

A Quadratic Bound on the Diameter of the Transportation Polytope ^{*}

Jan van den Heuvel [†] Leen Stougie [‡]

Abstract

We prove that the combinatorial diameter of the skeleton of the polytope of feasible solutions of any $m \times n$ transportation problem is less than $\frac{1}{2}(m+n)^2$.

The transportation problem (TP) is a classic problem in operations research. The problem was posed for the first time by Hitchcock in 1941 [8] and independently by Koopmans in 1947 [11], and appears in any standard introductory course on operations research.

The $m \times n$ TP has m supply points and n demand points. Each supply point i holds a quantity $r_i > 0$, and each demand point j wants a quantity $c_j > 0$, with $\sum_{i=1}^m r_i = \sum_{j=1}^n c_j$. A solution to the problem can be written as a $m \times n$ matrix X with entries decision variables x_{ij} having value equal to the amount transported from supply point i to demand point j . The objective is to minimize total transportation costs $\sum_{i=1}^m \sum_{j=1}^n t_{ij} x_{ij}$, where t_{ij} is the unit transportation cost from supply point i to demand point j . The set of feasible solutions of TP, the transportation polytope, is described by

$$\begin{aligned} \sum_{j=1}^n x_{ij} &= r_i, & i &= 1, 2, \dots, m; \\ \sum_{i=1}^m x_{ij} &= c_j, & j &= 1, 2, \dots, n; \\ x_{ij} &\geq 0, & i &= 1, 2, \dots, m, \quad j = 1, 2, \dots, n. \end{aligned}$$

The 1-skeleton (edge graph) of this polytope is defined as the graph with vertices the vertices of the polytope and edges its 1-dimensional faces. In 1957 W.M. Hirsch stated his famous conjecture (cf. [5]) saying that any d -dimensional polytope with n facets has diameter at most $n - d$. So far the best bound for any polytope is $O(n^{\log d+1})$ [9]. Any strongly polynomial bound is still lacking. Such bounds have been proved for some special classes of polytopes (for

^{*}Research for this paper was done while both authors were visiting the Isaac Newton Institute for Mathematical Sciences, Cambridge, U.K.

[†]Centre for Discrete and Applicable Mathematics and Department of Mathematics, London School of Economics, U.K., jan@maths.lse.ac.uk.

[‡]Technische Universiteit Eindhoven and CWI Amsterdam, The Netherlands, leen@win.tue.nl. Supported by the TMR Network DONET of the European Community ERB TMRX-CT98-0202.

examples, see [13]). Among those are some special classes of transportation polytopes [1, 3] and the polytope of the dual of TP [1].

A strongly polynomial time algorithm for TP follows directly from [14]. It is not a primal simplex type algorithm. In fact the existence of a strongly polynomial primal simplex algorithm is unknown. A weakly polynomial time algorithm has been presented in [12]. The first strongly polynomial bound on the diameter of the transportation polytope was given by Dyer and Frieze [7]. Actually, they prove a bound on the diameter of any polytope $\{x \in \mathbb{R}^n \mid Ax \leq b, b \in \mathbb{R}^m\}$ where A is a totally unimodular matrix. The proof is complicated and indirect, using the probabilistic method. Moreover, the bound is huge ($O(m^{16} n^3 (\ln(mn))^3)$ assuming $m \leq n$).

We will give a simple proof that the diameter of the transportation polytope is less than $\frac{1}{2}(m+n)^2$. The proof is constructive: it gives an algorithm that describes how to go from any vertex to any other vertex on the transportation polytope in less than $\frac{1}{2}(m+n)^2$ steps along the edges.

We first review some known facts about the transportation polytope. The dimension of the transportation polytope is $mn - m - n + 1$ [10]. Thus, according to the Hirsch Conjecture the diameter of TP is bounded by $m+n-1$.

We denote by $K_{m,n}$ the complete bipartite graph with vertices representing the supply points of TP as one color class and vertices representing the demand points of TP as the other color class. For any feasible solution X of TP, let $G(X)$ be the subgraph of $K_{m,n}$ with $E(X) = \{(i, j) \mid x_{ij} > 0, i = 1, \dots, m, j = 1, \dots, n\}$.

Lemma 1 [10]

- (a) Any feasible solution X of TP is a basic feasible solution (vertex of the polytope) if and only if $G(X)$ is a spanning forest.
- (b) The graph $G(X)$ has $q+1$ components if and only if X is a basic feasible solution of degeneracy q , i.e., a solution with $m+n-1-q$ non-zero variables.
- (c) A feasible solution X is a non-degenerate basic feasible solution if and only if $G(X)$ is a spanning tree.
- (d) There is a one-to-one relation between non-degenerate basic feasible solutions and spanning trees of $K_{m,n}$ corresponding to feasible solutions.

The proof of [10, Theorem 5] also implies that if a TP has degenerate basic feasible solutions, then appropriately chosen sufficiently small perturbations of supply and demands results in a non-degenerate transportation polytope with higher diameter than the original polytope. Therefore, we will concentrate from now on non-degenerate transportation problems only.

We explain a pivot operation (step from one vertex on the 1-skeleton of the transportation polytope to a neighbouring vertex) as an operation on the corresponding spanning trees. Given a basic feasible solution X , an edge $(a, b) \notin E(X)$ is inserted in $G(X)$, creating a unique cycle C . Since C is an even cycle we can label its edges alternately $+$ and $-$, starting with label $+$ for (a, b) . Let $E^+(C)$ and $E^-(C)$ be the edges of C with respectively label $+$ and $-$, and let (c, d) be the edge in $E^-(C)$ with x_{cd} minimal. Removing (c, d) from $G(X) \cup (a, b)$ finishes the pivot operation, which we call a pivot on (a, b) .

The above corresponds to increasing the value of all x_{ij} with $(i, j) \in E^+(C)$ with the amount x_{cd} and decreasing all x_{ij} with $(i, j) \in E^-(C)$ with the same amount. In particular,

x_{ab} is raised from 0 to x_{cd} (becomes non-zero variable), and x_{cd} gets value 0 (becomes zero variable). Since we assumed non-degeneracy, no other variable corresponding to an edge in $E^-(C)$ becomes zero.

Theorem 2

For any pair X, Y of basic feasible solutions of an $m \times n$ TP, at most $\sum_{k=4}^{m+n} (k-2)$ pivot steps suffice to go from X to Y .

PROOF. We use induction on $m+n$, the total number of supply and demand points. If $m=1$ or $n=1$, then $K_{m,n}$ is the graph of the only feasible solution, and the theorem is trivially true. Now assume $m > 2$ and $n > 2$ and take any two basic feasible solutions X and Y . Suppose $G(Y)$ has a leaf a incident to vertex b . Without loss of generality we assume that a corresponds to a supply point (hence b to a demand point). Thus, $y_{ab} = r_a \leq c_b$. Non-degeneracy implies that in fact $r_a < c_b$ and hence b cannot be a leaf incident to a in the graph of any basic feasible solution. We distinguish two cases.

CASE 1: a is also a leaf incident to b in $G(X)$.

Then $x_{ab} = y_{ab} = r_a$ and we may consider the transportation problem defined by removing supply point a and setting the demand of point b to $c_b - r_a$, leaving all other supplies and demands unchanged. The matrices X and Y with the row corresponding to supply point a deleted, are basic feasible solutions for this new transportation problem with $m+n-1$ supply and demand points. Hence the theorem is true by induction.

CASE 2: a is not a leaf incident to b in $G(X)$.

We assume that $(a, b) \in E(X)$, else one pivot step inserting (a, b) suffices to obtain this property. Since, as stated before, (a, b) cannot be the only edge incident to b , both a and b must have degree at least 2 in $G(X)$. Let $D(X)$ be the sum of their degrees in $G(X)$. There must exist $i \neq a$ and $j \neq b$ such that $(a, j) \in E(X)$ and $(i, b) \in E(X)$. This gives a path (j, a, b, i) of length three. By performing a pivot on (i, j) we obtain a new solution X_1 . Because (a, b) also received label $+$ on the cycle (j, a, b, i, j) , we have that $(a, b) \in E(X_1)$. Since (a, j) and (i, b) are the only edges with label $-$ on (j, a, b, i, j) , one of them is deleted in the pivot operation. Hence $D(X_1) = D(X) - 1$.

We repeat this procedure on consecutive solutions until arriving at a solution X' in which the degree of a or b has dropped to one. This occurs after at most $D(X) - 3$ pivot operations. Clearly, $D(X) \leq m+n$. Thus, including the pivot step to make x_{ab} non-zero if necessary, at most $m+n-2$ pivot steps are required to arrive at X' . By the choice of pivot operations, $(a, b) \in E(X')$. Using again that b cannot be a leaf incident to a in any basic feasible solution, a must be a leaf of $G(X')$ incident to b . Hence, X' belongs to *Case 1*, whence we can reduce the transportation problem to a problem with one supply point less. Applying induction proves the theorem. \square

Corollary 3

The transportation polytope has diameter less than $\frac{1}{2}(m+n)^2$.

It would be interesting to investigate if the result and its proof could help in the design of a strongly polynomial time primal simplex algorithm.

Special edges of the 1-skeleton of the transportation polytope are those that correspond to pivot operations in which the cycle C used in the pivot has length four. We call these 2×2 -pivot steps. In the proof of Theorem 2, once we assumed that the edge (a, b) is in $E(X)$, we used this type of pivot only. To get (a, b) into $E(X)$ we used one other pivot step, but the following lemma shows this could also have been achieved using 2×2 -pivot steps only.

Lemma 4

Let X be a basic feasible solution with $(a, b) \notin E(X)$. Then at most $\frac{1}{2}(m+n) - 1$ 2×2 -pivots are needed to obtain a basic feasible solution X' with $(a, b) \in E(X')$.

PROOF. Let $P = (a, j_1, i_1, j_2, \dots, j_k, i_k, b)$ be the unique path between a and b in $G(X)$. Then $(a, j_2) \notin E(X)$, and we do a 2×2 -pivot on (a, j_2) to obtain the solution X' with $(a, j_2) \in E(X')$. Since none of the edges of P between j_2 and b is involved in this pivot, $G(X')$ contains the path $P' = (a, j_2, i_2, j_3, \dots, j_k, i_k, b)$ of length two less than P . Repeating this procedure will decrease the length of the path between a and b by two at every pivot, until (a, b) is inserted. Since P has length at most $2 \min\{m, n\} - 1$, the procedure will use at most $\frac{1}{2}(m+n) - 1$ steps. \square

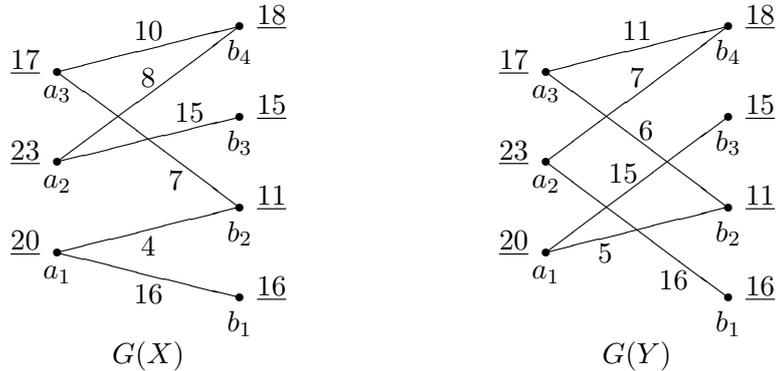
Thus, adding at most $\frac{1}{4}(m+n)^2$ extra 2×2 -pivot steps to the $\frac{1}{2}(m+n)^2$ steps from Theorem 2 yields a bound of $\frac{3}{4}(m+n)^2$ on the diameter of the restricted 1-skeleton of the transportation polytope in which only edges exist that correspond to 2×2 -pivot steps.

This opens the possibility that a random walk on the vertices of the transportation polytope using only 2×2 -pivots mixes rapidly. The analysis of such a walk seems easier than one that allows steps along any edge of the polytope. This would be a crucial step in devising a polynomial randomized approximation scheme for counting the vertices of the transportation polytope, a $\#P$ -complete problem [6]¹. So far rapid mixing on the transportation polytope has been shown only for problems with a fixed number of rows or a fixed number of columns [4].

It remains open if the transportation polytope has diameter $m+n-1$ according to the Hirsch Conjecture. It is unlikely that our algorithm underlying the quadratic bound will, if subjected to a more subtle analysis, lead to that result. The desired result requires, e.g., that between two vertices on the polytope, whose graphs have no edges in common, each of the pivot steps must bring the difference between the two down. However, from an example on [2, pages 141,142] we know that it is not true that for any two basic feasible solutions that differ in k non-zero variables there exists a pivot step that reduces the number of non-zero variables in which the solutions differ. In the example $k=2$. We can prove that such a pivot step exists in case $k=1$, and it trivially exists if $k=m+n-1$. The example does not disprove the Hirsch Conjecture, but it shows that pivot steps need to be chosen carefully to obtain the result.

Below we present a simplified version of the example in [2]:

¹In fact, [6] only claims NP-hardness, but the proof establishes $\#P$ -completeness.



The values of the supply points (on the left) and the demand points (on the right) are indicated by the underlined values. The two trees $G(X), G(Y)$ are both feasible, as is shown by the values on their edges. There are two edges $(a_1, b_3), (a_2, b_1)$ in $E(Y) \setminus E(X)$. But pivoting both on (a_1, b_3) and on (a_2, b_1) in $G(X)$ leads to trees that still differ in two edges from $G(Y)$.

Acknowledgement : We gratefully acknowledge discussions with Martin Dyer.

References

- [1] M. L. Balinski, On two special classes of transportation polytopes, *Mathematical Programming Studies 1*, 1974, 43–58.
- [2] M. L. Balinski and F. J. Rispoli, Signature classes of transportation polytopes, *Mathematical Programming 60*, 1993, 127–144.
- [3] E. D. Bolker, Transportation polytopes, *Journal of Combinatorial Theory B 13*, 1972, 251–262.
- [4] M. Cryan, M. Dyer, H. Müller and L. Stougie, Random walks on the vertices of transportation polytopes with constant number of sources, in *Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, Baltimore, Maryland, 2003.
- [5] G. B. Dantzig, *Linear programming and extensions*, Princeton University Press, Princeton, 1963.
- [6] M. E. Dyer, The complexity of vertex enumeration methods, *Mathematics of Operations Research 8*, 1983, 381–402.
- [7] M. E. Dyer and A. M. Frieze, Random walks, totally unimodular matrices and a randomized dual simplex method, *Mathematical Programming 64*, 1994, 1–16.
- [8] F. L. Hitchcock, The distribution of a product from several sources to numerous localities, *Journal of Mathematics and Physics 20*, 1941, 224–230.

- [9] G. Kalai and D. J. Kleitman, A quasi-polynomial bound for the diameter of graphs of polyhedra, *Bulletin of the American Mathematical Society* 26, 1992, 315–316.
- [10] V. Klee and C. Witzgall, Facets and vertices of transportation polytopes, in G. B. Dantzig and A. F. Veinott (eds.), *Mathematics of the decision sciences, Part 1*, American Mathematical Society, Providence, Richmond, 1968, 257–282.
- [11] Tj. C. Koopmans, Optimum utilization of the transportation system, in D. H. Leavens (ed.), *The Econometric Society Meeting, Washington DC, 1947*, 1948, 136–146.
- [12] J. B. Orlin, A polynomial time primal network simplex algorithm for minimum cost flows, *Mathematical Programming* 78, 1997, 109–129.
- [13] A. Schrijver, Polyhedral combinatorics, in R. Graham, M. Grötschel and L. Lovász (eds.), *Handbook of combinatorics, Volume II*, North-Holland Elsevier, 1995, 1649–1704.
- [14] E. Tardos, A strongly polynomial minimum cost circulation algorithm, *Combinatorica* 5, 1985, 247–255.