Consequences of the Provability of $\mathbf{NP} \subseteq \mathbf{P/poly}$

Stephen Cook*
University of Toronto

Jan Krajíček †
Academy of Sciences and Charles University
Prague

Abstract

We prove the following results: (i) **PV** proves $\mathbf{NP} \subseteq \mathbf{P/poly}$ iff **PV** proves $\mathbf{coNP} \subseteq \mathbf{NP}/O(1)$. (ii) If **PV** proves $\mathbf{NP} \subseteq \mathbf{P/poly}$ then **PV** proves that the Polynomial Hierarchy collapses to the Boolean Hierarchy. (iii) $\mathbf{S}^1_2$ proves $\mathbf{NP} \subseteq \mathbf{P/poly}$ iff $\mathbf{S}^1_2$ proves $\mathbf{coNP} \subseteq \mathbf{NP}/O(\log n)$. (iv) If $\mathbf{S}^1_2$ proves $\mathbf{NP} \subseteq \mathbf{P/poly}$ then $\mathbf{S}^1_2$ proves that the Polynomial Hierarchy collapses to $\mathbf{P}^{\mathbf{NP}}[\log n]$. (v) If $\mathbf{S}^2_2$ proves $\mathbf{NP} \subseteq \mathbf{P/poly}$ then $\mathbf{S}^2_2$ proves that the Polynomial Hierarchy collapses to $\mathbf{P}^{\mathbf{NP}}$.

1 Introduction

The theory **PV** [Coo75] formalizes reasoning that uses only polynomial time concepts. Similar theories (the so called bounded arithmetic theories) exist for many other complexity classes [Coo05, CN06]. Separating theories (at appropriate levels of quantifier complexity) corresponding to two complexity

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classes may not separate the classes, but it could still carry great significance. In fact, the problem of separating two such theories is often closely linked with the problem of whether the distinctness of two complexity classes is consistent with a suitable bounded arithmetic theory. For example, theories $\mathbf{PV}$ and $\mathbf{S}^1_2$ are different if $\mathbf{coNP} \not\subseteq \mathbf{NP/poly}$ is consistent with $\mathbf{PV}$. Another prominent example is this: Theory $\mathbf{S}_2$ is not finitely axiomatizable (i.e. theories $\mathbf{S}^i_2$ are all different) iff it is consistent with $\mathbf{S}_2$ that $\mathbf{PH}$ does not collapse [KPT91, Bus95, Zam96].

It is therefore interesting to study the consistency with bounded arithmetic theories of various standard conjectures separating complexity classes. A good example is the conjecture $\mathbf{P} \neq \mathbf{NP}$. Showing this is consistent with $\mathbf{PV}$ is a major open problem. In more detail, $\mathbf{P} = \mathbf{NP}$ is equivalent to the existence of a polynomial time function F such that $F(A)$ is a satisfying assignment for A whenever A is a satisfiable propositional formula. This can be formalized in the language of $\mathbf{PV}$ by

$$\forall T, A,\ \ SAT(T, A) \supset SAT(F(A), A) \tag{1}$$

where F is a $\mathbf{PV}$ function and $SAT(T, A)$ is an open $\mathbf{PV}$ formula expressing that T is a satisfying assignment for formula A. Showing consistency of $\mathbf{P} \neq \mathbf{NP}$ with $\mathbf{PV}$ is the same as showing that (1) is not provable in $\mathbf{PV}$ for any polynomial time F. (A natural way to do this would be, for every F, construct a model of $\mathbf{PV}$ in which (1) is false.) This would mean that even if a polytime F satisfying (1) exists, its correctness could not be proved using only polynomial time concepts.

It is known that PV proves $\mathbf{P} = \mathbf{NP}$ iff $\mathbf{PV}$ proves $\mathbf{NP} = \mathbf{coNP}$ iff $\mathbf{PV}$ defines a polynomial time algorithm that assigns to a propositional formula either a satisfying assignment or an Extended Frege proof of its unsatisfiability. The same holds for S^1_2. In particular, the consistency of $\mathbf{P} \neq \mathbf{NP} \neq \mathbf{coNP}$ with these theories would follow from a super-polynomial lower bound for Extended Frege proofs. This is one of the reasons why lower bounds for Extended Frege systems would be so important (and presumably why they seem so difficult).

In this paper we are concerned with whether the conjecture $\mathbf{NP} \not\subseteq \mathbf{P/poly}$ (some $\mathbf{NP}$ problem cannot be solved by any polynomial size family of Boolean circuits) is consistent with the theories $\mathbf{PV}$, $\mathbf{S}^1_2$, $\mathbf{S}^2_2$. A well-known consequence of $\mathbf{NP} \subseteq \mathbf{P/poly}$ is that the polynomial hierarchy $\mathbf{PH}$ collapses to

the second level [KL80, KL82], i.e.

$$\mathbf{NP} \subseteq \mathbf{P/poly} \implies \mathbf{PH} = \mathbf{\Sigma}_2^p \cap \mathbf{\Pi}_2^p \ .$$

Here we show that stronger collapses can be inferred from stronger assumptions of the form $\mathbf{NP} \not\subseteq \mathbf{P/poly}$ is inconsistent with various theories. In particular, if the theory is **PV**, then the collapse is to the Boolean hierarchy (i.e. the bounded query hierarchy). If the theory is $\mathbf{S}_2^1$, then the collapse is to $\mathbf{P}^{\mathbf{NP}}[O(\log n)]$ (polynomial time with $O(\log n)$ queries to an **NP** oracle). If the theory is $\mathbf{S}_2^2$, then the collapse is to $\mathbf{P}^{\mathbf{NP}}$. In all cases, the collapses are provable in the corresponding theories.

We also show two other intriguing results: **PV** proves $\mathbf{NP} \subseteq \mathbf{P/poly}$ iff **PV** proves $\mathbf{coNP} \subseteq \mathbf{NP}/O(1)$, and $\mathbf{S}_2^1$ proves $\mathbf{NP} \subseteq \mathbf{P/poly}$ iff $\mathbf{S}_2^1$ proves $\mathbf{coNP} \subseteq \mathbf{NP}/O(\log n)$. (Here the notation $/O(1)$ refers to a constant number of advice bits, and $/O(\log n)$ refers to $O(\log n)$ advice bits.)

2 Preliminaries

We presuppose basic knowledge of bounded arithmetic (see for example [Kra95] or [Bus98]). However we formulate our results in the two-sorted setting [Coo05, CN06], where **PV** becomes **VPV**, $\mathbf{S}_2^1$ becomes $\mathbf{V}^1$, and $\mathbf{S}_2^2$ becomes $\mathbf{V}^2$. In this setting lower case letters $x, y, z, m, n, ...$ range over $\mathbb{N}$, and upper case letters $A, B, C, X, Y, Z, ...$ range over finite bit strings (technically over finite subsets of $\mathbb{N}$). The two-sorted vocabulary includes the symbols $0, 1, +, \cdot, =, \leq$ of first-order arithmetic, and also the length function $|X|$, where the length of the finite set $X \subseteq \mathbb{N}$ is 1 plus the largest element of X, or 0 if X is empty.

For the two-sorted theory **VPV**, the vocabulary also includes symbols for all polynomial time functions, both number-valued functions $f(\vec{x}, \vec{X})$ and string-valued functions $F(\vec{x}, \vec{X})$. (Here "polynomial time" means time polynomial in the length of the inputs, where inputs and outputs of the number sort are written in unary notation: e.g. 5 is written 11111). The axioms for **VPV** include definitions for all of these functions, based on Cobham's Theorem. Also **VPV** proves the number induction scheme for open formulas.

It is an important fact that **VPV** is a universal theory (i.e. it can be axiomatized by purely universal formulas).

Bounded quantifiers for strings have the form $\exists X \leq t\varphi$, which stands for $\exists X(|X| \leq t \wedge \varphi)$, or $\forall X \leq t\varphi$, which stands for $\forall X(|X| \leq t \supset \varphi)$. Here t is

a number term which does not involve X.

$\mathbf{\Sigma}_1^B$ formulas have the form $\exists X_1 \leq t_1 \cdots \exists X_k \leq t_k \varphi$ where φ has no string quantifiers, but may have bounded number quantifiers. More generally $\mathbf{\Sigma}_i^B$ formulas begin with i blocks of bounded string quantifiers, in which the first block is existential, the second is universal, etc. $\mathbf{\Pi}_i^B$ formulas are the same, except the first block is universal. Note that $\mathbf{\Sigma}_i^B$ formulas correspond to *strict* $\mathbf{\Sigma}_i^b$ formulas in the single-sorted case, because we require that all string quantifiers are in front.

The theories $\mathbf{V}^i, i \geq 1$, have the finite two-sorted vocabulary mentioned above. For these theories, $\mathbf{\Sigma}_i^B$ and $\mathbf{\Pi}_i^B$ formulas are restricted to this vocabulary, but in the context of **VPV** we allow these formulas to have the full vocabulary of **VPV**. The theory $\mathbf{V}^i$ proves the number induction scheme for $\mathbf{\Sigma}_i^B$ formulas. The theory $\mathbf{V}^i(\mathbf{VPV})$ has the vocabulary of **VPV** and proves the number induction scheme for $\mathbf{\Sigma}_i^B$ formulas over this larger vocabulary. $\mathbf{V}^i(\mathbf{VPV})$ is a conservative extension of $\mathbf{V}^i$, but $\mathbf{V}^i(\mathbf{VPV})$ is not conservative over **VPV** unless the polynomial hierarchy collapses.

An important fact is that a set of strings is in **NP** iff it is represented by a $\mathbf{\Sigma}_1^B$ formula $\varphi(X)$ with one free string variable X. This is true whether or not we allow $\mathbf{\Sigma}_1^B$ formulas to have the full vocabulary of **VPV**.

The class **P/poly** consists of all problems solvable in polynomial time with polynomial "advice". That is, for each input length n there exists a string Y_n of length $n^{O(1)}$ (the advice) such that when the polytime algorithm is supplied with Y_n in addition to the input string of length n, the algorithm obtains the correct answer. An equivalent definition of **P/poly** is the class of all problems L solvable by a family $\langle C_n, n \in \mathbb{N} \rangle$ of Boolean circuits such that C_n has size $n^{O(1)}$ and solves the restriction of L to inputs of length n.

It is straightforward to verify that **VPV** can prove the equivalence of the two definitions of **P/poly**.

To formalize the assertion **NP** $\subseteq$ **P/poly** as a **VPV** formula, we start by noting that **VPV** can prove the **NP**-completeness of the propositional satisfiability problem in the following sense. For every $\mathbf{\Sigma}_1^B$ formula $\varphi(X)$ there is a **VPV** string function $F_\varphi(X)$ which takes a string X to a string $F_\varphi(X)$ encoding a propositional formula such that

$$\mathbf{VPV} \vdash \varphi(X) \leftrightarrow \exists T \; SAT(T, F_\varphi(X)) \tag{2}$$

where $SAT(T, A)$ is an open **VPV** formula which holds iff truth assignment T satisfies formula A.

Thus **NP** $\subseteq$ **P/poly** is equivalent in **VPV** to the assertion that the satisfiability problem can be solved by a polynomial size family of Boolean circuits. By the "self-reducibility" of satisfiability, it follows that this is again equivalent to the existence of a polynomial size family of Boolean circuits such that the n-th circuit C'_n in the family, given a satisfiable propositional formula A (coded as a bit string of length at most n) as input, outputs $C'_n(A)$, a satisfying assignment for A.

Further, **VPV** proves self-reducibility in the following sense. We define a **VPV** formula $Correct(C, n)$, asserting that the Boolean circuit C correctly solves the satisfiability problem for formulas A of length at most n, as follows:

$$Correct(C, n) \ \equiv \ \forall A \leq n, \ C(A) = 1 \leftrightarrow \exists T \ SAT(T, A) \ .$$

Then there is a **VPV** function $CIRC(C, n)$ which, given a circuit C satisfying $Correct(C, n)$ gives a Boolean circuit C' which outputs a satisfying assignment $C'(A)$ for every satisfiable formula A of length at most n. We claim that

$$\mathbf{VPV} \vdash Correct(C, n) \supset \forall T, A \leq n, \ SAT(T, A) \supset SAT(CIRC(C, n)(A), A) \ . \quad (3)$$

For each i, the circuit $CIRC(C, n)$ finds the i-th bit of the satisfying assignment by asking C whether A remains satisfiable when the $i - th$ variable of A is set to 1, given the values it has previously found for the first $i - 1$ variables. Then (assuming $Correct(C, n)$ and $SAT(T, A)$) **VPV** proves by induction on i that A instantiated by the first i truth values is satisfiable, according to C. The claim (3) follows.

The claim (3) justifies coding the assertion **NP** $\subseteq$ **P**/*poly* by the following $\mathbf{\Sigma}^B_2$ formula:

$$\forall n \exists C \leq t \forall A \leq n \forall T \leq n, \ SAT(T, A) \ \supset \ SAT(C(A), A) \quad (4)$$

where $t = t(n)$ is a **VPV** number term (we may assume it has the form n^ℓ), $C(A)$ is a **VPV** term expressing the output of circuit C on input A, and $SAT(T, A)$ is a **VPV** open formula which asserts that truth assignment T satisfies formula A.

3 Results for Universal Theories

A useful tool for studying universal theories is the KPT witnessing theorem [KPT91]. This is a form of the Herbrand Theorem, and can be stated in a

general first-order context as follows.

Theorem 3.1 (KPT). *Let $\mathcal{T}$ be a universal theory over a vocabulary $\mathcal{L}$ which contains at least one constant or function symbol. Let $\varphi(x, y, z)$ be an open $\mathcal{L}$-formula and suppose $\mathcal{T}$ proves $\forall x \exists y \forall z \varphi(x, y, z)$. Then there exists a finite sequence $t_1(x), t_2(x, z_1), \ldots, t_k(x, z_1, \ldots, z_{k-1})$ of $\mathcal{L}$-terms (containing only the displayed variables) such that*

$$\mathcal{T} \vdash \forall x \forall \vec{z},\ \varphi(x, t_1(x), z_1) \vee \varphi(x, t_2(x, z_1), z_2) \\ \vee \ldots \vee \varphi(x, t_k(x, z_1, \ldots, z_{k-1}), z_k).$$

Proof: [from [CT06]] Let $b, c_1, c_2, ...$ be a list of new constants, and let $u_1, u_2, ...$ be an enumeration of all variable-free terms built from symbols of $\mathcal{L}$ together with $b, c_1, c_2, ...$, where the only new constants in u_k are among $\{b, c_1, ..., c_{k-1}\}$. It suffices to show that

$$\mathcal{T} \cup \{\neg\varphi(b, u_1, c_1), \neg\varphi(b, u_2, c_2), \ldots, \neg\varphi(b, u_k, c_k)\}$$

is unsatisfiable for some k.

Suppose otherwise. Then by compactness

$$\mathcal{T} \cup \{\neg\varphi(b, u_1, c_1), \neg\varphi(b, u_2, c_2), ...\} \tag{5}$$

has a model M. Since $\mathcal{T}$ is universal, the substructure M' consisting of the denotations of the terms $u_1, u_2, ...$ is also a model for (5). It is easy to see that

$$M' \models \mathcal{T} + \forall y \exists z \neg\varphi(b, y, z)$$

and hence $\mathcal{T} \nvdash \forall x \exists y \forall z \varphi(x, y, z)$. □

The following is an easy consequence of this KPT theorem, in the two-sorted setting.

Corollary 3.2. *Let* **TVPV** *be* **VPV**, *or any universal theory with the same vocabulary as* **VPV**. *If* **TVPV** *proves (4) then there are* **VPV** *functions $C_1, \cdots, C_k$ whose values are Boolean circuits such that* **TVPV** *proves*

$$\forall n \forall A_1 \cdots A_k, T_1 \cdots T_k \leq n,\ [SAT(T_1, A_1) \supset SAT(C_1(n)(A_1), A_1)] \vee \\ [SAT(T_2, A_2) \supset SAT(C_2(n, A_1, T_1)(A_2), A_2)] \vee \cdots \vee \\ [SAT(T_k, A_k) \supset SAT(C_k(n, A_1, \cdots, A_{k-1}, T_1, \cdots, T_{k-1})(A_k), A_k)]\ . \tag{6}$$

A language $L \subseteq \{0,1\}^*$ is in the class $\mathbf{NP}/O(1)$ ($\mathbf{NP}$ with constant advice) iff there is a polynomial time relation $R(X,Y,i)$ and a constant $k \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ there exists $i \leq k$ (the advice) so for all $X \in \{0,1\}^n$

$$X \in L \Leftrightarrow \exists Y \leq n^{O(1)}\ R(X,Y,i)) \ . \tag{7}$$

The assertion $\mathbf{coNP} \subseteq \mathbf{NP}/O(1)$ can be formalized by saying that for every $\mathbf{\Sigma}_1^B$ formula $\varphi(X)$ there is a $\mathbf{\Sigma}_1^B$ formula $\psi(X,i)$ and a constant k such that

$$\forall n \bigvee_{i=1}^{k} \forall X,\ |X| = n \supset [\neg\varphi(X) \leftrightarrow \psi(X,i)] \ . \tag{8}$$

Theorem 3.3. *Let* $\mathbf{TVPV}$ *be* $\mathbf{VPV}$*, or any universal extension of* $\mathbf{VPV}$ *with the same vocabulary. Then* $\mathbf{TVPV}$ *proves* $\mathbf{NP} \subseteq \mathbf{P/poly}$ *as in (4) iff* $\mathbf{TVPV}$ *proves* $\mathbf{coNP} \subseteq \mathbf{NP}/O(1)$ *as in (8).*

Proof: Assume that $\mathbf{TVPV}$ proves $\mathbf{NP} \subseteq \mathbf{P/poly}$. It follows from Corollary 3.2 that $\mathbf{TVPV}$ proves (6). Since $\mathbf{VPV}$ proves that the propositional unsatisfiability problem is complete for $\mathbf{coNP}$ (see (2)), it suffices to prove (8) for the case $\varphi(X)$ is $\exists T \leq |X|\ SAT(T,X)$. Let the constant k be as in (6). Given n, the advice i_n for n is the smallest number $j \leq k$ such that (6) holds for this n and all $A_1, \cdots, A_k, T_1, \cdots, T_k$, with k replaced by j. Note that $\mathbf{VPV}$ easily proves the existence of i_n, since k is a constant and only order properties of $\mathbb{N}$ are needed.

We define the $\mathbf{\Sigma}_1^B$ formula $\psi(A,i)$ in (8) to assert the existence of $A_1, \cdots A_{i-1}$ and $T_1, \cdots, T_{i-1}$ which falsify the first $i-1$ disjuncts in (6) and also falsify

$$SAT(C_i(n, A_1, \cdots, A_{i-1}, T_1, \cdots, T_{i-1})(A), A) \ .$$

By the definition of the advice i_n given above, it is easy to verify in $\mathbf{TVPV}$ that if A is a propositional formula of length n, then $\psi(A, i_n)$ holds iff A is unsatisfiable.

Conversely, suppose that $\mathbf{TVPV}$ proves $\mathbf{coNP} \subseteq \mathbf{NP}/O(1)$ as in (8). For fixed n, define

$$\begin{aligned} U =& \{X \mid \varphi(X) \wedge |X| = n\} \\ V_i =& \{X \mid \psi(X,i) \wedge |X| = n\}, i = 1 \cdots k \ . \end{aligned}$$

Then by (8) we conclude in $\mathbf{TVPV}$ that at least one V_i is the complement of U:

$$\bigvee_i [U \cap V_i = \varnothing \wedge U \cup V_i = \{X \ : \ |X| = n\}] \ .$$

It follows that (writing $U(X)$ for $X \in U$ and $V_i(X)$ for $X \in V_i$) for any $I \subseteq \{1, \ldots, k\}$, **TVPV** proves (for $X_1, \cdots, X_k, Y$ ranging over strings of length n)

$$\bigwedge_{i \in I} U(X_i) \wedge V_i(X_i) \ \rightarrow \ \bigvee_{j \notin I} U(Y) \vee V_j(Y) \ . \tag{9}$$

This is because the conditions in the antecedent imply that V_i is not the complement of U while the condition in the succedent for j is implied if V_j is the complement.

The prenex form of this formula is $\mathbf{\Sigma}_1^B$(with additional free variables), so the Herbrand theorem provides us with polynomial time functions

$$F_I(\tilde{X}_{i_1}, \cdots, \tilde{X}_{i_\ell}, Y) \ ,$$

one for each $I = \{i_1 < \cdots < i_\ell\}$ which from witnesses $\tilde{X}_i$'s to the validity of the conjuncts in the antecedent (i.e. $\tilde{X}_i$ is $X_i \in U \cap V_i$ together with the two **NP** witnesses for the membership in the two sets) and from Y finds a $j \notin I$ and an **NP** witness of the membership of Y in either U or V_j.

Now we show how to compute U in polynomial time using polynomial size advice. For a given length n we get the following advice:

- $I = \{i \mid U \cap V_i \neq \varnothing\}$.
- For each $i \in I$ a witness $\tilde{A}_i$ to the fact that $U \cap V_i \neq \varnothing$, i.e. the string A_i of length n in the intersection and the two NP-witnesses of its membership in U and V_i.

If we want to decide whether or not $B \in U$, $|B| = n$, we simply compute

$$F_I(\tilde{A}_{i_1}, ..., \tilde{A}_{i_\ell}, B) \ .$$

The output is either a witness to $B \in U$ or a witness to $B \in V_j$ for some$j \notin I$. In the latter case, necessarily $B \notin U$ by definition of I.

□

The Boolean Hierarchy **BH** is the smallest class of sets that contains **NP** and is closed under the Boolean operations of intersection, union, and

complement. It coincides with the bounded query class $\mathbf{P}^{\mathbf{NP}}[O(1)]$ of sets which can be computed in polynomial time with a constant number of queries to an **NP** oracle (see [Bei91]). Thus

$$\mathbf{NP} \subseteq \mathbf{BH} \subseteq \mathbf{P}^{\mathbf{NP}}[\log n] \subseteq \mathbf{P}^{\mathbf{NP}} \subseteq \mathbf{\Sigma}_2^p \cap \mathbf{\Pi}_2^p \subseteq \mathbf{PH}$$

where **PH** is the polynomial hierarchy and $\mathbf{P}^{\mathbf{NP}}[\log n]$ is the class of sets solvable in polynomial time by making $O(\log n)$ queries to an **NP** oracle. It is not known whether any of these inclusions is proper.

Theorem 3.4. *Let* **TVPV** *be* **VPV** *or any universal extension of* **VPV** *with the same vocabulary. If* **TVPV** *proves* $\mathbf{NP} \subseteq \mathbf{P/poly}$ *as in (4), then* **TVPV** *proves that the polynomial hierarchy collapses to the Boolean Hierarchy.*

Proof: As in the proof of Theorem 3.3, we may assume that **TVPV** proves (6). To show $\mathbf{PH} = \mathbf{BH}$ it suffices to show that $\mathbf{\Sigma}_2^p \subseteq \mathbf{BH}$. Thus suppose that $L \in \mathbf{\Sigma}_2^p$, so

$$X \in L \;\Leftrightarrow\; \exists Y \leq f(|X|) \forall Z \leq g(|X|) \varphi(X, Y, Z) \tag{10}$$

where f and g are polynomials and $\varphi(X, Y, Z)$ is an open formula of **VPV**. Since the satisfiability problem is **NP**-complete, there is a **VPV** function $F(X, Y)$ such that $F(X, Y)$ is a satisfiable propositional formula iff $\neg \forall Z \leq g(|X|)\varphi(X, Y, Z)$. Further, the equivalence is provable in **VPV** (see (2)).

Now suppose C is a circuit such that for every satisfiable propositional formula A of length at most n, $C(A)$ is a satisfying assignment for A (n will be specified later). Then for all X, Y such that $|F(X, Y)| \leq n$ we have

$$\forall Z \leq g(|X|)\varphi(X, Y, Z) \;\Leftrightarrow\; \neg SAT(C(F(X, Y)), F(X, Y)) \; . \tag{11}$$

We can find a suitable circuit C by using (6) and computing the advice i_n used in the proof of Theorem 3.3. We can compute i_n by successive **NP** queries, for $j = 1, 2, ...k$

$$\exists T_1, A_1, \cdots T_j, A_j \leq n$$
$$\bigwedge_{i=1}^{j} [SAT(T_i, A_i) \wedge \neg SAT(C_i(n, A_1, \cdots, A_{i-1}, T_1, \cdots, T_{i-1})(A_i), A_i)] \; .$$

Then i_n is the smallest j such that the answer to the query is 'NO'. (**VPV** easily proves the existence of i_n because there are only a constant k number of possible choices for it.) Now one more **NP** query suffices. Thus, setting $j = i_n$, $X \in L$ iff

$$\begin{aligned}
&\exists Y \leq f(|X|) \exists T_1, A_1, \cdots T_{j-1}, A_{j-1} \leq n \\
&\quad \bigwedge_{i=1}^{j-1} [SAT(T_i, A_i) \wedge \neg SAT(C_i(n, A_1, \cdots, A_{i-1}, T_1, \cdots, T_{i-1})(A_i), A_i)] \\
&\qquad \wedge \neg SAT(C_j(n, A_1, \cdots, A_{j-1}, T_1, \cdots, T_{j-1})(F(X,Y)), F(X,Y)) \ .
\end{aligned}$$

Finally we can set $n = h(|X|)$ for a suitable polynomial h.

It is easy to see that this argument can be formalized in **TVPV**. □

It turns out that Theorem 3.4 can also be proved as an immediate consequence of Theorem 3.3 and the following complexity-theoretic result.

Theorem 3.5 (Jeřábek). [1] *If* **coNP** $\subseteq$ **NP**/$O(1)$ *then the Boolean Hierarchy collapses to the Polynomial Hierarchy. The proof can be formalized in* **VPV**.

Proof: As in the proof of Theorem 3.4, to show **PH** = **BH** it suffices to show that the language L satisfying (10) is in the Boolean Hierarchy. Let $F(X,Y)$ be as in that proof.

Assume that **coNP** $\subseteq$ **NP**/$O(1)$, so UNSAT is in **NP**/$O(1)$. Then there is a $\mathbf{\Sigma}_1^B$ formula $UNS(A,i)$ and a number k such that for every length n there is advice $i \leq k$ and for every propositional formula A of length at most n,

$$A \text{ is unsatisfiable iff } UNS(A,i) \tag{12}$$

Then we claim that

$$\begin{aligned}
X \in L \iff \bigvee_{i=0}^{k} [&\exists Y \leq f(|X|)\ UNS(F(X,Y), i) \\
&\wedge\ \forall A, T \leq h(|X|),\ UNS(A,i) \supset \neg SAT(T,A)]
\end{aligned} \tag{13}$$

[1] We are grateful to Emil Jeřábek for supplying the proof of this theorem, in response to an open problem stated in an earlier version of this paper.

where $h(m)$ is a polynomial upper bound on $|F(X,Y)|$ for all X of length m and all Y of length at most $f(m)$. It follows from this Claim that $L \in \mathbf{BH}$, since the RHS has the form $\bigvee_i [R_i \wedge S_i]$ where R_i is in $\mathbf{NP}$ and S_i is in $\mathbf{coNP}$.

To prove the Claim, first assume that $X \in L$. Let $n = h(|X|)$ and let $i \leq k$ be the advice such that (12) holds for all A of length at most n. Then the RHS of (13) holds by (10) and the stated property of $F(X,Y)$.

Conversely, suppose that X satisfies the RHS of (13), let i satisfy the disjunction, and let Y satisfy the existential quantifier for this i. Then the second conjunct implies that this i gives "sound advice" for $UNS(A,i)$, $|A| \leq h(|X|)$, and hence $X \in L$ by (10) and the stated property of $F(X,Y)$.

Note that $\mathbf{VPV}$ proves (13) from (10) and the properties of $F(X,Y)$ and $UNS(A,i)$. □

4 Witnessing Theorems

The notions surrounding definability of multivalued functions (which we call search problems) in bounded arithmetic were introduced in [BKT93].

A *search problem* Q_R is a multivalued function with graph $R(\vec{x},\vec{X},Z)$, so

$$Q_R(\vec{x},\vec{X}) = \{Z \mid R(\vec{x},\vec{X},Z)\} \ .$$

Here the arity of either or both of $\vec{x},\vec{X}$ may be zero. We assume here that the search problem is *total*, meaning that the set $Q_R(\vec{x},\vec{X})$ is non-empty for all $\vec{x},\vec{X}$. The search problem is a *function problem* if $|Q_R(\vec{x},\vec{X})| = 1$ for all $\vec{x},\vec{X}$.

A (single-valued) function $F(\vec{x},\vec{X})$ *solves* Q_R if

$$F(\vec{x},\vec{X}) \in Q_R(\vec{x},\vec{X})$$

for all $\vec{x},\vec{X}$. More generally, a search problem $Q_{R'}$ *solves* Q_R if $Q_{R'}$ is total and

$$Q_{R'}(\vec{x},\vec{X}) \subseteq Q_R(\vec{x},\vec{X})$$

for all $\vec{x},\vec{X}$.

We say that a search problem Q_R is $\mathbf{\Sigma}^B_i$-definable in a theory $\mathcal{T}$ if there is a $\mathbf{\Sigma}^B_i$-formula ψ_R such that

$$\psi_R(\vec{x},\vec{X},Z) \supset R(\vec{x},\vec{X},Z)$$

and

$$\mathcal{T} \vdash \exists Z \psi_R(\vec{x}, \vec{X}, Z) \ .$$

For example, a search problem is $\mathbf{\Sigma}_1^B$-definable in $\mathbf{V}^1$ iff it is solvable by a polynomial time function.

The standard notation $\mathbf{P}^{\mathbf{\Sigma}_i^p}$ refers to the class of decision problems solvable in polynomial time by accessing an oracle for a problem in $\mathbf{\Sigma}_i^p$. $\mathbf{P}^{\mathbf{\Sigma}_i^p}[q(n)]$ is the same, except that the number of oracle queries in a computation is limited to $O(q(n))$, where n is the length of the input. We use $\mathbf{FP}^{\mathbf{\Sigma_i^p}}$ and $\mathbf{FP}^{\mathbf{\Sigma_i^p}}$[q(n)] for the classes of *search problems* solvable in the same way. [BKT93] introduced the notation $\mathbf{FP}^{\mathbf{\Sigma_i^p}}[wit, q(n)]$ for the class of search problems solvable in polynomial time by making $O(q(n))$ witness queries to an oracle for some problem in $\mathbf{\Sigma}_i^p$. Here a *witness query* returns 1 together with a witness to the query if the answer is 'YES', and returns 0 if the answer is 'NO'. (A witness to an **NP** query is a certificate which can be used to verify a positive answer in polynomial time, and for $i > 1$ a witness to a $\mathbf{\Sigma}_i^p$ query allows a positive answer to be verified in $\mathbf{\Sigma}_{i-1}^p$.) When witness queries are allowed, the machine must output a solution to the search problem no matter which witnesses to the positive queries are returned.

Note that $\mathbf{FP}^{\mathbf{\Sigma_i^p}}[wit] = \mathbf{FP}^{\mathbf{\Sigma_i^p}}$; i.e. witness queries do not help if the number of queries is unrestricted. This is because by self reducibility, a witness for a positive query can be found using polynomially many decision queries.

In terms of two-sorted theories, the following results (among others) are known, or can be inferred from the corresponding single-sorted results. Here for $i \geq 1$ the theory $\mathbf{TV}^i$ [Coo05, CN06] is the two sorted analog of T_2^i, and $\mathbf{TV}^0$ is a finitely axiomatizable theory for polynomial time ($\mathbf{VPV}$ is a conservative extension of $\mathbf{TV}^0$) [Coo05, CN06].

Theorem 4.1. *(i) [Bus86] For $i \geq 1$, a search problem Q is $\mathbf{\Sigma}_i^B$-definable in $\mathbf{V}^i$ iff $Q \in \mathbf{FP}^{\mathbf{\Sigma_{i-1}^p}}$. For the only if direction, $\mathbf{V}^i$ proves the correctness of the witness oracle algorithm.*

(ii) [Kra93, Kra95] For $i \geq 1$, a search problem Q is $\mathbf{\Sigma}_{i+1}^B$-definable in $\mathbf{V}^i$ iff $Q \in \mathbf{FP}^{\mathbf{\Sigma_i^p}}[wit, \log n]$. For the only if direction, $\mathbf{V}^i$ proves the correctness of the witness oracle algorithm.

(iii) [Bus90, Coo05] For $i \geq 0$, a search problem Q is $\mathbf{\Sigma}_{i+1}^B$-definable in $\mathbf{TV}^i$ iff $Q \in \mathbf{FP}^{\mathbf{\Sigma_i^p}}$.

(iv) [Pol99] For $i \geq 0$, a search problem Q is $\mathbf{\Sigma}^B_{i+2}$-definable $\mathbf{TV}^i$ iff $Q \in \mathbf{FP}^{\mathbf{\Sigma^P_{i+1}}}[wit, 1]$.

The case (iv) in the above theorem follows from Theorem 54 in [Pol99], where the 'only if' direction is proved by a cut-elimination argument. We are interested in this direction for the case $i = 0$, so we give a simple proof of this case based on the KPT theorem. Since **VPV** is a conservative extension of $\mathbf{TV}^0$, it suffices to prove the theorem for **VPV**. Since both directions are interesting, we prove the 'if' direction also.

Theorem 4.2. *A search problem Q is $\mathbf{\Sigma}^B_2$-definable in* **VPV** *iff $Q \in \mathbf{FP}^{\mathbf{NP}}[wit, 1]$. For the only if direction,* **VPV** *proves the correctness of the witness oracle algorithm.*

Proof: For the direction $\Longrightarrow$, assume that $Q = Q_R$ is $\mathbf{\Sigma}^B_2$-definable in **VPV**, so

$$\mathbf{VPV} \vdash \exists Z \psi_R(\vec{x}, \vec{X}, Z) \tag{14}$$

where ψ_R is $\mathbf{\Sigma}^B_2$ and

$$\psi_R(\vec{x}, \vec{X}, Z) \supset R(\vec{x}, \vec{X}, Z) \ .$$

For some open formula φ, (14) can be written

$$\mathbf{VPV} \vdash \exists Z \exists Y \forall W \varphi(\vec{x}, \vec{X}, Z, Y, W) \tag{15}$$

where all quantifiers are bounded. By Theorem 3.1 (KPT) there are **VPV** functions $F_1, \ldots, F_k$ and $G_1, \ldots, G_k$ such that (thinking $Z = F_i()$ and $Y = G_i()$, and suppressing the arguments $\vec{x}, \vec{X}$)

$$\begin{aligned} \mathbf{VPV} \vdash \varphi(F_1, G_1, W_1) \vee \varphi(F_2(W_1), G_2(W_1), W_2) \vee \ldots \\ \vee \varphi(F_k(W_1, \ldots, W_{k-1}), G_k(W_1, \ldots, W_{k-1}), W_k) \ . \end{aligned} \tag{16}$$

Now a polynomial time witness oracle machine with an **NP** oracle, given inputs $\vec{x}, \vec{X}$, can compute Z satisfying $\psi_R(\vec{x}, \vec{X}, Z)$ as follows. First ask the oracle whether $\exists W_1 \neg \varphi(F_1, G_1, W_1)$. If 'NO', then output F_1. If 'YES', then let W_1 be a witness, and ask the oracle whether

$$\exists W_2 \neg \varphi(F_2(W_1), G_2(W_1), W_2) \ .$$

If 'NO', then output $F_2(W_1)$. If 'YES', then let W_2 be a witness, and continue. By (16) we are guaranteed a 'NO' answer after some number $i \leq k$ queries, so output $F_i(W_1, \ldots, W_{i-1})$. Then **VPV** proves that the output satisfies the quantifier $\exists Z$ in (15), and hence solves the search problem Q_R.

For the direction $\Longleftarrow$, assume that the oracle Turing machine M solves $Q(\vec{x}, \vec{X})$ in polynomial time with at most k witness queries to the **NP** language L, for some constant k. Let $Comp(\vec{x}, \vec{X}, W)$ be a $\mathbf{\Pi}_1^B$-formula asserting that W codes a halting computation of M on input $\vec{x}, \vec{X}$. Thus W codes the sequence of configurations of M on input $\vec{x}, \vec{X}$, and for each query to L it provides the answer to the query. If the answer is 'YES' it provides a witness for the query (the correctness of the witness can be checked by a $\mathbf{\Sigma}_0^B$-formula). Note that a 'NO' answer can be verified using the universal string quantifiers allowed for $\mathbf{\Pi}_1^B$-formulas.

Now define

$$\psi(\vec{x}, \vec{X}, Z) \equiv \exists W \leq t,\ Comp(\vec{x}, \vec{X}, W) \wedge Out(Z, W)$$

where $Out(Z, W)$ is a $\mathbf{\Sigma}_0^B$-formula asserting that Z is the output of the computation W and $t = t(\vec{x}, \vec{X})$ is a suitable bounding term. To show that Q is $\mathbf{\Sigma}_2^B$-definable in **VPV** is suffices to show that **VPV** proves $\exists Z \psi(\vec{x}, \vec{X}, Z)$. Since it is easy to show that every computation W has an output Z satisfying $Out(Z, W)$, it suffices to show

$$\mathbf{VPV} \vdash \exists W Comp(\vec{x}, \vec{X}, W)\ .$$

To do this, recall that k is an upper bound on the number of queries made by M during any computation. We define, for $0 \leq i \leq k+1$, the $\mathbf{\Pi}_1^B$-formula $Comp_i(\vec{x}, \vec{X}, W)$ to assert that W codes a partial computation of M on input $\vec{x}, \vec{X}$ which is either halting, or includes at least i queries, and ends on an unanswered query. It suffices to show that for each i,

$$\mathbf{VPV} \vdash \exists W Comp_i(\vec{x}, \vec{X}, W) \tag{17}$$

because by assumption $Comp_{k+1}$ is equivalent to $Comp$, so we may replace $Comp$ by $Comp_{k+1}$.

For $i = 0$, (17) follows from the fact that **VPV** proves the existence of a computation for any polytime (nonoracle) Turing machine. It suffices to show

$$\mathbf{VPV} \vdash Comp_i(\vec{x}, \vec{X}, W) \supset \exists W' Comp_{i+1}(\vec{x}, \vec{X}, W')$$

Arguing in **VPV**, assume $Comp_i(\vec{x}, \vec{X}, W)$. If W is a halting computation we are done. Otherwise the answer to the final query of W must be either 'YES' or 'NO'. If the answer is 'YES', then by definition there is a witness to the answer, and using this witness the computation can be continued until the next query. If the answer is 'NO', then again the computation can be continued until the next query. □

5 Results for $\mathbf{V}^1$ and $\mathbf{V}^2$

We now apply Theorem 4.1 to infer analogs of Theorems 3.3 and 3.4 for the theories $\mathbf{V}^1$ and $\mathbf{V}^2$. We also note that Theorem 3.4 and the the only if direction of Theorem 3.3 for **VPV** follow rather easily from Theorem 4.2, using the proof method of the next theorem.

Theorem 5.1. *(i)* $\mathbf{V}^1$ *proves* $\mathbf{NP} \subseteq \mathbf{P/poly}$ *iff* $\mathbf{V}^1$ *proves* $\mathbf{coNP} \subseteq \mathbf{NP}/O(\log n)$.

(ii) If $\mathbf{V}^1$ *proves* $\mathbf{NP} \subseteq \mathbf{P/poly}$ *then* $\mathbf{V}^1$ *proves that the Polynomial Hierarchy collapses to* $\mathbf{P}^{\mathbf{NP}}[\log n]$.

(iii) If $\mathbf{V}^2$ *proves* $\mathbf{NP} \subseteq \mathbf{P/poly}$ *then* $\mathbf{V}^2$ *proves that the Polynomial Hierarchy collapses to* $\mathbf{P}^{\mathbf{NP}}$.

Proof: (i) ($\Longrightarrow$): Assume that $\mathbf{V}^1$ proves $\mathbf{NP} \subseteq \mathbf{P/poly}$ as in (4). Let $\eta(n, C)$ be the formula

$$\forall A \leq n \forall T \leq n,\ SAT(T, A) \ \supset\ SAT(C(A), A)\ . \tag{18}$$

Then $\mathbf{V}^1$ proves $\exists C \eta(n, C)$, and hence the search problem Q given by

$$Q(n, C) \Leftrightarrow \eta(n, C)$$

is $\mathbf{\Sigma}_2^B$-definable in $\mathbf{V}^1$. Hence by Theorem 4.1 part (ii) for $i = 1$, Q is in $\mathbf{FP}^{\mathbf{NP}}[wit, \log n]$, provably in $\mathbf{V}^1$. Let M be a polynomial time witness oracle Turing machine solving Q by making $O(\log n)$ witness queries to some **NP** problem, such that $\mathbf{V}^1$ proves that any circuit C output by M on input n satisfies (18). Here is how $\mathbf{V}^1$ proves the existence of a correct computation of M for each input n. We distinguish between the decision part of a witness query, whose answer is 0 or 1, and the witness part of the answer in case the decision answer is 1. Note that the sequence of queries (and their answers) may not be determined by the input n, because more than one witness answer to a positive witness query may be possible. By the $\mathbf{\Sigma}_1^B$ number-MAX

principle, $\mathbf{V}^1$ proves for each input n the existence of a computation Z of M in which the sequence of 0-1 answers to the decision part of the queries is lexicographically the largest possible (here the $\mathbf{\Sigma}_1^B$ formula verifies the 1 answers and their witnesses, but does not verify the 0 answers). It follows that the 0 answers for this computation are correct, even though they have not been verified (the first wrong 0 answer would yield a lexicographically larger sequence of answers).

To formalize $\mathbf{coNP} \subseteq \mathbf{NP}/O(\log n)$ we adapt the formalization (8) of $\mathbf{coNP} \subseteq \mathbf{NP}/O(1)$ as follows: For every $\mathbf{\Sigma}_1^B$ formula $\varphi(X)$ there is a $\mathbf{\Sigma}_1^B$ formula $\psi(X)$ and a polynomial $f(n)$ such that

$$\forall n \exists i \leq f(n), |X| = n \supset [\neg\varphi(X) \leftrightarrow \psi(X, i)] \ . \tag{19}$$

As before, since the unsatisfiability problem is complete for $\mathbf{coNP}$, it suffices to show that $\mathbf{V}^1$ proves (19) for the case that $\varphi(X)$ is $\exists T \leq |X|\ SAT(T, X)$.

Given n, the advice string needed to show that a given unsatisfiable formula A of length n is indeed unsatisfiable is the lexicographically largest possible string of 0-1 query answers described above for the computation of M on input n. We define the $\mathbf{\Sigma}_1^B$ formula $\psi(A, i)$ in (19) to assert that some computation Z of M on input n, in which the query answers are those coded by i in binary, computes a circuit C such that $C(A)$ is not a satisfying assignment for A. Then $\mathbf{V}^1$ proves that C satisfies (18), and hence A is unsatisfiable.

(i) ($\Longleftarrow$): Assume that $\mathbf{V}^1$ proves $\mathbf{coNP} \subseteq \mathbf{NP}/O(\log n)$ as in (19). We argue that $\mathbf{V}^1(\mathbf{VPV})$ proves $\mathbf{NP} \subseteq \mathbf{P/poly}$ by a slight modification of the proof of the 'if' direction of Theorem 3.3. (Note that $\mathbf{V}^1(\mathbf{VPV})$ is a conservative extension of $\mathbf{V}^1$.) The sets U and V_i are as in that proof, except that i ranges up to $O(\log n)$ rather than the constant k. The set I in (9) is specified by a string variable I listing its members. Instead of a separate function F_I for every I, we now have a single $\mathbf{VPV}$ function $F(I, \tilde{X}, Y)$, where $\tilde{X}$ is an array giving witnesses to the validity of all conjuncts in the antecedent of (9). (The existence of a polynomial time such F now follows from the Buss witnessing theorem for $\mathbf{V}^1$.) The advice required to compute U is the same as before, except longer (but still polynomial in length), since it requires information for $O(\log n)$ values of i instead of a constant k values.

(ii): Assume that $\mathbf{V}^1$ proves $\mathbf{NP} \subseteq \mathbf{P/poly}$ as in (4). Let M be the witness oracle machine described in the proof of (i) ($\Longrightarrow$) above. Thus on input n, M computes a circuit C satisfying (18).

To show that $\mathbf{V}^1(\mathbf{VPV})$ proves that the Polynomial Hierarchy collapses to $\mathbf{P}^{\mathbf{NP}}[\log n]$, it suffices to show $\mathbf{\Sigma}_2^p \subseteq \mathbf{P}^{\mathbf{NP}}[\log n]$. We argue as in the proof of Theorem 3.4, and assume that $L \in \mathbf{\Sigma}_2^p$, so (10) holds. The idea is to use the circuit C solving SAT computed by M, so that (11) holds for this C. However there is a difficulty in finding C, because we are trying to show L is in $\mathbf{P}^{\mathbf{NP}}[\log n]$, so that only decision queries are allowed, but M requires witness queries to find C. We proceed as follows. To determine whether a given string X is in L, we start by asking a sequence of $O(\log n)$ **NP** decision queries to determine the lexicographically largest possible sequence S of 0-1 answers to the witness queries of the computation of M (see the proof of (i) ($\Longrightarrow$)) on input n (where n is a suitable polynomial in $|X|$). Now just one more **NP** decision query is needed. By (10) and (11), X is in L iff there exists $Y \leq f(|X|)$ and there exists a computation of M on input n such that the decision part of the answers to the witness queries of the computation are the sequence S (only positive query answers and their witnesses need be verified) such that if C is the resulting circuit computed by M then $\neg SAT(C(F(X,Y)), F(X,Y))$.

(iii): The proof is similar to the proof of (ii), but instead of part (ii) of Theorem 4.1 we use part (i): If Q is $\mathbf{\Sigma}_2^B$-definable in $\mathbf{V}^2$ then Q is in $\mathbf{FP}^{\mathbf{NP}}$. □

Just as Theorem 3.4 follows from Theorem 3.3 and Theorem 3.5, an alternative proof of Theorem 5.1 (ii) (except possibly the provability of the conclusion) can be obtained from the conclusion $\mathbf{coNP} \subseteq \mathbf{NP}/O(\log n)$ of Theorem 5.1 (i) and the following complexity-theoretic result.

Theorem 5.2. *If* $\mathbf{coNP} \subseteq \mathbf{NP}/O(\log n)$ *then the Polynomial Hierarchy collapses to* $\mathbf{P}^{\mathbf{NP}}[\log n]$.

Proof: We argue as in the proof of Theorem 3.5, except now the formula $UNS(A, i)$ needs $O(\log n)$ bits of advice instead of constant advice. Thus the constant k in (13) becomes a function $k(n) = n^{O(1)}$ (where $n = |X|$). From this we see that L can be computed in polynomial time with polynomially many *parallel* queries to an **NP** oracle. From results in [BH91] it follows that $O(\log n)$ *serial* queries suffice, so $L \in \mathbf{P}^{\mathbf{NP}}[\log n]$ as required. □

Open Questions: Does Theorem 3.4 have a converse as in Theorem 3.3: If **VPV** proves $\mathbf{PH} = \mathbf{BH}$ can we conclude **VPV** proves $\mathbf{NP} \subseteq \mathbf{P/poly}$?

Is there a converse to either of Theorems 3.5 or 5.2? For example, does $\mathbf{PH} = \mathbf{BH}$ imply $\mathbf{coNP} \subseteq \mathbf{NP}/O(1)$, possibly with the additional assumption that $\mathbf{NP} \subseteq \mathbf{P/poly}$?

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