \tilde{\varphi}^l$, where $\tilde{\varphi}^l = \frac{\varphi^l}{C^k_n}$, with φ^l satisfies

$$||\varphi^l||_{C^2} \leq \tilde{C}, \quad ||\frac{1}{\varphi^l}||_{C^0} \leq \tilde{C}, \quad ||u^l||_{L^\infty} \geq l.$$

Let $v^l = \frac{u^l}{||u^l||_{L^\infty}}$, then

$$||v^l||_{L^\infty} = 1, \tag{3.12}$$

By Proposition 3.1, we have

$$0 \leq H^l := \Delta u^l + nu^l \leq C, \tag{3.13}$$

where the constant C independent of l. From (3.13) v^l satisfies the following estimates

$$0 \leq \Delta v^l + nv^l \leq \frac{C}{||u^l||_{L^\infty}} \longrightarrow 0. \tag{3.14}$$

On the other hand, by Lemma 3.2, (3.12) and (3.14), we have

$$||v^l||_{C^2} \leq C.$$

Hence, there exists a subsequence $\{v^{l_i}\}$ and a function $v \in C^{1,\alpha}(S^n)$ satisfying (3.1) such that

$$v^{l_i} \longrightarrow v \quad \text{in} \quad C^{1,\alpha}(S^n), \quad \text{with} \quad ||v||_{L^\infty} = 1. \tag{3.15}$$

Combining (3.14) and (3.15), in the distribution sense we have

$$\Delta v + nv = 0, \qquad \text{on} \quad S^n.$$

By linear elliptic theory, v is in fact smooth. Since v satisfied (3.1), we conclude that, $v \equiv 0$ on S^n. This is a contradiction to (3.15). □

Now, C^2 a priori bounds follows from Lemma 3.2, Proposition 3.1 and Proposition 3.3. By Evans-Krylov theorem and Schauder theory (e.g, see [15]), we have the following a priori estimates.

Theorem 3.4. *For each integer $l \geq 1$ and $0 < \alpha < 1$, there exist a constant C depending only on $n, l, \alpha, \min \varphi$, and $||\varphi||_{C^{l,1}}(S^n)$ such that*

$$||u||_{C^{l+1,\alpha}}(S^n) \leq C, \tag{3.16}$$

for all admissible solution of (1.1) satisfying the condition (3.1).

So far, we have obtained upper bounds for the principal radii of Christoffel-Minkowski problem. For Minkowski problem, a lower bound of the principal radii follows directly from the equation (1.1). In the next section, we will show that the principal radii of the general Christoffel-Minowski problem is bounded from below if φ satisfies the condition in Theorem 1.3. In the case of Christoffel problem, Firey's conditions [12] are necessary and sufficient. But, they are very cumbersome. It is desirable to have some simple sufficient conditions. Pogorelov in [28] established a lower bound of principal radii on S^2 under the condition,

$$\varphi(x) - \varphi_{ss}(x) > 0, \quad \text{on} \quad S^2, \tag{3.17}$$

where $\varphi(x)$ is differentiated at the point x with respect to arc length of the great circle on S^2.

To conclude this section, we derive a simple estimate which drops dimensionality restriction in (3.17). For the Christoffel problem, (1.1) can be written in the simple form, $\sum_{i=1}^n W_{ii} = \varphi$.

We may assume that the smallest eigenvalue of matrix $\{W_{ij}\}$ attains at some point $x_o \in S^n$ and along e_1 direction. Then we have

$$\nabla_i W_{11}(x_o) = 0, \quad i = 1, 2, ..., n,$$
$$\Delta W_{11}(x_o) \geq 0.$$

As $W_{11ii} = W_{ii11} + W_{11} - W_{ii}$, at the point x_o,

$$0 \leq \sum_{i=1}^n W_{11ii} = (\Delta W)_{11} + nW_{11} - \sum_{i=1}^n W_{ii} = \varphi_{11} + nW_{11} - \varphi.$$

Therefore at x_o, $nW_{11} \geq \varphi - \varphi_{11}$.

4. A Deformation Lemma

In this section, we establish the key deformation lemma. As in previous section, we let $W = \{u_{ij} + \delta_{ij}u\}$.

Lemma 4.1. (Deformation Lemma) *Let $O \subset S^n$ be an open subset, suppose $u \in C^4(O)$ be a solution of (1.1) in O and the matrix $W = \{W_{ij}\}$ is semi-positive definite. Suppose there is a positive constants $C_0 > 0$, such that for a fixed integer $(n-1) \geq l \geq k$, $\forall x \in O$, $S_l(W(x)) \geq C_0$. Let $\phi(x) = S_{l+1}(W(x))$ and let $\tau(x)$ be the largest eigenvalue of $\{-(\varphi^{-\frac{1}{k}})_{ij}(x) - \delta_{ij}\varphi^{-\frac{1}{k}}(x)\}$. Then, there are constants C_1, C_2*

depending only on $||u||_{C^3}$, $||\varphi||_{C^{1,1}}$, *n and* C_0, *the following differential inequality holds in* O,

$$(4.1)\sum_{\alpha,\beta}^{n} F^{\alpha\beta}(x)\phi_{\alpha\beta}(x) \le k(n-l)\varphi^{\frac{k+1}{k}}(x)S_l(W(x))\tau(x) + C_1|\nabla\phi(x)| + C_2\phi(x),$$

where $F^{\alpha\beta}$ *are defined in Proposition 2.2.*

Remark 4.2. The lemma generalizes results of Caffarelli-Friedman [7] and Korevaar-Lewis [23] to Hessian equation. We note that we made no assumptions on the size of S_{l+1} and the constants C_1, C_2 in Lemma 4.1 depend only on $||u||_{C^3}$, $||\varphi||_{C^{1,1}}$, n and C_0. This dependence is crucial for us in establishing Theorem 1.3 for $\varphi \in C^{1,1}$.

Proof of Deformation Lemma. Following the notation of Caffarelli and Friedman [7], for two functions defined in an open set $O \subset S^n$, $y \in O$, we say that $h(y) \lesssim k(y)$ provided there exist positive constants c_1 and c_2 such that

$$(4.2) \qquad (h-k)(y) \le (c_1|\nabla\phi| + c_2\phi)(y).$$

We also write $h(y) \sim k(y)$ if $h(y) \lesssim k(y)$ and $k(y) \lesssim h(y)$. Next, we write $h \lesssim k$ if the above inequality holds in O, with the constant c_1, and c_2 depending only on $||u||_{C^3}$, $||\varphi||_{C^2}$, n and C_0 (independent of y and O). Finally, $h \sim k$ if $h \lesssim k$ and $k \lesssim h$. We shall show that

$$(4.3) \qquad \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}\phi_{\alpha\beta} \lesssim k(n-l)\varphi^{\frac{k+1}{k}} S_l(W)\tau,$$

For any $z \in O$, let $\lambda_1 \ge \lambda_2 ... \ge \lambda_n$ be the eigenvalues of W at z. Since $S_l(W) \ge C_0 > 0$ and $u \in C^3$, for any $z \in S^n$, there is a positive constant $C > 0$ depending only on $||u||_{C^3}$, $||\varphi||_{C^2}$, n and C_0, such that $\lambda_1 \ge \lambda_2 ... \ge \lambda_l \ge C$. Let $G = \{1, 2, ..., l\}$ and $B = \{l+1, ..., n\}$ be the "good" and "bewared" sets of indices, and define $S_k(W|i) = S_k((W|i))$ where $(W|i)$ means that the matrix W exclude the i-column and i-row, and $(W|ij)$ means that the matrix W exclude the i, j columns and i, j rows. In the following, all calculations are at the point z using the relation "$\lesssim$", with the understanding that the constants in (4.2) are under control.

For each $z \in O$ fixed, we choose a local orthonormal frame $e_1, ..., e_n$ so that W is diagonal at z, and $W_{ii} = \lambda_i, \forall i = 1, ..., n$. Now we compute ϕ and its first and second derivative in the direction e_α. Let

$$S^{ij} = \frac{\partial S_{l+1}(W)}{\partial W_{ij}}, \qquad S^{ij,rs} = \frac{\partial^2 S_{l+1}(W)}{\partial W_{ij}\partial W_{rs}}.$$

As $\phi = S_{l+1}(W)$ and $\phi_\alpha = \sum_{i,j} S^{ij} W_{ij\alpha}$, we find that (as W is diagonal at z),

$$0 \sim \phi(z) \sim (\sum_{i\in B} W_{ii}) S_l(G) \sim \sum_{i\in B} W_{ii}, \quad (\text{so} \quad W_{ii} \sim 0, \quad i \in B), \tag{4.4}$$

$$0 \sim \phi_\alpha \sim S_l(G) \sum_{i\in B} W_{ii\alpha} \sim \sum_{i\in B} W_{ii\alpha} \tag{4.5}$$

and

$$S_{l-1}(W|ij) \sim \begin{cases} 0, & \text{if } i,j \in G; \\ S_{l-1}(G|j), & \text{if } i \in B, j \in G; \\ S_{l-1}(G), & \text{if } i,j \in B, i \neq j. \end{cases} \tag{4.6}$$

Also, by Proposition 2.2,

$$S^{ij} \sim \begin{cases} S_l(G), & \text{if } i = j \in B, \\ 0, & \text{otherwise.} \end{cases} \tag{4.7}$$

$$S^{ij,rs} = \begin{cases} S_{l-1}(W|ir), & \text{if } i = j, r = s, i \neq r; \\ -S_{l-1}(W|ij), & \text{if } i \neq j, r = j, s = i; \\ 0, & \text{otherwise.} \end{cases} \tag{4.8}$$

Since $\phi_{\alpha\alpha} = \sum_{i,j} [S^{ij,rs} W_{rs\alpha} W_{ij\alpha} + S^{ij} W_{ij\alpha\alpha}]$, from (4.4) and (4.5), it follows that for any $\alpha \in \{1, 2, ..., n\}$

$$\begin{aligned} \phi_{\alpha\alpha} &= \sum_{i\neq j} S_{l-1}(W|ij) W_{ii\alpha} W_{jj\alpha} - \sum_{i \neq j} S_{l-1}(W|ij) W_{ij\alpha}^2 + \sum_i S^{ii} W_{ii\alpha\alpha} \\ &= (\sum_{\substack{i\in G\\ j\in B}} + \sum_{\substack{i\in B\\ j\in G}} + \sum_{\substack{i,j\in B\\ i\neq j}} + \sum_{\substack{i,j\in G\\ i\neq j}}) S_{l-1}(W|ij) W_{ii\alpha} W_{jj\alpha} \\ &- (\sum_{\substack{i\in G\\ j\in B}} + \sum_{\substack{i\in B\\ j\in G}} + \sum_{\substack{i,j\in B\\ i\neq j}} + \sum_{\substack{i,j\in G\\ i\neq j}}) S_{l-1}(W|ij) W_{ij\alpha}^2 + \sum_i S^{ii} W_{ii\alpha\alpha}. \end{aligned} \tag{4.9}$$

From (4.5) and (4.6), we have

$$\sum_{\substack{i\in B\\ j\in G}} S_{l-1}(W|ij) W_{ii\alpha} W_{jj\alpha} \sim [\sum_{j\in G} S_{l-1}(G|j) W_{jj\alpha}] \sum_{i\in B} W_{ii\alpha} \sim 0. \tag{4.10}$$

Similarily,

$$\sum_{\substack{i\in G\\ j\in B}} S_{l-1}(W|ij) W_{ii\alpha} W_{jj\alpha} \sim 0. \tag{4.11}$$

Again, by (4.5) and (4.6),

$$\sum_{\substack{i,j\in B\\ i\neq j}} S_{l-1}(W|ij)W_{ii\alpha}W_{jj\alpha} \sim -S_{l-1}(G)\sum_{i\in B} W_{ii\alpha}^2. \tag{4.12}$$

and

$$\sum_{i\in G, j\in B} S_{l-1}(W|ij)W_{ij\alpha}^2 \sim \sum_{i\in B, j\in G} S_{l-1}(G|j)W_{ij\alpha}^2. \tag{4.13}$$

Inserting (4.6), (4.10)-(4.13) into (4.9), we obtain

$$\phi_{\alpha\alpha} \sim \sum_i S^{ii}W_{ii\alpha\alpha} - 2\sum_{\substack{i\in B\\ j\in G}} S_{l-1}(G|j)W_{ij\alpha}^2 - S_{l-1}(G)\sum_{i,j\in B} W_{ij\alpha}^2. \tag{4.14}$$

Set $F^{\alpha\beta} := \frac{\partial S_k(W)}{\partial W_{\alpha\beta}}$. By Proposition 2.2 and (4.4), we have for any $\alpha \in \{1,2,...,n\}$

$$F^{\alpha\beta} \sim \begin{cases} S_{k-1}(G|\alpha), & \text{if } \alpha\in G, \alpha=\beta;\\ S_{k-1}(G), & \text{if } \alpha\in B, \alpha=\beta;\\ 0, & \text{if } \alpha\neq\beta, \end{cases} \tag{4.15}$$

and,

$$F^{ij,rs} = \begin{cases} S_{k-2}(W|ir), & \text{if } i=j, r=s, i\neq r;\\ -S_{k-2}(W|ij), & \text{if } i\neq j, r=j, s=i;\\ 0, & \text{otherwise.} \end{cases} \tag{4.16}$$

From (4.14)-(4.16) we obtain

$$\begin{aligned} \sum_{\alpha=1}^n F^{\alpha\alpha}\phi_{\alpha\alpha} &\sim \sum_{\alpha=1}^n\sum_i S^{ii}F^{\alpha\alpha}W_{ii\alpha\alpha}\\ &-2\sum_{\alpha=1}^n\sum_{\substack{i\in B\\ j\in G}} S_{l-1}(G|j)F^{\alpha\alpha}W_{ij\alpha}^2 - S_{l-1}(G)\sum_{\alpha=1}^n\sum_{i,j\in B} F^{\alpha\alpha}W_{ij\alpha}^2. \end{aligned} \tag{4.17}$$

By (4.4), (4.7) and homogeneity of S_k and S_{l+1},

$$\begin{aligned} \sum_{\alpha=1}^n\sum_{i=1}^n S^{ii}F^{\alpha\alpha}W_{ii\alpha\alpha} &= \sum_{\alpha=1}^n\sum_{i=1}^n S^{ii}F^{\alpha\alpha}(W_{\alpha\alpha ii} + W_{ii} - W_{\alpha\alpha})\\ &\sim \sum_{\alpha=1}^n\sum_{i=1}^n S^{ii}F^{\alpha\alpha}W_{\alpha\alpha ii} - (n-l)k\varphi S_l(G). \end{aligned} \tag{4.18}$$

Differentiating the equation (1.1), we get

$$\varphi_{ii} = F^{\alpha\beta,rs}W_{\alpha\beta i}W_{rsi} + F^{\alpha\beta}W_{\alpha\beta ii}.$$

(4.16), (4.7) and Propositon 2.2 yield,

$$\begin{aligned}\sum_\alpha\sum_i S^{ii}F^{\alpha\alpha}W_{\alpha\alpha ii} &= \sum_i S^{ii}\{\varphi_{ii}-\sum_{\alpha,\beta}F^{\alpha\beta,rs}W_{\alpha\beta i}W_{rsi}\}\\ &\sim \sum_{i\in B}\{-(\sum_{\substack{\alpha\in G\\ \beta\in B}}+\sum_{\substack{\alpha\in B\\ \beta\in G}}+\sum_{\substack{\alpha,\beta\in B\\ \alpha\neq\beta}}+\sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}})S_{k-2}(W|\alpha\beta)W_{\alpha\alpha i}W_{\beta\beta i}\\ &+\ \varphi_{ii}+\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^n S_{k-2}(W|\alpha\beta)W^2_{\alpha\beta i}\}S_l(G). \qquad (4.19)\end{aligned}$$

It follows from (4.6) and (4.5) that,

$$\sum_{\substack{\alpha\in B\\ \beta\in G}} S_{l-1}(W|\alpha\beta)W_{\alpha\alpha i}W_{\beta\beta i}\sim[\sum_{\beta\in G}S_{l-1}(G|\beta)W_{\beta\beta i}]\sum_{\alpha\in B}W_{\alpha\alpha i}\sim 0. \qquad (4.20)$$

In turn,

$$\begin{aligned}\sum_{\alpha=1}^n\sum_{i=1}^n S^{ii}F^{\alpha\alpha}W_{\alpha\alpha ii}&\sim S_l(G)\sum_{i\in B}\{\varphi_{ii}-\sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}}S_{k-2}(G|\alpha\beta)W_{\beta\beta i}W_{\alpha\alpha i}\\ &-\sum_{\substack{\alpha,\beta\in B\\ \alpha\neq\beta}}S_{k-2}(G)W_{\beta\beta i}W_{\alpha\alpha i}+\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^n S_{k-2}(W|\alpha\beta)W^2_{\alpha\beta i}\}. \qquad (4.21)\end{aligned}$$

Inserting (4.21) and (4.18) to (4.17), we have

$$\begin{aligned}\sum_{\alpha=1}^n F^{\alpha\alpha}\phi_{\alpha\alpha}&\sim S_l(G)\sum_{i\in B}(\varphi_{ii}-k\varphi)-S_l(G)\sum_{i\in B}\sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}}S_{k-2}(G|\alpha\beta)W_{\alpha\alpha i}W_{\beta\beta i}\\ &-S_l(G)\sum_{i\in B}\sum_{\substack{\alpha,\beta\in B\\ \alpha\neq\beta}}S_{k-2}(G)W_{\alpha\alpha i}W_{\beta\beta i}-2\sum_{\alpha=1}^n\sum_{i\in B,\beta\in G}S_{l-1}(G|\beta)S_{k-1}(W|\alpha)W^2_{i\beta\alpha}\\ &+S_l(G)\sum_{i\in B}\sum_{\alpha\neq\beta}S_{k-2}(W|\alpha\beta)W^2_{\alpha\beta i}-\sum_{\alpha=1}^n S_{l-1}(G)\sum_{i,\beta\in B}S_{k-1}(W|\alpha)W^2_{i\beta\alpha}. \qquad (4.22)\end{aligned}$$

We need further simplifications for the terms in (4.22). For the fourth and fifth terms on the right hand side of (4.22), $i\in B$, we regroup the summations as $\sum_{\alpha\neq\beta}=2\sum_{\substack{\alpha\in B\\ \beta\in G}}+\sum_{\substack{\alpha,\beta\in B\\ \alpha\neq\beta}}+\sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}}$, $\sum_{\alpha=1}^n\sum_{\beta\in G}=\sum_{\substack{\alpha\in B\\ \beta\in G}}+\sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}}+\sum_{\alpha=\beta\in G}$.

Since W is semi-positive definite,

$$W_{\beta\beta}S_{k-2}(G|\beta)\le S_{k-1}(G).$$

For any $\alpha \in B, \beta \in G$, by nominality of $S_l(G)$,

$$\sum_{\substack{i,\alpha\in B\\ \beta\in G}} S_l(G)S_{k-2}(W|\alpha\beta)W^2_{\alpha\beta i} \sim \sum_{\substack{i,\alpha\in B\\ \beta\in G}} S_{l-1}(G|\beta)W_{\beta\beta}S_{k-2}(G|\beta)W^2_{\alpha\beta i}$$

$$\leq \sum_{\substack{i,\alpha\in B\\ \beta\in G}} S_{l-1}(G|\beta)S_{k-1}(G)W^2_{\alpha\beta i}. \tag{4.23}$$

Also, when $\alpha, \beta \in G, \alpha \neq \beta$,

$$S_{l-1}(G|\beta)S_{k-1}(G|\alpha) = S_{l-1}(G|\beta)[S_{k-1}(G|\alpha\beta) + W_{\beta\beta}S_{k-2}(G|\alpha\beta)]$$

$$\geq S_{l-1}(G|\beta)W_{\beta\beta}S_{k-2}(G|\alpha\beta) = S_l(G)S_{k-2}(G|\alpha\beta). \tag{4.24}$$

From (4.24), we get

$$\sum_{i\in B}\sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}} S_l(G)S_{k-2}(W|\alpha\beta)W^2_{\alpha\beta i} - 2\sum_{i\in B}\sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}} S_{l-1}(G|\beta)S_{k-1}(G|\alpha)W^2_{\alpha\beta i}$$

$$\lesssim -\sum_{i\in B}\sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}} S_{l-1}(G|\beta)S_{k-1}(G|\alpha)W^2_{\alpha\beta i} \leq 0. \tag{4.25}$$

Combining (4.23) and (4.25), we obtain the following inequality,

$$S_l(G)\sum_{i\in B}\sum_{\alpha\neq\beta} S_{k-2}(W|\alpha\beta)W^2_{\alpha\beta i} - 2\sum_{\alpha=1}^{n}\sum_{i\in B,\beta\in G} S_{l-1}(G|\beta)S_{k-1}(W|\alpha)W^2_{i\beta\alpha}$$

$$\lesssim \sum_{i\in B}\sum_{\substack{\alpha,\beta\in B\\ \alpha\neq\beta}} S_l(G)S_{k-2}(W|\alpha\beta)W^2_{\alpha\beta i} - 2\sum_{i\in B}\sum_{\alpha\in G} S_{l-1}(G|\alpha)S_{k-1}(G|\alpha)W^2_{\alpha\alpha i}.$$

Putting above to (4.22),

$$\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \lesssim S_l(G)[\sum_{i\in B}(\varphi_{ii} - k\varphi) - \sum_{i\in B}\sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}} S_{k-2}(G|\alpha\beta)W_{\alpha\alpha i}W_{\beta\beta i}]$$

$$-2\sum_{i\in B}\sum_{\alpha\in G} S_{l-1}(G|\alpha)S_{k-1}(G|\alpha)W^2_{\alpha\alpha i} - S_l(G)\sum_{i\in B}\sum_{\substack{\alpha,\beta\in B\\ \alpha\neq\beta}} S_{k-2}(G)W_{\alpha\alpha i}W_{\beta\beta i}$$

$$-\sum_{i=1}^{n} S_{l-1}(G)\sum_{\alpha,\beta\in B} S_{k-1}(G|\alpha)W^2_{\alpha\beta i} + \sum_{i\in B}\sum_{\substack{\alpha,\beta\in B\\ \alpha\neq\beta}} S_l(G)S_{k-2}(G|\alpha\beta)W^2_{\alpha\beta i}. \tag{4.26}$$

We set,

$$I_1 = S_l(G)S_{k-2}(G)\sum_{\substack{i,\alpha,\beta\in B\\ \alpha\neq\beta}}[W_{\alpha\beta i}^2 - W_{\alpha\alpha i}W_{\beta\beta i}] - \sum_{i=1}^n S_{l-1}(G)\sum_{\alpha,\beta\in B} S_{k-1}(G|\alpha)W_{\alpha\beta i}^2,$$

and $\forall i\in B$,

$$I_2 = \frac{S_l(G)\varphi_i^2}{k\varphi} - \sum_{\alpha\in G} S_{l-1}(G|\alpha)S_{k-1}(G|\alpha)W_{\alpha\alpha i}^2,$$

and

$$I_3 = S_l(G)[\frac{\varphi_i^2}{\varphi} - \sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}} S_{k-2}(G|\alpha\beta)W_{\alpha\alpha i}W_{\beta\beta i}] - \sum_{\alpha\in G} S_{l-1}(G|\alpha)S_{k-1}(G|\alpha)W_{\alpha\alpha i}^2$$

Claim: $I_1 \lesssim 0$, $I_2 \lesssim 0$ and $I_3 \lesssim 0$.

If the **Claim** is true, we will have the following critical formula.

$$\sum_{\alpha=1}^n F^{\alpha\alpha}\phi_{\alpha\alpha} \lesssim S_l(G)\sum_{i\in B}[\varphi_{ii} - \frac{k+1}{k}\frac{\varphi_i^2}{\varphi} - k\varphi]. \tag{4.27}$$

Then (4.3) follows from (4.27).

PROOF of the CLAIM. We observe that,

$$-\sum_{i=1}^n S_{l-1}(G)\sum_{\alpha,\beta\in B} S_{k-1}(W|\alpha)W_{\alpha\beta i}^2 \sim -(\sum_{i\in B}+\sum_{i\in G})S_{l-1}(G)\sum_{\alpha,\beta\in B} S_{l-1}(G)W_{\alpha\beta i}^2$$

$$\lesssim -S_{l-1}(G)S_{k-1}(G)\{\sum_{i\in B}\sum_{\substack{\alpha,\beta\in B\\ \alpha\neq\beta}} W_{\alpha\beta i}^2 - \sum_{i\in B}\sum_{\alpha\in B} W_{\alpha\alpha i}^2\}. \tag{4.28}$$

If we put (4.28) into I_1, by (4.5) and Newton-Maclaurin inequality, we get

$$I_1 \lesssim -\{S_l(G)S_{k-2}(G)\sum_{i,\alpha\in B} W_{\alpha\alpha i}(\sum_{\beta\in B,\beta\neq\alpha} W_{\beta\beta i}) + S_{l-1}(G)S_{k-1}(G)\sum_{i,\alpha\in B} W_{\alpha\alpha i}^2\}$$

$$+\sum_{i\in B}\sum_{\substack{\alpha,\beta\in B\\ \alpha\neq\beta}}[S_l(G)S_{k-2}(W|\alpha\beta) - S_{l-1}(G)S_{k-1}(G)]W_{\alpha\beta i}^2$$

$$\sim [S_l(G)S_{k-2}(G) - S_{l-1}(G)S_{k-1}(G)][\sum_{i,\alpha\in B} W_{\alpha\alpha i}^2 + \sum_{i\in B}\sum_{\substack{\alpha,\beta\in B\\ \alpha\neq\beta}} W_{\alpha\beta i}^2] \leq 0.$$

To treat I_2, by (4.5) and Proposition 2.2, $\forall i \in B$,

$$\varphi_i = (\sum_{\alpha\in B} + \sum_{\alpha\in G}) S_{k-1}(W|\alpha)W_{\alpha\alpha i} \sim \sum_{\alpha\in G} S_{k-1}(G|\alpha)W_{\alpha\alpha i}. \tag{4.29}$$

This yields,

$$\begin{aligned}
I_2 &\sim \frac{1}{k\varphi}(\sum_{\alpha\in G} S_l^{\frac{1}{2}}(G)S_{k-1}(G|\alpha)W_{\alpha\alpha i})^2 - \sum_{\alpha\in G} S_{l-1}(G|\alpha)S_{k-1}(G|\alpha)W_{\alpha\alpha i}^2 \\
&= \frac{1}{k\varphi}[\sum_{\alpha\in G} S_{l-1}^{\frac{1}{2}}(G|\alpha)W_{\alpha\alpha}^{\frac{1}{2}}S_{k-1}(G|\alpha)W_{\alpha\alpha i}]^2 - \sum_{\alpha\in G} S_{l-1}(G|\alpha)S_{k-1}(G|\alpha)W_{\alpha\alpha i}^2 \\
&\le \frac{1}{k\varphi}\sum_{\alpha,\beta\in G} S_{l-1}(G|\alpha)S_{k-1}(G|\alpha)W_{\alpha\alpha i}^2 W_{\beta\beta}S_{k-1}(G|\beta) - \sum_{\alpha\in G} S_{l-1}(G|\alpha)S_{k-1}(G|\alpha)W_{\alpha\alpha i}^2 \\
&\sim \sum_{\alpha\in G} S_{l-1}(G|\alpha)S_{k-1}(G|\alpha)W_{\alpha\alpha i}^2 - \sum_{\alpha\in G} S_{l-1}(G|\alpha)S_{k-1}(G|\alpha)W_{\alpha\alpha i}^2 = 0.
\end{aligned}$$

Now we deal with I_3. It follows from (4.29) that for any $i \in B$,

$$\varphi_i{}^2 \sim \sum_{\alpha\in G} S_{k-1}^2(G|\alpha)W_{\alpha\alpha i}^2 + \sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}} S_{k-1}(G|\alpha)S_{k-1}(G|\beta)W_{\alpha\alpha i}W_{\beta\beta i}.$$

By Lemma 2.3 and Lemma 2.4,

$$\begin{aligned}
\phi I_3 &\sim \sum_{\alpha\in G}[S_l(G)S_{k-1}^2(G|\alpha) - S_k(G)S_{l-1}(G|\alpha)S_{k-1}(G|\alpha)]W_{\alpha\alpha i}^2 \\
&+ S_l(G)\sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}}[S_{k-1}(G|\alpha)S_{k-1}(G|\beta) - S_k(G)S_{k-2}(G|\alpha\beta)]W_{\alpha\alpha i}W_{\beta\beta i} \\
&= -\sum_{\alpha\in G} S_k(G|\alpha)S_{l-1}(G|\alpha)S_{k-1}(G|\alpha)W_{\alpha\alpha i}^2 \\
&+ S_l(G)\sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}}[S_{k-1}^2(G|\alpha\beta) - S_k(G|\alpha\beta)S_{k-2}(G|\alpha\beta)]W_{\alpha\alpha i}W_{\beta\beta i} \le 0.
\end{aligned}$$

The **Claim** is verified. The proof of the Deformation Lemma is complete. □

5. The Existence and Convexity

First, we prove Theorem 1.2.

Proof of Theorem 1.2. By Evan-Krylov theorem and Schauder theorem, $u \in C^{3,\alpha}(S^n), \forall 0 < \alpha < 1$. In fact, u is in Hölder-Zygmund space $\Lambda^4(S^n)$ by linear elliptic theory. If W is not of full rank at some point x_0, then there is $n-1 \ge l \ge k$

such that $S_l(W(x)) > 0, \forall x \in S^n$ and $\phi(x_0) = S_{l+1}(W(x_0)) = 0$. By (4.1) in the Deformation Lemma 4.1, as $\varphi \in \mathcal{C}_{-\frac{1}{k}}$,

$$\sum_{\alpha,\beta}^{n} F^{\alpha\beta}(x)\phi_{\alpha\beta}(x) \leq C_1|\nabla\phi(x)| + C_2\phi(x).$$

The strong minimum principle implies $\phi = S_{l+1}(W) \equiv 0$. On the other hand, we may assume u satisfies (3.1), so u is nonnegative on S^n. By Minkowski type formula (e.g., [30]),

$$(n-l)\int_{S^n} uS_l(W) = (l+1)\int_{S^n} S_{l+1}(W).$$

We conclude that $u \equiv 0$. This is a contradiction to (1.1). □

Now we proceed to prove Theorem 1.3.

Since φ is connected to 1 in $\mathcal{C}_{-\frac{1}{k}}$, there is a continuous function $h(t,x)$ in $[0,1]\times S^n$, such that $h(0,x) = 1$, $h(1,x) = \varphi(x)$ and h is in $\mathcal{C}_{-\frac{1}{k}}$ for each fixed t. Now, we approximate h by a sequence of positive functions h^m satisfying

Properties:

(i), h^m is continuous in $[0,1] \times S^n$, and $h^m(0,x) = 1$;

(ii), for each t fixed, h^m is smooth in x variables and satisfies (1.2);

(iii), for each m and l fixed, h^m in $C([0,1] \times C^l(S^n))$;

(iv), $h^m \longrightarrow h$ uniformly in $C([0,1] \times C^{1,1}(S^n))$.

Such a sequence can be easily obtained by the operations of smoothing and projecting in x variables (so to make (1.2) satisfied) on the function $h-1$. We point out that we do not require $h^m(t,.)$ to be in $\mathcal{C}_{-\frac{1}{k}}$ (note that there is no direct assumption on $S_k(W_m)$ in Theorem 1.2).

We consider the following equation:

$$S_k(u^{t,m}_{ij}(x) + \delta_{ij}u^{t,m}(x)) = h^m(t,x), \qquad \forall x \in S^n. \tag{5.1}$$

Proposition 5.1. *For sufficient large m, the equation (5.1) has a unique smooth strictly convex solution $u^{t,m}$ satisfying (3.1) for all $t \in [0,1]$.*

Proof: The uniqueness follows from Alexandrov-Fenchel-Jessen Theorem. The regularity follows from Theorem 3.4. We use continuity method for the existence.

For each m fixed, let

$$I_m = \{t \in [0,1] | (5.1)\text{has strictly convex solution}\}.$$

Since for is strictly convex u satisfying(1.1), the linearized operator L_u at u is self-adjoint and $Span\{x_1, ..., x_{n+1}\}$ is the exact kernel . By standard implicit function theorem, I_m is open and non-empty (as $0 \in I_m$).

We claim I_m is closed when m sufficiently large. Suppose this is not true, then by Theorem 3.4 and continuity method, there is a sequence of smooth functions $\{u^{t_m}\}$, and $t_m > 0, x_m \in S^n$ such that $W_t = \{u^{t,m}_{ij} + \delta_{ij}u^{t,m}\}$ positive definite for $t < t_m$, u_{t_m} satisfying (5.1) with

$$\det(W_{t_m}(x_m)) = 0.$$

Since $h^m \longrightarrow h$ uniformly in $C^{1,1}$, by Theorem 3.4 there is a subsequence $\{t_{m_j}\}$ converges to t_0, $h^{m_j}(t_{m_j}, x)$ converges to $h(t_0, x)$ in $C^{1,1}$, and $u^{t_{m_j}}$ converges to a function u in $C^{3,\alpha}$ for every $0 \leq \alpha < 1$. The Hessian matrix $W = \{u_{ij}+\delta_{ij}u\}$ is semi-positive definite on S^n, but W is degenerate at some point. This is a contradiction to Theorem 1.2 as $h(t_0, .) \in \mathcal{C}_{-\frac{1}{k}}$. □

Proof of Theorem 1.3. The uniqueness result follows from Alexandrov-Fenchel-Jessen theorem. By Proposition 5.1, there is a sequence of strictly convex functions u^m satisfying

$$S_k(W_m(x)) = h^m(1, x), \qquad \text{on} \quad S^n.$$

By Theorem 3.4, there is a subsequence of smooth strictly convex function u^{m_j} converges to u in $C^{3,\alpha}$ for every $0 \leq \alpha < 1$. And u satisfies (1.1). By Theorem 1.2, u is strictly convex. The higher regularity and the analyticity of u follows from the standard elliptic theory. □

Remark 5.2. The convexity of $\varphi^{-\frac{1}{k}}$ is dual to the concavity of the differential operator $S_k^{\frac{1}{k}}$. From the proof, the condition on φ in Theorem 1.3 can also be replaced by the condition φ connected to $S_k(v_{ij} + v\delta_{ij}) \in \mathcal{C}_{-\frac{1}{k}}$ for some arbitrary smooth strictly convex body with support function v in Theorem 1.3. It is easy to verify that if φ satisfies (1.2) and $\{-\varphi_{ij} + k\delta_{ij}\varphi\} \geq 0$, then φ satisfies the condition in Theorem 1.3.

Remark 5.3. If G is an automorphic group of S^n which has no fixed point (e.g., G a symmetry action with respect to the origin) and if φ_1 and φ_2 are invariant under G, one may connect φ_1 and φ_2 in $\mathcal{C}_{-\frac{1}{k}}$ by the function $h(t, x) = (t\varphi(x)_1{}^{-\frac{1}{k}} + (1-t)\varphi_2{}^{-\frac{1}{k}})^{-k}$. $h(t, x)$ satisfies (1.2) automatically as it is invariant under G (see [17]). In particular, every G-invariant function $\varphi \in \mathcal{C}_{-\frac{1}{k}}$ is connected to 1 in $\mathcal{C}_{-\frac{1}{k}}$. In this special situation, $\varphi \in C_{-\frac{1}{k}}$ if and only if $\varphi = v^{-k}$ for some positive support function v of a G-invariant convex body. By Theorem 1.3, for any G-invariant convex bodies K_1 and K_2 with support functions v_1 and v_2 respectively, $\forall\lambda \in [0,1]$, $\forall p \in \mathbb{R}$, there is a unique G-invariant convex body $\tilde{K}^p_\lambda$ with support function u such that $S_k(\{u_{ij}+\delta_{ij}u\}) = (\lambda v_1^p + (1-\lambda)v_2^p)^{-k}$. The relation defines an operation for such G-invariant convex bodies. The observation exhibits that the class of functions which satisfy the condition in Theorem 1.3 is quite large.

Remark 5.4. In the case of the figures of revolution, the intermediate Christoffel-Minkowski problems was solved in Firey [14]. Set $g_t = (t\varphi(x) + 1 - t)^{\frac{1}{k}}$, and let $\eta_1(x), ..., \eta_n(x)$ be the eigenvalues of the matrix $\{\delta_{ij}g_t - (g_t)_{ij}\}$ at the point x. Set $\tau = \max_{i,x}(\eta_i(x))$. Pogorelov in [29] obtained a sufficient condition for the intermediate Christoffel-Minkowski probelms. The condition is that, $(\frac{n-1}{n})^{\frac{1}{2(k-1)}}\tau < \min_{S^n} g_t$. We note that, at any maximum point of g_t, it yields $(\frac{n-1}{n})^{\frac{1}{2(k-1)}} \max_{S^n} g_t < \min_{S^n} g_t$. This puts the restriction on g_t that φ is close to a constant.

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