The Isospectral Dirac Operator on the 4-dimensional Quantum Euclidean Sphere

Francesco D'Andrea[1], Ludwik Dąbrowski[1], Giovanni Landi[2]

[1] Scuola Internazionale Superiore di Studi Avanzati,
Via Beirut 2-4, I-34014, Trieste, Italy

[2] Dipartimento di Matematica e Informatica, Università di Trieste,
Via Valerio 12/1, I-34127, Trieste, Italy
and INFN, Sezione di Trieste, Trieste, Italy

20th December 2006

Abstract

Equivariance under the action of $U_q(so(5))$ is used to compute the left regular and (chiral) spinorial representations of the algebra of the quantum Euclidean 4-sphere S^4_q. These representations are the constituents of a spectral triple on S^4_q with a Dirac operator which is isospectral to the canonical one of the spin structure of the round sphere S^4 and which gives 4^+-summability. Non-triviality of the geometry is proved by pairing the associated Fredholm module with an 'instanton' projection. A real structure which satisfies all required properties modulo a suitable ideal of 'infinitesimals' is also introduced.

Keywords: Noncommutative geometry, quantum group symmetries, quantum spheres, spectral triples, isospectral deformations.

1 Introduction

The recent constructions of spectral triples – with the consequent analysis of the corresponding spectral geometry – for the manifold of the quantum $SU(2)$ group in [7, 5, 11, 12] and for its quantum homogeneous spaces (the Podleś spheres) in [13, 10, 8, 9], have provided a number of examples showing that a marriage between noncommutative geometry and quantum groups theory is indeed possible. A common feature of most of these examples is that the dimension spectrum is the same as in the commutative ($q = 1$) limit. Furthermore, with the only known exception of the 0^+-summable 'exponential' spectral triple on the standard Podleś sphere given in [13], in order to have a real spectral triple one is forced to weaken the usual requirements that the real structure should satisfy.

It is then only natural to try and construct additional examples wondering in particular if these properties are common to all quantum spaces or are rather coincidences which happen for low dimensional examples (all related to the quantum group $SU_q(2)$). In this paper we present an example in 'dimension four' given by a spectral triple on the quantum Euclidean sphere S^4_q which is isospectral to the canonical spectral triple on the classical sphere with the round metric. We construct also a real structure modulo an ideal which is larger than the ideal of smoothing operators used for the modified real structures of the examples mentioned above.

There are a few reasons why in dimension greater or equal than four the quantum Euclidean sphere S^4_q is most interesting to study. Firstly, all the relevant irreducible representations of the symmetry algebra $U_q(so(5))$ are known [2] and both the algebra $\mathcal{A}(S^4_q)$ of polynomial functions as well as the modules of chiral spinors carry representations of $U_q(so(5))$ which are multiplicity free. Secondly, the spectrum of the Dirac operator $D\!\!\!\!/$ for the round metric on the undeformed sphere S^4 is known [1]. All this allows us to apply the already tested methods of isospectral deformations and indeed to construct an $U_q(so(5))$-equivariant spectral triple on S^4_q.

The sphere S^4_q could also be relevant for noncommutative physical models. In particular, on S^4_q there is a canonical 'instantonic vector bundle' [17] and the study of the noncommutative geometry of S^4_q could be a first step for the construction of $SU_q(2)$ instantons on this space.

In Section 2 we recall all generalities about spectral triples that we need. We give also some properties of finitely generated projective modules over algebras having quantum group symmetries. The rest of the paper is organized as follows. Sections 3 and 4 are devoted to the symmetry Hopf algebra $U_q(so(5))$ and its fundamental $*$-algebra module, the quantum Euclidean sphere S^4_q. In Section 5 we describe the $\mathcal{A}(S^4_q)$-modules of chiral spinors over S^4_q. Section 6 is devoted to the left regular representation of the algebra $\mathcal{A}(S^4_q)$ of polynomial functions over S^4_q and to the representations of $\mathcal{A}(S^4_q)$ which in the $q = 1$ limit corresponds to the modules of chiral spinors. These representations are $U_q(so(5))$-equivariant, that is they correspond to representations of the crossed product algebra $\mathcal{A}(S^4_q) \rtimes U_q(so(5))$. In Section 7 we use the isospectral Dirac operator to construct a spectral triple on S^4_q; it will be $U_q(so(5))$-equivariant, regular, even and of metric dimension 4. We also prove that is non-trivial by pairing the Fredholm module canonically associated to the spectral triple to an 'instanton' projection e. It turns out that the projection e has charge 1, as in the classical case. In Section 8, quotienting $\mathcal{A}(S^4_q)$ by a suitable ideal of 'infinitesimals' $\mathcal{J}$ we compute the part of the dimension spectrum contained in the right half plane $\{s \in \mathbb{C} \,|\, \mathrm{Re}\, s > 2\}$, as well as the top residue (which in the commutative case is proportional to the Riemann integral). At the moment we are unable to comment on the part of the dimension spectrum which is in the left half plane $\mathrm{Re}\, s \leq 2$, and whose analysis requires a less drastic approximation which we are lacking. Finally, in Section 9 we produce an equivariant real structure for which both the 'commutant property' and the 'first order condition' are satisfied modulo the ideal $\mathcal{J}$. We stress that the latter is larger than the ideal of smoothing operators employed in the 'modified' real structure used recently for the Podleś spheres in [10, 9] and for the manifold of $SU_q(2)$ in [11].

2 Some useful preliminaries

In this section, we collect same basic notions concerning equivariant spectral triples. We also give some general properties of finitely generated projective modules over algebras having quantum group symmetries.

2.1 Generalities about Spectral Triples

We start with the notion of finite summable spectral triples [3].

Definition 2.1. *A* spectral triple $(\mathcal{A}, \mathcal{H}, D)$ *is the datum of a complex associative unital* $*$-*algebra* $\mathcal{A}$, *a* $*$-*representation* $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ *by bounded operators on a (separable) Hilbert space* $\mathcal{H}$ *and a self-adjoint (unbounded) operator* $D = D^*$ *such that,*

- $(D + i)^{-1}$ *is a compact operator;*
- $[D, \pi(a)]$ *is a bounded operator for all* $a \in \mathcal{A}$.

We refer to D as the 'generalized' Dirac operator, or the Dirac operator 'tout court' and for simplicity we assume that it is invertible. Usually, the representation symbol π is removed when no risk of confusion arises.

With $n \in \mathbb{R}^+$, D is called n^+-*summable* if the operator $(D^2 + 1)^{1/2}$ is in the Dixmier ideal $\mathcal{L}^{n+}(\mathcal{H})$. We shall also call n the *metric dimension* of the spectral triple.

A spectral triple is called *even* if there exists a grading γ, i.e. a bounded operator satisfying $\gamma = \gamma^*$ and $\gamma^2 = 1$, such that the Dirac operator is odd and the algebra is even:

$$\gamma D + D\gamma = 0 \,, \qquad a\gamma = \gamma a \,, \quad \forall\, a \in \mathcal{A} \,.$$

We recall from [6] a few analytic properties of spectral triples. To the unbounded operator D on $\mathcal{H}$ one associates an unbounded derivation δ on $\mathcal{B}(\mathcal{H})$ by,

$$\delta(a) = [|D|, a] \,,$$

for all $a \in \mathcal{B}(\mathcal{H})$. A spectral triple is called *regular* if the following inclusion holds,

$$\mathcal{A} \cup [D, \mathcal{A}] \subset \bigcap\nolimits_{j \in \mathbb{N}} \mathrm{dom}\, \delta^j \,,$$

and we refer to $\mathrm{OP}^0 := \bigcap_{j\in\mathbb{N}} \mathrm{dom}\,\delta^j$ as the 'smooth domain' of the operator δ. For a regular spectral triple, the class Ψ^0 of pseudodifferential operators of order less or equal that zero is defined as the algebra generated by $\bigcup_{k\in\mathbb{N}} \delta^k(\mathcal{A}\cup[D,\mathcal{A}])$. If the triple has finite metric dimension n, the 'zeta-type' function

$$\zeta_a(s) := \mathrm{Trace}_{\mathcal{H}}(a|D|^{-s})$$

associated to $a \in \Psi^0$ is defined (and holomorphic) for $s \in \mathbb{C}$ with $\mathrm{Re}\, s > n$ and the following definition makes sense.

Definition 2.2. *A spectral triple has* dimension spectrum Σ *iff* $\Sigma \subset \mathbb{C}$ *is a countable set, for all* $a \in \Psi^0$ *the function* $\zeta_a(s)$ *extends to a meromorphic function on* $\mathbb{C}$ *with poles as unique singularities, and the union of such singularities is the set* Σ*.*

If Σ is made only of simple poles, the Wodzicki-type residue functional

$$⨍ T := \mathrm{Res}_{s=0}\mathrm{Trace}(T|D|^{-s}) \tag{2.1}$$

is tracial on Ψ^0. We also recall the definition of 'smoothing operators' $\mathrm{OP}^{-\infty}$,

$$\mathrm{OP}^{-\infty} := \{T \in \mathrm{OP}^0 \,|\, |D|^k T \in \mathrm{OP}^0 \;\forall\; k \in \mathbb{N}\}\ .$$

The class $\mathrm{OP}^{-\infty}$ is a two-sided $*$-ideal in the $*$-algebra OP^0, is δ-invariant and then in the smooth domain of δ. If T is a smoothing operator, $\zeta_T(s)$ is holomorphic on $\mathbb{C}$. Also, the integral (2.1) vanishes if T is a smoothing operator. Thus, elements in $\mathrm{OP}^{-\infty}$ can be neglected when computing the dimension spectrum and residue. Finally, we note that if the metric dimension is finite, rapid decay matrices – in a basis of eigenvectors for D with eigenvalues in increasing order – are smoothing operators.

In analogy with the notion of spin manifold, one asks for the existence of a real structure J on a spectral triple $(\mathcal{A},\mathcal{H},D)$. Motivated by the examples of real spectral triples on Podleś spheres [10, 9] and on $SU_q(2)$ [11], we use the following weakened definition of real structure.

Definition 2.3. *A* real structure *is an antilinear isometry* J *on* $\mathcal{H}$ *such that* $\forall\, a,b \in \mathcal{A}$*,*

$$J^2 = \pm 1\ , \qquad JD = \pm DJ\ , \qquad [a, JbJ^{-1}] \subset \mathcal{I}\ , \qquad [[D,a], JbJ^{-1}] \subset \mathcal{I}\ .$$

If the spectral triple is even with grading γ*, we impose the further relation* $J\gamma = \pm\gamma J$*.*

The signs '$\pm$' are determined by the dimension of the geometry [4]. A real spectral triple of dimension 4 corresponds to the choices $J^2 = -1$, $JD = DJ$ and $J\gamma = \gamma J$.
The set $\mathcal{I}$ is a suitable two-sided ideal in the algebra OP^0 of 'order zero' operators which is made of 'infinitesimals'. The original definition [4] corresponds to $\mathcal{I} = 0$; while in examples coming from quantum groups [10, 11, 9] one usually takes $\mathcal{I} = \mathrm{OP}^{-\infty}$. We shall see that for the spectral triple of the present paper we are forced to take an ideal which is strictly larger than $\mathrm{OP}^{-\infty}$, the ideal $\mathcal{J}$ defined in (8.1).

Let $F := D|D|^{-1}$ be the sign of D; if $(\mathcal{A}, \mathcal{H}, D)$ is a regular even spectral triple, the datum $(\mathcal{A}, \mathcal{H}, F, \gamma)$ is an even Fredholm module. We say that the Fredholm module is *p-summable* if $p \geq 1$ and, for all $a \in \mathcal{A}$, $[F, a]$ belongs to the p-th Schatten-von Neumann ideal $\mathcal{L}^p(\mathcal{H})$ of compact operators T such that $|T|^p$ is of trace class. Associated with a p-summable even Fredholm module there are cyclic cocycles defined by

$$\mathrm{ch}_n^F(a_0, \ldots, a_n) = \frac{\Gamma(\frac{n}{2}+1)}{2n!} \mathrm{Trace}(\gamma F [F, a_0] \ldots [F, a_n]) \ , \tag{2.2}$$

for all even integer $n \geq p-1$. By composing with a matrix trace, ch_n^F is canonically extended to matrices with entries in $\mathcal{A}$. The pairings with elements $[e] \in K_0(\mathcal{A})$, given by $\mathrm{ch}_n^F(e, e, \ldots, e)$ build up to an integer-valued map $\mathrm{ch}^F([e])$ which depends only on the class $[e]$ and which yields the index of the Dirac operator D twisted with the projection e (for further details see [3]).

Finally, we turn now to symmetries; these will be implemented by an action of a Hopf $*$-algebra. Firstly, let $\mathcal{V}$ be a dense linear subspace of a Hilbert space $\mathcal{H}$ with inner product $\langle\, , \rangle$, and let $\mathcal{U}$ be a $*$-algebra. An (unbounded) $*$-representation of $\mathcal{U}$ on $\mathcal{V}$ is a homomorphism $\lambda : \mathcal{U} \to \mathrm{End}(\mathcal{V})$ such that $\langle \lambda(h)v, w \rangle = \langle v, \lambda(h^*)w \rangle$ for all $v, w \in \mathcal{V}$ and all $h \in \mathcal{U}$. From now on, the symbol λ will be omitted. Next, let $\mathcal{U} = (\mathcal{U}, \Delta, \varepsilon, S)$ be a Hopf $*$-algebra and let $\mathcal{A}$ be a left $\mathcal{U}$-module $*$-algebra, i.e., there is a left action $\triangleright$ of $\mathcal{U}$ on $\mathcal{A}$ satisfying

$$h \triangleright ab = (h_{(1)} \triangleright a)(h_{(2)} \triangleright b) \ , \qquad h \triangleright 1 = \varepsilon(h) 1 \ , \qquad h \triangleright a^* = \{S(h)^* \triangleright a\}^* \ ,$$

for all $h \in \mathcal{U}$ and $a, b \in \mathcal{A}$. As customary, $\Delta(h) = h_{(1)} \otimes h_{(2)}$.

A $*$-representation of $\mathcal{A}$ on $\mathcal{V}$ is called $\mathcal{U}$-equivariant if there exists a $*$-representation of $\mathcal{U}$ on $\mathcal{V}$ such that, for all $h \in \mathcal{U}$, $a \in \mathcal{A}$ and $v \in \mathcal{V}$, it happens that

$$hav = (h_{(1)} \triangleright a)\, h_{(2)} v \ .$$

Given $\mathcal{U}$ and $\mathcal{A}$ as above, the left crossed product $*$-algebra $\mathcal{A} \rtimes \mathcal{U}$ is defined as the $*$-algebra generated by the two $*$-subalgebras $\mathcal{A}$ and $\mathcal{U}$ with crossed commutation relations

$$ha = (h_{(1)} \triangleright a) h_{(2)} \ , \quad \forall\, h \in \mathcal{U},\ a \in \mathcal{A} .$$

Thus, $\mathcal{U}$-equivariant $*$-representations of $\mathcal{A}$ correspond to $*$-representations of $\mathcal{A} \rtimes \mathcal{U}$.

A linear operator D defined on $\mathcal{V}$ is said to be equivariant if it commutes with $\mathcal{U}$, i.e.,

$$Dhv = hDv \tag{2.3}$$

for all $h \in \mathcal{U}$ and $v \in \mathcal{V}$. On the other hand, an antilinear operator T defined on $\mathcal{V}$ is called equivariant if it satisfies the relation

$$Thv = S(h)^* Tv \ , \tag{2.4}$$

for all $h \in \mathcal{U}$ and $v \in \mathcal{V}$, where S denotes the antipode of $\mathcal{U}$. Notice that if T is an equivariant antilinear operator, its square T^2 is an equivariant linear operator, but T^*T is not an equivariant linear operator unless $S^2 = 1$.

We collect all these equivariance requirements in the following definition (see also [19]).

Definition 2.4. *Let $\mathcal{U}$ be a Hopf $*$-algebra and $\mathcal{A}$ a left $\mathcal{U}$-module $*$-algebra. A (real, even) spectral triple $(\mathcal{A}, \mathcal{H}, D, \gamma, J)$ is called* equivariant *if $\mathcal{U}$ is represented on a dense subspace $\mathcal{V}$ of $\mathcal{H}$, $\mathcal{V} \subset \operatorname{dom} D$, the representation of $\mathcal{U}$ commutes with the grading γ, the restriction of the representation of $\mathcal{A}$ on $\mathcal{V}$ is $\mathcal{U}$-equivariant, the operator D is equivariant and J is the antiunitary part of the polar decomposition of an equivariant antilinear operator.*

2.2 Projective module description of equivariant representations

In order to construct the analogues of the modules of chiral spinors on the sphere S^4_q we need some properties of finitely generated projective modules over algebras having quantum group symmetries.

Let $\mathcal{U}$ be a Hopf $*$-algebra, $\mathcal{A}$ be an $\mathcal{U}$-module $*$-algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ be an invariant faithful state (i.e. φ is linear, $\varphi(a^*a) > 0$ for all nonzero $a \in \mathcal{A}$, and $\varphi(h \triangleright a) = \epsilon(h)\varphi(a)\ \forall\ a \in \mathcal{A}$ and $h \in \mathcal{U}$). Suppose also that there exists $\kappa \in \operatorname{Aut}(\mathcal{A})$ such that the 'twisted' cyclicity

$$\varphi(ab) = \varphi\big(b\,\kappa(a)\big)$$

holds for all $a, b \in \mathcal{A}$. Instances of this situation are provided by subalgebras of compact quantum group algebras with φ the Haar state and κ the modular involution[1]. In particular, for the case $\mathcal{A} = \mathcal{A}(S^4_q)$ and $\mathcal{U} = U_q(so(5))$, φ comes from the Haar functional of $\mathcal{A}(SO_{q^2}(5))$ and the modular automorphism is $\kappa(a) = K_1^8 K_2^6 \triangleright a$ [16, Sec. 11.3.4].

For $N \in \mathbb{N}$, let $\mathcal{A}^N := \mathcal{A} \otimes \mathbb{C}^N$ be the linear space with elements $v = (v_1, \ldots, v_N)$, $v_i \in \mathcal{A}$, and $\mathbb{C}$-valued inner product given by

$$\langle v, w \rangle := \sum\nolimits_{i=1}^{N} \varphi(v_i^* w_i)\ . \tag{2.5}$$

Lemma 2.5. *Let $\sigma : \mathcal{U} \to \operatorname{Mat}_N(\mathbb{C})$ be a $*$-representation. The formulæ:*

$$(a.v)_i := a v_i\ , \qquad (h.v)_i := \sum\nolimits_{j=1}^{N} (h_{(1)} \triangleright v_j)\sigma_{ij}(h_{(2)})\ , \tag{2.6}$$

for all $a, v \in \mathcal{A}$ and $h \in \mathcal{U}$ (and $i = 1, \ldots, N$), define a $*$-representation of the crossed product algebra $\mathcal{A} \rtimes \mathcal{U}$ on the linear space $\mathcal{A}^N$.

Proof. The inner product allows us to define the adjoint of an element of $\mathcal{A} \rtimes \mathcal{U}$ in the representation on $\mathcal{A}^N$. For $x \in \operatorname{End}(\mathcal{A}^N)$, its adjoint denoted with $x^\dagger$, is defined

$$\langle x^\dagger.v, w \rangle := \langle v, x.w \rangle\ , \qquad \forall\ , v, w \in \mathcal{A}^N\ .$$

Recal that being a $*$-representation means that $x^\dagger.v = x^*.v$ for any operator x and any $v \in \mathcal{A}^N$. The nontrivial part of the proof consists in showing that $h^\dagger.v = h^*.v$ for all $h \in \mathcal{U}$ and $v \in \mathcal{A}$. It is enough to take $N = 1$. For $N > 1$ we are considering the Hopf tensor product of the $N = 1$ representation with a matrix representation that is a $*$-representation by hypothesis.

[1] KMS states in Thermal Quantum Field Theory provide additional examples.

The $\mathcal{U}$-invariance of φ implies:

$$\epsilon(h)\,\langle v, w\rangle = \varphi\big(h \triangleright (v^* w)\big) = \varphi\big((h_{(1)} \triangleright v^*)(h_{(2)} \triangleright w)\big) \ .$$

But $h_{(1)} \triangleright v^* = \{S(h_{(1)})^* \triangleright v\}^*$ by definition of module $*$-algebra. Then,

$$\epsilon(h)\,\langle v, w\rangle = \big\langle S(h_{(1)})^*.v, h_{(2)}.w\big\rangle = \big\langle v, S(h_{(1)})^{*\dagger} h_{(2)}.w\big\rangle \ .$$

We deduce that for all $h \in \mathcal{U}$ one has that

$$S(h_{(1)})^{*\dagger} h_{(2)} = \epsilon(h) \ . \tag{2.7}$$

Recall that the convolution product '$\star$' for any $F, G \in \mathrm{End}(\mathcal{U})$ is defined by

$$(F \star G)(h) := F(h_{(1)}) G(h_{(2)}) \qquad \forall\, h \in \mathcal{U} \ ;$$

and $(\mathrm{End}(\mathcal{U}), \star)$ is an associative algebra with unity given by the endomorphism $h \mapsto \epsilon(h) 1_{\mathcal{U}}$, with S a left and right inverse for $id_{\mathcal{U}}$ in $(\mathrm{End}(\mathcal{U}), \star)$, that is

$$S \star id_{\mathcal{U}} = 1_{\mathcal{U}} \epsilon = id_{\mathcal{U}} \star S \ .$$

Let $S' \in \mathrm{End}(\mathcal{U})$ be the composition $S' := \dagger \circ * \circ S$. Equation (2.7) implies that S' is a left inverse for $id_{\mathcal{U}}$:

$$S' \star id_{\mathcal{U}} = 1_{\mathcal{U}} \epsilon \ .$$

Applying $\star S$ to the right of both members of this equation and using $id_{\mathcal{U}} \star S = 1_{\mathcal{U}} \epsilon$ we get $S' = S$ as endomorphisms of $\mathcal{U}$, i.e. $S(h)^{*\dagger} = S(h)$ for all $h \in \mathcal{U}$.

Now, the antipode of an Hopf $*$-algebra is invertible, with $S^{-1} = * \circ S \circ *$, thus we arrive at $h^{*\dagger} = h$ for all $h \in \mathcal{U}$. Replacing h with h^* we prove that $h^\dagger = h^*$ for all $h \in \mathcal{U}$, and this concludes the proof. □

Now, let $e = (e_{ij}) \in \mathrm{Mat}_N(\mathcal{A})$ be an $N \times N$ matrix with entries $e_{ij} \in \mathcal{A}$. Let $\pi : \mathcal{A}^N \to \mathcal{A}^N$ be the (linear) endomorphism defined by:

$$\pi(v)_j := \sum\nolimits_{i=1}^{N} v_i e_{ij} \ , \tag{2.8}$$

for all $v \in \mathcal{A}^N$ and $j = 1, \ldots, N$. Since $\mathcal{A}$ is associative, left and right multiplication commute and $\pi(av) = a\pi(v)$ for all $a \in \mathcal{A}$ and $v \in \mathcal{A}^N$. Thus we have the following Lemma.

Lemma 2.6. *The map π defined by* (2.8) *is an $\mathcal{A}$-module map.*

Recall that an endomorphism p of an inner product space V is a projection (not necessarily orthogonal) if $p \circ p = p$. A projection p is *orthogonal* if the image of p and $id_V - p$ are orthogonal with respect to the inner product of V, and this happens exactly when $p^\dagger = p$.

A simple computation shows that the map π in (2.8) is a projection iff $e^2 = e$, that is the matrix $e \in \mathrm{Mat}_N(\mathcal{A})$ is an idempotent. Now we use the twisted-cyclicity of φ to deduce:

$$\big\langle v, \pi^\dagger(w)\big\rangle = \langle \pi(v), w\rangle = \sum\nolimits_{ij} \varphi(e^*_{ij} v^*_i w_j) = \sum\nolimits_{ij} \varphi\big(v^*_i w_j \kappa(e^*_{ij})\big) \ ,$$

for all $v, w \in \mathcal{A}^N$. Hence the adjoint $\pi^\dagger$ of the endomorphism π is given by

$$\pi^\dagger(w)_i = \sum\nolimits_{j=1}^N w_j \kappa(e^*_{ij}) \ .$$

Let e^* be the matrix with entries $(e^*)_{jk} := e^*_{kj}$. We have the following Lemma.

Lemma 2.7. *The endomorphism π in* (2.8) *is an orthogonal projection iff $e^2 = e = \kappa(e^*)$.*

Next, we determine a sufficient condition for the endomorphism π to be not only an $\mathcal{A}$-module map, but also an $\mathcal{U}$-module map.

Lemma 2.8. *With '$\,^t\,$' denoting transposition, if*

$$h \triangleright e = \sigma(h_{(1)})^t \, e \, \sigma(S^{-1}(h_{(2)}))^t \ , \tag{2.9}$$

for all $h \in \mathcal{U}$, the endomorphism π in (2.8) *is an $\mathcal{U}$-module map.*

Proof. Equation (2.9) can be rewritten as,

$$h \triangleright e_{ij} = \sum\nolimits_{kl} \sigma_{ki}(h_{(1)}) \, e_{kl} \, \sigma_{jl}(S^{-1}(h_{(2)})) \ ;$$

by using it into the definition (2.6) one check that $\pi(h.v) = h.\pi(v)$ for all $h \in \mathcal{U}$ and $v \in \mathcal{A}^N$. □

When Lemma 2.7 and Equation (2.9) are satisfied, the orthogonal projections π and $\pi^\perp = 1 - \pi$ split $\mathcal{A}^N$ into the orthogonal sum of two sub $*$-representations $\pi(\mathcal{A}^N)$ and $\pi^\perp(\mathcal{A}^N)$ of $\mathcal{A} \rtimes \mathcal{U}$. Next Lemma gives a (quite obvious) sufficient condition for $\pi(\mathcal{A}^N)$ and $\pi^\perp(\mathcal{A}^N)$ to be not equivalent as representations of $\mathcal{A}$. Recall that an isomorphism of $\mathcal{A}$-modules is an $\mathcal{A}$-linear map, so isomorphic modules correspond to equivalent representations.

Lemma 2.9. *Let $(\mathcal{A}, \mathcal{H}, F, \gamma)$ be an even Fredholm module over $\mathcal{A}$. If $\mathrm{ch}^F([e]) \neq 0$, the $\mathcal{A}$-modules $\pi(\mathcal{A}^N)$ and $\pi^\perp(\mathcal{A}^N)$ are not equivalent.*

Proof. The map $K_0(\mathcal{A}) \to \mathbb{Z}$, $[e] \mapsto \mathrm{ch}^F([e])$ is an homomorphism. Suppose $\pi(\mathcal{A}^N)$ and $\pi^\perp(\mathcal{A}^N)$ are isomorphic $\mathcal{A}$-modules, then $[e] = [1 - e]$ and $\mathrm{ch}^F([1 - e]) = \mathrm{ch}^F([e])$.

But from Equation (2.2), $\mathrm{ch}^F([1 - e]) = -\mathrm{ch}^F([e])$ (since $[F, 1 - e] = -[F, e]$ and n is even).

Hence $\mathrm{ch}^F([e]) = 0$, and this concludes the proof by contraddiction. □

3 The symmetry Hopf algebra $U_q(so(5))$

Let $0 < q < 1$. We call $U_q(so(5))$ the real form of the Drinfeld-Jimbo deformation of $so_{\mathbb{C}}(5)$, corresponding to the Euclidean signature $(+,+,+,+,+)$; it is a real form of the Hopf algebra called $\breve{U}_q(so(5))$ in [16, Section 6.1.2]. As a $*$-algebra, $U_q(so(5))$ is generated by $\{K_i = K_i^*, K_i^{-1}, E_i, F_i := E_i^*\}_{i=1,2}$ ($i \to 3-i$ with respect to the notations of [16]), with relations:

$$[K_1, K_2] = 0 \ , \qquad K_i K_i^{-1} = K_i^{-1} K_i = 1 \ ,$$
$$[E_i, F_j] = \delta_{ij} \tfrac{K_j^2 - K_j^{-2}}{q^j - q^{-j}} \ ,$$
$$K_i E_i K_i^{-1} = q^i E_i \ , \qquad K_i E_j K_i^{-1} = q^{-1} E_j \ \text{ if } i \neq j \ ,$$

together with the ones obtained by conjugation and Serre relations, explicitly, given by

$$E_1 E_2^2 - (q^2 + q^{-2}) E_2 E_1 E_2 + E_2^2 E_1 = 0 \ , \tag{3.1a}$$
$$E_1^3 E_2 - (q^2 + 1 + q^{-2})(E_1^2 E_2 E_1 - E_1 E_2 E_1^2) - E_2 E_1^3 = 0 \ , \tag{3.1b}$$

together with their adjoints. These relations can be written in a more compact form by defining $[a, b]_q := q^2 ab - ba$. Then, (3.1) are equivalent to

$$[E_2, [E_1, E_2]_q]_q = 0 \ , \qquad [E_1, [E_1, [E_2, E_1]_q]_q] = 0 \ .$$

The Hopf algebra structure (Δ, ϵ, S) of $U_q(so(5))$ is given by:

$$\Delta K_i = K_i \otimes K_i \ , \quad \Delta E_i = E_i \otimes K_i + K_i^{-1} \otimes E_i \ ,$$
$$\epsilon(K_i) = 1 \ , \quad \epsilon(E_i) = 0 \ ,$$
$$S(K_i) = K_i^{-1} \ , \quad S(E_i) = -q^i E_i \ .$$

For each non negative n_1, n_2 such that $n_2 \in \frac{1}{2}\mathbb{N}$ and $n_2 - n_1 \in \mathbb{N}$ there is an irreducible representation of $U_q(so(5))$ whose representation space we denote $V_{(n_1,n_2)}$. We call it "the representation with highest weight (n_1, n_2)" since the highest weight vector is an eigenvector of K_1 and $K_1 K_2$ with eigenvalues q^{n_1} and q^{n_2}, respectively.

Irreducible representations with highest weight $(0, l)$ and $(\frac{1}{2}, l)$ (the ones that we need explicitly) can be found in [2] and are recalled presently. Let us use the shorthand notation $V_l := V_{(0,l)}$ if $l \in \mathbb{N}$ and $V_l := V_{(\frac{1}{2},l)}$ if $l \in \mathbb{N} + \frac{1}{2}$. The vector space V_l, for all $l \in \frac{1}{2}\mathbb{N}$, has orthonormal basis $|l, m_1, m_2; j\rangle$, where the labels (j, m_1, m_2) satisfy the following constraints. For $l \in \mathbb{N}$:

$$j = 0, 1, \ldots, l \ , \qquad j - |m_1| \in \mathbb{N} \ , \qquad l - j - |m_2| \in 2\mathbb{N} \ ,$$

while for $l \in \mathbb{N} + \frac{1}{2}$:

$$j = \tfrac{1}{2}, \tfrac{3}{2}, \ldots, l-1, l \ , \qquad j - |m_1| \in \mathbb{N} \ , \qquad l + \tfrac{1}{2} - j - |m_2| \in \mathbb{N} \ .$$

Notice that for any admissible (l, m_1, m_2, j) there exists a unique $\epsilon \in \{0, \pm\frac{1}{2}\}$ such that $l + \epsilon - j - m_2 \in 2\mathbb{N}$ (that is, $\epsilon = 0$ if $l \in \mathbb{N}$ and $\epsilon = \frac{1}{2}(-1)^{l+\frac{1}{2}-j-m_2}$ if $l \in \mathbb{N} + \frac{1}{2}$). We shall need the coefficients,

$$a_l(j, m_2) = \frac{1}{[2]} \sqrt{\frac{[l - j - m_2 + \epsilon][l + j + m_2 + 3 + \epsilon]}{[2(j + |\epsilon|) + 1][2(j - |\epsilon|) + 3]}} \ , \tag{3.2a}$$

$$b_l(j, m_2) = 2|\epsilon| \, \frac{\sqrt{[l - \epsilon(2j+1) - m_2 + 1][l - \epsilon(2j+1) + m_2 + 2]}}{[2j][2j+2]} \ , \tag{3.2b}$$

$$c_l(j,m_2) = \frac{(-1)^{2\epsilon}}{[2]}\sqrt{\frac{[l-j+m_2+2-\epsilon][l+j-m_2+1-\epsilon]}{[2(j+|\epsilon|)-1][2(j-|\epsilon|)+1]}} \ , \tag{3.2c}$$

where, as usual, $[z] := (q^z - q^{-z})/(q-q^{-1})$ denotes the q-analogue of $z \in \mathbb{C}$.

The $*$-representation $\sigma_l : U_q(so(5)) \to \mathrm{End}(V_l)$ is defined by the rules,

$$\begin{aligned}
\sigma_l(K_1)\,|l,m_1,m_2;j\rangle &= q^{m_1}\,|l,m_1,m_2;j\rangle \ , \\
\sigma_l(K_2)\,|l,m_1,m_2;j\rangle &= q^{m_2-m_1}\,|l,m_1,m_2;j\rangle \ , \\
\sigma_l(E_1)\,|l,m_1,m_2;j\rangle &= \sqrt{[j-m_1][j+m_1+1]}\,|l,m_1+1,m_2;j\rangle \ , \\
\sigma_l(E_2)\,|l,m_1,m_2;j\rangle &= \sqrt{[j-m_1+1][j-m_1+2]}\,a_l(j,m_2)\,|l,m_1-1,m_2+1;j+1\rangle \\
&\quad + \sqrt{[j+m_1][j-m_1+1]}\,b_l(j,m_2)\,|l,m_1-1,m_2+1;j\rangle \\
&\quad + \sqrt{[j+m_1][j+m_1-1]}\,c_l(j,m_2)\,|l,m_1-1,m_2+1;j-1\rangle \ .
\end{aligned}$$

When there is no risk of ambiguity the representation symbol σ_l will be suppressed.

For $l \in \mathbb{N}$ the representation σ_l is *real*. That is, there is an antilinear map $C : V_l \to V_l$, which satisfies $C^2 = 1$ and $C\sigma_l(h)C = \sigma_l(S(h)^*)$. This map is explicitly given by

$$C\,|l,m_1,m_2;j\rangle := (-q)^{m_1}q^{3m_2}\,|l,-m_1,-m_2;j\rangle \ . \tag{3.3}$$

The operator

$$\mathcal{C}_1 := q^{-1}K_1^2 + qK_1^{-2} + (q-q^{-1})^2E_1F_1 \ , \tag{3.4}$$

is a Casimir for the subalgebra generated by $(K_1, K_1^{-1}, E_1, F_1)$. For future reference, we note the action of $\mathcal{C}_1$ on a vector of V_l, with $l \in \frac{1}{2}\mathbb{N}$; it is

$$\mathcal{C}_1\,|l,m_1,m_2;j\rangle = (q^{2j+1} + q^{-2j-1})\,|l,m_1,m_2;j\rangle \ . \tag{3.5}$$

4 The quantum Euclidean 4-Sphere

Definition 4.1 ([14]). *We call* quantum Euclidean 4-sphere *the virtual space underlying the algebra* $\mathcal{A}(S^4_q)$ *generated by* $x_0 = x_0^*, x_i$ *and* x_i^* *(with* $i = 1,2$*), with commutation relations:*

$$\begin{aligned}
&x_ix_j = q^2x_jx_i \ , && \forall\ 0 \le i < j \le 2 \ , \\
&x_i^*x_j = q^2x_jx_i^* \ , && \forall\ i \neq j \ , \\
&[x_1^*, x_1] = (1-q^4)x_0^2 \ , && \\
&[x_2^*, x_2] = x_1^*x_1 - q^4x_1x_1^* \ , && \\
&x_0^2 + x_1x_1^* + x_2x_2^* = 1 \ . &&
\end{aligned}$$

The original notations of Fadeev-Reshetikhin-Takhtadzhyan [14, Eq. (1.14)] can be obtained by defining $x_1' := x_2^*$, $x_2' := x_1^*$, $x_3' := \sqrt{q(1+q^2)}\,x_0$, $x_4' := x_1$, $x_5' := x_2$ and $q' := q^2$. The notations in [17, Eq. (2.1)] can be obtained by the replacement $x_i \mapsto x_i^*$ and $q^2 \mapsto q^{-1}$.

In the next Propositions we summarize some well known facts.

Proposition 4.2. *The algebra $\mathcal{A}(S^4_q)$ is an $U_q(so(5))$-module $*$-algebra for the action given by:*

$$K_i \triangleright x_i = q x_i \ , i = 1, 2 \ ,$$

$$K_2 \triangleright x_1 = q^{-1} x_1 \ ,$$

$$E_1 \triangleright x_0 = q^{-1/2} x_1 \ , \qquad E_2 \triangleright x_1 = x_2 \ ,$$

$$F_1 \triangleright x_1 = q^{1/2}[2] x_0 \ , \quad F_1 \triangleright x_0 = -q^{-3/2} x_1^* \qquad F_2 \triangleright x_2 = x_1 \ ,$$

while $K_i \triangleright x_j = x_j$, $E_i \triangleright x_j = 0$ and $F_i \triangleright x_j = 0$ in all other cases.

Notice that the action on the x_i^*'s is determined by compatibility with the involution:

$$K_i \triangleright a^* = \{K_i^{-1} \triangleright a\}^* \ , \qquad E_1 \triangleright a^* = \{-q F_1 \triangleright a\}^* \ , \qquad E_2 \triangleright a^* = \{-q^2 F_2 \triangleright a\}^* \ .$$

Proof. The bijective linear map from the linear span of $\{x_i, x_i^*\}$ to the representation space V_1 defined (modulo a global proportionality constant) by

$$x_2 \mapsto |0,1;0\rangle \ , \ x_1 \mapsto |1,0;1\rangle \ , \ x_0 \mapsto (q[2])^{-1/2} |0,0;1\rangle \ , \ x_1^* \mapsto -q |-1,0;1\rangle \ , \ x_2^* \mapsto q^3 |0,-1;0\rangle \ ,$$

is a unitary equivalence of $U_q(so(5))$-modules (here unitary means that the real structure C on V_1 is implemented by the $*$ operation on x_i's). This guarantee that the free $*$-algebra $\mathbb{C}\langle x_i, x_i^* \rangle$ generated by $\{x_i, x_i^*\}$ is an $U_q(so(5))$-module $*$-algebra.

The degree ≤ 2 polynomials generating the ideal which defines $\mathcal{A}(S^4_q)$ span the real representations V_0 and $V_{(1,1)}$, inside the tensor product $V_1 \otimes V_1$. The quotient $*$-algebra of $\mathbb{C}\langle x_i, x_i^* \rangle$ by this ideal, $\mathcal{A}(S^4_q)$, is then an $U_q(so(5))$-module $*$-algebra. □

Proposition 4.3. *There is an isomorphism $\mathcal{A}(S^4_q) \simeq \bigoplus_{l \in \mathbb{N}} V_l$ of $U_q(so(5))$ left modules.*

Proof. A linear basis for $\mathcal{A}(S^4_q)$ is made of monomials $x_0^{n_0} x_1^{n_1} (x_1^*)^{n_2} x_2^{n_3}$ with $n_0, n_1, n_2 \in \mathbb{N}$, $n_3 \in \mathbb{Z}$ and with the notation $x_2^{n_3} := (x_2^*)^{|n_3|}$ if $n_3 < 0$. Using this basis one proves that a weight vector of $\mathcal{A}(S^4_q)$ is annihilated by both E_1 and E_2 if and only if it is of the form x_2^l, $l \in \mathbb{N}$. Thus, highest weight vectors are proportional to x_2^l and the algebra decomposes as multiplicity free direct sum of highest weight representations with weights $(0, l)$. □

The algebra $\mathcal{A}(S^4_q)$ has two inequivalent irreducible infinite dimensional representations. The representation space is the Hilbert space $\ell^2(\mathbb{N}^2)$ and the representations are given by

$$\begin{aligned} x_0 |k_1, k_2\rangle_\pm &:= \pm q^{2(k_1+k_2)} |k_1, k_2\rangle_\pm \ , \\ x_1 |k_1, k_2\rangle_\pm &:= q^{2k_2} \sqrt{1 - q^{4(k_1+1)}} \, |k_1 + 1, k_2\rangle_\pm \ , \\ x_2 |k_1, k_2\rangle_\pm &:= \sqrt{1 - q^{4(k_2+1)}} \, |k_1, k_2 + 1\rangle_\pm \ . \end{aligned} \tag{4.1}$$

The direct sum of these representations, with obvious grading γ and operator F given by $F |k_1, k_2\rangle_\pm := |k_1, k_2\rangle_\mp$, constitutes a 1-summable Fredholm module over $\mathcal{A}(S^4_q)$.

In the sequel we shall need both the quantum space $SU_q(2)$ as well as the equatorial Podleś sphere, whose algebras are in [20] and [18] respectively.

Definition 4.4. *The algebra* $\mathcal{A}(SU_q(2))$ *of polynomial functions on* $SU_q(2)$ *is the* $*$*-algebra generated by* α, β *and their adjoints, with relations:*

$$\beta\alpha = q\alpha\beta \ , \quad \beta^*\alpha = q\alpha\beta^* \ , \quad [\beta, \beta^*] = 0 \ , \quad \alpha\alpha^* + \beta\beta^* = 1 \ , \quad \alpha^*\alpha + q^2\beta^*\beta = 1 \ .$$

We call equatorial Podleś sphere *the virtual space underlying the* $*$*-algebra* $\mathcal{A}(S^2_q)$ *generated by* $A = A^*$, B *and* B^* *with relations:*

$$AB = q^2 BA \ , \qquad BB^* + A^2 = 1 \ , \qquad B^*B + q^4 A^2 = 1 \ .$$

Proposition 4.5. *There is a* $*$*-algebra morphism* $\varphi : \mathcal{A}(S^4_q) \to \mathcal{A}(SU_q(2)) \otimes \mathcal{A}(S^2_q)$ *defined by:*

$$\begin{aligned} \varphi(x_0) &= -(\alpha\beta + \beta^*\alpha^*) \otimes A \ , \\ \varphi(x_1) &= \big(-\alpha^2 + (\beta^*)^2\big) \otimes A \ , \\ \varphi(x_2) &= 1 \otimes B \ . \end{aligned} \tag{4.2}$$

Proof. One proves by direct computation that the five $\varphi(x_i), \varphi(x_i)^*$ satisfy all the defining relations of $\mathcal{A}(S^4_q)$. □

5 The modules of chiral spinors

We apply the general theory of Section 2.2, to the case $\mathcal{A} = \mathcal{A}(S^4_q)$ and $\mathcal{U} = U_q(so(5))$. Recall that in this case $\kappa(a) = K_1^8 K_2^6 \triangleright a$ is the modular automorphism. We shall use the notations of Section 3 for the irreducible representations (V_l, σ_l) of $U_q(so(5))$.

By Proposition 4.3 we have the equivalence $\mathcal{A}(S^4_q) \simeq \bigoplus_{l\in\mathbb{N}} V_l$ as left $U_q(so(5))$-modules. Using Lemma 2.5 for $N = 1$, we deduce that on the vector space $\bigoplus_{l\in\mathbb{N}} V_l$ there exists at least one $*$-representation of the crossed product $\mathcal{A}(S^4_q) \rtimes U_q(so(5))$ that extends the $*$-representation $\bigoplus_{l\in\mathbb{N}} \sigma_l$ of $U_q(so(5))$.

Let $e \in \mathrm{Mat}_4(\mathcal{A}(S^4_q))$ be the following idempotent:

$$e := \frac{1}{2}\begin{pmatrix} 1 + x_0 & q^3 x_2 & -q x_1 & 0 \\ q^{-3} x_2^* & 1 - q^2 x_0 & 0 & q^3 x_1 \\ -q^{-1} x_1^* & 0 & 1 - q^2 x_0 & q^3 x_2 \\ 0 & q x_1^* & q^{-3} x_2^* & 1 + q^4 x_0 \end{pmatrix} . \tag{5.1}$$

By direct computation one proves that $K_1^8 K_2^6 \triangleright e^* = e = e^2$ and then, by Lemma 2.7, e defines an orthogonal projection π, by Equation (2.8), on the linear space $\mathcal{A}(S^4_q)^4$ with inner product (2.5). Next, let $\sigma : U_q(so(5)) \to \mathrm{Mat}_4(\mathbb{C})$ be the $*$-representation defined by

$$\sigma(K_1) = \begin{pmatrix} q^{1/2} & 0 & 0 & 0 \\ 0 & q^{1/2} & 0 & 0 \\ 0 & 0 & q^{-1/2} & 0 \\ 0 & 0 & 0 & q^{-1/2} \end{pmatrix} , \qquad \sigma(K_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q^{-1} & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{5.2a}$$

$$\sigma(E_1) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \qquad \sigma(E_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{5.2b}$$

Again, by direct computation one proves that:

$$K_i \triangleright e = \sigma(K_i)\, e\, \sigma(K_i)^{-1} \ , \tag{5.3a}$$

$$E_i \triangleright e = \sigma(F_i)\, e\, \sigma(K_i)^{-1} - q^{-i}\sigma(K_i)^{-1} e\, \sigma(F_i) \ . \tag{5.3b}$$

Since $\sigma(K_i) = \sigma(K_i)^t$ and $\sigma(F_i) = \sigma(E_i)^t$, we conclude that condition (2.9) is satisfied and that π and $\pi^\perp = 1 - \pi$ project $\mathcal{A}(S^4_q)^4$ onto sub $*$-representations of $\mathcal{A}(S^4_q) \rtimes U_q(so(5))$.

We state the main Proposition of this section.

Proposition 5.1. *There exists two inequivalent representations of the crossed product algebra $\mathcal{A}(S^4_q) \rtimes U_q(so(5))$ on $\bigoplus_{l\in\mathbb{N}+\frac{1}{2}} V_l$ that extend the representation $\bigoplus_{l\in\mathbb{N}+\frac{1}{2}} \sigma_l$ of $U_q(so(5))$.*

The proof is in two steps. We first prove (in Lemma 5.2) that $\pi(\mathcal{A}(S^4_q)^4)$ and $\pi^\perp(\mathcal{A}(S^4_q)^4)$ are not equivalent as representations of the algebra $\mathcal{A}(S^4_q)$. Then we prove (in Lemma 5.3) that as $U_q(so(5))$ representations they are both equivalent to $\bigoplus_{l\in\mathbb{N}+\frac{1}{2}} V_l$.

Lemma 5.2. *The idempotent e in (5.1) splits $\mathcal{A}(S^4_q)^4$ into two* inequivalent *$*$-representations of the crossed product algebra $\mathcal{A}(S^4_q) \rtimes U_q(so(5))$.*

Proof. To prove the statement we apply Lemma 2.9. We use the Fredholm module associated to the representation on $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ defined by Equation (4.1). One has

$$\begin{aligned}\mathrm{ch}^F([e]) &= \tfrac{1}{2}\mathrm{Trace}_{\ell^2(\mathbb{N})\otimes\mathbb{C}^8}(\gamma F[F,e]) \\ &= \tfrac{1}{4}(1-q^2)^2\mathrm{Trace}_{\ell^2(\mathbb{N})\otimes\mathbb{C}^2}(\gamma F[F,x_0]) \\ &= (1-q^2)^2 \textstyle\sum_{k_1,k_2\in\mathbb{N}} q^{2(k_1+k_2)} = 1 \ .\end{aligned}$$

The statement of the Proposition 5.1 follows from the obvious observation that if the two representations of the crossed product algebra were equivalent, their restrictions to representations of $\mathcal{A}(S^4_q)$ would be equivalent too. □

Lemma 5.3. $\pi(\mathcal{A}(S^4_q)^4) \simeq \pi^\perp(\mathcal{A}(S^4_q)^4) \simeq \bigoplus_{l\in\mathbb{N}+\frac{1}{2}} V_l$ *as $U_q(so(5))$ representations.*

Proof. In this proof, '$\simeq$' always means equivalence of representations of $U_q(so(5))$.

Since σ in (5.2) is unitary equivalent to the spin representation $V_{1/2}$, the representation of $U_q(so(5))$ on $\mathcal{A}(S^4_q)^4$ is the Hopf tensor product of the representation over $\mathcal{A}(S^4_q)$ with the representation $V_{1/2}$. From $\mathcal{A}(S^4_q) \simeq \bigoplus_{l\in\mathbb{N}} V_l$ and from the decomposition $V_l \otimes V_{1/2} \simeq V_{l-\frac{1}{2}} \oplus V_{l+\frac{1}{2}}$ for all $l \in \{1,2,3,\dots\}$, we deduce that $\mathcal{A}(S^4_q)^4 \simeq \bigoplus_{l\in\mathbb{N}+\frac{1}{2}}(V_l \oplus V_l)$ and then,

$$\pi(\mathcal{A}(S^4_q)^4) \simeq \bigoplus\nolimits_{l\in\mathbb{N}+\frac{1}{2}} m_l^+ V_l \ , \qquad \pi^\perp(\mathcal{A}(S^4_q)^4) \simeq \bigoplus\nolimits_{l\in\mathbb{N}+\frac{1}{2}} m_l^- V_l \ ,$$

with multiplicities $m_l^\pm$ to be determined, such that $m_l^+ + m_l^- = 2$. For $l \in \mathbb{N} + \frac{1}{2}$, the vectors

$$v_l^\pm := x_2^{l-\frac{1}{2}}(1 \pm x_0, \pm q^3 x_2, \mp q x_1, 0) \ .$$

are highest weight vectors, being annihilated by both E_1 and E_2, and have weight $(\frac{1}{2}, l)$. Furthermore, $v_l^+(1-e) = v_l^- e = 0$. Thus, $v_l^+ \in \pi(\mathcal{A}(S^4_q)^4)$ and $v_l^- \in \pi^\perp(\mathcal{A}(S^4_q)^4)$.

Then in both modules $\pi(\mathcal{A}(S^4_q)^4)$ and $\pi^\perp(\mathcal{A}(S^4_q)^4)$ each representation V_l, $l \in \mathbb{N} + \frac{1}{2}$, appears with multiplicity $m_l^\pm \geq 1$. Since $m_l^+ + m_l^- = 2$, we deduce that $m_l^\pm = 1$ for all $l \in \mathbb{N} + \frac{1}{2}$. □

6 Equivariant representations of $\mathcal{A}(S^4_q)$

Next, we construct $U_q(so(5))$-equivariant representations of $\mathcal{A}(S^4_q)$ which classically correspond to the left regular and chiral spinor representations. The representation spaces will be (the closure of) $\bigoplus_{l\in\mathbb{N}} V_l$ and $\bigoplus_{l\in\mathbb{N}+\frac{1}{2}} V_l$.

Equivariance of a representation means that it is a representation of the crossed product algebra $\mathcal{A}(S^4_q) \rtimes U_q(so(5))$. The latter is defined by the crossed relations $ha = (h_{(1)} \triangleright a)h_{(2)}$ for all $a \in \mathcal{A}(S^4_q)$ and $h \in U_q(so(5))$; explicitly, the relations between the generators read:

$$\begin{aligned}
&[K_1, x_0] = 0 \,, && K_1x_1 = qx_1K_1 \,, && K_1x_2 = x_2K_1 \,, \\
&[K_2, x_0] = 0 \,, && K_2x_1 = q^{-1}x_1K_2 \,, && K_2x_2 = qx_2K_2 \,, \\
&[E_1, x_0] = q^{-1/2}x_1K_1 \,, && E_1x_1 = q^{-1}x_1E_1 \,, && E_1x_2 = x_2E_1 \,, \\
&[F_1, x_0] = -q^{-1/2}K_1x_1^* \,, && F_1x_1 = q^{-1}x_1F_1 + q^{1/2}[2]x_0K_1 \,, && F_1x_2 = x_2F_1 \,, \\
&[E_2, x_0] = 0 \,, && E_2x_1 = qx_1E_2 + x_2K_2 \,, && E_2x_2 = q^{-1}x_2E_2 \,, \\
&[F_2, x_0] = 0 \,, && F_2x_1 = qx_1F_2 \,, && F_2x_2 = q^{-1}x_2F_2 + x_1K_2 \,.
\end{aligned} \tag{6.1}$$

In previous Section we proved that on $\bigoplus_{l\in\mathbb{N}} V_l$ there is at least an equivariant representation, the left regular one, and that on $\bigoplus_{l\in\mathbb{N}+\frac{1}{2}} V_l$ there are at least two equivariant representations, corresponding to the projective modules $\mathcal{A}(S^4_q)^4e$ and $\mathcal{A}(S^4_q)^4(1-e)$. In this section we'll prove that on such spaces there are no other equivariant representations besides the ones just mentioned.

Let us denote with $|l, m_1, m_2; j\rangle$ the basis of the representation space V_l of $U_q(so(5))$ as discussed in Section 3. From the first two lines of (6.1) we deduce that

$$x_0\,|l, m_1, m_2; j\rangle = \sum_{l',j'} A^{m_1,m_2}_{j,j',l,l'}\,|l', m_1, m_2; j'\rangle \;, \tag{6.2a}$$

$$x_1\,|l, m_1, m_2; j\rangle = \sum_{l',j'} B^{m_1,m_2}_{j,j',l,l'}\,|l', m_1+1, m_2; j'\rangle \;, \tag{6.2b}$$

$$x_2\,|l, m_1, m_2; j\rangle = \sum_{l',j'} C^{m_1,m_2}_{j,j',l,l'}\,|l', m_1, m_2+1; j'\rangle \;, \tag{6.2c}$$

with coefficients to be determined. Notice that from the crossed relations

$$\begin{aligned}
x_1\,|l, m_1, m_2; j\rangle &= (F_2x_2 - q^{-1}x_2F_2)K_2^{-1}\,|l, m_1, m_2; j\rangle \;, \\
x_0\,|l, m_1, m_2; j\rangle &= q^{-1/2}[2]^{-1}(F_1x_1 - q^{-1}x_1F_1)K_1^{-1}\,|l, m_1, m_2; j\rangle \;,
\end{aligned}$$

the matrix coefficients of x_0 and x_1 can be expressed in term of the coefficients of x_2.

Lemma 6.1. *Let $k \in \mathbb{N}$. The following formulæ hold:*

$$F_1^k\,|l, m_1, m_2; j\rangle = \begin{cases} = 0 & \text{if } k > j + m_1 \\ \neq 0 & \text{if } k \leq j + m_1 \end{cases} \tag{6.3a}$$

$$E_1^k\,|l, m_1, m_2; j\rangle = \begin{cases} = 0 & \text{if } k > j - m_1 \\ \neq 0 & \text{if } k \leq j - m_1 \end{cases} \;. \tag{6.3b}$$

Proof. By direct computation:

$$F_1^k\,|l,m_1,m_2;j\rangle = \sqrt{[j+m_1][j+m_1-1]\dots[j+m_1-k+1]}\times$$
$$\times\sqrt{[j-m_1+1][j-m_1+2]\dots[j-m_1+k]}\;|l,m_1-k,m_2;j\rangle\ .$$

The second square root is always different from zero since the q-analogues are in increasing order and $j-m_1+1\geq 1$. In the first square root q-analogues are in decreasing order and are all different from zero if and only if $j+m_1-k+1\geq 1$. This proves equation (6.3a).

In the same way one establishes (6.3b) by computing that

$$E_1^k\,|l,m_1,m_2;j\rangle = \sqrt{[j-m_1][j-m_1-1]\dots[j-m_1-k+1]}\times$$
$$\times\sqrt{[j+m_1+1][j+m_1+2]\dots[j+m_1+k]}\;|l,m_1+k,m_2;j\rangle\ .$$

□

Lemma 6.2. *The coefficients in (6.2) satisfy:*

$$A^{m_1,m_2}_{j,j',l,l'} = B^{m_1,m_2}_{j,j',l,l'} = 0 \ \text{ if } \ |j-j'|>1\ , \qquad C^{m_1,m_2}_{j,j',l,l'} = 0 \ \text{ if } \ j'\neq j\ .$$

Proof. From (6.1), (6.3a) and (6.3b) we derive:

$$E_1^{j-m_1+1}x_1\,|l,m_1,m_2;j\rangle = q^{-j+m_1-1}x_1E_1^{j-m_1+1}\,|l,m_1,m_2;j\rangle = 0\ ,$$
$$F_1^{j'+m_1+2}x_1^*\,|l',m_1+1,m_2;j'\rangle = q^{j'+m_1+2}x_1^*F_1^{j'+m_1+2}\,|l',m_1+1,m_2;j'\rangle = 0\ .$$

We expand the left hand sides and use the independence of the vectors $E_1^{j-m_1+1}\,|l',m_1+1,m_2;j'\rangle$ and $F_1^{j'+m_1+2}\,|l,m_1,m_2;j\rangle$ to arrive at the conditions:

$$B^{m_1,m_2}_{j,j',l,l'}\Big\{E_1^{j-m_1+1}\,|l',m_1+1,m_2;j'\rangle\Big\} = 0\ ,$$
$$\bar{B}^{m_1,m_2}_{j,j',l,l'}\Big\{F_1^{j'+m_1+2}\,|l,m_1,m_2;j\rangle\Big\} = 0\ .$$

By (6.3b) the graph parenthesis in the first line is different from zero if $j-m_1+1\leq j'-m_1-1$, i.e. $B^{m_1,m_2}_{j,j',l,l'}$ must be zero if $j'\geq j+2$. By (6.3a) the graph parenthesis in the second line is different from zero if $j'+m_1+2\leq j+m_1$, i.e. $\bar{B}^{m_1,m_2}_{j,j',l,l'}$ must be zero if $j'\leq j-2$. This proves 1/3 of the statement

$$B^{m_1,m_2}_{j,j',l,l'} = 0 \quad \forall\, j'\notin\{j-1,j,j+1\}\ .$$

A similar argument applies to x_0. From the coproduct of E_1^n we deduce:

$$E_1^nx_0 = \sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}(E_1^k\triangleright x_0)E_1^{n-k}K_1^k = x_0E_1^n - [n]q^{-1/2}x_1E_1^{n-1}K_1\ .$$

This implies that $E_1^{j-m_1+2}x_0\,|l,m_1,m_2;j\rangle = 0$ and $F_1^{j'+m_1+2}x_0\,|l',m_1,m_2;j\rangle = 0$. From these conditions we deduce that also x_0 shift j by $\{0,\pm1\}$ only.

Finally, let $\mathcal{C}_1$ be the Casimir element in equation (3.4). Then $[\mathcal{C}_1,x_2]=0$ and from (3.5) we deduce that x_2 is diagonal on the index j. □

Lemma 6.3. *The coefficients in (6.2c) satisfy*

$$C^{m_1,m_2}_{j,j',l,l'} = 0 \ \text{ if } \ |l-l'|>1 \ \text{ or if } \ |l-l'|=0 \ \text{ and } \ l\in\mathbb{N}\ .$$

Proof. The elements $\{x_i, x_i^*\}$ are a basis of the irreducible representation V_1. Covariance of the action tells that $x_i\,|l,m_1,m_2;j\rangle$ and $x_i^*\,|l,m_1,m_2;j\rangle$ are a basis of the tensor representation $V_1\otimes V_l$. Equations (14–15) in Chapter 7 of [16] tells that $V_1\otimes V_l\simeq V_{l-1}\oplus V_{l+1}$ if $l\in\mathbb{N}$ and that $V_1\otimes V_l\simeq V_{l-1}\oplus V_l\oplus V_{l+1}$ if $l\in\mathbb{N}+\frac{1}{2}$ (with V_{l-1} omitted if $l-1<0$). This Clebsh-Gordan decomposition tells that $x_2\,|l,m_1,m_2;j\rangle$ is in the linear span of the basis vectors $|l',m_1,m_2+1;j\rangle$ with $l'-l=\pm1$ if $l\in\mathbb{N}$ or with $l'-l=0,\pm1$ if $l\in\mathbb{N}+\frac{1}{2}$. This concludes the proof of the Lemma. □

6.1 Computing the coefficients of x_2

From Lemma 6.3, we have to consider only the cases $j'=j$, $|l'-l|\leq 1$ if $l\in\mathbb{N}+\frac{1}{2}$ or $|l'-l|=1$ if $l\in\mathbb{N}$. The condition $[E_1,x_2]=0$ implies that $C^{m_1,m_2}_{j,j,l,l'}=:C^{m_2}_{j,l,l'}$ is independent on m_1. Equations $(E_2x_2-q^{-1}x_2E_2)\,|l,-j,m_2;j\rangle=0$ and $(F_2x_2^*-qx_2^*F_2)\,|l',j,m_2+1;j\rangle=0$ imply, respectively:

$$\begin{aligned}
C^{m_2}_{j,l,l'}\sqrt{[l'-j-m_2-1+\epsilon'][l'+j+m_2+4+\epsilon']} &= C^{m_2+1}_{j+1,l,l'}q^{-1}\sqrt{[l-j-m_2+\epsilon][l+j+m_2+3+\epsilon]}\ ,\\
C^{m_2}_{j,l,l'}\sqrt{[l+j-m_2+3-\epsilon][l-j+m_2-\epsilon]} &= C^{m_2-1}_{j+1,l,l'}q\sqrt{[l'+j-m_2+2-\epsilon'][l'-j+m_2+1-\epsilon']}\ ,
\end{aligned}$$

with $\epsilon,\epsilon'\in\{0,\pm\frac{1}{2}\}$ determined by the conditions $l+\epsilon-j-m_2\in 2\mathbb{N}$ and $l'-\epsilon'-j-m_2\in 2\mathbb{N}$. Notice that if $l'-l\in 2\mathbb{N}+1$ then $\epsilon'=\epsilon$, while if $l'-l\in 2\mathbb{N}$ then $\epsilon'=-\epsilon$. Looking at the cases $l'-l=\pm1$, we deduce that

$$\frac{q^{-\frac{1}{2}(j+m_2)}}{\sqrt{[l+j+m_2+3+\epsilon]}}\,C^{m_2}_{j,l,l+1} \qquad\text{and}\qquad \frac{q^{-\frac{1}{2}(j+m_2)}}{\sqrt{[l-j-m_2+\epsilon]}}\,C^{m_2}_{j,l,l-1}$$

depend on $j+m_2$ only through their parity (i.e. they depend only on the value of ϵ). Similarly,

$$\frac{q^{\frac{1}{2}(j-m_2)}}{\sqrt{[l-j+m_2+2-\epsilon]}}\,C^{m_2}_{j,l,l+1} \qquad\text{and}\qquad \frac{q^{\frac{1}{2}(j-m_2)}}{\sqrt{[l+j-m_2+1-\epsilon]}}\,C^{m_2}_{j,l,l-1}$$

depend on $j-m_2$ only through their parity. Combining these informations, we deduce that the following elements do not depend on the exact value of j, m_2, but only on the value of ϵ,

$$\begin{aligned}
\frac{q^{-m_2}}{\sqrt{[l+j+m_2+3+\epsilon][l-j+m_2+2-\epsilon]}}\,C^{m_2}_{j,l,l+1} &=: C_{l,l+1}(\epsilon)\ ,\\
\frac{q^{-m_2}}{\sqrt{[l-j-m_2+\epsilon][l+j-m_2+1-\epsilon]}}\,C^{m_2}_{j,l,l-1} &=: C_{l,l-1}(\epsilon)\ .
\end{aligned}$$

If $l\in\mathbb{N}$ there are no other coefficients $C^{m_2}_{j,l,l'}$ to compute. If $l\notin\mathbb{N}$, we have to compute also $C^{m_2}_{j,l,l}$. In this case $\epsilon'=-\epsilon$ and we get:

$$C^{m_2}_{j,l,l}\sqrt{[l-j-m_2-1-\epsilon][l+j+m_2+4-\epsilon]} = C^{m_2+1}_{j+1,l,l}q^{-1}\sqrt{[l-j-m_2+\epsilon][l+j+m_2+3+\epsilon]}\ ,$$

$$C^{m_2}_{j,l,l}\sqrt{[l+j-m_2+3-\epsilon][l-j+m_2-\epsilon]} = C^{m_2-1}_{j+1,l,l}q\sqrt{[l+j-m_2+2+\epsilon][l-j+m_2+1+\epsilon]}\ .$$

Again, looking at the two cases $\epsilon=\pm\frac{1}{2}$ we deduce that

$$\frac{q^{-\frac{1}{2}(j+m_2)}}{\sqrt{[l+\frac{1}{2}-j-m_2]}}\,C^{m_2}_{j,l,l}\quad\text{if } \epsilon=\tfrac{1}{2}\quad\text{and}\quad\frac{q^{-\frac{1}{2}(j+m_2)}}{\sqrt{[l+\frac{1}{2}+j+m_2+2]}}\,C^{m_2}_{j,l,l}\quad\text{if } \epsilon=-\tfrac{1}{2}$$

do not depend on $j+m_2$ (this time ϵ is fixed, then the parity of $j+m_2$ is fixed). Similarly,

$$\frac{q^{\frac{1}{2}(j-m_2)}}{\sqrt{[l+\frac{1}{2}-j+m_2+1]}}\,C^{m_2}_{j,l,l}\quad\text{if } \epsilon=\tfrac{1}{2}\quad\text{and}\quad\frac{q^{\frac{1}{2}(j-m_2)}}{\sqrt{[l+\frac{1}{2}+j-m_2+1]}}\,C^{m_2}_{j,l,l}\quad\text{if } \epsilon=-\tfrac{1}{2}$$

do not depend on $j-m_2$. Combining these informations, we deduce that the following element do not depend on the exact value of j, m_2, but only on the value of ϵ:

$$\frac{q^{-m_2}}{\sqrt{[l-2\epsilon j-m_2+1-\epsilon][l-2\epsilon j+m_2+2-\epsilon]}}\,C^{m_2}_{j,l,l} =: C_{l,l}(\epsilon)\ .$$

The denominator of the left hand side is just $[2j][2j+2]b_l(j,m_2)$ with b_l the coefficient in equation (3.2b). The formula $C^{m_2}_{j,l,l} = q^{m_2}[2j][2j+2]b_l(j,m_2)C_{l,l}(\epsilon)$ is valid for all l, since $b_l(j,m_2)$ vanish for l integer.

Summarizing, we find that

$$C^{m_2}_{j,l,l+1} = q^{m_2}\sqrt{[l+j+m_2+3+\epsilon][l-j+m_2+2-\epsilon]}\,C_{l,l+1}(\epsilon)\ , \tag{6.4a}$$

$$C^{m_2}_{j,l,l} = q^{m_2}[2j][2j+2]b_l(j,m_2)\,C_{l,l}(\epsilon)\ , \tag{6.4b}$$

$$C^{m_2}_{j,l,l-1} = q^{m_2}\sqrt{[l-j-m_2+\epsilon][l+j-m_2+1-\epsilon]}\,C_{l,l-1}(\epsilon)\ , \tag{6.4c}$$

with coefficients $C_{l,l'}(\epsilon)$ to be determined.

6.2 Computing the coefficients of x_1

From Lemma 6.2, we have to consider only the three cases $j'=j,j\pm1$. Using equation $E_1x_1=q^{-1}x_1E_1$ we get,

$$\frac{q^{-m_1}}{\sqrt{[j+m_1+1][j+m_1+2]}}\,B^{m_1,m_2}_{j,j+1,l,l'} = \frac{q^{-m_1-1}}{\sqrt{[j+m_1+2][j+m_1+3]}}\,B^{m_1+1,m_2}_{j,j+1,l,l'}\ ,$$

$$\frac{q^{-m_1}}{\sqrt{[j-m_1][j+m_1+1]}}\,B^{m_1,m_2}_{j,j,l,l'} = \frac{q^{-m_1-1}}{\sqrt{[j-m_1-1][j+m_1+2]}}\,B^{m_1+1,m_2}_{j,j,l,l'}\ ,$$

$$\frac{q^{-m_1}}{\sqrt{[j-m_1][j-m_1-1]}}\,B^{m_1,m_2}_{j,j-1,l,l'} = \frac{q^{-m_1-1}}{\sqrt{[j-m_1-1][j-m_1-2]}}\,B^{m_1+1,m_2}_{j,j-1,l,l'}\ .$$

We see that the left hand sides of these three equations are independent of m_1, and call:

$$B^{m_1,m_2}_{j,j+1,l,l'} =: q^{m_1}\sqrt{[j+m_1+1][j+m_1+2]}\,B^{m_2}_{j,j+1,l,l'}\ , \tag{6.5a}$$

$$B^{m_1,m_2}_{j,j,l,l'} =: q^{m_1}\sqrt{[j-m_1][j+m_1+1]}\, B^{m_2}_{j,j,l,l'} \ , \tag{6.5b}$$
$$B^{m_1,m_2}_{j,j-1,l,l'} =: q^{m_1}\sqrt{[j-m_1][j-m_1-1]}\, B^{m_2}_{j,j-1,l,l'} \ . \tag{6.5c}$$

Imposing the condition $x_1K_2 = F_2x_2 - q^{-1}x_2F_2$ on the subspace spanned by $|l,j,m_2;j\rangle$ (so $m_1=j$ and $B^{m_1,m_2}_{j,j,l,l'} = B^{m_1,m_2}_{j,j-1,l,l'} = 0$ on this subspace) we get:

$$q^{m_2}B^{m_2}_{j,j+1,l,l'} = c_{l'}(j+1,m_2)C^{m_2}_{j,l,l'} - q^{-1}c_l(j+1,m_2-1)C^{m_2-1}_{j+1,l,l'} \ .$$

From this we deduce that, that since coefficients $C^{m_2}_{j,l,l'}$ vanishes for $|l-l'|>1$, also $B^{m_2}_{j,j+1,l,l'}$ is zero in these cases. In the remaining three cases $l'=l,l\pm1$, using equation (6.4) we get:

$$B^{m_2}_{j,j+1,l,l+1} = (-1)^{2\epsilon}q^{l-j+m_2-\epsilon}\sqrt{\frac{[l+j+m_2+3+\epsilon][l+j-m_2+3-\epsilon]}{[2(j+|\epsilon|)+1][2(j-|\epsilon|)+3]}}\, C_{l,l+1}(\epsilon) \ , \tag{6.6a}$$
$$B^{m_2}_{j,j+1,l,l} = -2\epsilon q^{2\epsilon\, l-j+m_2-2+3\epsilon}\frac{\sqrt{[l+\frac{1}{2}+j-2\epsilon\, m_2+2][l+\frac{1}{2}-j-2\epsilon\, m_2]}}{[2j+2]}\, C_{l,l}(\epsilon) \ , \tag{6.6b}$$
$$B^{m_2}_{j,j+1,l,l-1} = (-1)^{2\epsilon+1}q^{-l-j+m_2-3+\epsilon}\sqrt{\frac{[l-j+m_2-\epsilon][l-j-m_2+\epsilon]}{[2(j+|\epsilon|)+1][2(j-|\epsilon|)+3]}}\, C_{l,l-1}(\epsilon) \ . \tag{6.6c}$$

Imposing $qx_1^*K_2 = x_2^*E_2 - q^{-1}E_2x_2^*$ on the subspace spanned by $|l',-j+1,m_2;j-1\rangle$ (so $B^{m_1,m_2}_{j,j,l,l'} = B^{m_1,m_2}_{j,j+1,l,l'} = 0$ on this subspace) we get:

$$q^{m_2}B^{m_2}_{j,j-1,l,l'} = a_{l'}(j-1,m_2)C^{m_2}_{j,l,l'} - q^{-1}a_l(j-1,m_2-1)C^{m_2-1}_{j-1,l,l'} \ .$$

We deduce that $B^{m_2}_{j,j-1,l,l'}$ vanishes if $|l-l'|>1$, while in the three remaining cases $l'=l,l\pm1$ using equation (6.4) we get:

$$B^{m_2}_{j,j-1,l,l+1} = q^{l+j+m_2+1+\epsilon}\sqrt{\frac{[l-j-m_2+2+\epsilon][l-j+m_2+2-\epsilon]}{[2(j+|\epsilon|)-1][2(j-|\epsilon|)+1]}}\, C_{l,l+1}(\epsilon) \ , \tag{6.7a}$$
$$B^{m_2}_{j,j-1,l,l} = -2\epsilon q^{-2\epsilon\, l+j+m_2-1-3\epsilon}\frac{\sqrt{[l+\frac{1}{2}+j+2\epsilon\, m_2+1][l+\frac{1}{2}-j+2\epsilon\, m_2+1]}}{[2j]}\, C_{l,l}(\epsilon) \ , \tag{6.7b}$$
$$B^{m_2}_{j,j-1,l,l-1} = -q^{-l+j+m_2-2-\epsilon}\sqrt{\frac{[l+j+m_2+1+\epsilon][l+j-m_2+1-\epsilon]}{[2(j+|\epsilon|)-1][2(j-|\epsilon|)+1]}}\, C_{l,l-1}(\epsilon) \ . \tag{6.7c}$$

Moreover, the condition $\langle l',j,m_2;j|x_1K_2+q^{-1}x_2F_2-F_2x_2|l,j-1,m_2;j\rangle = 0$ implies that

$$q^{m_2}B^{m_2}_{j,j,l,l'} = b_{l'}(j,m_2)C^{m_2}_{j,l,l'} - q^{-1}b_l(j,m_2-1)C^{m_2-1}_{j,l,l'} \ . \tag{6.8}$$

A further elaboration on these coefficients is postponed to after the following section.

6.3 Computing the coefficients of x_0

The condition $q^{1/2}[2]x_0K_1 = F_1x_1 - q^{-1}x_1F_1$ implies:

$$q^{m_1+\frac{1}{2}}[2]A^{m_1,m_2}_{j,j',l,l'} = \sqrt{[j'-m_1][j'+m_1+1]}\, B^{m_1,m_2}_{j,j',l,l'} - q^{-1}\sqrt{[j+m_1][j-m_1+1]}\, B^{m_1-1,m_2}_{j,j',l,l'} \ .$$

In the three non-trivial cases $j'-j = 1, 0, -1$, using (6.5), we get:

$$A^{m_1,m_2}_{j,j+1,l,l'} = q^{j+m_1-\frac{1}{2}}\sqrt{[j+m_1+1][j-m_1+1]}\, B^{m_2}_{j,j+1,l,l'} \ , \tag{6.9a}$$

$$A^{m_1,m_2}_{j,j,l,l'} = [2]^{-1}q^{-2-\frac{1}{2}}\big(q^{j+m_1+1}[2][j-m_1] - [2j]\big)B^{m_2}_{j,j,l,l'} \ , \tag{6.9b}$$

$$A^{m_1,m_2}_{j,j-1,l,l'} = -q^{-j+m_1-1-\frac{1}{2}}\sqrt{[j+m_1][j-m_1]}\, B^{m_2}_{j,j-1,l,l'} \ . \tag{6.9c}$$

The hermiticity condition $x_0 = x_0^*$ means that $A^{m_1,m_2}_{j,j+1,l,l'} = \bar{A}^{m_1,m_2}_{j+1,j,l'l}$ and $A^{m_1,m_2}_{j,j,l,l'} = \bar{A}^{m_1,m_2}_{j,j,l'l}$. Thus, from (6.9) follows that:

$$B^{m_2}_{j+1,j,l'l} = -q^{2j+2}\bar{B}^{m_2}_{j,j+1,l,l'} \ , \qquad B^{m_2}_{j,j,l,l'} = \bar{B}^{m_2}_{j,j,l'l} \ .$$

Using (6.6), the first equation turns out to be equivalent to the following conditions:

$$C_{l+1,l}(\epsilon) = (-1)^{2\epsilon}q^{2l+4}\bar{C}_{l,l+1}(\epsilon) \ , \qquad C_{l,l}(\epsilon) = \bar{C}_{l,l}(-\epsilon) \ . \tag{6.10a}$$

The second of equation together with (6.8) implies:

$$b_{l'}(j,m_2)C^{m_2}_{j,l,l'} - q^{-1}b_l(j,m_2-1)C^{m_2-1}_{j,l,l'} = b_l(j,m_2)C^{m_2}_{j,l'l} - q^{-1}b_{l'}(j,m_2-1)C^{m_2-1}_{j,l'l} \ .$$

That is, using (6.4):

$$C_{l,l+1}(\epsilon) = C_{l,l+1}(-\epsilon) \ , \qquad C_{l,l}(\epsilon) = \bar{C}_{l,l}(\epsilon) \ . \tag{6.10b}$$

6.4 Again the coefficients of x_1

Now, using (6.10) together with (6.8) we are able to compute the last coefficients. Notice that from (3.2b) the coefficients b_l vanish if $\epsilon = 0$ (i.e. in the left regular representation), and then from (6.8) $B^{m_2}_{j,j,l,l'}$ vanish too if $\epsilon = 0$. Moreover, from Lemma 6.2 $B^{m_2}_{j,j,l,l'}$ vanish also if $|l-l'| > 1$. In the three cases $l' = l, l\pm 1$, using equation (6.4) we get:

$$B^{m_2}_{j,j,l,l+1} = 2|\epsilon|[2]q^{l+m_2+1+\epsilon(2j+1)}\frac{\sqrt{[l+2\epsilon\, j-m_2+2+\epsilon][l-2\epsilon\, j+m_2+2-\epsilon]}}{[2j][2j+2]}C_{l,l+1}(\epsilon) \ ,$$

$$\begin{aligned} B^{m_2}_{j,j,l,l} &= \frac{2|\epsilon|}{[2j][2j+2]}\Big\{[l-\epsilon(2j+1)-m_2+1][l-\epsilon(2j+1)+m_2+2]+ \\ &\qquad - q^{-2}[l+\epsilon(2j+1)-m_2+2][l+\epsilon(2j+1)+m_2+1]\Big\} \\ &= -\frac{2|\epsilon|}{[2j][2j+2]}\frac{q^{-2\epsilon(2j+1)}[2l+4] - q^{2\epsilon(2j+1)}[2l+2] - [2]q^{2m_2}}{1-q^2}C_{l,l}(\epsilon) \ , \end{aligned}$$

$$B^{m_2}_{j,j,l,l-1} = -2|\epsilon|[2]q^{-l+m_2-2-\epsilon(2j+1)}\frac{\sqrt{[l+2\epsilon\, j+m_2+1+\epsilon][l-2\epsilon\, j-m_2+1-\epsilon]}}{[2j][2j+2]}C_{l,l-1}(\epsilon) \ .$$

We have inserted the factor $2|\epsilon|$, so that the expressions remain valid also when $\epsilon = 0$.

6.5 The condition on the radius

Orbits for $SO(5)$ are spheres of arbitrary radius, equivariance alone not imposing constraints on the radius. Similarly, for the quantum spheres one has to impose a constraint on the radius to determine the coefficients of the representation. In fact, this will determine $C_{l,l+1}(0)$, $C_{l,l+1}(\frac{1}{2})$ and $C_{l,l}(\frac{1}{2})$ only up to a phase. Different choices of the phases correspond to unitary equivalent representations and without loosing generality we choose $C_{l,l'}(\epsilon) \in \mathbb{R}$. A possible expression for the radius is $q^8 x_0^2 + q^4 x_1^* x_1 + x_2^* x_2$ which we constrain to be equal to 1. Let then,

$$r(l, m_1, m_2; j) := \big\langle l, m_1, m_2; j | q^8 x_0^2 + q^4 x_1^* x_1 + x_2^* x_2 | l, m_1, m_2; j \big\rangle \ .$$

All these matrix coefficients must be 1. In particular, for $l \in \mathbb{N}$ the condition $r(l, 0, l; 0) = 1$ implies (up to a phase) that

$$C_{l,l+1}(0) = \frac{q^{-l-3/2}}{\sqrt{[2l+3][2l+5]}} \ . \tag{6.11}$$

For $l \in \mathbb{N} + \frac{1}{2}$ we first require that $r(l, \frac{1}{2}, l; \frac{1}{2}) = r(l, -\frac{1}{2}, l; \frac{1}{2})$ obtaining two possibilities:

$$C_{l,l}(\tfrac{1}{2}) = \pm \frac{[2]q^{l+2}}{[2l+2]} C_{l,l+1}(\tfrac{1}{2}) \ .$$

Then imposing $r(l, \frac{1}{2}, l; \frac{1}{2}) = 1$, yields (up to a phase)

$$C_{l,l+1}(\tfrac{1}{2}) = \frac{q^{-l-3/2}}{[2l+4]} \ , \tag{6.12}$$

hence,

$$C_{l,l}(\tfrac{1}{2}) = \pm \frac{q^{1/2}[2]}{[2l+2][2l+4]} \ . \tag{6.13}$$

With these, all coefficients are completely determined. If a representation on these spaces exists, the coefficients are the ones we computed. But we know that it exists at least one representation for integer l, and at least two for $l \in \mathbb{N} + \frac{1}{2}$. We can conclude that these define really representations of the crossed product, that there are no others on these spaces (modulo unitary equivalence) and that the signs (6.13) correspond to inequivalent representations.

An independent check has been performed using a computer program.

6.6 Explicit form of the representations

Let us summarize the main results of this section in the following two theorems, which correspond to the scalar (i.e. left regular) and chiral spinor representations.

Theorem 6.4. *The vector space $\mathcal{A}(S^4_q)$ has orthonormal basis $|l, m_1, m_2; j\rangle$ with,*

$$j = 0, 1, \ldots, l \ , \qquad j - |m_1| \in \mathbb{N} \ , \qquad l - j - |m_2| \in 2\mathbb{N} \ .$$

We call $L^2(S^4_q)$ the Hilbert space completion of $\mathcal{A}(S^4_q)$. Modulo a unitary equivalence, the left regular representation is given by

$$\begin{aligned}
x_0\,|l,m_1,m_2;j\rangle &= A_{j,m_1}C^+_{l,j,m_2}\,|l+1,m_1,m_2;j+1\rangle \\
&+ A_{j,m_1}C^-_{l,j,m_2}\,|l-1,m_1,m_2;j+1\rangle \\
&+ A_{j-1,m_1}C^-_{l+1,j-1,m_2}\,|l+1,m_1,m_2;j-1\rangle \\
&+ A_{j-1,m_1}C^+_{l-1,j-1,m_2}\,|l-1,m_1,m_2;j-1\rangle \ , \\
x_1\,|l,m_1,m_2;j\rangle &= B^+_{j,m_1}C^+_{l,j,m_2}\,|l+1,m_1+1,m_2;j+1\rangle \\
&+ B^+_{j,m_1}C^-_{l,j,m_2}\,|l-1,m_1+1,m_2;j+1\rangle \\
&+ B^-_{j,m_1}C^-_{l+1,j-1,m_2}\,|l+1,m_1+1,m_2;j-1\rangle \\
&+ B^-_{j,m_1}C^+_{l-1,j-1,m_2}\,|l-1,m_1+1,m_2;j-1\rangle \ , \\
x_2\,|l,m_1,m_2;j\rangle &= D^+_{l,j,m_2}\,|l+1,m_1,m_2+1;j\rangle \\
&+ D^-_{l,j,m_2}\,|l-1,m_1,m_2+1;j\rangle \ ,
\end{aligned}$$

with coefficients

$$\begin{aligned}
A_{j,m_1} &= q^{m_1-1}\sqrt{\frac{[j+m_1+1][j-m_1+1]}{[2j+1][2j+3]}} \ , \\
B^+_{j,m_1} &= q^{-j+m_1-1/2}\sqrt{\frac{[j+m_1+1][j+m_1+2]}{[2j+1][2j+3]}} \ , \\
B^-_{j,m_1} &= -q^{j+m_1+1/2}\sqrt{\frac{[j-m_1][j-m_1-1]}{[2j-1][2j+1]}} \ .
\end{aligned}$$

and

$$\begin{aligned}
C^+_{l,j,m_2} &= q^{m_2-1}\sqrt{\frac{[l+j+m_2+3][l+j-m_2+3]}{[2l+3][2l+5]}} \ , \\
C^-_{l,j,m_2} &= -q^{m_2-1}\sqrt{\frac{[l-j+m_2][l-j-m_2]}{[2l+1][2l+3]}} \ , \\
D^+_{l,j,m_2} &= q^{-l+m_2-3/2}\sqrt{\frac{[l+j+m_2+3][l-j+m_2+2]}{[2l+3][2l+5]}} \ , \\
D^-_{l,j,m_2} &= q^{l+m_2+3/2}\sqrt{\frac{[l-j-m_2][l+j-m_2+1]}{[2l+1][2l+3]}} \ .
\end{aligned}$$

The two chiral spinorial representations (corresponding to the sign $\pm$ in equation (6.13)) are described in the following Theorem.

Theorem 6.5. *Let $\mathcal{H}_\pm$ be two Hilbert spaces with orthonormal basis $|l,m_1,m_2;j\rangle_\pm$, where*

$$j=\tfrac{1}{2},\tfrac{3}{2},\dots,l\ , \qquad j-|m_1|\in\mathbb{N}\ , \qquad l+\tfrac{1}{2}-j-|m_2|\in\mathbb{N}\ .$$

Let $\epsilon = \pm\frac{1}{2}$ be defined by $l + \epsilon - j - m_2 \in 2\mathbb{N}$. On each space $\mathcal{H}_\pm$ there is an equivariant $*$-representation of $\mathcal{A}(S^4_q)$ defined by:

$$
\begin{aligned}
x_0 \, |l, m_1, m_2; j\rangle_\pm &= A^+_{j,m_1} C^+_{l,j,m_2} \, |l+1, m_1, m_2; j+1\rangle_\pm \\
&\mp A^+_{j,m_1} C^0_{l,j,m_2} \, |l, m_1, m_2; j+1\rangle_\pm \\
&+ A^+_{j,m_1} C^-_{l,j,m_2} \, |l-1, m_1, m_2; j+1\rangle_\pm \\
&+ A^0_{j,m_1} H^+_{l,j,m_2} \, |l+1, m_1, m_2; j\rangle_\pm \\
&\pm A^0_{j,m_1} H^0_{l,j,m_2} \, |l, m_1, m_2; j\rangle_\pm \\
&+ A^0_{j,m_1} H^+_{l-1,j,m_2} \, |l-1, m_1, m_2; j\rangle_\pm \\
&+ A^+_{j-1,m_1} C^-_{l+1,j-1,m_2} \, |l+1, m_1, m_2; j-1\rangle_\pm \\
&\mp A^+_{j-1,m_1} C^0_{l,j-1,m_2} \, |l, m_1, m_2; j-1\rangle_\pm \\
&+ A^+_{j-1,m_1} C^+_{l-1,j-1,m_2} \, |l-1, m_1, m_2; j-1\rangle_\pm \ , \\
x_1 \, |l, m_1, m_2; j\rangle_\pm &= B^+_{j,m_1} C^+_{l,j,m_2} \, |l+1, m_1+1, m_2; j+1\rangle_\pm \\
&\mp B^+_{j,m_1} C^0_{l,j,m_2} \, |l, m_1+1, m_2; j+1\rangle_\pm \\
&+ B^+_{j,m_1} C^-_{l,j,m_2} \, |l-1, m_1+1, m_2; j+1\rangle_\pm \\
&+ B^0_{j,m_1} H^+_{l,j,m_2} \, |l+1, m_1+1, m_2; j\rangle_\pm \\
&\pm B^0_{j,m_1} H^0_{l,j,m_2} \, |l, m_1+1, m_2; j\rangle_\pm \\
&+ B^0_{j,m_1} H^+_{l-1,j,m_2} \, |l-1, m_1+1, m_2; j\rangle_\pm \\
&+ B^-_{j,m_1} C^-_{l+1,j-1,m_2} \, |l+1, m_1+1, m_2; j-1\rangle_\pm \\
&\mp B^-_{j,m_1} C^0_{l,j-1,m_2} \, |l, m_1+1, m_2; j-1\rangle_\pm \\
&+ B^-_{j,m_1} C^+_{l-1,j-1,m_2} \, |l-1, m_1+1, m_2; j-1\rangle_\pm \ , \\
x_2 \, |l, m_1, m_2; j\rangle_\pm &= D^+_{l,j,m_2} \, |l+1, m_1, m_2+1; j\rangle_\pm \\
&\pm D^0_{l,j,m_2} \, |l, m_1, m_2+1; j\rangle_\pm \\
&+ D^-_{l,j,m_2} \, |l-1, m_1, m_2+1; j\rangle_\pm \ ,
\end{aligned}
$$

with coefficients

$$
\begin{aligned}
A^+_{j,m_1} &= q^{m_1-1} \frac{\sqrt{[j+m_1+1][j-m_1+1]}}{[2j+2]} \ , \\
A^0_{j,m_1} &= q^{-2} \frac{q^{j+m_1+1}[2][j-m_1] - [2j]}{[2j][2j+2]} \ , \\
B^+_{j,m_1} &= q^{-j+m_1-1/2} \frac{\sqrt{[j+m_1+1][j+m_1+2]}}{[2j+2]} \ , \\
B^0_{j,m_1} &= (1+q^2) q^{m_1-1/2} \frac{\sqrt{[j-m_1][j+m_1+1]}}{[2j][2j+2]} \ , \\
B^-_{j,m_1} &= -q^{j+m_1+1/2} \frac{\sqrt{[j-m_1][j-m_1-1]}}{[2j]} \ .
\end{aligned}
$$

and

$$C^{+}_{l,j,m_2} = -q^{m_2-1-\epsilon}\frac{\sqrt{[l+j+m_2+3+\epsilon][l+j-m_2+3-\epsilon]}}{[2l+4]}\ ,$$

$$C^{0}_{l,j,m_2} = [4\epsilon]\, q^{2\epsilon\, l+m_2-1+3\epsilon}\frac{\sqrt{[l+\frac{1}{2}+j-2\epsilon\, m_2+2][l+\frac{1}{2}-j-2\epsilon\, m_2]}}{[2l+2][2l+4]}\ ,$$

$$C^{-}_{l,j,m_2} = -q^{m_2-1+\epsilon}\frac{\sqrt{[l-j+m_2-\epsilon][l-j-m_2+\epsilon]}}{[2l+2]}\ ,$$

$$H^{+}_{l,j,m_2} = q^{m_2-1+\epsilon(2j+1)}\frac{\sqrt{[l+2\epsilon j-m_2+2+\epsilon][l-2\epsilon j+m_2+2-\epsilon]}}{[2l+4]}\ ,$$

$$H^{0}_{l,j,m_2} = \frac{[l-\epsilon(2j+1)-m_2+1][l-\epsilon(2j+1)+m_2+2]-q^{-2}[l+\epsilon(2j+1)-m_2+2][l+\epsilon(2j+1)+m_2+1]}{[2l+2][2l+4]}\ ,$$

$$D^{+}_{l,j,m_2} = q^{-l+m_2-3/2}\frac{\sqrt{[l+j+m_2+3+\epsilon][l-j+m_2+2-\epsilon]}}{[2l+4]}\ ,$$

$$D^{0}_{l,j,m_2} = [2]q^{m_2+1/2}\frac{\sqrt{[l-2\epsilon j-m_2+1-\epsilon][l-2\epsilon j+m_2+2-\epsilon]}}{[2l+2][2l+4]}\ ,$$

$$D^{-}_{l,j,m_2} = -q^{l+m_2+3/2}\frac{\sqrt{[l-j-m_2+\epsilon][l+j-m_2+1-\epsilon]}}{[2l+2]}\ .$$

These two representations are inequivalent and correspond to the projective modules $\mathcal{A}(S^4_q)^4 e$ and $\mathcal{A}(S^4_q)^4(1-e)$, with e the idempotent in Equation (5.1).

7 The Dirac operator on the quantum Euclidean 4-sphere

We start by constructing a non-trivial Fredholm module on the quantum Euclidean sphere. With different representations a non-trivial Fredholm module was already constructed in [17].

Proposition 7.1. *Consider the representations of $\mathcal{A}(S^4_q)$ on $\mathcal{H}_\pm$ given in Theorem 6.5. Then, the datum $(\mathcal{A}(S^4_q), \mathcal{H}, F, \gamma)$ is a 1-summable even Fredholm module, where $\mathcal{H} := \mathcal{H}_+ \oplus \mathcal{H}_-$, γ is the natural grading and $F \in \mathcal{B}(\mathcal{H})$ is defined by*

$$F\,|l,m_1,m_2;j\rangle_\pm := |l,m_1,m_2;j\rangle_\mp\ .$$

This Fredholm module is non-trivial. In particular,

$$\mathrm{ch}^F([e]) := \tfrac{1}{2}\mathrm{Trace}_{\mathcal{H}\otimes\mathbb{C}^2}(\gamma F[F,P]) = 1\ , \qquad (7.1)$$

with e the idempotent defined by Equation (5.1).

Proof. That $F = F^*$, $F^2 = 1$ and $\gamma F + F\gamma = 0$ is obvious. Then, it is enough to show that $[F, x_i] \in \mathcal{L}^1(\mathcal{H})$ for $i = 0, 1, 2$. From this and Leibniz rule it follows that $[F, a]$ is trace class, and then compact, for all $a \in \mathcal{A}(S^4_q)$.

Now, notice that

$$[F, x_0]\,|l,m_1,m_2;j\rangle_\pm = \mp 2A^{+}_{j,m_1}C^{0}_{l,j,m_2}\,|l,m_1,m_2;j+1\rangle_\mp$$

$$
\begin{aligned}
& \pm 2A^0_{j,m_1} H^0_{l,j,m_2} \, |l, m_1, m_2; j\rangle_\mp \\
& \mp 2A^+_{j-1,m_1} C^0_{l,j-1,m_2} \, |l, m_1, m_2; j-1\rangle_\mp \ , \\
[F, x_1] \, |l, m_1, m_2; j\rangle_\pm = & \mp 2B^+_{j,m_1} C^0_{l,j,m_2} \, |l, m_1+1, m_2; j+1\rangle_\mp \\
& \pm 2B^0_{j,m_1} H^0_{l,j,m_2} \, |l, m_1+1, m_2; j\rangle_\mp \\
& \mp 2B^-_{j,m_1} C^0_{l,j-1,m_2} \, |l, m_1+1, m_2; j-1\rangle_\mp \ , \\
[F, x_2] \, |l, m_1, m_2; j\rangle_\pm = & \pm 2D^0_{l,j,m_2} \, |l, m_1, m_2+1; j\rangle_\mp \ .
\end{aligned} \tag{7.2}
$$

All the coefficients appearing in these equations are bounded by q^{2l}. Thus the commutators are trace class and this concludes the first part of the proof.

To prove non-triviality it is enough to prove (7.1). Substituting (5.1) into (7.1) yields

$$
\mathrm{ch}^F([e]) = \tfrac{(1-q^2)^2}{4} \mathrm{Trace}_{\mathcal{H}}(\gamma F [F, x_0]) \ .
$$

and in turn, using equation (7.2),

$$
\mathrm{ch}^F([e]) = (1-q^2)^2 \sum_{l,j,m_1,m_2} A^0_{j,m_1} H^0_{l,j,m_2} \ .
$$

Summing over m_1 from $-j$ to j we obtain that

$$
\begin{aligned}
\mathrm{ch}^F([e]) &= q^{-3}(1-q^2)^2 \sum_{l,j,m_2} \frac{[l+\epsilon(2j+1)-m_2+2][l+\epsilon(2j+1)+m_2+1]}{[2l+2][2l+4][2j][2j+2]} \times \\
&\qquad\qquad \times \sum_{m_1} \left\{ q^{2j+2} + q^{-2j} - [2] q^{2m_1+1} \right\} \\
&= \sum_{l,j} \frac{(2j+1)(q^{2j+1}+q^{-2j-1}) - [2][2j+1]}{[2l+2][2l+4][2j][2j+2]} \times \\
&\qquad\qquad \times \sum_{m_2} \left\{ q^{2l+2\epsilon(2j+1)+3} + q^{-2l-2\epsilon(2j+1)-3} - q^{2m_2-1} - q^{-2m_2+1} \right\} \ .
\end{aligned}
$$

The sum over m_2 requires additional care. For ϵ fixed, $l-\epsilon-j+m_2 = 0, 2, 4, \ldots, 2(l-j)$. If we call $2i := l-\epsilon-j+m_2$ and sum first over $i = 0, 1, \ldots, l-j$ and then over $\epsilon = \pm 1/2$ we get:

$$
\begin{aligned}
\mathrm{ch}^F([e]) &= \sum_{l,j} \frac{(2j+1)(q^{2j+1}+q^{-2j-1}) - [2][2j+1]}{[2l+2][2l+4][2j][2j+2]} \times \\
&\quad \times \sum_{2\epsilon=\pm 1} \left\{ (l-j+1)(q^{2l+2\epsilon(2j+1)+3} + q^{-2l-2\epsilon(2j+1)-3}) - (q^{2\epsilon-1}+q^{-2\epsilon+1})[2]^{-1}[2(l-j+1)] \right\} \\
&= \sum_{l,j} \frac{(2j+1)(q^{2j+1}+q^{-2j-1}) - [2][2j+1]}{[2l+2][2l+4][2j][2j+2]} \times \\
&\qquad\qquad \times \left\{ (l-j+1)(q^{2l+3}+q^{-2l-3})(q^{2j+1}+q^{-2j-1}) - [2][2(l-j+1)] \right\} \\
&=: \sum_{l,j} f_{lj}(q) =: f(q) \ .
\end{aligned}
$$

We call $f_{lj}(q)$ the generic term of last series, explicitly written as

$$f_{lj}(q) = (1-q^2)^4 \, \frac{(2j+1)(1+q^{4j+2}) - \frac{1+q^2}{1-q^2}(1-q^{4j+2})}{(1-q^{4l+4})(1-q^{4l+8})(1-q^{4j})(1-q^{4j+4})} \times \\ \times q^{2l-1}\Big\{(l-j+1)(1+q^{4l+6})(1+q^{4j+2}) - \tfrac{1+q^2}{1-q^2}\, q^2(q^{4j}-q^{4l+4})\Big\} ,$$

and consider it as a function of $q \in [0,1[$. Notice that each $f_{lj}(q)$ is a C^∞ function of q (they are rational functions whose denominators never vanish for $0 \leq q < 1$). From the inequality

$$0 \leq f_{lj}(q) \leq 4(2j+1)q^{2l-1}$$

we deduce (using the Weierstrass M-test) that the series is absolutely (hence uniformly) convergent in each interval $[0,q_0] \subset [0,1[$. Then, it converges to a function $f(q)$ which is continuous in $[0,1[$. Being the index of a Fredholm operator, $f(q)$ is integer valued in $]0,1[$; by continuity it is constant and can be computed in the limit $q \to 0$. In this limit we have $f_{lj}(q) = 2j(l-j+1)q^{2l-1} + O(q^{2l})$. Thus, $f_{lj}(0) = \delta_{l,1/2}\delta_{j,1/2}$ and $\mathrm{ch}^F([e]) = f(0) = 1$. □

The next step is to define a spectral triple whose Fredholm module is the one described in Proposition 7.1.

Proposition 7.2. *Let D be the (unbounded) operator on $\mathcal{H} := \mathcal{H}_+ \oplus \mathcal{H}_-$ defined by*

$$D\,|l,m_1,m_2;j\rangle_\pm := (l+\tfrac{3}{2})\,|l,m_1,m_2;j\rangle_\mp \ .$$

Then, the datum $(\mathcal{A}(S^4_q),\mathcal{H},D,\gamma)$ is a $U_q(so(5))$-equivariant regular even spectral triple of metric dimension 4.

Remark: The operator D is isospectral to the classical Dirac operator on S^4 (whose spectrum has been computed in [1]). When $q=1$, this spectral triple becomes the canonical one associated to the spin structure of S^4.

Proof. Clearly the representation of the algebra is even, D is odd, with compact resolvent and 4^+-summable (being isospectral to the classical Dirac operator on S^4).

Let δ be the unbounded derivation on $\mathcal{B}(\mathcal{H})$ defined by $\delta(T) := [|D|,T]$. Each generator of $\mathcal{A}(S^4_q)$ is the sum of a finite number of weighted shifts; each of these weighted shifts is a bounded operator (the coefficients are all bounded by 1) and is an eigenvector of δ, i.e., if T shifts the index l by k, then $\delta(T) = kT$. Thus, such weighted shifts are not only bounded but also in the smooth domain of δ, which we denote by $\mathrm{OP}^0 := \bigcap_{j\in\mathbb{N}} \mathrm{dom}\,\delta^j$. As a consequence $\mathcal{A}(S^4_q) \subset \mathrm{OP}^0$.

Recall that $[F,x_i]$ has coefficients decaying faster than q^l; thus $|D|[F,x_i]$ is a matrix of rapid decay. In particular, $|D|[F,x_i] \in \mathrm{OP}^{-\infty} \subset \mathrm{OP}^0$. The identity

$$[D,x_i] = \delta(x_i)F + |D|[F,x_i] \ ,$$

tells us that $[D,x_i]$ is not only bounded but even in OP^0 – being the sum of two bounded operators contained in the $*$-algebra OP^0. Then, D defines a spectral triple and such a spectral triple is regular.

Finally, since D is proportional to the identity in any irreducible subrepresentation V_l of $U_q(so(5))$, it commutes with all $h \in U_q(so(5))$ and it is equivariant. □

As a preparation for the study of the dimension spectrum in Section 8, let us explicitly verify the 4-summability of D. As one can easily check, the dimension of V_l is [1]

$$\dim V_l = \tfrac{2}{3}(l+\tfrac{5}{2})(l+\tfrac{3}{2})(l+\tfrac{1}{2}) \; .$$

From this we get

$$\mathrm{Trace}(|D|^{-s}) = \textstyle\sum_{l\in\mathbb{N}+\frac{1}{2}} 2(l+\frac{3}{2})^{-s}\dim V_l = \frac{4}{3}\sum_{n=1}^{\infty}(n^2-1)n^{-s+1} \; ,$$

where $n = l + \frac{3}{2}$ (and we added the term with $n = 1$ since it is identically zero). The above series is convergent in the right half-plane $\{s \in \mathbb{C} \,|\, \mathrm{Re}\, s > 4\}$, thus D has metric dimension 4.

Notice that $\mathrm{Trace}(|D|^{-s})$ has meromorphic extention on $\mathbb{C}$ given by

$$\mathrm{Trace}(|D\!\!\!/|^{-s}) = \tfrac{4}{3}\big\{\zeta(s-3) - \zeta(s-1)\big\} \; , \tag{7.3}$$

where $\zeta(s)$ is the Riemann zeta-function. We recall that $\zeta(s)$ has a simple pole in $s = 1$ as unique singularity and that $\mathrm{Res}_{s=1}\zeta(s) = 1$.

8 The dimension spectrum and residues

We need the following class of operators:

$$\mathcal{J} := \big\{T \in \mathrm{OP}^0 \,\big|\, |D|^{-p}T \in \mathcal{L}^1(\mathcal{H}) \;\forall\, p > 2\big\} \; . \tag{8.1}$$

Lemma 8.1. *The collection $\mathcal{J}$ is a two-sided ideal in* OP^0.

Proof. $\mathcal{J}$ is clearly a vector space.

Let $T_1 \in \mathrm{OP}^0$ and $T_2 \in \mathcal{J}$. Then $T_2T_1 \in \mathcal{J}$ since trace class operators are a right ideal in $\mathcal{B}(\mathcal{H})$. For the same reason $T_1|D|^{-p}T_2$ is trace class for all $p > 2$, and $T_1T_2 \in \mathcal{J}$ if we can prove that $|D|^{-p}T_1T_2 - T_1|D|^{-p}T_2$ is trace class for all $p > 2$. Using the asymptotic expansion of [6], this term can be rewritten as:

$$|D|^{-p}T_1T_2 - T_1|D|^{-p}T_2 \sim \sum_{n=1}^{\infty} \frac{(-1)^n p(p+1)\dots(p+n-1)}{n!}\,\delta^n(T_1)|D|^{-(p+n)}T_2 \; .$$

Each summand on the right hand side is the product of a compact operator with $|D|^{-p}T_2$, which is trace class. This concludes the proof. □

Next we define $L_q \in \mathcal{B}(\mathcal{H})$ by

$$L_q\,|l, m_1, m_2; j\rangle := q^{j+\frac{1}{2}}\,|l, m_1, m_2; j\rangle \;\; ,$$

for which we have the following Lemma.

Lemma 8.2. *For any $s \in \mathbb{C}$ with $\operatorname{Re} s > 2$ one has that*

$$\zeta_{L_q}(s) := \sum_{l,j,m_1,m_2} (l+\tfrac{3}{2})^{-s} q^{j+\frac{1}{2}} = \frac{4q}{(1-q)^2}\Big(\zeta(s-1) - \frac{1+q}{1-q}\,\zeta(s)\Big) + \text{holomorphic function} ,$$

where $\zeta(s)$ is the Riemann zeta-function.

Proof. Calling $n := l + \frac{3}{2}$, $k := j + \frac{1}{2}$, we have

$$\zeta_{L_q}(s) = 4 \sum_{n=2}^{\infty} n^{-s} \sum_{k=1}^{n-1} k(n-k) q^k ,$$

We can sum starting from $n = 1$ and for $k = 0, \dots, n$ (we simply add zero terms) to get

$$\zeta_{L_q}(s) = 4 \sum_{n=1}^{\infty} n^{-s} \big\{ nq\partial_q - (q\partial_q)^2 \big\} \sum_{k=0}^{n} q^k = 4 \sum_{n=1}^{\infty} n^{-s} \big\{ nq\partial_q - (q\partial_q)^2 \big\} \frac{1-q^{n+1}}{1-q} .$$

Terms decaying as q^n give an holomorphic function of s, thus modulo holomorphic functions,

$$\zeta_{L_q}(s) \sim 4 \sum_{n=1}^{\infty} n^{-s} \Big\{ n \tfrac{q}{(1-q)^2} - \tfrac{q(1+q)}{(1-q)^3} \Big\} .$$

Last series is summable for all s with $\operatorname{Re} s > 2$, and its sum can be written in term of the Riemann zeta-function as in the statement of the Lemma. □

As a consequence of Lemma 8.2 we have the following Corollary.

Corollary 8.3. *The operator L_q belongs to the ideal $\mathcal{J}$.*

8.1 An approximated representation

Let $\hat{\mathcal{H}}$ be an Hilbert space with orthonormal basis $\|l, m_1, m_2; j\rangle\!\rangle_\pm$ labelled by,

$$l \in \tfrac{1}{2}\mathbb{Z} , \qquad l + j \in \mathbb{Z} , \qquad j + m_1 \in \mathbb{N} , \qquad l + \tfrac{1}{2} - j + m_2 \in \mathbb{N} .$$

Let I be the labelling set of the Hilbert space $\mathcal{H}_\pm$ as in Theorem 6.5, and given by

$$I := \big\{ (j, m_1, m_2, j) \;\big|\; l \in \mathbb{N} + \tfrac{1}{2} ,\;\; j = \tfrac{1}{2}, \tfrac{3}{2}, ..., l ,\;\; j - |m_1| \in \mathbb{N} ,\;\; l + \tfrac{1}{2} - j - |m_2| \in \mathbb{N} \big\} .$$

Notice that I is the subset of labels of $\hat{\mathcal{H}}$ satisfying $l \in \mathbb{N} + \frac{1}{2}$, $m_1 \leq j \leq l$ and $m_2 \leq l + \frac{1}{2} - j$. Define the inclusion $Q : \mathcal{H} \to \hat{\mathcal{H}}$ and the adjoint projection $P : \hat{\mathcal{H}} \to \mathcal{H}$ by,

$$Q\,|l, m_1, m_2; j\rangle_\pm := \|l, m_1, m_2; j\rangle\!\rangle_\pm \quad \text{for all } (l, m_1, m_2, j) \in I ,$$

$$P\,\|l, m_1, m_2; j\rangle\!\rangle_\pm := \begin{cases} |l, m_1, m_2; j\rangle_\pm & \text{if } (l, m_1, m_2, j) \in I , \\ 0 & \text{otherwise} . \end{cases}$$

Clearly, $PQ = id_{\mathcal{H}}$. The Hilbert space $\hat{\mathcal{H}}$ carries a bounded $*$-representation of the algebra $\mathcal{A}(SU_q(2)) \otimes \mathcal{A}(S^2_q)$ defined by,

$$\begin{aligned}
\alpha\, \|l, m_1, m_2; j\rangle\!\rangle_\pm &= \sqrt{1 - q^{2(j+m_1+1)}}\; \|l + \tfrac{1}{2}, m_1 + \tfrac{1}{2}, m_2; j + \tfrac{1}{2}\rangle\!\rangle_\pm \ , \\
\beta\, \|l, m_1, m_2; j\rangle\!\rangle_\pm &= q^{j+m_1}\; \|l + \tfrac{1}{2}, m_1 - \tfrac{1}{2}, m_2; j + \tfrac{1}{2}\rangle\!\rangle_\pm \ , \\
A\, \|l, m_1, m_2; j\rangle\!\rangle_\pm &= q^{l-j+m_2-\epsilon}\; \|l, m_1, m_2; j\rangle\!\rangle_\pm \ , \\
B\, \|l, m_1, m_2; j\rangle\!\rangle_\pm &= \sqrt{1 - q^{2(l-j+m_2+2-\epsilon)}}\; \|l + 1, m_1, m_2 + 1; j\rangle\!\rangle_\pm \ ,
\end{aligned}$$

where, as before, $\epsilon := \frac{1}{2}(-1)^{l+\frac{1}{2}-j-m_2}$. Composition of such a representation with the algebra embedding $\mathcal{A}(S^4_q) \hookrightarrow \mathcal{A}(SU_q(2)) \otimes \mathcal{A}(S^2_q)$ given in equation (4.2) results into a $*$-representation $\pi : \mathcal{A}(S^4_q) \to \mathcal{B}(\hat{\mathcal{H}})$. The sandwich $\tilde{\pi}(a) := P\pi(a)Q$ defines a $*$-linear map $\tilde{\pi} : \mathcal{A}(S^4_q) \to \mathcal{B}(\mathcal{H})$.

Proposition 8.4. *With $\mathcal{J}$ the class of operators defined in equation (8.1), one has that the difference $a - \tilde{\pi}(a) \in \mathcal{J}$ for all $a \in \mathcal{A}(S^4_q)$.*

Proof. Define $\hat{\mathcal{J}}$ as the collection of bounded operators $T : \hat{\mathcal{H}} \to \mathcal{H}$ such that $|D|^{-p}T$ is trace class for all $p > 2$. Since trace class operators are a two sided ideal in bounded operators, the space $\hat{\mathcal{J}}$ is stable when multiplied from the right by bounded operators: $T_1 \in \hat{\mathcal{J}}$ and $T_2 \in \mathcal{B}(\hat{\mathcal{H}}) \ \Rightarrow \ T_1T_2 \in \hat{\mathcal{J}}$.

Next, suppose that a, b satisfy $a - \tilde{\pi}(a) \in \mathcal{J}$ and $b - \tilde{\pi}(b) \in \mathcal{J}$ and consider the following algebraic identity,

$$ab - \tilde{\pi}(ab) = a\big\{b - \tilde{\pi}(b)\big\} + \big\{aP - P\pi(a)\big\}\pi(b)Q \ .$$

Since $\mathcal{J}$ is a two-sided ideal in OP^0, the first summand is in $\mathcal{J}$. The stability of $\hat{\mathcal{J}}$ discussed above implies that $\big\{aP - P\pi(a)\big\}\pi(b) \in \hat{\mathcal{J}}$, but if $T \in \hat{\mathcal{J}}$ clearly $TQ \in \mathcal{J}$. Hence the second summand in $\mathcal{J}$ too. Thus, $ab - \tilde{\pi}(ab) \in \mathcal{J}$ whenever this property holds for each of the operators a, b. We conclude that it is enough to show that $a - \tilde{\pi}(a) \in \mathcal{J}$ when a is a generator of $\mathcal{A}(S^4_q)$.

Now, for any $a \in \{x_i, x_i^*\}$ the operator $a - \tilde{\pi}(a)$ is the sum of a finite number of weighted shifts whose weights, as one proves by direct computation, are bounded in modulus by $q^{j+\frac{1}{2}}$. Each of such shifts is then the product of an operator in OP^0 (an eigenvector of δ) with the operator L_q of Corollary 8.3, which we know is contained in $\mathcal{J}$. This concludes the proof. □

8.2 The dimension spectrum and the top residue

The approximation modulo $\mathcal{J}$ allows considerable simplifications when getting information on the part of the dimension spectrum contained in the half plane $\mathrm{Re}\, s > 2$. To study the part of the dimension spectrum in the left half plane $\mathrm{Re}\, s \leq 2$ would require a less drastic approximation which we are lacking at the moment.

Proposition 8.5. *In the region $\mathrm{Re}\, s > 2$ the dimension spectrum Σ of the spectral triple $(\mathcal{A}(S^4_q), \mathcal{H}, D, \gamma)$ given in Proposition 7.2 consits of the two points $\{3, 4\}$, which are simple poles* *of the zeta-functions. The top residue coincides with the integral on the subspace of classical points of S^4_q, that is*

$$⨍ a|D|^{-4} = \frac{2}{3\pi}\int_0^{2\pi} \sigma(a)(\theta)\mathrm{d}\theta \ , \tag{8.2}$$

with $\sigma : \mathcal{A}(S^4_q) \to \mathcal{A}(S^1)$ the $*$-algebra morphism defined by $\sigma(x_0) = \sigma(x_1) = 0$ and $\sigma(x_2) = u$, where u, given by $u(\theta) := e^{i\theta}$, is the unitary generator of $\mathcal{A}(S^1)$.

Proof. Let Ψ^0 be the $*$-algebra generated by $\mathcal{A}(S^4_q)$, by $[D, \mathcal{A}(S^4_q)]$ and by iterated applications of the derivation δ. Let $\mathfrak{A} \subset \mathcal{A}(SU_q(2)) \otimes \mathcal{A}(S^2_q) \otimes \mathrm{Mat}_2(\mathbb{C})$ be the $*$-algebra generated by α, β, α^*, β^*, A, B, B^* and F. By Proposition 8.4 there is an inclusion $\mathcal{A}(S^4_q) \subset P\mathfrak{A}Q + \mathcal{J}$.

A linear basis for $\mathfrak{A}$ is given by,

$$T := \alpha^{k_1}\beta^{n_1}(\beta^*)^{n_2}A^{n_3}B^{k_2}F^h \ , \tag{8.3}$$

where $h \in \{0, 1\}$, $n_i \in \mathbb{N}$, $k_i \in \mathbb{Z}$ and with the notation $\alpha^{k_1} := (\alpha^*)^{-k_1}$ if $k_1 < 0$ and $B^{k_2} := (B^*)^{-k_2}$ if $k_2 < 0$. For this operator,

$$\delta(PTQ) = \big(\tfrac{1}{2}(k_1 + n_1 - n_2) + k_2\big)PTQ \qquad \text{and} \qquad [D, PTQ] = \delta(PTQ)F \ .$$

Thus, $P\mathfrak{A}Q$ is invariant under application of δ and $[D, \, . \,]$ and hence $\Psi^0 \subset P\mathfrak{A}Q + \mathcal{J}$.

For the part of the dimension spectrum in the right half plane $\mathrm{Re}\, s > 2$, we can neglect $\mathcal{J}$ and consider only the singularities of zeta-functions associated with elements in $P\mathfrak{A}Q$. By linearity of the zeta-functions, it is enough to consider the generic basis element in equation (8.3).

Such a T shifts l by $\frac{1}{2}(k_1 + n_1 - n_2) + k_2$, m_1 by $\frac{1}{2}(k_1 - n_1 + n_2)$, m_2 by k_2, j by $\frac{1}{2}(k_1 + n_1 - n_2)$, and flips the chirality if $h = 1$. Thus it is off-diagonal unless $h = k_i = 0$ and $n_1 = n_2$. The zeta-function associated with a bounded off-diagonal operator is identically zero in the half-plane $\mathrm{Re}\, z > 4$, and so is its holomorphic extension to the entire complex plane. It remains to consider the cases $T = P(\beta\beta^*)^k A^n Q$, with $n, k \in \mathbb{N}$.

If n and k are both different from zero, one finds

$$\zeta_T(s) = 2\sum_{l,j,m_1,m_2} (l + \tfrac{3}{2})^{-s} q^{n(l-j+m_2-\epsilon)+2k(j+m_1)} = 2\sum_{l,j,m_2} (l + \tfrac{3}{2})^{-s} q^{n(l-j+m_2-\epsilon)} \frac{1 - q^{2k(2j+1)}}{1 - q^{2k}} \ .$$

For ϵ fixed, set $2i := l - \epsilon - j + m_2 = 0, 2, \ldots, 2(l - j)$. Then,

$$\begin{aligned} \zeta_T(s) &= 2\sum_{l,j} (l + \tfrac{3}{2})^{-s} \frac{1 - q^{2k(2j+1)}}{1 - q^{2k}} \sum_{\epsilon=\pm 1/2} \sum_{i=0}^{l-j} q^{2ni} = 4\sum_{l,j} (l + \tfrac{3}{2})^{-s} \frac{1 - q^{2k(2j+1)}}{1 - q^{2k}} \frac{1 - q^{2n(l-j+1)}}{1 - q^{2n}} \\ &= 4\zeta(s-1) - 4\,\frac{1 + (1 - q^{4k})^{-1} + (1 - q^{2n})^{-1}}{(1 - q^{2k})(1 - q^{2n})}\,\zeta(s) + \text{holomorphic function} \ , \end{aligned}$$

which has meromorphic extention on $\mathbb{C}$ with simple pole in $s = \{1, 2\}$.

If $n = 0$ and $k \neq 0$,

$$\zeta_T(s) = 4\sum_{l,j}(l+\tfrac{3}{2})^{-s}(l-j+1)\frac{1-q^{2k(2j+1)}}{1-q^{2k}}$$
$$= \tfrac{4}{1-q^{2k}}\left(\tfrac{1}{2}\,\zeta(s-2) - \left(\tfrac{1}{2}+\tfrac{1}{1-q^{4k}}\right)\zeta(s-1) + \tfrac{q^{4k}}{(1-q^{4k})^2\log q^{4k}}\,\zeta(s)\right) + \text{hol. function}\ ,$$

which has meromorphic extention on $\mathbb{C}$ with simple pole in $s = \{1,2,3\}$.

If $n \neq 0$ and $k = 0$,

$$\zeta_T(s) = 4\sum_{l,j}(l+\tfrac{3}{2})^{-s}(2j+1)\frac{1-q^{2n(l-j+1)}}{1-q^{2n}}$$
$$= \tfrac{4}{1-q^{2n}}\left\{\zeta(s-2) - \left(1+\tfrac{2q^{2n}}{1-q^{2n}}\right)\zeta(s-1) + \tfrac{2q^{2n}}{1-q^{2n}}\left(1+\tfrac{q^{2n}}{(1-q^{2n})\log q^{2n}}\right)\zeta(s)\right\} + \text{hol. fun.}\ ,$$

which has meromorphic extention on $\mathbb{C}$ with simple pole in $s = \{1,2,3\}$.

Finally, if both n and k are zero we get (cf. equation (7.3)),

$$\zeta_T(s) = \tfrac{4}{3}\big\{\zeta(s-3) - \zeta(s-1)\big\}\ ,$$

and this is meromorphic with simple poles in $\{2,4\}$. Thus, the part of the dimension spectrum in the region $\operatorname{Re} s > 2$ consists at most of the two points $\{3,4\}$ and both are simple poles.

Since we have considered the enlarged algebra $P\mathfrak{A}Q + \mathcal{J}$, it sufficies to prove that there exists an $a \in \Psi^0$ whose zeta-function is singular in both points $s = 3$ and $s = 4$. We take $a = x_2x_2^*$. From the definition

$$\tilde{\pi}(x_2x_2^*)\,|l,m_1,m_2;j\rangle_\pm = (1-q^{2(l-\epsilon-j+m_2)})\,|l,m_1,m_2;j\rangle_\pm\ .$$

Then, modulo functions that are holomorphic when $\operatorname{Re} s > 2$, we have

$$\zeta_{x_2x_2^*}(s) \sim \zeta_{\tilde{\pi}(x_2x_2^*)}(s) = \zeta_1(s) - 2\sum_{l,j,m_1,m_2}(l+\tfrac{3}{2})^{-s}q^{2(l-\epsilon-j+m_2)} \sim \tfrac{4}{3}\,\zeta(s-3) - \tfrac{4}{1-q^4}\,\zeta(s-2)\ .$$

This proves the first part of the proposition, that is $\Sigma \cap \{\operatorname{Re} s > 2\} = \{3,4\}$.

The proof of equation (8.2) is based on the observation that the residue in $s = 4$ of ζ_T, for T a basis element of $P\mathfrak{A}Q$, is zero unless $T = 1$. That is, it depends only on the image of T under the map sending β, A and F to 0 while $\alpha \mapsto e^{i\phi}$ and $B \mapsto e^{i\theta}$. Composing this map with $\tilde{\pi}$ we get the morphism $\sigma : \mathcal{A}(S^4_q) \to \mathcal{A}(S^1)$ of the proposition and that

$$⨍ a|D|^{-4} \propto \int_0^{2\pi}\sigma(a)\mathrm{d}\theta\ .$$

The equality $⨍ |D|^{-4} = \frac{4}{3}$ fixes the proportionality constant. □

9 The Real structure

Classically, if $(\mathcal{A}(M), \mathcal{H}, D, \gamma)$ is the canonical spectral triple associated with a 4-dimensional spin manifold M, there exists an antilinear isometry J on $\mathcal{H}$, named the *real structure*, satisfying the following compatibility condition

$$J^2 = -1 \ , \qquad J\gamma = \gamma J \ , \qquad JD = DJ \ . \tag{9.1}$$

There are also two additional conditions involving the coordinate algebra $\mathcal{A}(M)$:

$$[a, JbJ^{-1}] = 0 \ , \qquad [[D,a], JbJ^{-1}] = 0 \ , \qquad \forall\, a, b \in \mathcal{A}(M) \ . \tag{9.2}$$

The real structure on S^4 is equivariant and equivariance is sufficient to determine J.

In the deformed situation one has to be careful on how to implement equivariance. Let us start with the working hypothesis that equivariance for J is the requirement that it satisfies $Jh = S(h)^* J$ for all $h \in U_q(so(5))$. Then, consider the Casimir operator $\mathcal{C}_1$ given in equation (3.4). This operator commutes with with J since $S(\mathcal{C}_1)^* = \mathcal{C}_1$ and from its expression, $\mathcal{C}_1 \,|l, m_1, m_2; j\rangle = (q^{2j+1} + q^{-2j-1})\,|l, m_1, m_2; j\rangle$, we conclude that J leaves the index j invariant. Compatibility with γ and D in Equation (9.1) and equivariance with respect to $h = K_1$ and $h' = K_2$ yields

$$J\,|l, m_1, m_2; j\rangle_\pm = c_\pm(l, m_1, m_2; j)\,|l, -m_1, -m_2; j\rangle_\pm \ ,$$

with some constants $c_\pm$ to be determined. Equivariance with respect to $h = E_1$ implies

$$c_\pm(l, m_1, m_2; j) = (-1)^{m_1+1/2} q^{m_1} c_\pm(l, m_2; j) \ .$$

For $h = E_2$, looking at the piece diagonal in j we deduce that the dependence on m_2 is through a factor q^{3m_2}; and looking at the piece shifting j by ± 1 we conclude that

$$c_\pm(l, m_1, m_2; j) = (-1)^{j+m_1} q^{m_1+3m_2} c_\pm(l) \ .$$

Such an operator J cannot be antiunitary unless $q = 1$. At $q = 1$ the antiunitarity condition requires that $c_\pm(l) \in U(1)$ and modulo a unitary equivalence we can choose $c_\pm(l) = i^{2l+1}$. In conclusion for $q = 1$ the operator

$$J\,|l, m_1, m_2; j\rangle_\pm = i^{2l+1}(-1)^{j+m_1}\,|l, -m_1, -m_2; j\rangle_\pm \ , \tag{9.3}$$

is *the* real structure on S^4 (modulo a unitary equivalence).

For $q \neq 1$ we keep (9.3) as the real structure and notice that conditions (9.1) are satisfied, but J does no longer satisfies the requirement $Jh = S(h)^* J$ for all $h \in U_q(so(5))$. Nevertheless, J is the antiunitary part of an antilinear operator T that has this property. The antilinear operator T defined by

$$T\,|l, m_1, m_2; j\rangle_\pm = i^{2l+1}(-1)^{j+m_1} q^{m_1+3m_2}\,|l, -m_1, -m_2; j\rangle_\pm \ ,$$

has the J in (9.3) as antiunitary part and it is *equivariant*, i.e. it is such that $Th = S(h)^* T$ for all $h \in U_q(so(5))$.

Next, we turn to the conditions (9.2) that in the present case we can satisfy only modulo the ideal $\mathcal{J}$ defined by (8.1). We stress that $\mathcal{J}$ is strictly larger that the ideal of smoothing operators used in the cases of $SU_q(2)$ in [11] and of Podleś spheres in [10, 9].

Proposition 9.1. *Let J be the antilinear isometry given by (9.3). Then,*

$$[a, JbJ] \in \mathcal{J} \ , \quad [[D, a], JbJ] \in \mathcal{J} \ , \qquad \forall \, a, b \in \mathcal{A}(S^4_q) \ .$$

Proof. We lift J and D to the Hilbert space $\hat{\mathcal{H}}$ defined in Section 8.1, as follows:

$$\begin{aligned} \hat{J} \, \|l, m_1, m_2; j\rangle\!\rangle_{\pm} &= i^{2l+1}(-1)^{j+m_1} \, \|l, m_1, m_2; j\rangle\!\rangle_{\pm} \ , \\ \hat{D} \, \|l, m_1, m_2; j\rangle\!\rangle_{\pm} &= (l + \tfrac{3}{2}) \, \|l, m_1, m_2; j\rangle\!\rangle_{\mp} \ . \end{aligned}$$

Notice that $\hat{J}^2 = -1$ on $\hat{\mathcal{H}}$ (thanks to the phase i^{2l+1} that is irrelevant when restricted to $\mathcal{H}$).

Let now $\{\alpha, \beta, \alpha^*, \beta^*, A, B, B^*\}$ be the operators defined in Section 8.1, generators of the algebra $\mathcal{A}(SU_q(2)) \otimes \mathcal{A}(S^2_q)$. Due to Proposition 8.4 it is enough to prove that for all pairs (a, b) of such generators, the commutators $[a, \hat{J}b\hat{J}]$ and $[[\hat{D}, a], \hat{J}b\hat{J}]$ are weighted shifts with weight which are bounded by q^{2j}. From

$$[\hat{D}, \alpha] = \tfrac{1}{2} \, \alpha \hat{F} \ , \quad [\hat{D}, \beta] = \tfrac{1}{2} \, \beta \hat{F} \ , \quad [\hat{D}, A] = 0 \ , \quad [\hat{D}, B] = B \hat{F} \ ,$$

the condition on $[[\hat{D}, a], \hat{J}b\hat{J}]$ follows from the same condition on $[a, \hat{J}b\hat{J}]$, and we have to compute only the latter commutators.

Since $[a, \hat{J}b^*\hat{J}] = -[a^*, (\hat{J}b\hat{J})^*]^*$ and $[b, \hat{J}a\hat{J}] = \hat{J}[a, \hat{J}b\hat{J}]\hat{J}$, we have to check the 16 combinations in the following table.

$b \backslash a$	α	α^*	β	β^*	A	B	B^*
α	•	•	×	×	•	•	•
β			•	×	•	•	•
B					•	•	•
A					•		

By direct computations one shows that bullets in the table correspond to vanishing commutators. On the other hand, the commutators corresponding to the crosses in the table are given, on the subspace with $j - |m_1| \in \mathbb{N}$, by

$$\begin{aligned} [\beta^*, \hat{J}\alpha\hat{J}] \, \|l, m_1, m_2; j\rangle\!\rangle_{\pm} &= q^{j+m_1} \Big\{ \sqrt{1 - q^{2(j-m_1+1)}} - \sqrt{1 - q^{2(j-m_1)}} \Big\} \, \|l, m_1, m_2; j\rangle\!\rangle_{\pm} \\ [\beta, \hat{J}\alpha\hat{J}] \, \|l, m_1, m_2; j\rangle\!\rangle_{\pm} &= -[\beta^*, \hat{J}\alpha\hat{J}] \, \|l+1, m_1 - 1, m_2; j+1\rangle\!\rangle_{\pm} \ , \\ [\beta^*, \hat{J}\beta\hat{J}] \, \|l, m_1, m_2; j\rangle\!\rangle_{\pm} &= -[2] q^{2j} \, \|l, m_1 + 1, m_2; j\rangle\!\rangle_{\pm} \ . \end{aligned}$$

Since $1 - u \leq \sqrt{1 - u} \leq 1$ for all $u \in [0, 1]$, we have that

$$0 \leq q^{j+m_1} \Big\{ \sqrt{1 - q^{2(j-m_1+1)}} - \sqrt{1 - q^{2(j-m_1)}} \Big\} \leq q^{j+m_1}(1 - 1 + q^{2(j-m_1)}) \leq q^{2j} \ .$$

Then, all three non-zero commutators are weighted shifts with weights bounded by q^{2j}. □

The impossibility to restrict the ideal $\mathcal{J}$ to the ideal of smoothing operators in the previous Proposition follows from the following Lemma.

Lemma 9.2. *The operator* $[[D, x_2], Jx_2J]$ is *not* a smoothing operator.

Proof. Define

$$f(l, j, m_1, m_2) := {}_{\mp}\langle l, m_1, m_2; j|[[D, x_2], Jx_2J]|l, m_1, m_2; j\rangle_{\pm} \ .$$

If $[[D, x_2], Jx_2J]$ were smoothing, then $\lim_{l\to\infty} f(l, \frac{1}{2}, \frac{1}{2}, -l)$ should be zero. By direct computation one finds instead that

$$\lim_{l\to\infty} f(l, \tfrac{1}{2}, \tfrac{1}{2}, -l) = -2q^{-3} \neq 0 \ .$$

□

Acknowledgements

G.L. wishes to thank the I. Newton Institute for Mathematical Sciences in Cambridge for hospitality and support during the final stage of this project.

References

[1] M. Cahen and S. Gutt, *Spin structures on compact simply connected Riemannian symmetric spaces*, S. Stevin Journal 62 (1988) 209-242.

[2] A. Chakrabarti, *$SO(5)_q$ and contraction: Chevalley basis representations for q-generic and root of unity*, J. Math. Phys. 35 (1994) 4247-4267.

[3] A. Connes, *Noncommutative Geometry*, Academic Press (1994).

[4] A. Connes, *Gravity coupled with matter and the foundation of non-commutative geometry*, Commun. Math. Phys. 182 (1996) 155-176.

[5] A. Connes, *Cyclic Cohomology, Quantum group Symmetries and the Local Index Formula for $SU_q(2)$*, J. Inst. Math. Jussieu 3 (2004) 17-68.

[6] A. Connes and H. Moscovici, *The local index formula in noncommutative geometry*, Geom. Funct. Anal. 5 (1995), no. 2, 174-243.

[7] P.S. Chakraborty and A. Pal, *Equivariant spectral triples on the quantum $SU(2)$ group*, K-Theory 28 (2003) 107-126.

[8] F. D'Andrea and L. Dąbrowski, *Local Index Formula on the Equatorial Podleś Sphere*, Lett. Math. Phys. 75 (2006) 235-254.

[9] L. Dąbrowski, F. D'Andrea, G. Landi and E. Wagner, *Dirac operators on all Podleś spheres*, `math.QA/0606480`; J. Noncomm. Geom., in press.

[10] L. Dąbrowski, G. Landi, M. Paschke and A. Sitarz, *The spectral geometry of the equatorial Podles sphere*, Comptes Rendus Acad. Sci. Paris, Ser. I 340 (2005) 819-822.

[11] L. Dąbrowski, G. Landi, A. Sitarz, W. van Suijlekom and J.C. Várilly, *The Dirac operator on* $SU_q(2)$, Comm. Math. Phys. 259 (2005) 729-759.

[12] L. Dąbrowski, G. Landi, A. Sitarz, W. van Suijlekom and J.C. Várilly, *The local index formula for* $SU_q(2)$, K-Theory 35 (2005) 375-394.

[13] L. Dąbrowski and A. Sitarz, *Dirac operator on the standard Podleś quantum sphere*, Noncommutative geometry and quantum groups (Warsaw, 2001), 49-58, Banach Center Publ. 61, Polish Acad. Sci. (2003).

[14] N.Yu. Reshetikhin, L. Takhtadzhyan and L.D. Fadeev, *Quantization of Let groups and Lie algebras*, Leningrad Math. J. 1 (1990) 193-225.

[15] J.M. Gracia-Bondía, J.C. Várilly and H. Figueroa, *Elements of noncommutative geometry*, Birkhäuser (2001).

[16] A. Klimyk and K. Schmüdgen, *Quantum groups and their representations*, Springer (1997).

[17] E. Hawkins and G. Landi, *Fredholm Modules for Quantum Euclidean Spheres*, J. Geom. Phys. 49 (2004) 272-293.

[18] P. Podleś, *Quantum spheres*, Lett. Math. Phys. 14 (1987) 193-202.

[19] A. Sitarz, *Equivariant spectral triples*, in: Noncommutative Geometry and Quantum Groups, Banach Centre Publications 61 (Warszawa, 2003) 231-263.

[20] S.L. Woronowicz, *Twisted* $SU(2)$ *group. An example of a non-commutative differential calculus*, Publ. Res. Inst. Math. Sci. 23 (1987) 117-181.