

# KÄHLER-RICCI SOLITONS ON COMPACT COMPLEX MANIFOLDS WITH $c_1(M) > 0$

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ABSTRACT. In this paper, we discuss the relation between the existence of Kähler-Ricci solitons and certain functional associated to some complex Monge-Ampère equation on compact complex manifolds with positive first Chern class. In particular, we obtain a strong inequality of Moser-Trudinger type on a compact complex manifold admitting a Kähler-Ricci soliton. Our result also improves the one obtained in [Ti] and [TZ1].

## 0. Introduction.

In this paper, we study the existence of Kähler-Ricci solitons by using properness of certain functionals. Our approach is similar to that of [Ti] for Kähler-Einstein metrics. A Kähler metric  $g$  on a compact complex manifold  $M$  with first Chern class  $c_1(M) > 0$  is called a (homothetically shrinking) Kähler-Ricci soliton if there is a holomorphic vector field  $X$  on  $M$  such that the Kähler form  $\omega_g$  satisfies

$$\text{Ric}(\omega_g) - \omega_g = L_X \omega_g,$$

where  $\text{Ric}(\omega_g)$  denotes the Ricci form of  $\omega_g$  and  $L_X$  is the Lie derivative operator along  $X$ . In particular, if  $X = 0$ , such a  $g$  is a Kähler-Einstein metric. So Kähler-Ricci solitons can be regarded as a generalization of Kähler-Einstein metrics of positive scalar curvature. Ricci solitons have been studied extensively in the recent years (cf. [Ko], [Ha], [C1], [C2], [Ti], [TZ2], [TZ3], [CH], etc.). One motivation is that they are very closely related to the limiting singular behavior of solutions of certain PDEs which arise from geometric analysis, such as Hamilton's Ricci flow ([Ha]) and certain complex Monge-Ampère equations associated to Kähler-Einstein metrics ([Ti]). Kähler-Ricci solitons are special Ricci solitons. It was proved recently in ([TZ3], [TZ4]) that there exists at most one Kähler-Ricci soliton on any compact Kähler manifold with positive first Chern class, modulo holomorphic automorphisms. This extends Bando and Mabuchi's theorem on the uniqueness of Kähler-Einstein metrics with positive scalar curvature ([BM]).

Let  $g$  be a Kähler metric on  $M$  with its Kähler form

$$\omega_g = \frac{\sqrt{-1}}{2\pi} \sum_{i,j} g_{i\bar{j}} dz_i \wedge d\bar{z}_j,$$

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representing  $c_1(M)$ . Then, by the Hodge theory, there is a smooth real-valued function  $h_g$  such that

$$\text{Ric}(\omega_g) - \omega_g = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}h_g.$$

Suppose that  $X$  is a holomorphic vector field on  $M$  so that the integral curve of  $\text{Im}(X)$  consists of isometries of  $g$ . By the Hodge Decomposition Theorem, there is a unique smooth real-valued function  $\theta_X = \theta_X(\omega_g)$  on  $M$  such that

$$\begin{cases} i_X \omega_g = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\theta_X, \\ \int_M e^{\theta_X} \omega_g^n = \int_M \omega_g^n = V. \end{cases}$$

Set

$$\mathcal{M}_X(\omega_g) = \{\psi \in C^\infty(M) \mid \omega_\psi = \omega_g + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\psi > 0, \text{Im}(X)(\psi) = 0\}.$$

The following functional on  $\mathcal{M}_X(\omega_g)$  was introduced in [TZ4],

$$\tilde{F}_{\omega_g}(\psi) = \tilde{J}_{\omega_g}(\psi) - \frac{1}{V} \int_M \psi e^{\theta_X} \omega_g^n - \log\left(\frac{1}{V} \int_M e^{h_g - \psi} \omega_g^n\right),$$

where

$$\omega_\psi = \omega_g + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\psi,$$

and

$$\begin{aligned} & \tilde{J}_{\omega_g}(\psi) \\ &= \frac{n\sqrt{-1}}{2\pi V} \sum_{k=0}^{n-1} C_{n-1}^k \int_M \left( \int_0^1 \int_0^1 t(st)^k (1-st)^{n-1-k} e^{\theta_X + stX(\psi)} dt ds \right) \\ & \times \partial\psi \wedge \bar{\partial}\psi \wedge \omega_\psi^k \wedge \omega_g^{n-1-k}. \end{aligned} \quad (0.1)$$

Let  $K_0$  be a maximal compact subgroup of the automorphism group of  $M$  such that  $\sigma \cdot \eta = \eta \cdot \sigma$  for any  $\eta \in K_0$  and any isometry  $\sigma$  in the subgroup  $K_X$  generated by  $\text{Im}(X)$ . Then a Kähler-Ricci soliton with respect to  $X$  on  $M$  must be  $K_0$ -invariant by the uniqueness theorem in [TZ3]. We introduce

**Definition 0.1.** *The functional  $\tilde{F}_{\omega_g}$  is said to be proper with respect to some holomorphic vector field  $X$  on  $M$  if there is a monotone increasing function  $\mu(t) : (a, +\infty) \rightarrow \mathbb{R}$  such that*

$$\lim_{t \rightarrow +\infty} \mu(t) = +\infty,$$

and

$$\tilde{F}_{\omega_g}(\psi) \geq \mu(I_{\omega_g}(\psi)),$$

for any  $K_0$ -invariant function  $\psi \in \mathcal{M}_X(\omega_g)$ , where

$$I_{\omega_g}(\psi) = \frac{1}{V} \int_M \psi (\omega_g^n - \omega_\psi^n).$$

It was essentially proved in [TZ4] that the properness of  $\tilde{F}_{\omega_g}$  is a sufficient condition for the existence of Kähler-Ricci soliton, that is,

**Theorem 0.1.** *Suppose that  $\tilde{F}_\omega$  is proper with respect to a holomorphic vector field  $X$  on  $M$ . Then there is a Kähler-Ricci soliton with respect to  $X$  on  $M$ .*

The main purpose of this paper is to show that the properness of  $\tilde{F}_{\omega_g}$  is also a necessary condition for the existence of Kähler-Ricci solitons (cf. Section 5). Suppose that  $M$  admits a Kähler-Ricci soliton  $\omega_{KS}$  with respect to some holomorphic vector field  $X$ . We define a weighted inner product on  $C^\infty(M)$  by

$$(\phi, \psi) = \int_M \phi \psi e^{\theta_X(\omega_{KS})} \omega_{KS}^n,$$

and denote by

$$\Lambda_1(M, \omega_{KS}) = \{u \in C^\infty(M) \mid \Delta_{\omega_{KS}} u + X(u) = -u\}.$$

**Theorem 0.2.** *Let  $M$  be a compact complex manifold admitting a Kähler-Ricci soliton  $\omega_{KS}$  with respect to some holomorphic vector field  $X$  and  $G(\subseteq K_0)$  a compact subgroup of  $\text{Aut}(M)$  with  $\sigma \cdot \eta = \eta \cdot \sigma$  for any  $\sigma \in K_X$  and  $\eta \in G$ . Suppose that any  $G$ -invariant smooth function on  $M$  is perpendicular to the space  $\Lambda_1(M, \omega_{KS})$  with respect to the weighted inner product defined above. Then there are two positive numbers  $c$  and  $C$  such that for any  $G$ -invariant  $\psi$ ,*

$$\tilde{F}_{\omega_{KS}}(\psi) \geq c I_{\omega_{KS}}(\psi)^{\frac{1}{4n+5}} - C. \quad (0.2)$$

*In particular,  $\tilde{F}_{\omega_{KS}}$  is proper under the same assumption.*

The inequality (0.2) is equivalent to the following non-linear inequality of Moser-Trudinger type,

$$\int_M e^{-\psi} \omega_{KS}^n \leq C \exp\{\tilde{J}_{\omega_{KS}}(\psi) - c \tilde{J}_{\omega_{KS}}(\psi)^{\frac{1}{4n+5}} - \frac{1}{V} \int_M \psi \omega_{KS}^n\}. \quad (0.3)$$

The idea of our proof of Theorem 0.2 is inspired by [Ti] and [TZ1], where the authors proved a fully non-linear inequality for Kähler-Einstein manifolds. The inequality (0.3) generalizes such an inequality in [Ti] and [TZ1]. A weak version of (0.2) was obtained in [TZ4].

The organization of this paper is as follows. In Section 1, we give a proof of Theorem 0.1 following [TZ4]. In Section 2, we give an estimate for certain heat kernel on manifolds with modified positive Ricci curvature. In Section 3, a  $C^0$ -estimate for certain complex Monge-Ampère equations is obtained. In Section 4, we prove a smoothing lemma by using Hamilton's Ricci flow. Theorem 0.2 will be proved in Section 5.

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## 1. An Analytic Criterion for Kähler-Ricci Solitons.

Let  $(M, g)$  be an  $n$ -dimensional compact Kähler manifold with the first Chern class  $c_1(M) > 0$ . Denote by  $\text{Aut}^\circ(M)$  the connected component of the automorphism group of  $M$ . Let  $K$  be a maximal compact subgroup of  $\text{Aut}^\circ(M)$ , then the Chevalley decomposition allows us to write  $\text{Aut}^\circ(M)$  as a semi-direct decomposition ([FM]),

$$\text{Aut}^\circ(M) = \text{Aut}_r(M) \ltimes R_u,$$

where  $\text{Aut}_r(M) \subset \text{Aut}^\circ(M)$  is a reductive algebraic subgroup and the complexification of  $K$ , and  $R_u$  is the unipotent radical of  $\text{Aut}^\circ(M)$ . Let  $\eta_r(M)$  be the Lie subalgebra of  $\text{Aut}_r(M)$ .

Let  $X \in \eta_r(M)$  such that the one-parameter subgroup  $K_X$  generated by  $\text{Im}(X)$  is contained in  $K$ , where  $\text{Im}(X)$  denotes the imaginary part of  $X$ . Choose a  $K_X$ -invariant Kähler metric  $g$  on  $M$  with the Kähler form  $\omega_g$ . Then by the Hodge Decomposition Theorem, there is a smooth real-valued function  $\theta_X = \theta_X(\omega_g)$  of  $M$  such that  $i_X \omega_g = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \theta_X$ , and consequently,

$$L_X \omega_g = di_X(\omega_g) = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \theta_X.$$

We consider the following complex Monge-Ampère equations with parameter  $t \in [0, 1]$ :

$$\begin{cases} \det(g_{i\bar{j}} + \phi_{i\bar{j}}) = \det(g_{i\bar{j}}) \exp\{h - \theta_X - X(\phi) - t\phi\} \\ (g_{i\bar{j}} + \phi_{i\bar{j}}) > 0, \end{cases} \quad (1.1)$$

where  $h = h_{\omega_g}$  is a smooth real-valued function on  $M$  defined by

$$\begin{cases} \text{Ric}(\omega_g) - \omega_g = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} h \\ \int_M e^h \omega_g^n = \int_M \omega_g^n = V. \end{cases}$$

Here  $\omega_g^n = \omega_g \wedge \dots \wedge \omega_g$ . Then one can check that  $\omega_\phi = \omega_g + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi$  is a Kähler-Ricci soliton with respect to  $X$ , i.e.,  $\omega_\phi$  satisfies

$$\text{Ric}(\omega_\phi) - \omega_\phi = L_X \omega_\phi,$$

if and only if  $\phi$  is a solution of equations (1.1)<sub>t</sub> for  $t = 1$ . In fact, (1.1)<sub>t</sub> is equivalent to

$$\text{Ric}(\omega_{\phi_t}) - L_X(\omega_{\phi_t}) = t\omega_{\phi_t} + (1-t)\omega_g, \quad (1.2)$$

where  $\phi_t$  is a solution of (1.1)<sub>t</sub>. Furthermore, we can normalize  $\theta_X$  by

$$\int_M e^{\theta_X} \omega_g^n = \int_M \omega_g^n = V. \quad (1.3)$$

Since

$$\frac{d}{dt} \left( \int_M e^{\theta_X + X(\phi_t)} \omega_{\phi_t}^n \right) = \int_M (\Delta' \dot{\phi}_t + X(\dot{\phi}_t)) e^{\theta_X + X(\phi_t)} \omega_{\phi_t}^n = 0,$$

then by (1.3), we have

$$\int_M e^{\theta_X + X(\phi_t)} \omega_{\phi_t}^n = \int_M e^{\theta_X} \omega_g^n = V.$$

Integrating (1.1)<sub>t</sub> after multiplied by  $e^{\theta_X + X(\phi)}$ , it follows

$$\int_M e^{h-t\phi_t} \omega_g^n = V.$$

Therefore, differentiating the above identities, we get

$$\int_M \dot{\phi}_t e^{h-t\phi_t} \omega_g^n = -\frac{1}{t} \int_M \phi_t e^{h-t\phi_t} \omega_g^n. \quad (1.4)$$

Set

$$\mathcal{M}(\omega_g) = \{\phi \in C^\infty(M) \mid \omega_\phi = \omega_g + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\phi > 0\}.$$

and

$$\mathcal{M}_X(\omega_g) = \{\phi \in \mathcal{M}(\omega_g) \mid \text{Im}(X)(\phi) = 0\}.$$

In [TZ3], the following two functionals on  $\mathcal{M}_X(\omega_g)$  were introduced:

$$\tilde{I}_{\omega_g}(\phi) = \frac{1}{V} \int_M \phi (e^{\theta x} \omega_g^n - e^{\theta x + X(\phi)} \omega_\phi^n)$$

and

$$\tilde{J}_{\omega_g}(\phi) = \frac{1}{V} \int_0^1 \int_M \dot{\phi}_s (e^{\theta x} \omega_g^n - e^{\theta x + X(\phi_s)} \omega_{\phi_s}^n) \wedge ds, \quad (1.5)$$

where  $\phi_s$  is a path from 0 to  $\phi$  in  $\mathcal{M}_X(\omega_g)$ . It is proved in [TZ3] that there are two uniform positive constants  $c_1$  and  $c_2$  such that

$$c_1 I_{\omega_g}(\phi) \leq \tilde{I}_{\omega_g}(\phi) - \tilde{J}_{\omega_g}(\phi) \leq c_2 I_{\omega_g}(\phi), \quad (1.6)$$

where  $I_{\omega_g}(\phi) = \frac{1}{V} \int_M \phi (\omega_g^n - \omega_\phi^n)$ .

Let

$$\begin{aligned} \hat{F}_{\omega_g}(\phi) &= \tilde{J}_{\omega_g}(\phi) - \frac{1}{V} \int_M \phi e^{\theta x} \omega_g^n \\ &= -\frac{1}{V} \int_0^1 \int_M \dot{\phi}_s e^{\theta x + X(\phi_s)} \omega_{\phi_s}^n \wedge ds. \end{aligned}$$

By simple computations, one can show that for any two functions  $\phi$  and  $\psi$  in  $\mathcal{M}_X(\omega_g)$ , the following co-cycle condition is satisfied,

$$\hat{F}_{\omega_g}(\psi) = \hat{F}_{\omega_g}(\phi) + \hat{F}_\phi(\psi),$$

where  $\hat{F}_\phi(\psi) = -\frac{1}{V} \int_0^1 \int_M \dot{\phi}_s e^{\theta x + X(\phi_s)} \omega_{\phi_s}^n \wedge ds$  and  $\phi_s$  is a path from  $\phi$  to  $\psi$  in  $\mathcal{M}_X(\omega_g)$ .

**Proposition 1.1.** *Let  $\phi_t$  be a solution of (1.1)<sub>s</sub> for  $s = t$ . Then*

$$\hat{F}_{\omega_g}(\phi_t) = -\frac{1}{t} \int_0^t (\tilde{I}_{\omega_g}(\phi_s) - \tilde{J}_{\omega_g}(\phi_s)) ds.$$

*Proof.* Differentiating (1.1)<sub>t</sub> with respect to  $t$ , we have

$$\Delta' \dot{\phi}_t + X(\dot{\phi}_t) = -(t\dot{\phi}_t + \phi_t). \quad (1.7)$$

Then by using (1.7) and (1.4), we get

$$\begin{aligned} & \frac{d}{dt}(\tilde{I}_{\omega_g}(\phi_t) - \tilde{J}_{\omega_g}(\phi_t)) \\ &= -\frac{1}{V} \int_M \phi_t \frac{d}{dt}(e^{\theta x + X(\phi_t)} \omega_{\phi_t}^n) \\ &= -\frac{1}{V} \int_M \phi_t (\Delta' \dot{\phi}_t + X(\dot{\phi}_t)) e^{\theta x + X(\phi_t)} \omega_{\phi_t}^n \\ &= \frac{1}{V} \int_M \phi_t (t\dot{\phi}_t + \phi_t) e^{\theta x + X(\phi_t)} \omega_{\phi_t}^n \\ &= \frac{1}{V} \int_M \phi_t (t\dot{\phi}_t + \phi_t) e^{f - t\phi_t} \omega_g^n \\ &= \frac{1}{V} \frac{d}{dt} \left( \int_M (-\phi_t) e^{f - t\phi_t} \omega_g^n \right) + \frac{1}{V} \int_M \dot{\phi}_t e^{f - t\phi_t} \omega_g^n \\ &= \frac{1}{tV} \frac{d}{dt} \left( \int_M t(-\phi_t) e^{f - t\phi_t} \omega_g^n \right) \\ &= \frac{1}{tV} \frac{d}{dt} \left( \int_M t(-\phi_t) e^{\theta x + X(\phi_t)} \omega_{\phi_t}^n \right). \end{aligned}$$

It follows

$$\begin{aligned} & \frac{d}{dt}(t(\tilde{I}_{\omega_g}(\phi_t) - \tilde{J}_{\omega_g}(\phi_t))) - (\tilde{I}_{\omega_g}(\phi_t) - \tilde{J}_{\omega_g}(\phi_t)) \\ &= \frac{1}{V} \frac{d}{dt} \left( \int_M t(-\phi_t) e^{\theta x + X(\phi_t)} \omega_{\phi_t}^n \right), \end{aligned} \quad (1.8)$$

and consequently,

$$\begin{aligned} \hat{F}_{\omega_g}(\phi_t) &= -\frac{1}{V} \int_M \phi e^{\theta x + X(\phi_t)} \omega_t^n - (\tilde{I}_{\omega_g}(\phi_t) - \tilde{J}_{\omega_g}(\phi_t)) \\ &= -\frac{1}{t} \int_0^t (\tilde{I}_{\omega_g}(\phi_s) - \tilde{J}_{\omega_g}(\phi_s)) ds. \end{aligned}$$

□

Recall that the functional  $\tilde{F}_{\omega_g}$  is defined as

$$\begin{aligned} \tilde{F}_{\omega_g}(\psi) &= \hat{F}_{\omega_g}(\psi) - \log\left(\frac{1}{V} \int_M e^{h-\psi} \omega_g^n\right) \\ &= \tilde{J}_{\omega_g}(\psi) - \frac{1}{V} \int_M \psi e^{\theta x} \omega_g^n - \log\left(\frac{1}{V} \int_M e^{h-\psi} \omega_g^n\right), \end{aligned}$$

and  $\tilde{F}_{\omega_g}$  is called proper with respect to some holomorphic vector field  $X$  on  $M$  if there is a monotone increasing function  $\mu(t) : (a, +\infty) \rightarrow \mathbb{R}$  such that

$$\lim_{t \rightarrow +\infty} \mu(t) = +\infty,$$

and

$$\tilde{F}_{\omega_g}(\psi) \geq C_1 \mu(I_{\omega_g}(\psi)) - C_2$$

for any  $K_0$ -invariant function  $\psi$ , where  $K_0(\supseteq K_X)$  is a maximal compact subgroup of automorphism group  $\text{Aut}(M)$  such that  $\sigma \cdot \eta = \eta \cdot \sigma$  for any  $\sigma \in K_X$  and  $\eta \in K_0$ .

*Proof of Theorem 0.1.* Assuming the functional  $\tilde{F}_{\omega_g}$  is proper, we shall prove the existence of a Kähler-Ricci soliton with respect to  $X$  on  $M$ . This is equivalent to prove that there is a solution of (1.1)<sub>t</sub> for  $t = 1$ . It suffices to prove that  $I_{\omega_g}(\phi_t)$  is uniformly bounded for any solution of (1.1)<sub>t</sub> for  $0 \leq t \leq 1$ . This is because  $C^3$ -norm of  $\phi_t$  can be uniformly bounded by  $I_{\omega_g}(\phi_t)$  (cf. [TZ3], [Ya]). By the Implicit Function Theorem, one can show the solution of (1.1)<sub>t</sub> varies smoothly with  $t < 1$ . Without the loss of generality, we may assume that the Kähler form  $\omega_g$  is  $G_0$ -invariant, where  $G_0(\supseteq K_0)$  is a maximal compact subgroup of  $\text{Aut}(M)$ . Then solutions  $\phi_t$  are all  $K_0$ -invariant.

By Proposition 1.1, we have

$$\begin{aligned} & \tilde{F}_{\omega_g}(\phi_t) \\ &= -\frac{1}{t} \int_0^t (\tilde{I}_{\omega_g}(\phi_s) - \tilde{J}_{\omega_g}(\phi_s)) ds - \log\left(\frac{1}{V} \int_M e^{h\omega_g - \phi_t \omega_g^n}\right) \\ &\leq -\log\left(\frac{1}{V} \int_M e^{h - \phi_t \omega_g^n}\right). \end{aligned} \quad (1.9)$$

On the other hand, by using (1.1)<sub>t</sub> and concavity of the logarithmic function, one can deduce

$$\begin{aligned} -\log\left(\frac{1}{V} \int_M e^{h - \phi_t \omega_g^n}\right) &= \frac{1-t}{V} \int_M \phi_t e^{\theta_x + X(\phi_t) \omega_{\phi_t}^n} \\ &\leq \frac{1-t}{V} \int_M \phi_t e^{h-t\phi_t \omega_g^n} \leq C. \end{aligned} \quad (1.10)$$

Combining (1.9) and (1.10), we get

$$\tilde{F}_{\omega_g}(\phi_t) \leq C.$$

Therefore, the assumption of properness on  $\tilde{F}_{\omega_g}$  implies

$$I_{\omega_g}(\phi_t) \leq C,$$

and consequently, there is a Kähler-Ricci soliton metric on  $M$  with respect to  $X$ .  $\square$

## 2. A Heat Kernel Estimate.

In this section, we give an estimate on the heat kernel of the linear elliptic operator  $P = P_\omega = \Delta + \text{Re}(X(\cdot))$  associated to a Kähler form  $\omega$  and a holomorphic vector field  $X$  on  $M$ . As a consequence, we derive a lower bound of the Green function of  $P$ . The method here follows that of T. Mabuchi in [M1] with modifications which in turn is inspired by Li-Yau [LY]. Note that  $P$  is a self-adjoint elliptic operator acting on  $C^\infty(M)$  with respect to the inner product,

$$(\phi, \psi) = \int_M \phi \bar{\psi} e^{\theta_x} \omega^n.$$

**Lemma 2.1.** *Let  $\omega$  be a Kähler form on  $M$  and  $X$  a holomorphic vector field on  $M$  with*

$$\text{Ric}(\omega) - L_X\omega \geq 0,$$

and

$$\Delta\theta_X \leq k,$$

for some positive number  $k$ . Let  $v(x, t)$  be a positive smooth solution on  $M \times (0, \infty)$  of equation  $(P - \frac{\partial}{\partial t})v = 0$ . Suppose that

$$\limsup_{t \rightarrow 0} \sup_{M \times \{t\}} t(v^{-2} \langle \bar{\partial}v, \bar{\partial}v \rangle - v^{-1}v_t) \leq 2n,$$

where  $v_t = \frac{\partial v}{\partial t}$ . Then there is a positive number  $C$  depending only on

$$m_1 = -\max_M \theta_X \text{ and } m_2 = -\min_M \theta_X$$

such that

$$v(x, t_1) \leq v(y, t_2) \left(\frac{t_2}{t_1}\right)^{\frac{n}{C}} \exp\{(t_2 - t_1)^{-1}r(x, y)^2/2 + C^{-1}k(t_2 - t_1)\}, \quad (2.1)$$

where  $r(x, y)$  denotes the distance between  $x$  and  $y$  associate to metric  $\omega$ .

*Proof.* Let  $f = \ln v$  and  $\hat{F} = t(\langle \bar{\partial}f, \bar{\partial}f \rangle - f_t)$ . Then

$$\begin{aligned} P\hat{F} + tPf_t &= t(\langle \text{tr}_\omega(\bar{\nabla}\bar{\nabla})\bar{\partial}f, \bar{\partial}f \rangle + \langle \text{tr}_\omega(\bar{\nabla}\bar{\nabla})\partial f, \partial f \rangle \\ &\quad + \text{Re}(X(\langle \bar{\partial}f, \bar{\partial}f \rangle)) + \|\nabla\bar{\nabla}f\|^2 + \|\nabla\nabla f\|^2). \end{aligned} \quad (2.2)$$

For simplicity, we choose a local holomorphic coordinate system  $(z_1, \dots, z_n)$  near each point  $p$  such that  $g_{i\bar{j}}(p) = \delta_{i\bar{j}}$  and  $f_{i\bar{j}}(p) = \delta_{i\bar{j}}f_{i\bar{i}}(p)$ . By a direct computation, one sees,

$$\begin{aligned} f_{\bar{i}}f_{i\bar{j}\bar{j}} &= f_{\bar{i}}(f_{j\bar{j}i} + f_l R_{i\bar{l}}), \\ f_i f_{i\bar{j}\bar{j}} &= f_i f_{j\bar{j}i}, \end{aligned}$$

and

$$\begin{aligned} X_{\bar{i}}(f_j f_{\bar{j}})_i + f_{\bar{i}}f_j X_{\bar{j}i} \\ = f_i(X_{\bar{j}}f_j)_{\bar{i}} + f_{\bar{i}}(X_{\bar{j}}f_j)_i. \end{aligned}$$

Inserting the above identities into (2.2), we obtain

$$\begin{aligned} P\hat{F} &\geq t(\langle \partial f, \partial(Pf) \rangle + \langle \partial(Pf), \partial f \rangle) - tPf_t + \frac{t}{n}(\Delta f)^2 \\ &= 2t\text{Re} \langle \partial(Pf), \partial f \rangle - tPf_t + \frac{t}{n}(\Delta f)^2 \end{aligned} \quad (2.3)$$

Since

$$Pf - f_t = -\langle \bar{\partial}f, \bar{\partial}f \rangle,$$

we have

$$\hat{F} = -tPf,$$

and

$$\frac{\partial}{\partial t}\hat{F} - \frac{1}{t}\hat{F} = -tPf_t.$$

Hence by (2.3), we get

$$\begin{aligned} & (P - \frac{\partial}{\partial t})\hat{F} \\ & \geq -2\text{Re} \langle \partial\hat{F}, \partial f \rangle - \frac{1}{t}\hat{F} + \frac{t}{n}(\frac{1}{t}\hat{F} + \text{Re}(Xf))^2. \end{aligned} \quad (2.4)$$

As in [M1], we define a monotone-increasing function  $\eta$  on  $[m_1, m_2]$  by

$$\eta(s) = \exp \int_m^s \frac{1}{b_0 e^{-y} - 1} dy,$$

where  $b_0 = e^{m_2}(1+n)$ . Then  $\eta$  is a solution of the ODE on  $[m_1, m_2]$ ,

$$\frac{\eta''}{\eta} - \frac{\eta'}{\eta} = 2(\frac{\eta'}{\eta})^2.$$

Moreover, one can check that the number  $C$  defined by

$$C = \min_{s \in [m_1, m_2]} \eta(s)^{-1} (1 - (1 - \eta(s)^{-1} \eta'(s) n)^2)$$

is positive, and

$$0 < \frac{\eta'}{\eta} \leq \frac{1}{n}.$$

Let  $F = \eta(-\theta_X)\hat{F}$ . Then by (2.4), one can show that

$$\begin{aligned} & (P - \frac{\partial}{\partial t})F \\ & \geq -2\text{Re} \langle \partial F, \partial f \rangle - \frac{1}{t}F \\ & \quad + (\frac{\eta''}{\eta} - \frac{\eta'}{\eta} - 2(\frac{\eta'}{\eta})^2) \|X\|^2 F \\ & \quad + \eta^{-1} \eta' (-2\text{Re}(XF) - F \Delta \theta_X) \\ & \quad + n^{-1} t^{-1} F^2 \eta^{-1} (1 - (1 - \eta^{-1} \eta' n)^2) \\ & \quad + \frac{t}{n} \eta (-\frac{1}{t} \eta^{-1} F (1 - n \eta^{-1} \eta')^2 - \text{Re}(Xf))^2. \end{aligned}$$

By the assumption in the lemma, we have

$$\begin{aligned} (P - \frac{\partial}{\partial t})F & \geq -2\text{Re} \langle \partial F, \partial f \rangle - 2\eta^{-1} \eta' \text{Re}(X(F)) \\ & \quad - (\frac{1}{t} + \frac{k}{n})F + \frac{C}{nt} F^2. \end{aligned} \quad (2.5)$$

Applying the Maximal Principle for the function  $F(x, t)$  on  $M \times (0, T]$ , we get from (2.5),

$$F(x, t) \leq C^{-1}(2n + kt),$$

for any  $(x, t) \in M \times (0, T]$ , and consequently,

$$v(x, t)^{-2} < \bar{\partial}v(x, t), \bar{\partial}v(x, t) > -v(x, t)^{-1}v_t(x, t) \leq C^{-1}\left(\frac{2n}{t} + k\right),$$

for any  $(x, t) \in M \times (0, \infty)$ . Now by integrating the above estimate as in [LY], we can immediately obtain (2.1).  $\square$

Let  $H = H(x, y, t)$  be a fundamental solution on  $M \times M \times [0, \infty)$  of equation

$$\left(P - \frac{\partial}{\partial t}\right)v(x, y, t) = 0, \quad (2.6)$$

i.e.  $H$  is a smooth solution of (2.6) satisfying

$$\begin{cases} H(x, y, t) = H(y, x, t), \\ H(x, y, t) = \int_M H(x, z, t-s)H(z, y, s)e^{\theta x} \omega^n, \\ \lim_{t \rightarrow 0} H(x, y, t) = \delta_x(y). \end{cases}$$

By using the asymptotic behaviour of  $H$ , for each fixed  $x \in M$ , one sees

$$\lim_{t \rightarrow 0} \sup_{M \times \{t\}} t(H^{-2} < \bar{\partial}H(x, \cdot, t), \bar{\partial}H(x, \cdot, t) > -H^{-1}H_t) \leq 2n.$$

Moreover, by using an argument in [LY] and Lemma 3.2, we can deduce

**Lemma 2.2.** *Let  $Z_1$  and  $Z_2$  be two measurable subset of  $M$ . Let  $T, \delta, \tau$  be three positive numbers with  $\tau < (1 + 2\delta)T$ . For each  $x, y \in M$  and  $0 < t \leq \tau$ , denote*

$$F_{x,T}(y, t) = \int_{Z_1} H(y, \cdot, t)H(x, \cdot, T)e^{\theta x} \omega^n.$$

Then

$$\int_{Z_2} F_{x,T}(\cdot, t)^2 e^{\theta x} \omega^n \leq \exp\left\{\frac{-r(x, Z_1)^2}{2(1 + 2\delta)T} + \frac{R(x, Z_2)^2}{2(1 + 2\delta)T - 2t}\right\} F_{x,T}(x, T), \quad (2.7)$$

where  $r(x, Z_1) = \inf_{z \in Z_1} r(x, z)$  and  $R(x, Z_2) = \sup_{z \in Z_2} r(x, z)$ .

**Proposition 2.1.** *Let  $H(x, y, t)$  be a fundamental solution on  $M \times M \times [0, \infty)$  of equation (2.6). Suppose*

$$\text{Ric}(\omega) - L_X \omega \geq 0,$$

and

$$\Delta \theta_x \leq k,$$

for some positive number  $k$ . Then we have

$$\begin{aligned} & \text{vol}(B_x(\sqrt{t}))^{\frac{1}{2}} \text{vol}(B_y(\sqrt{t}))^{\frac{1}{2}} H(x, y, t) \\ & \leq (1 + \delta)^{4n/C} \exp\left\{-\frac{\{r(x, y) - \sqrt{t}\}_+^2}{4t(1 + 3\delta + 2\delta^2)} + t\delta(2 + \delta)C^{-1}k + \frac{3}{4\delta(1 + \delta)} + \frac{1}{2\delta}\right\}, \end{aligned}$$

where  $\text{vol}(B_y(\sqrt{t})) = \int_{B_y(\sqrt{t})} e^{\theta x} \omega^n$ , and  $\{r(x, y) - \sqrt{t}\}_+ = \max\{0, r(x, y) - \sqrt{t}\}$ .

*Proof.* First applying Lemma 2.1 to the function  $F_{x,T}(y, t)$  with  $(t_1, t_2) = (T, \tau = (1 + \delta)T)$ , we have

$$F_{x,T}(x, T) \leq F_{x,T}(y, \tau)(1 + \delta)^{2n/C} \exp\{T^{-1}\delta^{-1}r(x, y)^2/2 + C^{-1}kT\delta\}. \quad (2.8)$$

Let  $Z_1 = B_y(\sqrt{t})$  and  $Z_2 = B_x(\sqrt{t})$ . Then integrating the square of the above inequality over all  $y \in Z_2$  and using (2.8) and Lemma 2.2, it follows

$$\begin{aligned} & \text{vol}(B_x(\sqrt{t})) F_{x,T}(x, T)^2 \\ & \leq (1 + \delta)^{4n/C} \exp\left\{\frac{-r(x, Z_1)^2}{2(1 + 2\delta)T} + 2C^{-1}kT\delta + \frac{3t}{2T\delta}\right\} F_{x,T}(x, T), \end{aligned}$$

and consequently,

$$\begin{aligned} & \text{vol}(B_x(\sqrt{t})) \int_{Z_1} H(x, \cdot, T)^2 e^{\theta x} \omega^n \\ & = \text{vol}(B_x(\sqrt{t})) F_{x,T}(x, T) \\ & \leq (1 + \delta)^{4n/C} \exp\left\{\frac{-r(x, Z_1)^2}{2(1 + 2\delta)T} + 2C^{-1}kT\delta + \frac{3t}{2T\delta}\right\}. \end{aligned} \quad (2.9)$$

On the other hand, applying Lemma 2.1 to the function  $H(x, z, t)$  in  $z$  with  $(t_1, t_2) = (t, T = (1 + \delta)t)$ , we have for any  $x, y, z \in M$ ,

$$H(x, y, t)^2 \leq (1 + \delta)^{4n/C} H(x, z, T)^2 \exp\{T^{-1}\delta^{-1}r(y, z)^2 + 2C^{-1}kT\delta\}.$$

Integrating this inequality over all  $z \in Z_1$ , and using (2.9), we can get (2.7).  $\square$

**Theorem 2.1.** *Let  $\phi \in \mathcal{M}_X(\omega)$ . Suppose that*

$$\text{Ric}(\omega_\phi) - L_X \omega_\phi \geq \lambda \omega_\phi, \quad (2.10)$$

and

$$\Delta \theta_X(\omega_\phi) \leq k,$$

for some positive numbers  $\lambda$  and  $k$ . Then there is a uniform constant  $C$  depending only on  $\lambda$  and  $k$  such that

$$\sup_M(-\phi) \leq \frac{1}{V} \int_M (-\phi) e^{\theta x(\omega_\phi)} \omega_\phi^n + C. \quad (2.11)$$

*Proof.* Let  $\mu_i (\mu_0 = 0), i = 0, 1, \dots$ , be the increasing sequence of eigenvalues of operator  $-P$  associated to metric  $\omega_\phi$ . Then by using the standard Bochner technique, one can obtain  $\mu_1 \geq \lambda$  (cf. [TZ3]). Let  $G(x, y)$  be the Green function with  $\int_M G(x, \cdot) e^{\theta x(\omega_\phi)} \omega_\phi^n = 0$  associated to operator  $P$ . Then

$$G(x, y) = \int_0^\infty (H(x, y, t) - \frac{1}{V}) dt.$$

Since

$$H_0(x, y, t) = H(x, y, t) - \frac{1}{V} = \sum_{i=1}^\infty e^{-\mu_i t} f_i(x) f_i(y),$$

we have

$$H_0(x, x, t + t_0) \leq e^{-\mu_1 t} H_0(x, x, t_0), \quad (2.12)$$

for any  $t_0, t > 0$ , where  $f_i(x)$  denote the eigenfunctions of  $\mu_i$ .

In [M2], it was proved under the condition (2.10) that there is a uniform constant  $C_1$  such that

$$\text{Diam}(M, \omega_\phi) \leq \frac{C_1}{\sqrt{\lambda}}.$$

Choose  $t_0 = \frac{1}{4} \text{Diam}(M, \omega_\phi)^2$ . Then by Proposition 2.1 we have

$$H_0(x, x, t_0) \leq C_2, \quad (2.13)$$

for some uniform constant  $C_2$  depending only on  $\lambda$  and  $k$ .

By using (2.12) and (2.13), we get

$$\begin{aligned} G(x, y) &\geq - \int_0^{t_0} \frac{1}{V} dt - \int_{t_0}^\infty e^{-\mu_1(t-t_0)} (H_0(x, x, t_0) H_0(y, y, t_0))^{1/2} dt \\ &\geq -C_3. \end{aligned}$$

Therefore applying the Green formula to function  $-\phi$ , together with the fact (cf. [Zh]),

$$\sup_{\mathcal{M}_X(\omega_g)} |X(\phi)| \leq c \quad (2.14)$$

for some uniform constant  $c = c(\omega_g, X)$ , we prove

$$\begin{aligned} \sup_M (-\phi) &= \frac{1}{V} \int_M (-\phi) e^{\theta x(\omega_\phi)} \omega_\phi^n \\ &\quad + \sup_M \int_M P(\phi(\cdot)) G(x, \cdot) e^{\theta x(\omega_\phi)} \omega_\phi^n \\ &\leq \frac{1}{V} \int_M (-\phi) e^{\theta x(\omega_\phi)} \omega_\phi^n + V(n + \sup_{x \in M} |X(\phi)(x)|) C_3 \\ &\leq \frac{1}{V} \int_M (-\phi) e^{\theta x(\omega_\phi)} \omega_\phi^n + C. \end{aligned}$$

□

### 3. An $C^0$ -Estimate for Solutions of (1.1)<sub>t</sub>.

**Lemma 3.1.** *There are two positive numbers  $c_1$  and  $c_2 < 1$  such that for any  $\phi \in \mathcal{M}_X(\omega_g)$ ,*

$$c_1 \bar{I}_{\omega_g}(\phi) \leq \bar{I}_{\omega_g}(\phi) - \bar{J}_{\omega_g}(\phi) \leq c_2 \bar{I}_{\omega_g}(\phi). \quad (3.1)$$

*Proof.* Let  $\omega_{s\phi} = \omega + s \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi$ . Then one can compute

$$\begin{aligned} & \bar{I}_{\omega_g}(\phi) \\ &= \frac{1}{V} \int_M \phi \int_0^1 \frac{d}{ds} (e^{\theta x + sX(\phi)} \omega_{s\phi}^n) \wedge ds \\ &= \frac{n\sqrt{-1}}{2\pi V} \int_0^1 ds \int_M \partial \phi \wedge \bar{\partial} \phi e^{\theta x + sX(\phi)} \wedge \omega_{s\phi}^{n-1} \\ &= \frac{n\sqrt{-1}}{2\pi V} \sum_{k=0}^{n-1} C_{n-1}^k \int_M \left( \int_0^1 s^k (1-s)^{n-1-k} e^{\theta x + sX(\phi)} ds \right) \\ & \quad \times \partial \phi \wedge \bar{\partial} \phi \wedge \omega_\phi^k \wedge \omega_g^{n-1-k} \\ &\leq \frac{n\sqrt{-1}}{2\pi V} C_1 \sum_{k=0}^{n-1} C_{n-1}^k \partial \phi \wedge \bar{\partial} \phi \wedge \omega_\phi^k \wedge \omega_g^{n-1-k}. \end{aligned} \quad (3.2)$$

and

$$\begin{aligned} & \bar{J}_{\omega_g}(\phi) \\ &= \frac{n\sqrt{-1}}{2\pi V} \int_0^1 dt \int_0^1 ds \int_M t \partial \phi \wedge \bar{\partial} \phi e^{\theta x + stX(\phi)} \wedge \omega_{st\phi}^{n-1} \\ &= \frac{n\sqrt{-1}}{2\pi V} \sum_{k=0}^{n-1} C_{n-1}^k \int_M \left( \int_0^1 \int_0^1 t(st)^k (1-st)^{n-1-k} e^{\theta x + stX(\phi)} dt \wedge ds \right) \\ & \quad \times \partial \phi \wedge \bar{\partial} \phi \wedge \omega_\phi^k \wedge \omega_g^{n-1-k} \\ &\geq \frac{n\sqrt{-1}}{2\pi V} C'_1 \sum_{k=0}^{n-1} C_{n-1}^k \partial \phi \wedge \bar{\partial} \phi \wedge \omega_\phi^k \wedge \omega_g^{n-1-k}. \end{aligned} \quad (3.3)$$

Combining (3.2) and (3.3), we get

$$\bar{J}_{\omega_g}(\phi) \geq \frac{C'_1}{C_1} \bar{I}_{\omega_g}(\phi),$$

and consequently, prove the second inequality of (3.1).

On the other hand, we have (cf. [TZ3]),

$$\begin{aligned} & \bar{I}_{\omega_g}(\phi) - \bar{J}_{\omega_g}(\phi) \\ &= \frac{n\sqrt{-1}}{2\pi V} \sum_{k=0}^{n-1} C_{n-1}^k \int_M \left( \int_0^1 s^{k+1} (1-s)^{n-1-k} e^{\theta x + sX(\phi)} ds \right) \\ & \quad \partial \phi \wedge \bar{\partial} \phi \wedge \omega_\phi^k \wedge \omega_g^{n-1-k} \\ &\geq \frac{n\sqrt{-1}}{2\pi V} C_2 \sum_{k=0}^{n-1} C_{n-1}^k \partial \phi \wedge \bar{\partial} \phi \wedge \omega_\phi^k \wedge \omega_g^{n-1-k}. \end{aligned} \quad (3.4)$$

Hence, combining (3.2) and (3.4), we also prove the first inequality of (3.1).  $\square$

**Proposition 3.1.** *Let  $\phi = \phi_t (t \geq t_0)$  be a solution of equation (1.1)<sub>t</sub>. Then there are two uniform constants  $C_1$  and  $C_2$  depending only on  $X$  and  $t_0$  such that*

$$\text{osc}_M \phi \leq C_1 \int_M \phi(\omega_g^n - \omega_\phi^n) + C_2. \quad (3.5)$$

*Proof.* Let  $\theta'_X = \theta_X(\omega_\phi)$ . First we note

$$\Delta \theta'_X = -\theta'_X - X(h_{\omega_\phi}) + c,$$

for some constant  $c$ . Clearly  $c \leq \text{osc}_M |\theta'_X|$ , since  $\theta'_X$  changes the sign. By using the fact (cf. (1.2)),

$$h_{\omega_\phi} = \theta'_X - (1-t)\phi + \text{const.},$$

we have

$$\begin{aligned} \Delta \theta'_X &= -\theta'_X - \|X\|_{\omega_\phi} + (1-t)X(\phi) + c \\ &\leq 2\text{osc}_M |\theta'_X| + 3|X(\phi)| \leq C'_1, \end{aligned}$$

for some uniform constant  $C'_1$ . Applying Theorem 2.1, we see that there is some uniform constant  $C'_2$  depending only on  $X$  and  $t_0$  such that

$$\sup_M (-\phi) \leq \frac{1}{V} \int_M (-\phi) e^{\theta'_X} \omega_\phi^n + C'_2. \quad (3.6)$$

On the other hand, by using the Green formula, we have (cf. [TZ4]),

$$\sup_M \phi \leq \frac{1}{V} \int_M \phi e^{\theta'_X} \omega_\phi^n + C'_3, \quad (3.7)$$

for some uniform constant  $C'_3$ . Hence, combining (3.6) and (3.7), we get

$$\text{osc}_M \phi \leq \frac{1}{V} \int_M \phi(e^{\theta'_X} \omega_\phi^n - e^{\theta'_X} \omega_g^n) + C'_2 + C'_3.$$

By using (1.6) and (3.1), one can prove (3.5).  $\square$

**Remark 3.1.** *Proposition 3.1 improves our previous result about  $C^0$ -estimate for equations (1.1)<sub>t</sub> obtained in [TZ3].*

#### 4. A Smoothing Lemma.

In this section, we prove a smoothing lemma by using Hamilton's Ricci flow. This lemma will be used in the proof of Theorem 0.2. Let  $\omega$  be any Kähler form in  $c_1(M) > 0$  such that

$$\begin{cases} \text{Ric}(\omega) - L_X \omega \geq (1-\epsilon)\omega \\ |X(h_\omega - \theta_X(\omega))| \leq \epsilon c_1, \end{cases} \quad (4.1)$$

for some constant  $c_1$  and  $0 < \epsilon < 1$ . We consider the heat equation

$$\begin{cases} \frac{\partial u}{\partial t} = \log\left(\frac{(\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u)^n}{\omega^n}\right) + u - h_\omega + \theta_X(\omega), \\ u|_{t=0} = 0. \end{cases} \quad (4.2)$$

Note that Eq. (4.2) is the scalar version of the modified Kähler-Ricci flow

$$\frac{\partial}{\partial t}\omega_t = -\text{Ric}(\omega_t) + \omega_t + L_X\omega_t.$$

Here  $\omega_t = \omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u_t$ , and  $u_t = u(x, \cdot)$ . Denote by  $h_t = h_{\omega_t}$  and  $\theta_t = \theta_X(\omega_t)$ , then it follows from the above equation and maximum principle that

$$h_t - \theta_t = -\frac{\partial u}{\partial t} + \tilde{c}_t$$

where  $\tilde{c}_t$  depends on  $t$  only. Also,  $u_0 = 0$  and hence  $\tilde{c}_0 = 0$ .

We list a few basic estimates for the solution  $u(x, t)$ . Differentiating (4.2), we get

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = (\Delta + X)\left(\frac{\partial u}{\partial t}\right) + \left(\frac{\partial u}{\partial t}\right). \quad (4.3)$$

Applying the maximum principle, we have

**Lemma 4.1.** *Let  $u_t$  be a solution of (4.2). Then*

$$\|u_t\|_{C^0} \leq e^t \|h_\omega - \theta_X(\omega)\|_{C^0},$$

and

$$\left\|\frac{\partial u}{\partial t}\right\|_{C^0} \leq e^t \|h_\omega - \theta_X(\omega)\|_{C^0}.$$

**Lemma 4.2.**

$$(\Delta + X)(h_t - \theta_t) \geq -(c_1 + n)\epsilon e^t.$$

*Proof.* From Eq.(4.4), we have

$$\begin{aligned} \frac{\partial}{\partial t}\left((\Delta + X)\left(\frac{\partial u}{\partial t}\right)\right) &= (\Delta + X)^2\left(\frac{\partial u}{\partial t}\right) + (\Delta + X)\left(\frac{\partial u}{\partial t}\right) - |\nabla\bar{\nabla}\left(\frac{\partial u}{\partial t}\right)|^2 \\ &\leq (\Delta + X)^2\left(\frac{\partial u}{\partial t}\right) + (\Delta + X)\left(\frac{\partial u}{\partial t}\right). \end{aligned}$$

It follows from the maximum principle that

$$(\Delta + X)(h_t - \theta_t) \geq e^t \inf_M (\Delta + X)(h_\omega - \theta_X(\omega)).$$

On the other hand, (4.1) implies that, at  $t = 0$ ,

$$(\Delta + X)(h_\omega - \theta_X(\omega)) \geq -(c_1 + n)\epsilon.$$

Then the lemma follows directly.  $\square$

**Lemma 4.3.**

$$\left\| \frac{\partial}{\partial t} u \right\|_{C^0}^2 + t \left\| \nabla \left( \frac{\partial}{\partial t} u \right) \right\|_{C^0}^2 \leq e^{2t} \|h_\omega - \theta_X(\omega)\|_{C^0}^2.$$

*Proof.* By direct computations, we have

$$\frac{\partial}{\partial t} \left( \frac{\partial u}{\partial t} \right)^2 = (\Delta + X) \left( \frac{\partial u}{\partial t} \right)^2 - \left| \nabla \left( \frac{\partial u}{\partial t} \right) \right|^2 + 2 \left( \frac{\partial u}{\partial t} \right)^2,$$

and

$$\begin{aligned} & \frac{\partial}{\partial t} \left( \left| \nabla \left( \frac{\partial u}{\partial t} \right) \right|^2 \right) \\ &= (\Delta + X) \left( \left| \nabla \left( \frac{\partial u}{\partial t} \right) \right|^2 \right) - \left| \nabla \nabla \left( \frac{\partial u}{\partial t} \right) \right|^2 - \left| \nabla \bar{\nabla} \left( \frac{\partial u}{\partial t} \right) \right|^2 + \left| \nabla \left( \frac{\partial u}{\partial t} \right) \right|^2. \end{aligned}$$

Hence

$$\begin{aligned} & \frac{\partial}{\partial t} \left( \frac{\partial u}{\partial t} \right)^2 + t \left| \nabla \left( \frac{\partial u}{\partial t} \right) \right|^2 \\ & \leq (\Delta + X) \left( \frac{\partial u}{\partial t} \right)^2 + t \left| \nabla \left( \frac{\partial u}{\partial t} \right) \right|^2 + 2 \left( \frac{\partial u}{\partial t} \right)^2 + t \left| \nabla \left( \frac{\partial u}{\partial t} \right) \right|^2. \end{aligned}$$

Lemma 4.3 follows from the maximum principle again.  $\square$

Set

$$v = h_1 - \theta_1 - \frac{1}{V} \int_M (h_1 - \theta_1) e^{\theta_1 \omega_1^n}.$$

**Lemma 4.4.**

$$\|v\|_{L^2}^2 \leq \frac{2(c_1 + n)e^2 V}{\lambda_1} \epsilon \|h_\omega - \theta_X(\omega)\|_{C^0}$$

*Proof.* By Lemma 4.2, we have

$$(\Delta + X)v + (c_1 + n)e\epsilon \geq 0.$$

It follows

$$\begin{aligned} \int_M |(\Delta + X)v + (c_1 + n)e\epsilon| e^{\theta_1 \omega_1^n} &= \int_M ((\Delta + X)v + (c_1 + n)e\epsilon) e^{\theta_1 \omega_1^n} \\ &= (c_1 + n)e\epsilon V. \end{aligned}$$

Hence, by applying the Poincaré inequality and Lemma 4.1, we have

$$\begin{aligned}
\frac{\lambda_1}{V} \int_M |v|^2 e^{\theta_1 \omega_1^n} &\leq \frac{1}{V} \int_M |\nabla v|^2 e^{\theta_1 \omega_1^n} \\
&= \frac{1}{V} \int_M (-v)(\Delta + X)v e^{\theta_1 \omega_1^n} \\
&= \frac{1}{V} \int_M (-v)[(\Delta + X)v + c_1 e \epsilon] e^{\theta_1 \omega_1^n} \\
&\leq \frac{1}{V} \|v\|_{C^0} \int_M ((\Delta + X)v + (c_1 + n)e \epsilon) e^{\theta_1 \omega_1^n} \\
&\leq 2e^2 (c_1 + n) \epsilon \|h_\omega - \theta_X(\omega)\|_{C^0}.
\end{aligned}$$

This shows the lemma is true.  $\square$

**Lemma 4.5.** *We have*

$$\|v\|_{C^0} \leq C(n, c_1 a, \lambda_1) (1 + \|h_\omega - \theta_X(\omega)\|_{C^0}) \epsilon^{\frac{1}{2(n+1)}}, \quad (4.4)$$

provided that the following condition holds: there exists a constant  $a > 0$  such that for any  $x_0 \in M$  and  $0 < r < 1$ ,

$$\text{vol}(B_r(x_0)) \geq ar^{2n} \quad (4.5)$$

with respect to the metric  $\omega_1$ .

*Proof.* Pick  $r = \epsilon^{\frac{1}{2(n+1)}}$  and cover  $M$  by geodesic balls of radius  $r$ . For any  $x \in M$ , we have  $x \in B_r(x_0)$  for some  $x_0 \in M$ . Now

$$\begin{aligned}
\inf_{B_r(x_0)} |v|^2 \epsilon^{\frac{n}{n+1}} &\leq \frac{1}{a} \int_{B_r(x_0)} |v|^2 e^{\theta_1 \omega_1^n} \\
&\leq \frac{2(c_1 + n)e^2 V^2}{a \lambda_1} \epsilon \|h_\omega - \theta_X(\omega)\|_{C^0}.
\end{aligned}$$

Hence

$$\inf_{B_r(x_0)} |v| \leq C(n, c_1, a, \lambda_1) \epsilon^{\frac{1}{2(n+1)}} \|h_\omega - \theta_X(\omega)\|_{C^0}^{\frac{1}{2}}.$$

Assuming  $\inf_{B_r(x_0)} |v| = v(x'_0)$ , then

$$\begin{aligned}
|v(x)| &\leq |v(x) - v(x'_0)| + v(x'_0) \leq r \sup_{B_r(x_0)} |\nabla v| + v(x'_0) \\
&\leq e \|h_\omega - \theta_X(\omega)\|_{C^0} \epsilon^{\frac{1}{2(n+1)}} + C \epsilon^{\frac{1}{2(n+1)}} \|h_\omega - \theta_X(\omega)\|_{C^0}^{\frac{1}{2}} \\
&\leq C'(1 + \|h_\omega - \theta_X(\omega)\|_{C^0}) \epsilon^{\frac{1}{2(n+1)}}.
\end{aligned}$$

This finishes the proof of Lemma 4.5.  $\square$

**Proposition 4.1 (Smoothing Lemma).** *Let  $\omega \in c_1(M) > 0$  be any Kähler metric satisfying (4.1). Then there is another Kähler form  $\omega' = \omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u$  such that*

$$(1) \|u\|_{C^0} \leq e \|h_\omega - \theta_X(\omega)\|$$

(2)  $\|h' - \theta'\|_{C^{\frac{1}{2}}} \leq C(n, c_1, a, \lambda_1)(1 + \|h_\omega - \theta_X(\omega)\|_{C^0})\epsilon^{\frac{1}{4(n+1)}}$ , where  $C(n, c_1, a, \lambda_1)$  is a constant depending only on the dimension  $n$ , the Poincare constant  $\lambda_1$ , constants  $c_1$  and  $a$  appeared in (4.5).

*Proof.* We shall prove that  $\omega_1$  satisfies the above two conditions of the proposition under the assumption (4.5). By Lemma 4.1, it suffices to check the second condition only.

Since

$$\frac{1}{V} \int_M e^{h' - \theta'} e^{\theta_1 \omega_1^n} = 1,$$

by (4.4) in Lemma 4.5, we have

$$\|h' - \theta'\|_{C^0} \leq C(n, c_1, a, \lambda_1)(1 + \|h_\omega - \theta_X(\omega)\|_{C^0})\epsilon^{\frac{1}{2(n+1)}}. \quad (4.6)$$

For any two points  $x, y$  in  $M$ , if the distance  $d(x, y) \leq \epsilon^{\frac{1}{2(n+1)}}$ , then Lemma 4.3 implies that  $|\nabla(h' - \theta')| \leq e\|h_\omega - \theta_X(\omega)\|_{C^0}$  and hence

$$\frac{|(h' - \theta')(x) - (h' - \theta')(y)|}{\sqrt{d(x, y)}} \leq e\|h_\omega - \theta_X(\omega)\|_{C^0}\epsilon^{\frac{1}{4(n+1)}}.$$

On the other hand, if  $d(x, y) \geq \epsilon^{\frac{1}{2(n+1)}}$  then (4.6) implies that

$$\frac{|(h' - \theta')(x) - (h' - \theta')(y)|}{\sqrt{d(x, y)}} \leq C(n, c_1, a, \lambda_1)(1 + \|h_\omega - \theta_X(\omega)\|_{C^0})\epsilon^{\frac{1}{4(n+1)}}.$$

This completes the proof of Proposition 4.1.  $\square$

## 5. Properness of $\tilde{F}_\omega(\psi)$ .

We are now ready to prove Theorem 0.2 stated in the introduction.

*Proof of Theorem 0.2.* Without the loss of generality, we may assume that the Kähler-Ricci soliton  $g_{KS}$  is  $K_0(\supseteq K)$ -invariant ([TZ3]). Let  $\omega_{KS}$  be the Kähler form of  $g_{KS}$  and  $\omega_g = \omega_\psi = \omega_{KS} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\psi$ . We consider the complex Monge-Ampère equations with parameter  $t \in [0, 1]$ :

$$\begin{cases} \det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = \det(g_{i\bar{j}})\exp\{h_g - \theta_X - X(\varphi) - t\varphi\} \\ (g_{i\bar{j}} + \varphi_{i\bar{j}}) > 0, \end{cases} \quad (5.1)$$

Clearly,  $-\psi + \text{const.}$  is a solution of (5.1) <sub>$t$</sub>  for  $t = 1$ . Since  $\psi$  is  $G$ -invariant, by the Implicit Function Theorem, there are  $G$ -invariant solutions of (5.1) <sub>$t$</sub>  for  $t$  sufficiently close to 1. In fact, in [TZ3], it is proved that there are  $G$ -invariant solutions of (5.1) <sub>$t$</sub>  for any  $t \in [0, 1]$ . This is because  $\tilde{I}_\omega(\varphi_t) - \tilde{J}_\omega(\varphi_t)$  is nondecreasing in  $t$ , and consequently the  $C^3$ -norm of  $\varphi_t$  can be uniformly bounded.

Put  $\omega_t = \omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_t$ . Then  $\omega_1 = \omega_{KS}$ . Moreover, by (1.2), we have

$$\begin{cases} h_{\omega_t} - \theta_X(\omega_t) = -(1-t)\varphi_t + c_t, \\ \text{Ric}(\omega_t) - L_X(\omega_t) = t\omega_t + (1-t)\omega \geq t\omega_t, \end{cases}$$

where  $c_t$  is determined by

$$\int_M e^{-(1-t)\varphi_t + c_t} \omega_t^n = V.$$

In particular,

$$|c_t| \leq (1-t)\|\varphi_t\|_{C^0}, \quad \|h_{\omega_t} - \theta_X(\omega_t)\|_{C^0} \leq 2(1-t)\|\varphi_t\|_{C^0},$$

and

$$\begin{aligned} |X(h_{\omega_t} - \theta_X(\omega_t))| &= |(1-t)X(\varphi_t)| \\ &\leq (1-t)(|X(\varphi_t - \varphi_1)| + |X(\psi)|) \leq (1-t)c_1(\omega_{KS}, X) \end{aligned}$$

Hence by applying Proposition 4.1 to each  $\omega_t$ , we obtain a Kähler form  $\omega'_t = \omega_t + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}u_t$  satisfying

$$\|u_t\|_{C^0} \leq 2e(1-t)\|\varphi_t\|_{C^0},$$

$$\|h_{\omega'_t}\|_{C^{\frac{1}{2}}} \leq C(n, c_1, a, \lambda_1)(1 + \|(1-t)\varphi_t\|_{C^0})(1-t)^{\frac{1}{4(n+1)}}.$$

As before, there are  $\psi_t$  such that  $\omega_{KS} = \omega'_t + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\psi_t$  and

$$\omega_{KS}^n = (\omega'_t)^n e^{h_{\omega'_t} - \theta_X(\omega'_t) - X(\psi_t) - \psi_t}$$

It follows from the maximum principle that

$$\phi_t = \phi_1 - \psi_t + \mu_t, \tag{5.2}$$

where  $\mu_t$  are constants with

$$|\mu_t| \leq 2(e+1)(1-t)\|\phi_t\|_{C^0} + c_1(\omega_{KS}, X). \tag{5.3}$$

Hence,  $\phi_t$  is uniformly equivalent to  $\phi_1$  as long as  $\psi_t$  is uniformly bounded. Consider the operator  $\Phi_t : C^{2, \frac{1}{2}}(M) \rightarrow C^{0, \frac{1}{2}}$  by

$$\Phi_t(\psi) = \log\left(\frac{\omega_{KS} - \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\psi}{\omega_{KS}^n}\right)^n + h_{\omega'_t} - \theta_X(\omega'_t) - X(\psi) - \psi.$$

Its linearization at  $t = 0$  is  $(-\Delta - 1 - X(\cdot))$ , so it is invertible in the space of  $G$ -invariant functions by the assumption of Theorem 0.2. Then by the Implicit Function Theorem, there is a  $\delta > 0$ , such that if the Hölder norm  $\|h_{\omega'_t}\|_{C^{\frac{1}{2}}(\omega_{KS})}$  with respect to  $\omega_{KS}$  is less than  $\delta$ , then there is a unique  $\psi$  such that  $\Phi_t(\psi) = 0$  and  $\|\psi\|_{C^{2, \frac{1}{2}}} \leq C(\delta)$ .

We observe that

$$\lambda_{1, \omega'} \geq 2^{-n-1} \lambda_{1, \omega_{KS}}, \quad a \geq \frac{1}{2^{2n}} a_0,$$

whenever  $\frac{1}{2}\omega_{KS} \leq \omega' \leq 2\omega_{KS}$ , where  $a$  is a constant appeared in (4.5) and  $a_0$  is a constant such that

$$a_0 r^{2n} \leq \text{vol}_{\omega_{KS}}(B_r(x)), \quad \forall x \in M.$$

Now we choose  $t_0$  such that  $(1 - t_0) \leq (\frac{\delta}{4C_0})^{4(n+1)}$  and

$$(1 - t_0)\|\varphi_t\|_{C^0}(1 - t_0)^{\frac{1}{4(n+1)}} = \frac{\delta}{4C_0}, \quad (5.4)$$

where  $C_0 = C_0(n, c_1, \frac{1}{2^{2n}}a_0, 2^{-n-1}\lambda_{1,\omega_{KS}})$ . Then by Proposition 4.1, one can prove that for any  $t \geq t_0$ , we have

$$\|u_t\|_{C^0} \leq 2e(1 - t)\|\varphi_t\|_{C^0}, \quad \|\psi_t\|_{C^0} \leq \frac{1}{4}.$$

Therefore, by using (5.2) and (5.3), we get

$$\|\varphi_1 - \varphi_t\|_{C^0} \leq 6e(1 - t)\|\varphi_t\|_{C^0} + 2c_1, \quad (5.5)$$

and

$$\frac{1}{2}\|\varphi_1\|_{C^0} - c_2 \leq \|\varphi_t\|_{C^0} \leq 2\|\phi_1\|_{C^0} + c_2, \quad (5.6)$$

for some uniform constants  $c_1$  and  $c_2$ , as long as  $1 - t \leq \min\{\frac{1}{12e}, 1 - t_0\}$ .

Since  $\tilde{I}(\varphi_t) - \tilde{J}(\varphi_t)$  is nondecreasing, by (5.5), we have

$$\begin{aligned} \tilde{F}_{\omega_{KS}}(\psi) &= -\tilde{F}_{\omega_g}(\varphi_1) \\ &= \int_0^1 (\tilde{I}_{\omega_g}(\varphi_t) - \tilde{J}_{\omega_g}(\varphi_t)) dt \\ &\geq (1 - t)(\tilde{I}_{\omega_g}(\varphi_t) - \tilde{J}_{\omega_g}(\varphi_t)) \\ &\geq C_1(1 - t)I_{\omega_g}(\varphi_t) \\ &\geq C_1(1 - t)J_{\omega_g}(\varphi_t) \\ &\geq C_1(1 - t)J_{\omega_g}(\varphi_1) - C_1(1 - t)\text{osc}_M(\varphi_t - \varphi_1) \\ &\geq C_1\frac{1 - t}{n}I_{\omega_g}(\varphi_1) - 8eC_1(1 - t)^2\|\varphi_1\|_{C^0} - C_2 \\ &= C_1\frac{1 - t}{n}I_{\omega_{KS}}(\psi) - 8eC_1(1 - t)^2\text{osc}_M\psi - C_2. \end{aligned} \quad (5.7)$$

In case

$$\text{osc}_M\psi \leq C'(1 + I_{\omega_{KB}}(\psi))$$

for some uniform constant  $C'$ , then by using (5.4) and (5.6), we see that there are two positive numbers  $C'_1$  and  $C'_2$  such that

$$\tilde{F}_{\omega_{KS}}(\psi) \geq C'_1I_{\omega_{KS}}(\psi)^{\frac{1}{4n+5}} - C'_2, \quad (5.8)$$

and consequently this would prove the theorem.

In the general case, we shall use a trick in [TZ1]. First by Proposition 3.1, we have for any  $t \geq \frac{1}{2}$ ,

$$\text{osc}_M(\phi_t - \phi_1) \leq C''(1 + I_{\omega_{KS}}(\phi_t - \phi_1)).$$

Set  $\psi' = \phi_t - \phi_1$ . Then by (5.8), we get

$$\begin{aligned}\tilde{F}_{\omega_g}(\phi_t) - \tilde{F}_{\omega_g}(\phi_1) &= \tilde{F}_{\omega_{KS}}(\psi') \\ &\geq C_3 I_{\omega_{KS}}(\psi')^{\frac{1}{4n+5}} - C_4.\end{aligned}\tag{5.9}$$

On the other hand, by integrating (1.6) from  $t$  to 1, we have

$$\begin{aligned}\hat{F}_{\omega_g}(\phi_1) - \hat{F}_{\omega_g}(\phi_t) &\geq \tilde{J}_{\omega_g}(\phi_1) - \frac{1}{V} \int_M \phi_1 e^{\theta x} \omega_g^n - t(\tilde{J}_{\omega_g}(\phi_t) - \frac{1}{V} \int_M \phi_t e^{\theta x} \omega_g^n) \\ &\geq -(1-t)(\tilde{I}_{\omega_g}(\phi_1) - \tilde{J}_{\omega_g}(\phi_1)) \\ &\geq -C_5(1-t)I_{\omega_g}(\phi_1) = -C_5(1-t)I_{\omega_{KS}}(\psi).\end{aligned}\tag{5.10}$$

By using the concavity of the logarithmic function and (3.6), we also have,

$$\begin{aligned}-\log\left(\frac{1}{V} \int_M e^{h-\phi_t} \omega_g^n\right) &\leq \frac{1-t}{V} \int_M \phi_t e^{\theta' x} \omega_{\phi_t}^n \\ &\leq -\frac{1-t}{V} \sup_M(-\phi_t) + C_6 \leq C_6.\end{aligned}\tag{5.11}$$

Hence combining (5.10) and (5.11), we get

$$\tilde{F}_{\omega_g}(\phi_t) - \tilde{F}_{\omega_g}(\phi_1) \leq C_5(1-t)I_{\omega_{KS}}(\psi) + C_6.\tag{5.12}$$

From (5.9) and (5.12), we deduce

$$(1-t)I_{\omega_{KS}}(\psi) \geq c_3 \text{osc}_M(\phi_t - \phi_1)^{\frac{1}{4n+5}} - c_4.$$

Then as in (5.7), we prove (cf. [TZ1]),

$$\begin{aligned}\tilde{F}_{\omega_{KS}}(\psi) &\geq C_1 \frac{1-t}{n} I_{\omega_{KS}}(\psi) - C_1(1-t) \text{osc}_M(\varphi_t - \varphi_1) \\ &\geq C_1 \frac{1-t}{n} I_{\omega_{KS}}(\psi) - C_1(1-t)(c_3^{-1})^{4n+5}((1-t)I_{\omega_{KS}}(\psi) + c_4)^{4n+5} \\ &\geq c I_{\omega_{KS}}(\psi)^{\frac{1}{4n+5}} - C,\end{aligned}\tag{5.13}$$

for some small positive number  $c$  and large number  $C$ . Thus Theorem 0.2 is proved.  $\square$

**Remark 5.1.** By (1.6) and (3.1), we see that the inequality (5.13) is equivalent to the following non-linear inequality of Moser-Trudinger type,

$$\int_M e^{-\psi} \omega_{KS}^n \leq C \exp\left\{\tilde{J}_{\omega_{KS}}(\psi) - c \tilde{J}_{\omega_{KS}}(\psi)^{\frac{1}{4n+5}} - \frac{1}{V} \int_M \psi \omega_{KS}^n\right\}$$

for some positive numbers  $c$  and  $C$ . The inequality of this type was first obtained in [Ti] and [TZ1] on Kähler-Einstein manifolds with positive scalar curvature. Inequality (5.13) improves the result in those two papers. A weak version of (5.13) was obtained in [TZ4].

**Remark 5.2.** *It seems that Theorem 0.2 can be improved as follows:*

$$\tilde{F}_{\omega_{KS}}(\psi) \geq cI_{\omega_{KS}}(\psi)^{\frac{1}{4n+5}} - C$$

*holds for any  $\psi \in \Lambda_1(M, \omega_{KS})^\perp$ . In the proof of Theorem 0.2, we used a technical assumption on subgroup  $G(\subseteq K_0)$  in order to apply the Implicit Theorem. We also notice from Theorem 0.2 that (5.13) holds for any almost plurisubharmonic function on a Kähler-Einstein manifold without any nontrivial holomorphic vector field.*

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