

QUANTISATION OF TWISTOR THEORY BY COCYCLE TWIST

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ABSTRACT. We present the main ingredients of twistor theory leading up to and including the Penrose-Ward transform in a coordinate algebra form which we can then ‘quantise’ by means of a functorial cocycle twist. The quantum algebras for the conformal group, twistor space \mathbb{CP}^3 , compactified Minkowski space $\mathbb{CM}^\#$ and the twistor correspondence space are obtained along with their canonical quantum differential calculi, both in a local form and in a global $*$ -algebra formulation which even in the classical commutative case provides a useful alternative to the formulation in terms of projective varieties. We outline how the Penrose-Ward transform then quantises. As an example, we show that the pull-back of the tautological bundle on $\mathbb{CM}^\#$ pulls back to the basic instanton on $S^4 \subset \mathbb{CM}^\#$ and that this observation quantises to obtain the Connes-Landi instanton on θ -deformed S^4 as the pull-back of the tautological bundle on our θ -deformed $\mathbb{CM}^\#$. We likewise quantise the fibration $\mathbb{CP}^3 \rightarrow S^4$ and use it to construct the bundle on θ -deformed \mathbb{CP}^3 that maps over under the transform to the θ -deformed instanton.

1. INTRODUCTION AND PRELIMINARIES

There has been a lot of interest in recent years in the ‘quantisation’ of space-time (in which the algebra of coordinates x_μ is noncommutative), among them one class of examples of the Heisenberg form

$$[x_\mu, x_\nu] = i\theta_{\mu\nu}$$

where the deformation parameter is an antisymmetric tensor or (when placed in canonical form) a single parameter θ . One of the motivations here is from the effective theory of the ends of open strings in a fixed D-brane[18] and in this context a lot of attention has been drawn to the existence of noncommutative instantons and other nontrivial noncommutative geometry that emerges, see [15] and references therein to a large literature. One also has θ -versions of S^4 coming out of considerations of cyclic cohomology in noncommutative geometry (used to characterise what a noncommutative 4-sphere should be), see notably [5, 10].

Date: January, 2007.

1991 Mathematics Subject Classification. 58B32, 58B34, 20C05.

Key words and phrases. Twistors, quantum group, flag varieties, instantons, noncommutative geometry.

The work was mainly completed while S.M. was visiting July-December 2006 at the Isaac Newton Institute, Cambridge, which both authors thank for support.

In the present paper we show that underlying and bringing together these constructions is in fact a systematic theory of what could be called θ -deformed or ‘quantum’ twistor theory. Thus we introduce noncommutative versions of conformal complexified space-time $\mathbb{CM}^\#$, of twistor space \mathbb{CP}^3 as well as of the twistor correspondence space \mathcal{F}_{12} of 1-2-flags in \mathbb{C}^4 used in the Penrose-Ward transform [17, 19]. Our approach is a general one but we do make contact for specific parameter values with some previous ideas on what should be noncommutative twistor space, notably with [9, 8] even though these works approach the problem entirely differently. In our approach we canonically find not just the noncommutative coordinate algebras but their algebras of differential forms, indeed because our quantisation takes the form of a ‘quantisation functor’ we find in principle the noncommutative versions of all suitably covariant constructions. Likewise, inside our θ -deformed $\mathbb{CM}^\#$ we find (again for certain parameter values) exactly the θ -deformed S^4 of [5] as well as its differential calculus.

While the quantisation of twistor theory is our main motivation, most of the present paper is in fact concerned with properly setting up the classical theory from the ‘right’ point of view after which quantisation follows functorially. We provide in this paper *two* classical points of view, both of interest. The first is purely local and corresponds in physics to ordinary (complex) Minkowski space as the flat ‘affine’ part of $\mathbb{CM}^\#$. Quantisation at this level gives the kind of noncommutative space-time mentioned above which can therefore be viewed as a local ‘patch’ of the actual noncommutative geometry. The actual varieties $\mathbb{CM}^\#$ and \mathbb{CP}^3 are however projective varieties and cannot therefore be simply described by generators and relations in algebraic geometry, rather one should pass to the ‘homogeneous coordinate algebras’ corresponding to the affine spaces $\widetilde{\mathbb{CM}^\#}$, $\widetilde{\mathbb{CP}^3} = \mathbb{C}^4$ that project on removing zero and quotienting by an action of \mathbb{C}^* to the projective varieties of interest. Let us call this the ‘conventional approach’. We explain the classical situation in this approach in Sections 1.1, 2 below, and quantise it (including the relevant quantum group of conformal transformations and the algebra of differential forms) in Sections 4,5. The classical Sections 1.1, 2 here are not intended to be anything new but to provide a lightning introduction to the classical theory and an immediate coordinate algebra reformulation for those unfamiliar either with twistors or with algebraic groups. The quantum Sections 4,5 contain the new results in this stream of the paper and provide a more or less complete solution to the basic noncommutative differential geometry at the level of the quantum homogeneous coordinate algebras $\mathbb{C}_F[\widetilde{\mathbb{CM}^\#}]$, $\mathbb{C}_F[\widetilde{\mathbb{CP}^3}]$ etc. Here F is a 2-cocycle which is the general quantisation data in the cocycle twisting method [11, 12] that we use.

Our second approach even to classical twistor theory is a novel one suggested in fact from quantum theory. We call this the unitary or $*$ -algebraic formulation

of our projective varieties $\mathbb{C}\mathbb{M}^\#$, $\mathbb{C}\mathbb{P}^3$ as real manifolds, setting aside that they are projective varieties. The idea is that mathematically $\mathbb{C}\mathbb{M}^\#$ is the Grassmannian of 2-planes in \mathbb{C}^4 and every point in it can therefore be viewed not as a 2-plane but as a self-adjoint rank two projector P that picks out the two-plane as the eigenspace of eigenvalue 1. Working directly with such projectors as a coordinatisation of $\mathbb{C}\mathbb{M}^\#$, its commutative coordinate $*$ -algebra is therefore given by 16 generators P^μ_ν with relations that $P.P = P$ as an algebra-valued matrix, $\text{Tr } P = 2$ and the $*$ -operation $P^\mu_\nu^* = P^\nu_\mu$. Similarly $\mathbb{C}\mathbb{P}^3$ is the commutative $*$ -algebra with a matrix of generators Q^μ_ν , the relations $Q.Q = Q$, $\text{Tr } Q = 1$ and the $*$ -operation $Q^\mu_\nu^* = Q^\nu_\mu$. One may proceed similarly for all classical flag varieties. The merit of this approach is that if one forgets the $*$ -structure one has affine varieties defined simply by generators and relations (they are the complexifications of our original projective varieties viewed as real manifolds), while the $*$ -structure picks out the real forms that are $\mathbb{C}\mathbb{M}^\#$, $\mathbb{C}\mathbb{P}^3$ as real manifolds in our approach (these cannot themselves be described simply by generators and relations). Finally, the complex structure of our projective varieties appears now in real terms as a structure on the cotangent bundle. This amounts to a new approach to projective geometry suggested by our theory for classical flag varieties and provides a second stream in the paper starting in Section 3. Note that there is no simple algebraic formula for change of coordinates from describing a 2-plane as a 2-form and as a rank 2 projector, so the projector coordinates have a very different flavour from those usually used for $\mathbb{C}\mathbb{M}^\#$, $\mathbb{C}\mathbb{P}^3$. For example the tautological vector bundles in these coordinates are now immediate to write down and we find that the pull-back of the tautological one on $\mathbb{C}\mathbb{M}^\#$ to a natural S^4 contained in it is exactly the instanton bundle given by the known projector for S^4 (it is the analogue of the Bott projector that gives the basic monopole bundle on S^2). We explain this calculation in detail in Section 3.1. The Lorentzian version is also mentioned and we find that Penrose's diamond compactification of Minkowski space arises very naturally in these coordinates. In Section 3.2 we explain the known fibration $\mathbb{C}\mathbb{P}^3 \rightarrow S^4$ in our new approach, used to construct an auxiliary bundle that maps over under the Penrose-Ward transform to the basic instanton.

The second merit of our approach is that just as commutative C^* -algebras correspond to (locally compact) topological spaces, quantisation has a precise meaning as a noncommutative $*$ -algebra with (in principle) C^* -algebra completion. Moreover, one does not need to consider completions but may work at the $*$ -algebra level, as has been shown amply in the last two decades in the theory of quantum groups [11]. The quantisation of all flag varieties, indeed of all varieties defined by 'matrix' type relations on a matrix of generators is given in Section 6, with the quantum tautological bundle looked at explicitly in Section 6.1. Our quantum algebra $\mathbb{C}_F[\mathbb{C}\mathbb{M}^\#]$ actually has three independent real parameters in the unitary

case and takes a ‘Weyl form’ with phase factor commutation relations (see Proposition 6.3). We also show that only a 1-parameter subfamily gives a natural quantum S^4 and in this case we recover exactly the θ -deformed S^4 and its instanton as in [5, 10], now from a different point of view as ‘pull back’ from our θ -deformed $\mathbb{CM}^\#$.

Finally, while our main results are about the coordinate algebras and differential geometry behind twistor theory in the classical and quantum cases, we look in Section 7,8 at enough of the deeper theory to see that our methods are compatible also with the Penrose-Ward transform and ADHM construction respectively. In these sections we concentrate on the classical theory but formulated in a manner that is then ‘quantised’ by our functorial method. Since their formulation in noncommutative geometry is not fully developed we avoid for example the necessity of the implicit complex structures. We also expect our results to be compatible with another approach to the quantum version based on groupoid C^* -algebras [4]. Although we only sketch the quantum version, we do show that our formulation includes for example the quantum basic instanton as would be expected. A full account of the quantum Penrose-Ward transform including an explicit treatment of the noncommutative complex structure is deferred to a sequel.

1.1. Conformal space-time. Classically, complex Minkowski space \mathbb{CM} is the four-dimensional affine vector space \mathbb{C}^4 equipped with the metric

$$ds^2 = 2(dzd\tilde{z} - dwd\tilde{w})$$

written in double null coordinates [14]. Certain conformal transformations, such as isometries and dilations, are defined globally on \mathbb{CM} , whereas others, such as inversions and reflections, may map a light cone to infinity and *vice versa*. In order to obtain a group of globally defined conformal transformations, we adjoin a light cone at infinity to obtain *compactified* Minkowski space, usually denoted $\mathbb{CM}^\#$.

This compactification is achieved geometrically as follows (and is just the Plücker embedding, see for example [14, 20, 3]). One observes that the exterior algebra $\Lambda^2\mathbb{C}^4$ can be identified with the set of 4×4 matrices as

$$x = \begin{pmatrix} 0 & s & -w & \tilde{z} \\ -s & 0 & -z & \tilde{w} \\ w & z & 0 & t \\ -\tilde{z} & -\tilde{w} & -t & 0 \end{pmatrix},$$

the points of $\Lambda^2\mathbb{C}^4$ being identified with the six entries $x^{\mu\nu}$, $\mu < \nu$. Then $\mathrm{GL}_4 = \mathrm{GL}(4, \mathbb{C})$ acts from the left on $\Lambda^2\mathbb{C}^4$ by conjugation,

$$x \mapsto axa^t, \quad a \in \mathrm{GL}_4.$$

We note that multiples of the identity act trivially, and that this action preserves the quadratic relation $\det x \equiv (st - z\tilde{z} + w\tilde{w})^2 = 0$. From the point of view of $\Lambda^2\mathbb{C}^4$ this quadric, which we shall denote $\widetilde{\mathbb{CM}^\#}$, is the subset of the form $\{a \wedge b : a, b \in \mathbb{C}^4\} \subset$

$\Lambda^2\mathbb{C}^4$, (the antisymmetric projections of rank-one matrices, i.e. of decomposable elements of the tensor product). We exclude $x = 0$. Note that x of the form

$$x = \begin{pmatrix} 0 & a_{11}a_{22} - a_{21}a_{12} & -(a_{31}a_{12} - a_{11}a_{32}) & a_{11}a_{42} - a_{41}a_{12} \\ -(a_{11}a_{22} - a_{21}a_{12}) & 0 & -(a_{31}a_{22} - a_{21}a_{32}) & a_{21}a_{42} - a_{41}a_{22} \\ a_{31}a_{12} - a_{11}a_{32} & a_{31}a_{22} - a_{21}a_{32} & 0 & a_{31}a_{42} - a_{41}a_{32} \\ -(a_{11}a_{42} - a_{41}a_{12}) & -(a_{21}a_{42} - a_{41}a_{22}) & -(a_{31}a_{42} - a_{41}a_{32}) & 0 \end{pmatrix},$$

or $x^{\mu\nu} = a^{[\mu}b^{\nu]}$ where $a = a_{\cdot 1}, b = a_{\cdot 2}$, automatically has determinant zero. Conversely, if the determinant vanishes then an antisymmetric matrix has this form over \mathbb{C} . To see this, we provide a short proof as follows. Thus, we have to solve

$$\begin{aligned} a_1b_2 - a_2b_1 &= s, & a_1b_3 - a_3b_1 &= -w, & a_1b_4 - a_4b_1 &= \tilde{z} \\ a_2b_3 - a_3b_2 &= -z, & a_2b_4 - a_4b_2 &= \tilde{w}, & a_3b_4 - a_4b_3 &= t. \end{aligned}$$

We refer to the first relation as the (12)-relation, the second as the (13)-relation and so forth. Now if a solution for a_i, b_i exists, we make use of a ‘cycle’ consisting of the (12) b_3 , (23) b_1 , (13) b_2 relations (multiplied as shown) to deduce that

$$a_1b_2b_3 = a_2b_1b_3 + sb_3 = a_3b_1b_2 + sb_3 - zb_1 = a_1b_3b_2 + sb_3 - zb_1 + wb_2$$

hence a linear equation for b . The cycles consisting of the (12) b_4 , (24) b_1 , (14) b_2 relations, the (13) b_4 , (34) b_1 , (14) b_3 relations, and the (23) b_4 , (34) b_2 , (24) b_3 relations give altogether the necessary conditions

$$\begin{pmatrix} 0 & -s & -w & z \\ s & 0 & -\tilde{z} & \tilde{w} \\ w & \tilde{z} & 0 & -t \\ -z & -\tilde{w} & t & 0 \end{pmatrix} \begin{pmatrix} b_4 \\ b_3 \\ b_2 \\ b_1 \end{pmatrix} = 0.$$

The matrix here is not the matrix x above but it has the same determinant. Hence if $\det x = 0$ we know that a nonzero vector b obeying these necessary conditions must exist. We now fix such a vector b , and we know that at least one of its entries must be non-zero. We treat each case in turn. For example, if $b_2 \neq 0$ then from the above analysis, the (12),(23) relations imply the (13) relations. Likewise (12),(24) \Rightarrow (14), (23),(24) \Rightarrow (34). Hence the six original equations to be solved become the three linear equations in four unknowns a_i :

$$a_1b_2 - a_2b_1 = s, \quad a_2b_3 - a_3b_2 = -z, \quad a_2b_4 - a_4b_2 = \tilde{w}$$

with general solution

$$a = \lambda b + b_2^{-1} \begin{pmatrix} s \\ 0 \\ z \\ -\tilde{w} \end{pmatrix}, \quad \lambda \in \mathbb{C}.$$

One proceeds similarly in each of the other cases where a single $b_i \neq 0$. Clearly, adding any multiple of b will not change $a \wedge b$, but we see that apart from this a is

uniquely fixed by a choice of zero mode b of a matrix with the same but permuted entries as x . It follows that every x defines a two-plane in \mathbb{C}^4 spanned by the obtained linearly independent vectors a, b .

Such matrices x with $\det x = 0$ are the orbit under GL_4 of the point where $s = 1, t = z = \tilde{z} = w = \tilde{w} = 0$. It is easily verified that this point has isotropy subgroup \tilde{H} consisting of elements of GL_4 such that $a_{3\mu} = a_{4\mu} = 0$ for $\mu = 1, 2$ and $a_{11}a_{22} - a_{21}a_{12} = 1$. Thus $\widetilde{\mathbb{CM}^\#} = GL_4/\tilde{H}$ where we quotient from the right.

Finally, we may identify conformal space-time $\mathbb{CM}^\#$ with the rays of the above quadric cone $st = z\tilde{z} - w\tilde{w}$ in $\Lambda^2\mathbb{C}^4$, identifying the finite points of space-time with the rays for which $t \neq 0$ (which have coordinates $z, \tilde{z}, w, \tilde{w}$ up to scale): the rays for which $t = 0$ give the light cone at infinity. It follows that the group $\text{PGL}(4, \mathbb{C}) = \text{GL}(4, \mathbb{C})/\mathbb{C}$ acts globally on $\mathbb{CM}^\#$ by conformal transformations and that every conformal transformation arises in this way. Observe that $\mathbb{CM}^\#$ is, in particular, the orbit of the point $s = 1, z = \tilde{z} = w = \tilde{w} = t = 0$ under the action of the conformal group $\text{PGL}(4, \mathbb{C})$. Moreover, by the above result we have that $\mathbb{CM}^\# = \mathcal{F}_2(\mathbb{C}^4)$, the Grassmannian of two-planes in \mathbb{C}^4 .

We may equally identify $\mathbb{CM}^\#$ with the resulting quadric in the projective space \mathbb{CP}^5 by choosing homogeneous coordinates $s, z, \tilde{z}, w, \tilde{w}$ and projective representatives with $t = 0$ and $t = 1$. In doing so, there is no loss of generality in identifying the conformal group $\text{PGL}(4, \mathbb{C})$ with $\text{SL}(4, \mathbb{C})$ by representing each equivalence class with a transformation of unit determinant. Observing that $\Lambda^2\mathbb{C}^4$ has a natural metric

$$\tilde{v} = 2(-dsdt + dzd\tilde{z} - dwd\tilde{w}),$$

we see that $\widetilde{\mathbb{CM}^\#}$ is the null cone through the origin in $\Lambda^2\mathbb{C}^4$. This metric may be restricted to this cone and moreover it descends to give a metric v on $\mathbb{CM}^\#$ [17]. Indeed, choosing a projective representative $t = 1$ of the coordinate patch corresponding to the affine piece of space-time, we have

$$v = 2(dzd\tilde{z} - dwd\tilde{w}),$$

thus recovering the original metric. Similarly, we find the metric on other coordinate patches of $\mathbb{CM}^\#$ by in turn choosing projective representatives $s = 1, z = 1, \tilde{z} = 1, w = 1, \tilde{w} = 1$.

Passing to the level of coordinates algebras let us denote by a_ν^μ the coordinate functions in $\mathbb{C}[GL_4]$ (where we have now rationalised indices so that they are raised and lowered by the metric \tilde{v}) and by $s, t, z, \tilde{z}, w, \tilde{w}$ the coordinates in $\mathbb{C}[\Lambda^2\mathbb{C}^4]$. The algebra $\mathbb{C}[\Lambda^2\mathbb{C}^4]$ is the commutative polynomial algebra on these six generators with no further relations, whereas the algebra $\widetilde{\mathbb{CM}^\#}$ is the quotient by the further relation $st - z\tilde{z} + w\tilde{w} = 0$. (Although we are ultimately interested in the projective geometry of the space described by this algebra, we shall put this point aside for

the moment). In the coordinate algebra (as an affine algebraic variety) we do not see the deletion of the zero point in $\widetilde{\mathbb{C}\mathbb{M}^\#}$.

As explained, $\mathbb{C}[\widetilde{\mathbb{C}\mathbb{M}^\#}]$ is essentially the algebra of functions on the orbit of the point $s = 1, t = z = \tilde{z} = w = \tilde{w} = 0$ in $\Lambda^2\mathbb{C}^4$ under the action of GL_4 . The specification of a GL_4/\tilde{H} element that moves the base point to a point of $\mathbb{C}\mathbb{M}^\#$ becomes at the level of coordinate algebras the map

$$\phi : \mathbb{C}[\widetilde{\mathbb{C}\mathbb{M}^\#}] \cong \mathbb{C}[\mathrm{GL}_4]^{\mathbb{C}[\tilde{H}]}, \quad \phi(x^{\mu\nu}) = a_1^\mu a_2^\nu - a_1^\nu a_2^\mu.$$

As shown, the relation $st = z\tilde{z} - w\tilde{w}$ in $\mathbb{C}[\Lambda^2\mathbb{C}^4]$ automatically holds for the image of the generators, so this map is well-defined. Also in these dual terms there is a left coaction

$$\Delta_L(x^{\mu\nu}) = a_\alpha^\mu a_\beta^\nu \otimes x^{\alpha\beta}$$

of $\mathbb{C}[\mathrm{GL}_4]$ on $\mathbb{C}[\Lambda^2\mathbb{C}^4]$. One should view the orbit base point above as a linear function on $\mathbb{C}[\Lambda^2\mathbb{C}^4]$ that sends $s = 1$ and the rest to zero. Then applying this to Δ_L defines the above map ϕ . By construction, and one may easily check if in doubt, the image of ϕ lies in the fixed subalgebra under the right coaction $\Delta_R = (\mathrm{id} \otimes \pi)\Delta$ of $\mathbb{C}[\tilde{H}]$ on $\mathbb{C}[\mathrm{GL}_4]$, where π is the canonical surjection to

$$\mathbb{C}[\tilde{H}] = \mathbb{C}[\mathrm{GL}_4]/\langle a_1^3 = a_2^3 = a_1^4 = a_2^4 = 0, \quad a_1^1 a_2^2 - a_1^2 a_2^1 = 1 \rangle$$

and Δ is the matrix coproduct of $\mathbb{C}[\mathrm{GL}_4]$.

Ultimately we want the same picture for the projective variety $\mathbb{C}\mathbb{M}^\#$. In order to do this the usual route in algebraic geometry is to work with rational functions instead of polynomials in the homogeneous coordinate algebra and make the quotient by \mathbb{C}^* as the subalgebra of total degree zero. Rational functions here may have poles so to be more precise, for any open set $U \subset X$ in a projective variety, we take the algebra

$$\mathcal{O}_X(U) = \{a/b \mid a, b \in \mathbb{C}[\tilde{X}], \quad a, b \text{ same degree, } b(x) \neq 0 \quad \forall x \in U\}$$

where $\mathbb{C}[\tilde{X}]$ denotes the homogeneous coordinate algebra (the coordinate algebra functions of the affine (i.e. non-projective) version \tilde{X}) and a, b are homogeneous. Doing this for any open set gives a sheaf of algebras. Of particular interest are principal open sets of the form $U_f = \{x \in X \mid f(x) \neq 0\}$ for any nonzero homogeneous f . Then

$$\mathcal{O}_X(U_f) = \mathbb{C}[\tilde{X}][f^{-1}]_0$$

where we adjoin f^{-1} to the homogeneous coordinate algebra and 0 denotes the degree zero part. In the case of PGL_4 we in fact have a coordinate algebra of regular functions

$$\mathbb{C}[\mathrm{PGL}_4] := \mathbb{C}[\mathrm{GL}_4]^{\mathbb{C}[\mathbb{C}^*]} = \mathbb{C}[\mathrm{GL}_4]_0$$

constructed as the affine algebra analogue of $\mathrm{GL}_4/\mathbb{C}^*$. It is an affine variety and not projective (yet one could view its coordinate algebra as $\mathcal{O}_{\mathbb{C}\mathbb{P}^{15}}(U_D)$ where D is the

determinant). In contrast, $\mathbb{C}\mathbb{M}^\#$ is projective and we have to work with sheaves. For example, U_t is the open set where $t \neq 0$. Then

$$\mathcal{O}_{\mathbb{C}\mathbb{M}^\#}(U_t) := \mathbb{C}[\widetilde{\mathbb{C}\mathbb{M}^\#}][t^{-1}]_0.$$

There is a natural inclusion

$$\mathbb{C}[\mathbb{C}\mathbb{M}] \rightarrow \mathcal{O}_{\mathbb{C}\mathbb{M}^\#}(U_t)$$

of the coordinate algebra of affine Minkowski space $\mathbb{C}\mathbb{M}$ (polynomials in the four coordinate functions x_1, x_2, x_3, x_4 on \mathbb{C}^4 with no further relations) given by

$$x_1 \mapsto t^{-1}z, \quad x_2 \mapsto t^{-1}\tilde{z}, \quad x_3 \mapsto t^{-1}w, \quad x_4 \mapsto t^{-1}\tilde{w}.$$

This is the coordinate algebra version of identifying the affine piece of space-time $\mathbb{C}\mathbb{M}$ with the patch of $\mathbb{C}\mathbb{M}^\#$ for which $t \neq 0$.

2. TWISTOR SPACE AND THE CORRESPONDENCE SPACE

Next we give the coordinate picture for twistor space $T = \mathbb{C}\mathbb{P}^3 = \mathcal{F}_1(\mathbb{C}^4)$ of lines in \mathbb{C}^4 . As a partial flag variety this is also known to be a homogeneous space. At the non-projective level we just mean $\tilde{T} = \mathbb{C}^4$ with coordinates $Z = (Z^\mu)$ and the origin deleted. This is of course a homogeneous space for GL_4 and may be identified as the orbit of the point $Z^1 = 1, Z^2 = Z^3 = Z^4 = 0$: the isotropy subgroup \tilde{K} consists of elements such that $a_1^1 = 1, a_1^2 = a_1^3 = a_1^4 = 0$, giving the identification $\tilde{T} = \mathrm{GL}_4/\tilde{K}$ (again we quotient from the right).

Again we pass to the coordinate algebra level. At this level we do not see the deletion of the origin, so we define $\mathbb{C}[\tilde{T}] = \mathbb{C}[\mathbb{C}^4]$. We have an isomorphism

$$\phi : \mathbb{C}[\tilde{T}] \rightarrow \mathbb{C}[\mathrm{GL}_4]^{\mathbb{C}[\tilde{K}]}, \quad \phi(Z^\mu) = a_1^\mu$$

according to a left coaction

$$\Delta_L(Z^\mu) = a_\alpha^\mu \otimes Z^\alpha.$$

One should view the principal orbit base point as a linear function on \mathbb{C}^4 that sends $Z^1 = 1$ and the rest to zero: as before, applying this to the coaction Δ_L defines ϕ as the dual of the orbit construction. It is easily verified that the image of this isomorphism is exactly the subalgebra of $\mathbb{C}[\mathrm{GL}_4]$ fixed under

$$\mathbb{C}[\tilde{K}] = \mathbb{C}[\mathrm{GL}_4]/\langle a_1^1 = 1, a_1^2 = a_1^3 = a_1^4 = 0 \rangle$$

by the right coaction on $\mathbb{C}[\mathrm{GL}_4]$ given by projection from the coproduct Δ of $\mathbb{C}[\mathrm{GL}_4]$.

Finally we introduce a new space \mathcal{F} , the ‘correspondence space’, as follows. For each point $Z \in T$ we define the associated ‘ α -plane’

$$\hat{Z} = \{x \in \mathbb{C}\mathbb{M}^\# \mid x \wedge Z = x^{[\mu\nu} Z^{\rho]} = 0\} \subset \mathbb{C}\mathbb{M}^\#.$$

The condition on x is independent both of the scale of x and of Z , so constructions may be done ‘upstairs’ in terms of matrices, but we also have a well-defined map at the projective level. The α -plane \hat{Z} contains for example all points in the quadric of the form $W \wedge Z$ as $W \in \widetilde{\mathbb{C}^4}$ varies. Any multiple of Z does not contribute, so \hat{Z} is a 3-dimensional space in $\mathbb{CM}^\#$ and hence a \mathbb{CP}^2 contained in $\mathbb{CM}^\#$ (the image of a two-dimensional subspace of \mathbb{CM} under the conformal compactification, hence the term ‘plane’).

Explicitly, the condition $x \wedge Z = 0$ in our coordinates is:

$$(1) \quad \begin{aligned} \tilde{z}Z^3 + wZ^4 - tZ^1 &= 0, & \tilde{w}Z^3 + zZ^4 - tZ^2 &= 0, \\ sZ^3 + wZ^2 - zZ^1 &= 0, & sZ^4 - \tilde{z}Z^2 + \tilde{w}Z^1 &= 0. \end{aligned}$$

If $t \neq 0$ one can check that the second pair of equations is implied by the first (given the quadric relation $\det x = 0$), so generically we have two equations for four unknowns as expected. Moreover, at each point of a plane \hat{Z} we have in the Lorentzian case the property that $\nu(A, B) = 0$ for any two tangent vectors to the plane (where ν is the aforementioned metric on $\mathbb{CM}^\#$). One may check that the plane \hat{Z} defined by $x \wedge Z = 0$ is null if and only if the bivector $\pi = A \wedge B$ defined at each point of the plane (determined up to scale) is self-dual with respect to the Hodge $*$ -operator. We note that one may also construct ‘ β -planes’, for which the tangent bivector is anti-self-dual: these are instead parameterized by 3-forms in the role of Z .

Conversely, given any point $x \in \mathbb{CM}^\#$ we define the ‘line’

$$\hat{x} = \{Z \in T \mid x \in \hat{Z}\} = \{Z \in T \mid x^{[\mu\nu} Z^{\rho]} = 0\} \subset T.$$

We have seen that we may write $x = a \wedge b$ and indeed $Z = \lambda a + \mu b$ solves this equation for all $\lambda, \mu \in \mathbb{C}$. This is a plane in $\tilde{T} = \mathbb{C}^4$ which projects to a \mathbb{CP}^1 contained in \mathbb{CP}^3 , thus each \hat{x} is a projective line in twistor space $T = \mathbb{CP}^3$.

We then define \mathcal{F} to be the set of pairs (Z, x) , where $x \in \mathbb{CM}^\#$ and $Z \in \mathbb{CP}^3 = T$ are such that $x \in \hat{Z}$ (or equivalently $Z \in \hat{x}$), i.e. such that $x \wedge Z = 0$. This space fibres naturally over both space-time and twistor space *via* the obvious projections

$$(2) \quad \begin{array}{ccc} & \mathcal{F} & \\ p \swarrow & & \searrow q \\ \mathbb{CP}^3 & & \mathbb{CM}^\#. \end{array}$$

Clearly we have

$$\hat{Z} = q(p^{-1}(Z)), \quad \hat{x} = p(q^{-1}(x)).$$

It is also clear that the defining relation of \mathcal{F} is preserved under the action of GL_4 .

From the Grassmannian point of view, $Z \in \mathbb{CP}^3 = \mathcal{F}_1(\mathbb{C}^4)$ is a line in \mathbb{C}^4 and $\hat{Z} \subset \mathcal{F}_2(\mathbb{C}^4)$ is the set of two-planes in \mathbb{C}^4 containing this line. Moreover,

$\hat{x} \subset \mathcal{F}_1(\mathbb{C}^4)$ consists of all one-dimensional subspaces of \mathbb{C}^4 contained in x viewed as a two-plane in \mathbb{C}^4 . Then \mathcal{F} is the partial flag variety

$$\mathcal{F}_{1,2}(\mathbb{C}^4)$$

of subspaces $\mathbb{C} \subset \mathbb{C}^2 \subset \mathbb{C}^4$. Here $x \in \mathbb{C}\mathbb{M}^\# = \mathcal{F}_2(\mathbb{C}^4)$ is a plane in \mathbb{C}^4 and $Z \in T = \mathbb{C}\mathbb{P}^3$ is a line in \mathbb{C}^4 contained in this plane. From this point of view it is known that the homology $H_4(\mathbb{C}\mathbb{M}^\#)$ is two-dimensional and indeed one of the generators is given by any \hat{Z} (they are all homologous and parameterized by $\mathbb{C}\mathbb{P}^3$). The other generator is given by a similar construction of ‘ β -planes’ [17, 14] with correspondence space $\mathcal{F}_{2,3}(\mathbb{C}^4)$ and with $\mathcal{F}_3(\mathbb{C}^4) = (\mathbb{C}\mathbb{P}^3)^*$. Likewise, the homology $H_2(\mathbb{C}\mathbb{P}^3)$ is one-dimensional and indeed any flag \hat{x} is a generator (they are all homologous and parameterized by $\mathbb{C}\mathbb{M}^\#$). For more details on the geometry of this construction, see [14]. More on the algebraic description can be found in [3].

This \mathcal{F} is known to be a homogeneous space. Moreover, $\tilde{\mathcal{F}}$ (the non-projective version of \mathcal{F}) can be viewed as a quadric in $(\Lambda^2\mathbb{C}^4) \otimes \mathbb{C}^4$ and hence the orbit under the action of GL_4 of the point in $\tilde{\mathcal{F}}$ where $s = 1, Z^1 = 1$ and all other coordinates are zero. The isotropy subgroup \tilde{R} of this point consists of those $a \in \mathrm{GL}_4$ such that $a_1^2 = 0, a_\mu^3 = a_\mu^4 = 0$ for $\mu = 1, 2$ and $a_1^1 = a_2^2 = 1$. As one should expect, $\tilde{R} = \tilde{H} \cap \tilde{K}$.

At the level of the coordinate rings, the identification of $\tilde{\mathcal{F}}$ with the quadric in $(\Lambda^2\mathbb{C}^4) \otimes \mathbb{C}^4$ gives the definition of $\mathbb{C}[\tilde{\mathcal{F}}]$ as the polynomials in the coordinate functions $x^{\mu\nu}, Z^\alpha$, modulo the quadric relations and the relations (1). That it is an affine homogeneous space is the isomorphism

$$\phi : \mathbb{C}[\tilde{\mathcal{F}}] \rightarrow \mathbb{C}[\mathrm{GL}_4]^{\mathbb{C}[\tilde{R}]}, \quad \phi(x^{\mu\nu} \otimes Z^\beta) = (a_1^\mu a_2^\nu - a_1^\nu a_2^\mu) a_1^\beta,$$

according to the left coaction

$$\Delta_L(x^{\mu\nu} \otimes Z^\sigma) = a_\alpha^\mu a_\beta^\nu a_\gamma^\sigma \otimes (x^{\alpha\beta} \otimes Z^\gamma).$$

The image of ϕ is the invariant subalgebra under the right coaction of

$$\mathbb{C}[\tilde{R}] = \mathbb{C}[\mathrm{GL}_4] / \langle a_1^2 = 0, \quad a_1^1 a_2^2 = 1, \quad a_1^3 = a_2^3 = a_1^4 = a_2^4 = 0 \rangle.$$

on $\mathbb{C}[\mathrm{GL}_4]$ given by projection from the coproduct Δ of $\mathbb{C}[\mathrm{GL}_4]$.

3. SL_4 AND UNITARY VERSIONS

As discussed, the group GL_4 acts on $\widetilde{\mathbb{C}\mathbb{M}^\#} = \{x \in \Lambda^2\mathbb{C}^4 \mid \det x = 0\}$ by conjugation, $x \mapsto axa^t$, and since multiples of the identity act trivially on $\widetilde{\mathbb{C}\mathbb{M}^\#}$, this picture descends to an action of the projective group PGL_4 on the quotient space $\mathbb{C}\mathbb{M}^\#$. Our approach accordingly was to work at the non-projective level in order for the algebraic structure to have an affine form and pass at the end to the projective spaces $\mathbb{C}\mathbb{M}^\#, T$ and \mathcal{F} as rational functions of total degree zero.

If one wants to work with these spaces directly as homogeneous spaces one may do this as well, so that $\mathbb{C}\mathbb{M}^\# = \mathrm{PGL}_4/P\tilde{H}$, and so on. From a mathematician's point of view one may equally well define

$$\mathbb{C}\mathbb{M}^\# = \mathcal{F}_2(\mathbb{C}^4) = \mathrm{GL}_4/H, \quad T = \mathcal{F}_1(\mathbb{C}^4) = \mathrm{GL}_4/K, \quad \mathcal{F} = \mathcal{F}_{1,2}(\mathbb{C}^4) = \mathrm{GL}_4/R,$$

$$H = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}, \quad K = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}, \quad R = H \cap K = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix},$$

where the overall GL_4 determinants are non-zero. Here H is slightly bigger than the subgroup \tilde{H} we had before. As homogeneous spaces, $\mathbb{C}\mathbb{M}^\#, T$ and \mathcal{F} carry left actions of GL_4 which are essentially identical to those given above at both the PGL_4 and at the non-projective level.

One equally well has

$$\mathbb{C}\mathbb{M}^\# = \mathcal{F}_2(\mathbb{C}^4) = \mathrm{SL}_4/H, \quad T = \mathcal{F}_1(\mathbb{C}^4) = \mathrm{SL}_4/K, \quad \mathcal{F} = \mathcal{F}_{1,2}(\mathbb{C}^4) = \mathrm{SL}_4/R$$

where H, K, R are as above but now viewed in SL_4 , and now $\mathbb{C}\mathbb{M}^\#, T$ and \mathcal{F} carry canonical left actions of SL_4 similar to those previously described.

These versions would be the more usual in algebraic geometry but at the coordinate level one does need to then work with an appropriate construction to obtain these projective or quasi-projective varieties. For example, if one simply computes the invariant functions $\mathbb{C}[\mathrm{SL}_4]^{\mathbb{C}[K]}$ etc. as affine varieties, one will not find enough functions.

As an alternative, we mention a version where we consider all our spaces in the double fibration as real manifolds, and express this algebraically in terms of $*$ -structures on our algebras. Thus for example $\mathbb{C}\mathbb{P}^3$ is a real 6-dimensional manifold which we construct by complexifying it to an affine 6-dimensional variety over \mathbb{C} , but we remember its real form by means of a $*$ -involution on the complex algebra. The $*$ -algebras in this approach can then in principle be completed to an operator-algebra setting and the required quotients made sense of in this context, though we shall not carry out this last step here.

In this case the most natural choice is

$$\mathbb{C}\mathbb{M}^\# = \mathrm{SU}_4/H, \quad T = \mathrm{SU}_4/K, \quad \mathcal{F} = \mathrm{SU}_4/R$$

$H = \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2)), \quad K = \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(3)), \quad R = H \cap K = \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(2)),$
 embedded in the obvious diagonal way into SU_4 . As homogeneous spaces one has canonical actions now of SU_4 from the left on $\mathbb{C}\mathbb{M}^\#, T$ and \mathcal{F} .

For the coordinate algebraic version one expresses SU_4 by generators a_ν^μ , the determinant relation and in addition the $*$ -structure

$$a^\dagger = Sa$$

where \dagger denotes transpose and $*$ on each matrix generator entry, $(a_\nu^\mu)^\dagger = (a_\nu^\mu)^*$, and S is the Hopf algebra antipode characterised by $aS(a) = (Sa)a = \text{id}$. This is as for any compact group or quantum group coordinate algebra. The coordinate algebras of the subgroups are similarly defined as $*$ -Hopf algebras.

There is also a natural $*$ -structure on twistor space. To see this let us write it in the form

$$T = \mathbb{CP}^3 = \{Q \in M_4(\mathbb{C}), \quad Q = Q^\dagger, \quad Q^2 = Q, \quad \text{Tr } Q = 1\}$$

in terms of Hermitian-conjugation \dagger . Thus \mathbb{CP}^3 is the space of Hermitian rank one projectors on \mathbb{C}^4 . Such projectors can be written explicitly in the form $Q_\nu^\mu = Z^\mu \bar{Z}^\nu$ for some complex vector Z of modulus 1 and determined only up to a $U(1)$ normalisation. Thus $\mathbb{CP}^3 = S^7/U(1)$ as a real 6-dimensional manifold. In this description the left action of SU_4 is given by conjugation in $M_4(\mathbb{C})$, i.e. by unitary transformation of the Z and its inverse on \bar{Z} . One can exhibit the identification with the homogeneous space picture, as the orbit of the projector $\text{diag}(1, 0, 0, 0)$ (i.e. $Z = (1, 0, 0, 0) = Z^*$). The isotropy group of this is the intersection of SU_4 with $U(1) \times U(3)$ as stated.

The coordinate $*$ -algebra version is

$$\mathbb{C}[\mathbb{CP}^3] = \mathbb{C}[Q_\nu^\mu] / \langle Q^2 = Q, \quad \text{Tr } Q = 1, \quad Q = Q^\dagger \rangle$$

with the last equation now as a definition of the $*$ -algebra structure via $\dagger = ()^{*t}$. We can also realise this as the degree zero subalgebra,

$$\mathbb{C}[\mathbb{CP}_3] = \mathbb{C}[S^7]_0, \quad \mathbb{C}[S^7] = \mathbb{C}[Z^\mu, Z^{\nu*}] / \langle \sum_\mu Z^{\mu*} Z^\mu = 1 \rangle,$$

where Z, Z^* are two sets of generators related by the $*$ -involution. The grading is given by $\text{deg}(Z) = 1$ and $\text{deg}(Z^*) = -1$, corresponding to the $U(1)$ action on Z and its inverse on Z^* . Finally, the left coaction is

$$\Delta_L(Z^\mu) = a_\alpha^\mu \otimes Z^\alpha, \quad \Delta_L(Z^{\mu*}) = S a_\mu^\alpha \otimes Z^{\alpha*},$$

as required for a unitary coaction of a Hopf $*$ -algebra on a $*$ -algebra, as well as to preserve the relation.

We have similar $*$ -algebra versions of \mathcal{F} and $\mathbb{CM}^\#$ as well. Thus

$$\mathbb{C}[\mathbb{CM}^\#] = \mathbb{C}[P_\nu^\mu] / \langle P^2 = P, \quad \text{Tr } P = 2, \quad P = P^\dagger \rangle$$

in terms of a rank two projector matrix of generators, while

$$\begin{aligned} \mathbb{C}[\mathcal{F}] &= \mathbb{C}[Q_\nu^\mu, P_\nu^\mu] / \langle Q^2 = Q, \quad P^2 = P, \quad \text{Tr } Q = 1, \quad \text{Tr } P = 2, \quad PQ = Q = QP \rangle \\ & \quad Q = Q^\dagger, \quad P = P^\dagger. \end{aligned}$$

We see that our $*$ -algebra approach to flag varieties has a ‘quantum logic’ form. In physical terms, the fact that P, Q commute as matrices (or matrices of generators in the coordinate algebras) means that they may be jointly diagonalised, while

$QP = Q$ says that the 1-eigenvectors of Q are a subset of the 1-eigenvectors of P (equivalently, $PQ = Q$ says that the 0-eigenvectors of P are a subset of the 0-eigenvectors of Q). Thus the line which is the image of Q is contained in the plane which is the image of P : this is of course the defining property of pairs of projectors $(Q, P) \in \mathcal{F} = \mathcal{F}_{1,2}(\mathbb{C}^4)$. Clearly this approach works for all flag varieties $\mathcal{F}_{k_1, \dots, k_r}(\mathbb{C}^n)$ of $k_1 < \dots < k_r$ -dimensional planes in \mathbb{C}^n as the $*$ -algebra with $n \times n$ matrices P_i of generators

$$\mathbb{C}[\mathcal{F}_{k_1, \dots, k_r}] = \mathbb{C}[P_i^\mu; i = 1, \dots, r] / \langle P_i^2 = P_i, \text{Tr } P_i = k_i, P_i P_{i+1} = P_i = P_{i+1} P_i, \rangle$$

$$P_i = P_i^\dagger.$$

In this setting all our algebras are now complex affine varieties (with $*$ -structure) and we can expect to be able to work algebraically. Thus one may expect for example that $\mathbb{C}[\mathbb{C}\mathbb{P}^3] = \mathbb{C}[\text{SU}_4]^{\mathbb{C}[\text{S}(\text{U}(1) \times \text{U}(3))]}$ (similarly for other flag varieties) and indeed we may identify the above generators and relations in the relevant invariant subalgebra of $\mathbb{C}[\text{SU}_4]$. This is the same approach as was successfully used for the Hopf fibration construction of $\mathbb{C}\mathbb{P}^1 = S^2 = \text{SU}(2)/\text{U}(1)$ as $\mathbb{C}[\text{SU}_2]^{\mathbb{C}[\text{U}(1)]}$, namely as $\mathbb{C}[\text{SL}_2]^{\mathbb{C}[\mathbb{C}^*]}$ with suitable $*$ -algebra structures [13]. Note that one should not confuse such Hopf algebra ('GIT') quotients with complex algebraic geometry quotients, which are more complicated to define and typically quasi-projective. Finally, we observe that in this approach the tautological bundle of rank k over a flag variety $\mathcal{F}_k(\mathbb{C}^n)$ appears tautologically as a matrix generator viewed as a projection $P \in M_n(\mathbb{C}[\mathcal{F}_k])$. The classical picture is that the flag variety with this tautological bundle is universal for rank k vector bundles.

3.1. Tautological bundle on $\mathbb{C}\mathbb{M}^\#$ and the instanton as its Grassmann connection. Here we conclude with a result that is surely known to some, but apparently not well-known even at the classical level, and yet drops out very naturally in our $*$ -algebra approach. We show that the tautological bundle on $\mathbb{C}\mathbb{M}^\#$ restricts in a natural way to $S^4 \subset \mathbb{C}\mathbb{M}^\#$, where it becomes the 1-instanton bundle, and for which the Grassmann connection associated to the projector is the 1-instanton.

We first explain the Grassmann connection for a projective module \mathcal{E} over an algebra A . We suppose that $\mathcal{E} = A^n e$ where $e \in M_n(A)$ is a projection matrix acting on an A -valued row vector. Thus every element $v \in \mathcal{E}$ takes the form

$$v = \tilde{v} \cdot e = \tilde{v}_j e_j = (\tilde{v}_k) e_{kj} e_j,$$

where $e_j = e_j \in A^n$ span \mathcal{E} over A and $\tilde{v}_i \in A$. The action of the Grassmann connection is the exterior derivative on components followed by projection back down to \mathcal{E} :

$$\nabla v = \nabla(\tilde{v} \cdot e) = (d(\tilde{v} \cdot e))e = ((d\tilde{v}_j) e_{jk} + \tilde{v}_j d e_{jk}) e_k = (d\tilde{v} + \tilde{v} d e) e.$$

One readily checks that this is both well-defined and a connection in the sense that

$$\nabla(av) = da.v + a\nabla v, \quad \forall a \in A, \quad v \in \mathcal{E},$$

and that its curvature operator $F = \nabla^2$ on sections is

$$F(v) = F(\tilde{v}.e) = (\tilde{v}.de.de).e$$

As a warm-up example we compute the Grassmann connection for the tautological bundle on $A = \mathbb{C}[\mathbb{C}\mathbb{P}^1]$. Here $e = Q$, the projection matrix of coordinates in our $*$ -algebraic set-up:

$$e = Q = \begin{pmatrix} a & z \\ z^* & 1-a \end{pmatrix}; \quad a(1-a) = zz^*, \quad a \in \mathbb{R}, \quad z \in \mathbb{C},$$

where $s = a - \frac{1}{2}$ and $z = x + iy$ describes a usual sphere of radius $1/2$ in Cartesian coordinates (x, y, s) . We note that

$$(1-2a)da = dz.z^* + zdz^*$$

allows to eliminate da in the open patch where $a \neq \frac{1}{2}$ (i.e. if we delete the north pole of S^2). Then

$$de.de = \begin{pmatrix} da & dz \\ dz^* & -da \end{pmatrix} \begin{pmatrix} da & dz \\ dz^* & -da \end{pmatrix} = dzdz^* \begin{pmatrix} 1 & -\frac{2z}{1-2a} \\ -\frac{2z^*}{1-2a} & -1 \end{pmatrix} = \frac{dzdz^*}{1-2a}(1-2e)$$

and hence

$$F(\tilde{v}.e) = -\frac{dzdz^*}{1-2a}\tilde{v}.e.$$

In other words, F acts as a multiple of the identity operator on $\mathcal{E} = A^2.e$ and this multiple has the standard form for the charge 1 monopole connection if one converts to usual Cartesian coordinates. We conclude that the Grassmann connection for the tautological bundle on $\mathbb{C}\mathbb{P}^1$ is the standard 1-monopole. This is surely well-known. The q -deformed version of this statement can be found in [7] provided one identifies the projector introduced there as the defining projection matrix of generators for $\mathbb{C}_q[\mathbb{C}\mathbb{P}^1] = \mathbb{C}_q[\text{SL}_2]^{\mathbb{C}[t, t^{-1}]}$ as a $*$ -algebra in the q -version of the above picture (the projector there obeys $\text{Tr}_q(e) = 1$ where we use the q -trace). Note that if one looks for any algebra A containing potentially non-commuting elements a, z, z^* and a projection e of the form above with $\text{Tr}(e) = 1$, one immediately finds that these elements commute and obey the sphere relation as above. If one performs the same exercise with the q -trace, one finds exactly the four relations of the standard q -sphere as a $*$ -algebra.

Next, we look in detail at $A = \mathbb{C}[\mathbb{C}\mathbb{M}^\#]$ in our projector $*$ -algebra picture. This has a 4×4 matrix of generators which we write in block form

$$P = \begin{pmatrix} A & B \\ B^\dagger & D \end{pmatrix}, \quad \text{Tr}A + \text{Tr}D = 2, \quad A^\dagger = A, \quad D^\dagger = D$$

$$(3) \quad A(1-A) = BB^\dagger, \quad D(1-D) = B^\dagger B$$

$$(4) \quad \left(A - \frac{1}{2}\right)B + B\left(D - \frac{1}{2}\right) = 0,$$

where we have written out the requirement that P be a Hermitian A -valued projection without making any assumptions on the $*$ -algebra A (so that these formulae also apply to any noncommutative version of $\mathbb{C}[\mathbb{C}\mathbb{M}^\#]$ in our approach).

To proceed further, it is useful to write

$$A = a + \alpha \cdot \sigma, \quad B = t + ix \cdot \sigma, \quad B^\dagger = t^* - ix^* \cdot \sigma, \quad D = 1 - a + \delta \cdot \sigma$$

in terms of usual Pauli matrices $\sigma_1, \sigma_2, \sigma_3$. We recall that these are traceless and Hermitian, so a, α, δ are self-adjoint, whilst $t, x_i, i = 1, 2, 3$ are not necessarily so and are subject to (3)-(4).

Proposition 3.1. *The commutative $*$ -algebra $\mathbb{C}[\mathbb{C}\mathbb{M}^\#]$ is defined by the above generators $a = a^*, \alpha = \alpha^*, \delta = \delta^*, t, t^*, x, x^*$ and the relations*

$$tt^* + xx^* = a(1-a) - \alpha \cdot \alpha, \quad (1-2a)(\alpha - \delta) = 2i(t^*x - tx^*), \quad (1-2a)(\alpha + \delta) = 2ix \times x^* \\ \alpha \cdot \alpha = \delta \cdot \delta, \quad (\alpha + \delta) \cdot x = 0, \quad (\alpha + \delta)t = (\alpha - \delta) \times x$$

Proof. This is a direct computation of (3)-(4) under the assumption that the generators commute. Writing our matrices in the form above, equations (3) become

$$a(1-a) - \alpha \cdot \alpha + (1-2a)\alpha \cdot \sigma = tt^* + x \cdot \sigma x^* \cdot \sigma + i(xt^* - tx^*) \cdot \sigma \\ (1-a)a - \delta \cdot \delta - (1-2a)\delta \cdot \sigma = t^*t + x^* \cdot \sigma x \cdot \sigma + i(t^*x - x^*t) \cdot \sigma.$$

Taking the sum and difference of these equations and in each case the parts proportional to 1 (which is the same on both right hand sides) and the parts proportional to σ (where the difference of the right hand sides is proportional to $x \times x^*$) gives four of the stated equations (all except those involving terms $(\alpha + \delta) \cdot x$ and $(\alpha + \delta)t$). We employ the key identity

$$\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk} \sigma_k,$$

where ϵ with $\epsilon_{123} = 1$ is the totally antisymmetric tensor used in the definition of the vector cross product. Meanwhile, (4) becomes

$$i\alpha \cdot \sigma x \cdot \sigma + \alpha t \cdot \sigma + ix \cdot \sigma \delta \cdot \sigma + t\delta \cdot \sigma = 0$$

after cancellations, and this supplies the remaining two relations using our key identity. \square

We see that in the open set where $a \neq \frac{1}{2}$ we have α, δ fully determined by the second and third relations, so the only free variables are the complex generators t, \vec{x} , with a determined from the first equation. The complex affine variety generated by the independent variables x, x^*, t, t^* modulo the first three equations is reducible; the second ‘auxiliary’ line of equations makes $\mathbb{C}_F[\mathbb{C}\mathbb{M}^\#]$ reducible (we conjecture this).

Proposition 3.2. *There is a natural $*$ -algebra quotient $\mathbb{C}[S^4]$ of $\mathbb{C}[\mathbb{C}\mathbb{M}^\#]$ defined by the additional relations $x^* = x$, $t^* = t$ and $\alpha = \delta = 0$. The tautological projector of $\mathbb{C}[\mathbb{C}\mathbb{M}^\#]$ becomes*

$$e = \begin{pmatrix} a & t + \imath x \cdot \sigma \\ t - \imath x \cdot \sigma & 1 - a \end{pmatrix} \in M_2(\mathbb{C}[S^4]).$$

The Grassmann connection on the projective module $\mathcal{E} = \mathbb{C}[S^4]^4 e$ is the 1-instanton with local form

$$(F \wedge F)(\tilde{\nu}.e) = -4! \frac{dtd^3x}{1-2a} \tilde{\nu}.e$$

Proof. All relations in Proposition 3.1 are trivially satisfied in the quotient except $tt^* + xx^* = a(1-a)$, which is that of a 4-sphere of radius $\frac{1}{2}$ in usual Cartesian coordinates (t, x, s) if we set $s = a - \frac{1}{2}$. The image e of the projector exhibits $S^4 \subset \mathbb{C}\mathbb{M}^\#$ as a projective variety in our $*$ -algebra projector approach. We interpret this as providing a projective module over $\mathbb{C}[S^4]$, the pull-back of the tautological bundle on $\mathbb{C}\mathbb{M}^\#$ from a geometrical point of view. To compute the curvature of its Grassmann connection we first note that

$$dx \cdot \sigma dx \cdot \sigma = \imath(dx \times dx) \cdot \sigma, \quad (dx \times dx) \cdot (dx \times dx) = 0,$$

$$(dx \times dx) \times (dx \times dx) = dx \times (dx \times dx) = 0, \quad dx \cdot (dx \times dx) = 3!d^3x,$$

since 1-forms anticommute and since any four products of the dx_i vanish. Now we have

$$\begin{aligned} dede &= \begin{pmatrix} da & dt + \imath dx \cdot \sigma \\ dt - \imath dx \cdot \sigma & -da \end{pmatrix} \begin{pmatrix} da & dt + \imath dx \cdot \sigma \\ dt - \imath dx \cdot \sigma & -da \end{pmatrix} \\ &= \begin{pmatrix} -2\imath dtdx \cdot \sigma + \imath dx \times dx \cdot \sigma & 2dadt + 2\imath dadx \cdot \sigma \\ -2dadt + 2\imath dadx \cdot \sigma & 2\imath dtdx \cdot \sigma + \imath dx \times dx \cdot \sigma \end{pmatrix} \end{aligned}$$

and we square this matrix to find that

$$(de)^4 = \begin{pmatrix} 1 & -\frac{2}{1-2a}(t + \imath x \cdot \sigma) \\ -\frac{2}{1-2a}(t - \imath x \cdot \sigma) & -1 \end{pmatrix} 4!dtd^3x = \frac{1-2e}{1-2a} 4!dtd^3x$$

after substantial computation. For example, since $(da)^2 = 0$, the 1-1 entry is

$$-(dx \times dx - 2dtdx) \cdot \sigma (dx \times dx - 2dtdx) \cdot \sigma = 2dtdx \cdot (dx \times dx) + 2(dx \times dx) \cdot dtdx = 4!dtd^3x$$

where only the cross-terms contribute on account of the second observation above and the fact that $(dt)^2 = 0$. For the 1-2 entry we have similarly that

$$\begin{aligned} &2\imath da(dx \times dx - 2dtdx) \cdot \sigma (dt + \imath dx \cdot \sigma) + 2\imath da(dt + \imath dx \cdot \sigma)(dx \times dx + 2dtdx) \cdot \sigma \\ &= 12\imath dadt(dx \times dx) \cdot \sigma - 4da(dx \times dx) \cdot \sigma dx \cdot \sigma \\ &= -\frac{24}{1-2a} 2\imath x \cdot \sigma dtd^3x - \frac{8}{1-2a} t 3!dtd^3x = -\frac{2}{1-2a} (t + \imath x \cdot \sigma) 4!dtd^3x, \end{aligned}$$

where at the end we substitute

$$da = \frac{2(tdt + x \cdot dx)}{1 - 2a}$$

and note that

$$x \cdot dx(dx \times dx) = x_i dx_i \epsilon_{jkm} dx_j dx_k = 2x_m d^3x,$$

since in the sum over i only $i = m$ can contribute for a nonzero 3-form. The 2-2 and 2-1 entries are analogous and left to the reader. We conclude that $(de)^4 e$ acts on $\mathbb{C}[S^4]^4$ from the right as a multiple of the identity as stated. \square

One may check that $F = dede.e$ is anti-self-dual with respect to the usual Euclidean Hodge $*$ -operator. Note also that the off-diagonal corners of e are precisely a general quaternion $q = t + ix \cdot \sigma$ and its conjugate, which relates our approach to the more conventional point of view on the 1-instanton. However, that is not our starting point as we come from $\mathbb{CM}^\#$, where the top right corner is a general 2×2 matrix B and the bottom left corner its adjoint.

If instead we let B be an arbitrary Hermitian matrix in the form

$$B = t + x \cdot \sigma$$

(i.e. replace ix above by x and let $t^* = t, x^* = x$) then the quotient $t^* = t, x^* = x, \alpha = -\delta = 2tx/(1 - 2a)$ gives us

$$s^2 + t^2 + \left(1 + \frac{t^2}{s^2}\right)x^2 = \frac{1}{4}$$

when $s = a - \frac{1}{2} \neq 0$ and $tx_i = 0, t^2 + x^2 + \alpha^2 = \frac{1}{4}$ when $s = 0$. We can also approach this case directly from (3)-(4). We have to find Hermitian A, D , or equivalently $S = A - \frac{1}{2}, T = D - \frac{1}{2}$ with $\text{Tr}(S + T) = 0$ and $S^2 = T^2 = \frac{1}{4} - B^2$. Since B is Hermitian it has real eigenvalues and indeed after conjugation we can rotate x to $|x|$ times a vector in the 3-direction, i.e. B has eigenvalues $t \pm |x|$. It follows that square roots S, T exist precisely when

$$(5) \quad |t| + |x| \leq \frac{1}{2}$$

and are diagonal in the same basis as was B , hence they necessarily commute with B . In this case (4) becomes that $(S + T).B = 0$. If B has two nonzero eigenvalues then $S + T = 0$. If B has one nonzero eigenvalue then $S + T$ has a zero eigenvalue, but the trace condition then again implies $S + T = 0$. If $B = 0$ our equations reduce to those for two self-adjoint 2×2 projectors A, D with traces summing to two. In summary, if $B \neq 0$ there exists a projector of the form required if and only if (t, x) lies in the diamond region (5), with $S = T$ and a fourfold choice (the choice of root for each eigenvalue) of S in the interior. These observations about the moduli of projectors with B Hermitian means that the corresponding quotient of $\mathbb{C}[\mathbb{CM}^\#]$ is a fourfold cover of a diamond region in affine Minkowski space-time (viewed as the space of 2×2 Hermitian matrices). The diamond is conformally equivalent

to a compactification of all of usual Minkowski space (the Penrose diagram for Minkowski space), while its fourfold covering reminds us of the Penrose diagram for a black-white hole pair. It is the analogue of the disk that one obtains by projecting S^4 onto its first two coordinates. One may in principle compute the connection associated to the pull-back of the tautological bundle to this region as well as the 4-dimensional object of which it is a projection. The most natural version of this is to slightly change the problem to two 2×2 Hermitian matrices S, B with $S^2 + B^2 = \frac{1}{4}$ (a ‘matrix circle’), a variety which will be described elsewhere.

Note also that in both cases $D = 1 - A$ and if we suppose this at the outset our equations including (4) and (3) simplify to

$$(6) \quad [A, B] = [B, B^\dagger] = 0, \quad A(1 - A) = BB^\dagger, \quad A = A^\dagger.$$

In fact this is the same calculation as for any potentially noncommutative $\mathbb{C}\mathbb{P}^1$ which (if we use the usual trace) is forced to be commutative as mentioned above.

Finally, returning to the general case of $\mathbb{C}[\mathbb{C}\mathbb{M}^\#]$, we have emphasised ‘Cartesian coordinates’ with different signatures. From a twistor point of view it is more natural to work with the four matrix entries of B as the natural twistor coordinates. This will also be key when we quantise. Thus equivalently to Proposition 3.1 we write

$$(7) \quad B = \begin{pmatrix} z & \tilde{w} \\ w & \tilde{z} \end{pmatrix}, \quad A = \begin{pmatrix} a + \alpha_3 & \alpha \\ \alpha^* & a - \alpha_3 \end{pmatrix}, \quad D = \begin{pmatrix} 1 - a + \delta_3 & \delta \\ \delta^* & 1 - a - \delta_3 \end{pmatrix},$$

where $a = a^*, \alpha_3 = \alpha_3^*, \delta_3 = \delta_3^*$ as before but all our other notations are different. In particular, $\alpha, \alpha^*, \delta, \delta^*, z, z^*, w, w^*, \tilde{z}, \tilde{z}^*, \tilde{w}, \tilde{w}^*$ are now complex generators.

Corollary 3.3. *The relations of $\mathbb{C}[\mathbb{C}\mathbb{M}^\#]$ in these new notations appear as*

$$\begin{aligned} zz^* + ww^* + \tilde{z}\tilde{z}^* + \tilde{w}\tilde{w}^* &= 2(a(1 - a) - \alpha\alpha^* - \alpha_3^2) \\ (1 - 2a)\alpha &= zw^* + \tilde{w}\tilde{z}^*, \quad (1 - 2a)\delta = -z^*\tilde{w} - \tilde{z}w^* \\ (1 - 2a)(\alpha_3 + \delta_3) &= \tilde{w}\tilde{w}^* - ww^*, \quad (1 - 2a)(\alpha_3 - \delta_3) = zz^* - \tilde{z}\tilde{z}^* \end{aligned}$$

and the auxiliary relations

$$\begin{aligned} \alpha\alpha^* + \alpha_3^2 &= \delta\delta^* + \delta_3^2 \\ (\alpha_3 + \delta_3) \begin{pmatrix} z \\ \tilde{z} \end{pmatrix} &= \begin{pmatrix} -\alpha & -\delta^* \\ \delta & \alpha^* \end{pmatrix} \begin{pmatrix} w \\ \tilde{w} \end{pmatrix}, \quad (\alpha_3 - \delta_3) \begin{pmatrix} w \\ \tilde{w} \end{pmatrix} = \begin{pmatrix} -\alpha^* & -\delta^* \\ \delta & \alpha \end{pmatrix} \begin{pmatrix} z \\ \tilde{z} \end{pmatrix}. \end{aligned}$$

Moreover, $S^4 \subset \mathbb{C}\mathbb{M}^\#$ appears as the $*$ -algebra quotient $\mathbb{C}[S^4]$ defined by $w^* = -\tilde{w}$, $z^* = \tilde{z}$ and $\alpha = \delta = 0$.

Proof. It is actually easier to recompute these, but of course this is just a change of generators from the equations in Proposition 3.1. \square

Note that these affine $*$ -algebra coordinates are more similar in spirit but not the same as those for $\mathbb{C}\mathbb{M}^\#$ as a projective quadric in Section 1.

3.2. Twistor space \mathbb{CP}^3 in the $*$ -algebra approach. For completeness, we also describe \mathbb{CP}^3 more explicitly in our affine $*$ -algebra approach. As a warm up we start with \mathbb{CP}^2 since \mathbb{CP}^1 is already covered above. Thus $\mathbb{C}[\mathbb{CP}^2]$ has a trace 1 matrix of generators

$$Q = \begin{pmatrix} a & x & y \\ x^* & b & z \\ y^* & z^* & c \end{pmatrix}, \quad a + b + c = 1$$

with a, b, c self-adjoint.

Proposition 3.4. $\mathbb{C}[\mathbb{CP}^2]$ is the algebra with the above matrix of generators with $a + b + c = 1$ and the projector relations

$$\begin{aligned} x^*x &= ab, & y^*y &= ac, & z^*z &= bc \\ cx &= yz^*, & by &= xz, & az &= x^*y. \end{aligned}$$

Proof. First of all, the ‘projector relations’ $Q^2 = Q$ come out as the second line of relations stated and the relations

$$a(1-a) = X + Y, \quad b(1-b) = X + Z, \quad c(1-c) = Y + Z$$

where we use the shorthand $X = x^*x, Y = y^*y, Z = z^*z$. We subtract these from each other to obtain

$$X - Z = (a - c)b, \quad Y - Z = (a - b)c, \quad X - Y = (b - c)a$$

(in fact there are only two independent ones here). Combining with the original relations allows to solve for X, Y, Z as stated. \square

Clearly, if $a \neq 0$ (say), i.e. if we look at $\mathbb{C}[\mathbb{CP}^2][a^{-1}]$, we can regard x, y (and their adjoints) and a, a^{-1}, b, c as generators with the relations

$$(8) \quad x^*x = ab, \quad y^*y = ac, \quad a + b + c = 1$$

and all the other relations become empty. Thus $az = x^*y$ is simply viewed as a definition of z and one may check for example that $z^*za^2 = y^*xx^*y = XY = bca^2$, as needed. Likewise, for example, $ayz^* = yy^*x = Yx = acx$ as required. We can further regard (8) as defining b, c , so the localisation viewed in this way is a punctured S^4 with complex generators x, y , real invertible generator a and the relations

$$x^*x + y^*y = a(1-a),$$

conforming to our expectations for \mathbb{CP}^2 as a complex 2-manifold.

We can also consider setting a, b, c to be real numbers with $a + b + c = 1$ and $b, c > 0, b + c < 1$. The inequalities here are equivalent to $ab, ac > 0$ and $a \neq 0$

(with $a > 0$ necessarily following since if $a < 0$ we would need $b, c < 0$ and hence $a + b + c < 0$, which is not allowed). We then have

$$\mathbb{C}[\mathbb{CP}^2] \Big|_{\substack{b, c > 0 \\ b + c < 1}} = \mathbb{C}[S^1 \times S^1],$$

so the passage to this quotient algebra is geometrically an inclusion $S^1 \times S^1 \subset \mathbb{CP}^2$ with (8) defining the two circles (recall that x, y are complex generators). As the parameters vary the circles vary in size so the general case with a inverted can be viewed in that sense as an inclusion $\mathbb{C}^* \times \mathbb{C}^* \subset \mathbb{CP}^2$. This holds classically as an open dense subset (since \mathbb{CP}^2 is a toric variety). We have the same situation for $\mathbb{C}[\mathbb{CP}^1]$ where there is only one relation $x^*x = a(1 - a)$, i.e. circles $S^1 \subset \mathbb{CP}^1$ of different size as $0 < a < 1$. They are the circles of constant latitude and as a varies in this range they map out \mathbb{C}^* (viewed as S^2 with the north and south pole removed).

We now find similar results for \mathbb{CP}^3 (the general \mathbb{CP}^n case is analogous). We now have a matrix of generators

$$Q = \begin{pmatrix} a & x & y & z \\ x^* & b & w & v \\ y^* & w^* & c & u \\ z^* & v^* & u^* & d \end{pmatrix}, \quad a^* = a, \quad b^* = b, \quad c^* = c, \quad d^* = d, \quad a + b + c + d = 1$$

and make free use of the shorthand notation

$$X = x^*x, \quad Y = y^*y, \quad Z = z^*z, \quad U = u^*u, \quad V = v^*v, \quad W = w^*w.$$

Proposition 3.5. $\mathbb{C}[\mathbb{CP}^3]$ is the commutative $*$ -algebra with generators Q of the form above with $a + b + c + d = 1$ and projector relations

$$a(1 - a) = X + Y + Z, \quad X - U = ab - cd, \quad Y - V = ac - bd, \quad Z - W = ad - bc,$$

$$au = y^*z, \quad av = x^*z, \quad aw = x^*y, \quad bu = w^*v, \quad cv = wu, \quad dw = vu^*,$$

$$cx = yw^*, \quad by = xw, \quad bz = xv, \quad dx = zv^*, \quad dy = zu^*, \quad cz = yu.$$

Proof. We first write out the relations $P^2 = P$ as

$$(9) \quad a(1 - a) = X + Y + Z, \quad b(1 - b) = X + V + W,$$

$$(10) \quad c(1 - c) = Y + U + W, \quad d(1 - d) = Z + U + V,$$

$$(11) \quad yw^* + zv^* = x(c + d), \quad xw + zu^* = y(b + d), \quad xv + yu = z(b + c),$$

$$(12) \quad y^*z + w^*v = u(a + b), \quad x^*z + wu = v(a + c), \quad x^*y + vu^* = w(a + d).$$

We add and subtract several combinations of (9)-(10) to obtain the equivalent four equations stated in the proposition. For example, subtracting (9) gives $(Y - V) + (Z - W) = (c + d)(a - b)$ while subtracting (10) gives $(Y - V) - (Z - W) = (c - d)(a + b)$

and combining these gives the $Y - V$ and $Z - W$ relations stated. Similarly, for the $X - U$ relation. We can also write our three equations as

$$(13) \quad (a+c)(a+d) + X - U = (a+b)(a+d) + Y - V = (a+b)(a+c) + Z - W = a$$

using $a + b + c + d = 1$.

Next, we compute (12) assuming (11), for example

$$\begin{aligned} (a+b)(a+c)u &= (a+c)(y^*z + w^*v) = (a+c)y^*z + w^*(x^*z + wu) \\ &= (a+c)y^*z + Wu + (y^*(b+d) - uz^*)z = y^*z + (W-Z)u \end{aligned}$$

which, using (13), becomes $au = y^*z$. We similarly obtain $av = x^*z, aw = x^*y$. Given these relations, clearly (12) is equivalent to the next three stated equations, which completes the first six equations of this type. Similarly for the remaining six. \square

Lemma 3.6. *In $\mathbb{C}[\mathbb{CP}^3]$ we have*

$$(X-ab)(Y-(ac-bd)) = 0, \quad (X-ab)(Z-(ad-bc)) = 0$$

$$(Y-ac)(X-(ab-cd)) = 0, \quad (Y-ac)(Z-(ad-bc)) = 0$$

$$(Z-ad)(X-(ab-cd)) = 0, \quad (Z-ad)(Y-(ac-bd)) = 0$$

$$(X-ab)(X-b(1-a)) = 0, \quad (Y-ac)(Y-c(1-a)) = 0, \quad (Z-ad)(Z-d(1-a)) = 0$$

$$(X-ab)(X-a(1-b)) = 0, \quad (Y-ac)(Y-a(1-c)) = 0, \quad (Z-ad)(Z-a(1-d)) = 0$$

Proof. For example, $adu = dy^*z = uz^*z = uZ$. In this way one has

$$(X-ab)v = (X-ab)w = (Y-ac)u = (Y-ac)w = (Z-ad)u = (Z-ad)v = 0.$$

Multiplying by u^*, v^*, w^* and replacing U, V, W using Proposition 3.5 give the first two lines of relations. Next, $by = xw$ in Proposition 3.5 implies $b^2Y = XW$ and similarly for bz gives $b^2(Y+Z) = X(V+W)$. We then use (9) to obtain $X^2 - bX + b^2a(1-a)$ which factorises to one of the quadratic equations stated. Similarly, the equations $a^2V = XZ, a^2W = XY$ imply $a^2(V+W) = X(Y+Z)$, which yields the other quadratic equation for X . Similarly for the other quadratic equations. \square

Lemma 3.7. *If we consider the trace one projection Q as a numerical Hermitian matrix of the form above, then*

$$X = ab, \quad Y = ac, \quad Z = ad$$

necessarily holds.

Proof. We use the preceding lemma but regarded as for real numbers (equivalently one can assume that our algebra has no zero divisors). Suppose without loss of generality that $X \neq ab$. Then by the lemma, $V = W = 0$ or $Y = ac - bd$, $Z = ad - bc$. We also have $v = w = 0$ and hence from Proposition 3.5 that $x^*z = x^*y = 0$. We can also deduce from the quadratic equations of X that $a = b$ and $X = a(1 - a)$ or $U = a(1 - 2a) + cd$. We distinguish two cases: (i) $x = 0$ in which case $a = b = 0$ (since $X = a(1 - a) \neq ab = a^2$) and (ii) $x \neq 0$, $y = z = 0$, in which case $a = b \neq 0, 1$, $c + d = 0$. In either case since $b \neq 1$ we have $Y \neq ac$ and $Z \neq ad$, hence $X = ab - cd$ or $U = 0$, $u = 0$ and hence $y^*z = 0$ while $c, d \neq 0$. This means that at most one of x, y, z is non-zero. We can now go through all of the subcases and find a contradiction in every case. Similar arguments prove that $Y = ac, Z = ad$. \square

Let us denote by $\mathbb{C}^-[\mathbb{CP}^3]$ the quotient of $\mathbb{C}[\mathbb{CP}^3]$ by the relations in the lemma. We call it the ‘regular form’ of the coordinate algebra for \mathbb{CP}^3 in our $*$ -algebraic approach and will work with it henceforth. The lemma means that there is no discernible difference (if any) between the $*$ -algebras $\mathbb{C}^-[\mathbb{CP}^3]$ and $\mathbb{C}[\mathbb{CP}^3]$ in the sense that if we were looking at \mathbb{CP}^3 as a set of projector matrices and the above variables as real or complex numbers, we would not see any distinction. (As long as the relevant intersections are transverse the same would then be true in the algebras also, but it is beyond our scope to prove this here.) Moreover, if either x, y, z or u, v, w are made invertible then one can show that the solution in the lemma indeed holds and does not need to be imposed, i.e. $\mathbb{C}[\mathbb{CP}^3]$ and $\mathbb{C}^-[\mathbb{CP}^3]$ have the same localisations in this respect.

Proposition 3.8. $\mathbb{C}^-[\mathbb{CP}^3]$ can be viewed as having generators a, b, c, x, y, z with $a + b + c + d = 1$ and the relations

$$X = ab, \quad Y = ac, \quad Z = ad$$

as well as auxiliary generators u, v, w and auxiliary relations

$$U = cd, \quad V = bd, \quad W = bc,$$

$$\begin{aligned} au = y^*z, \quad av = x^*z \quad aw = x^*y, \quad bu = w^*v, \quad cv = wu, \quad dw = vu^*, \\ cx = yw^*, \quad by = xw, \quad bz = xv, \quad dx = zv^*, \quad dy = zu^*, \quad cz = yu. \end{aligned}$$

If $a \neq 0$ these auxiliary variables and equations are redundant.

Proof. If $a \neq 0$ (i.e. if we work in the algebra with a^{-1} adjoined) we regard three of the auxiliary equations as a definition of u, v, w . We then verify that the other equations hold automatically. The first line is clear since these equations times a^2 were solved in the lemma above. For example, from the next line we have $a^2w^*v = y^*xx^*z = Xy^*z = aby^*z = a^2bu$, as required. Similarly $a(yw^* + zv^*) = yy^*x + zz^*x = (Y + Z)x = a(c + d)x$ as required. \square

From this we see that the ‘patch’ given by inverting a is described by just three independent complex variables x, y, z and one invertible real variable a with the single relation

$$x^*x + y^*y + z^*z = a(1 - a)$$

(the relations stated can be viewed as a definition of b, c, d but we still need $a + b + c + d = 1$), in other words a punctured S^6 where the point $x = y = z = a = 0$ is deleted. This conforms to our expectations for $\mathbb{C}\mathbb{P}^3$ as a complex 3-manifold or real 6-manifold. Of course, our original projector system was symmetric and we could have equally well analysed and presented our algebra in a form adapted to one of $b, c, d \neq 0$.

Finally, we also see that if we set a, b, c, d to actual real values with $a + b + c + d = 1$ and $b, c, d > 0, b + c + d < 1$ (the inequalities here are equivalent to $a \neq 0$ and $ab, ac, ad > 0$) then

$$\mathbb{C}^-[\mathbb{C}\mathbb{P}^3] \Big|_{\substack{b, c, d > 0 \\ b + c + d < 1}} = \mathbb{C}[S^1 \times S^1 \times S^1]$$

for three circles $x^*x = ab, y^*y = ac, z^*z = ad$ and no further relations. This is the analogue in our $*$ -algebra approach of inclusions $S^1 \times S^1 \times S^1 \subset \mathbb{C}\mathbb{P}^3$ and as the circles vary in radius we have part of the fact in the usual picture that $\mathbb{C}\mathbb{P}^3$ is a toric variety (namely that $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* \subset \mathbb{C}\mathbb{P}^3$ is open dense).

We now relate this description of twistor space to the space-time algebras in the previous section. In particular, we note that classically there is a fibration of twistor space over the Euclidean four-sphere, $\mathbb{C}\mathbb{P}^3 \rightarrow S^4$, whose fibre is $\mathbb{C}\mathbb{P}^1$ (see for example [20]). This fibration arises through the observation that each α -plane in $\mathbb{C}\mathbb{M}^\#$ intersects S^4 at a unique point (essentially because there are no null lines in Euclidean signature). The double fibration (2) thus collapses to a single fibration $\mathbb{C}\mathbb{P}^3 \rightarrow S^4$ (making the twistor theory of the real space-time S^4 much easier to study than that of its complex counterpart). To see this we make use of the following nondegenerate antilinear involution on \mathbb{C}^4 ,

$$J(Z) = J(Z^1, Z^2, Z^3, Z^4) := (-\bar{Z}^2, \bar{Z}^1, -\bar{Z}^4, \bar{Z}^3).$$

Once again we recall that points of twistor space are one-dimensional subspaces of \mathbb{C}^4 , whereas points of $\mathbb{C}\mathbb{M}^\#$ are two-dimensional subspaces. Of course, given a 1d subspace (spanned by $Z \in \mathbb{C}^4$), there are many 2d subspaces in which it lies, and these constitute exactly the set $\hat{Z} = \mathbb{C}\mathbb{P}^2$. However, the involution J serves to pick out a unique such 2d subspace, the one spanned by Z and $J(Z)$.

Now recall our ‘quantum logic’ interpretation of the correspondence space \mathcal{F} , as pairs of projectors (Q, P) on \mathbb{C}^4 with Q of rank one and P of rank two such that $QP = Q = PQ$. Then since we have

$$\mathbb{C}\mathbb{P}^3 = S^7/\mathrm{U}(1), \quad S^7 = \{Z^\mu, \bar{Z}^\nu \mid \sum Z^\mu \bar{Z}^\mu = 1\},$$

the involution J extends to one on $\mathbb{C}\mathbb{P}^3$, given by

$$J(Q^\mu{}_\nu) = J(Z^\mu \bar{Z}^\nu) = J(\bar{Z}^\nu)J(Z^\mu).$$

At the level of the coordinate algebra $\mathbb{C}^-[\mathbb{C}\mathbb{P}^3]$ we have the following interpretation.

Lemma 3.9. *There is an antilinear involution $J : \mathbb{C}^-[\mathbb{C}\mathbb{P}^3] \rightarrow \mathbb{C}^-[\mathbb{C}\mathbb{P}^3]$, given in the notation of Proposition 3.8 by*

$$J(a) = b, \quad J(b) = a, \quad J(c) = 1 - (a + b + c),$$

$$J(x) = -x, \quad J(y) = v^*, \quad J(z) = -w^*, \quad J(u) = -u, \quad J(v) = y^*, \quad J(w) = -z^*,$$

Proof. This is by direct computation, noting that if we write

$$Q = \begin{pmatrix} \bar{Z}^1 Z^1 & \bar{Z}^1 Z^2 & \bar{Z}^1 Z^3 & \bar{Z}^1 Z^4 \\ \bar{Z}^2 Z^1 & \bar{Z}^2 Z^2 & \bar{Z}^2 Z^3 & \bar{Z}^2 Z^4 \\ \bar{Z}^3 Z^1 & \bar{Z}^3 Z^2 & \bar{Z}^3 Z^3 & \bar{Z}^3 Z^4 \\ \bar{Z}^4 Z^1 & \bar{Z}^4 Z^2 & \bar{Z}^4 Z^3 & \bar{Z}^4 Z^4 \end{pmatrix}, \quad \text{Tr } Q = 1,$$

we see that

$$J(Q) = \begin{pmatrix} \bar{Z}^2 Z^2 & -\bar{Z}^1 Z^2 & \bar{Z}^4 Z^2 & -\bar{Z}^3 Z^2 \\ -\bar{Z}^2 Z^1 & \bar{Z}^1 Z^1 & -\bar{Z}^4 Z^1 & \bar{Z}^3 Z^1 \\ \bar{Z}^2 Z^4 & -\bar{Z}^1 Z^4 & \bar{Z}^4 Z^4 & -\bar{Z}^3 Z^4 \\ -\bar{Z}^2 Z^3 & \bar{Z}^1 Z^3 & -\bar{Z}^4 Z^3 & \bar{Z}^3 Z^3 \end{pmatrix}, \quad \text{Tr } J(Q) = \text{Tr } Q = 1,$$

and the result follows by comparing with the notation of Proposition 3.8. In particular we see that $J(X) = X$ and $J(U) = U$. The relations of Proposition 3.5 indicate that J extends to the full algebra as an antialgebra map (indeed this needs to be the case for J to be well-defined), since then we have

$$J(au) = J(u)J(a) = -ub = -bu = -w^*v = J(z)J(y^*) = J(y^*z),$$

similarly for the remaining relations. \square

We remark that since the algebra is commutative here, we may treat J as an algebra map rather than as an antialgebra map as required in the notion of an antilinear involution. This will no longer be the case when we come to quantise, when it is the notion of antilinear involution that will survive.

Now J extends further to an involution on $\mathbb{C}\mathbb{M}^\#$: given $P \in \mathbb{C}\mathbb{M}^\#$ we write $P = Q + Q'$ for $Q, Q' \in \mathbb{C}\mathbb{P}^3$ and define

$$J(P) := J(Q) + J(Q').$$

Indeed, we note that the 1d subspaces defined by a pair of rank one projectors Q_1, Q_2 are distinct if and only if $Q_1 Q_2 = Q_2 Q_1 = 0$, and it is easily checked that this is equivalent to the condition that $J(Q_1)J(Q_2) = J(Q_2)J(Q_1) = 0$ (computed either by direct calculation with matrices or by working with vectors $Z_1, Z_2 \in \mathbb{C}^4$ which define Q_1, Q_2 up to scale and using the inner product on \mathbb{C}^4 induced by

J). Thus if P is a rank two projector with $P = Q_1 + Q'_1 = Q_2 + Q'_2$, then if Q_1, Q_2 are distinct, so are $J(Q_1), J(Q_2)$. Elementary linear algebra then tells us that $J(Q_1) + J(Q'_1)$ and $J(Q_2) + J(Q'_2)$ must define the same projector $J(P)$, i.e. the map J is well-defined on $\mathbb{CM}^\#$.

Proposition 3.10. $P \in \mathbb{CM}^\#$ is invariant under J if and only if $P \in S^4$.

Proof. Writing $Q' = (\bar{W}^\mu W^\nu)$, we have that

$$P = Q + Q' = (\bar{Z}^\mu Z^\nu + \bar{W}^\mu W^\nu),$$

and that this supposed to be identified with the 2×2 block decomposition

$$P = \begin{pmatrix} A & B \\ B^\dagger & D \end{pmatrix}, \quad A = A^\dagger, D = D^\dagger, \text{Tr } A + \text{Tr } D = 2.$$

Here we shall use the notation of Proposition 3.1. Examining A , we have

$$A = a + \alpha \cdot \sigma = \begin{pmatrix} \bar{Z}^1 Z^1 + \bar{W}^1 W^1 & \bar{Z}^1 Z^2 + \bar{Z}^2 Z^1 \\ \bar{Z}^2 Z^1 + \bar{Z}^1 Z^2 & \bar{Z}^2 Z^2 + \bar{W}^2 W^2 \end{pmatrix},$$

and hence an identification

$$a = \frac{1}{2}(\bar{Z}^1 Z^1 + \bar{Z}^2 Z^2 + \bar{W}^1 W^1 + \bar{W}^2 W^2),$$

$$\alpha_3 = \frac{1}{2}(\bar{Z}^1 Z^1 - \bar{Z}^2 Z^2 + \bar{W}^1 W^1 - \bar{W}^2 W^2),$$

as well as the off-diagonal entries

$$\alpha_1 = \frac{1}{2}(\bar{Z}^1 Z^2 - \bar{Z}^2 Z^1 + \bar{W}^1 W^2 - \bar{W}^2 W^1),$$

$$\alpha_2 = \frac{1}{2i}(\bar{Z}^1 Z^2 + \bar{Z}^2 Z^1 + \bar{W}^1 W^2 + \bar{W}^2 W^1).$$

Clearly the relations $a = a^*$, $\alpha = \alpha^*$ hold under this identification. Under the involution J we calculate that

$$J(a) = a, \quad J(\alpha) = -\alpha.$$

Similarly we look at the block D ,

$$D = d + \delta \cdot \sigma = \begin{pmatrix} \bar{Z}^3 Z^3 + \bar{W}^3 W^3 & \bar{Z}^3 Z^4 + \bar{W}^3 W^4 \\ \bar{Z}^4 Z^3 + \bar{W}^4 W^3 & \bar{Z}^4 Z^4 + \bar{W}^4 W^4 \end{pmatrix}.$$

The same computation as above shows that the relations $d = d^*$, $\delta = \delta^*$ hold here, and moreover the trace relation implies that $d = 1 - a$, in agreement with Section 3.1. Under the involution J we also see that

$$J(d) = d, \quad J(\delta) = -\delta.$$

Finally we look at the matrix B ,

$$B = t + ix \cdot \sigma = \begin{pmatrix} \bar{Z}^1 Z^3 + \bar{W}^1 W^3 & \bar{Z}^1 Z^4 + \bar{W}^1 W^4 \\ \bar{Z}^2 Z^3 + \bar{W}^2 W^3 & \bar{Z}^2 Z^4 + \bar{W}^2 W^4 \end{pmatrix}.$$

Solving, we have the identification of generators

$$t = \frac{1}{2}(\bar{Z}^1 Z^3 + \bar{Z}^2 Z^4 + \bar{W}^1 W^3 + \bar{W}^2 W^4),$$

$$x_3 = \frac{1}{2i}(\bar{Z}^1 Z^3 - \bar{Z}^2 Z^4 + \bar{W}^1 W^3 - \bar{W}^2 W^4)$$

for the diagonal entries, as well as

$$x_1 = \frac{1}{2i}(\bar{Z}^2 Z^3 + \bar{Z}^1 Z^4 + \bar{W}^2 W^3 + \bar{W}^1 W^4),$$

$$x_2 = \frac{1}{2}(\bar{Z}^2 Z^3 - \bar{Z}^1 Z^4 + \bar{W}^2 W^3 - \bar{W}^1 W^4)$$

on the off-diagonal. This is in agreement with the fact as in Proposition 3.1 that the generators t, x are not necessarily Hermitian. Moreover, it is a simple matter to compute that under the involution J we have

$$J(t) = t^*, \quad J(x) = x^*.$$

Overall, we see that J has fixed points in $\mathbb{C}\mathbb{M}^\#$ consisting of those with coordinates subject to the additional constraints $\alpha = \delta = 0, t = t^*, x = x^*$. Thus (upon verification of the extra relations) the fixed points of $\mathbb{C}\mathbb{M}^\#$ under J are exactly those lying in S^4 , in accordance with proposition 3.2. \square

In the notation of Proposition 3.3, the action of J on $\mathbb{C}[\mathbb{C}\mathbb{M}^\#]$ is to map

$$J(a) = a, \quad J(\alpha_3) = \alpha_3, \quad J(\delta_3) = \delta_3, \quad J(\alpha) = -\alpha, \quad J(\delta) = -\delta,$$

$$J(w) = -\tilde{w}^*, \quad J(z) = \tilde{z}^*, \quad J(\tilde{w}) = -w^*, \quad J(\tilde{z}) = z^*.$$

This may either be recomputed, or obtained simply by making the same change of variables as was made in going from Proposition 3.1 to Proposition 3.3. The fixed points in these coordinates are those with $\alpha = \delta = 0, w^* = -\tilde{w}, z^* = \tilde{z}$, in agreement with Proposition 3.3.

Proposition 3.11. *For each $P \in \mathbb{C}\mathbb{M}^\#$ we have $P \in S^4$ if and only if there exists $Q \in \mathbb{C}\mathbb{P}^3$ such that $P = Q + J(Q)$.*

Proof. By the previous proposition, $P \in S^4$ if and only if $J(P) = P$. Of course, the reverse direction of the claim is easy, since if $P = Q + J(Q)$, we have $J(P) = J(Q) + J^2(Q) = P$. Conversely, given $P \in S^4$ with $J(P) = P$ we may write $P = Q + Q'$ for some $Q, Q' \in \mathbb{C}\mathbb{P}^3$ (as remarked already, this is not a unique decomposition but given Q we have $Q' = P - Q$, and J acts independently of this decomposition). The result is now obvious since J is nondegenerate, so $Q' = J(Q'')$ for some Q'' , but we must have $Q'' = Q$ since J is an involution. \square

As promised, there is a fibration of $\mathbb{C}\mathbb{P}^3$ over S^4 given at the coordinate algebra level by an inclusion $\mathbb{C}[S^4] \hookrightarrow \mathbb{C}^-[\mathbb{C}\mathbb{P}^3]$. In terms of the $\mathbb{C}^-[\mathbb{C}\mathbb{P}^3]$ coordinates

a, b, c, x, y, z used in Proposition 3.8 and the $\mathbb{C}[S^4]$ coordinates z, w, a of Proposition 3.3 we have the following (we note the overlap in notation between these propositions and rely on the context for clarity).

Proposition 3.12. *There is an algebra inclusion*

$$\eta : \mathbb{C}[S^4] \hookrightarrow \mathbb{C}^-[\mathbb{CP}^3]$$

given by

$$\eta(a) = a + b, \quad \eta(z) = y + v^*, \quad \eta(w) = w - z^*.$$

Proof. That this is an algebra map is a matter of rewriting the previous proposition in our explicit coordinates, for example that $\eta(z) = y + v^* = y + J(y)$. The sole relation to investigate is the image of the sphere relation $zz^* + ww^* = a(1 - a)$. Applying η to the left hand side, we obtain

$$\eta(zz^* + ww^*) = yy^* + yv + v^*y^* + v^*v + ww^* - wz - z^*w^* + z^*z.$$

Now using the relations of Proposition 3.5 we compute that

$$ayv = yav = yx^*z = x^*yz = awz,$$

where we have relied upon the commutativity of the algebra. Similarly one computes that $byv = bwz$, $cyv = cwz$, $dyv = dwz$, so that adding these four relations yields that $yv = wz$ in $\mathbb{C}^-[\mathbb{CP}^3]$. Then finally using the relations in Proposition 3.8 we see that

$$\eta(zz^* + ww^*) = Y + V + W + Z = (a + b)(c + d) = (a + b)(1 - (a + b)) = \eta(a(1 - a)),$$

as required. \square

We now look at the typical fibre \mathbb{CP}^1 of the fibration $\mathbb{CP}^3 \rightarrow S^4$, but now in the coordinate algebra picture.

Proposition 3.13. *The quotient of the algebra $\mathbb{C}^-[\mathbb{CP}^3]$ obtained by setting $\eta(a)$, $\eta(z)$, $\eta(w)$ to be constant numerical values is isomorphic to the coordinate algebra of a \mathbb{CP}^1 .*

Proof. If we suppose that we are in the patch where $a \neq 0$ in $\mathbb{C}^-[\mathbb{CP}^3]$ then we can view x, y, z as the variables and $X = ab$, $Y = ac$, $Z = ad$ as the relations. The generators u, v, w are defined by the equations in Proposition 3.8 and the rest are redundant.

Now suppose that $a + b = A$, a fixed real number, and $y + v^* = B$, $w - z^* = C$, fixed complex numbers, such that

$$BB^* + CC^* = A(1 - A)$$

(an element of S^4). Then we have just one equation

$$X = a(A - a) = (A/2)^2 - s^2$$

if we set $s = a - A/2$. This is a $\mathbb{C}\mathbb{P}^1$ of radius $A/2$ in place of the usual radius $1/2$.

The equation $Y = ac$ is viewed as a definition of c . The equation $Z = ad$ is then equivalent to $Y + Z = a(1 - A)$. We'll see that this is automatic and that y, z are uniquely determined by x, a and our fixed parameters A, B, C so are not in fact free variables.

Indeed, $av = x^*z$ and $aw = x^*y$ determine v and w as mentioned above, so our quotient is

$$aB = ay + z^*x, \quad aC = x^*y - az^*,$$

which implies that $a^2(BB^* + CC^*) = (a^2 + X)(Y + Z) = aA(Y + Z)$, so $Y + Z = a(1 - A)$ necessarily holds if A, B, C lie in S^4 .

We also combine the equations to find $ax^*B = a^2C + z^*aA$ and $aCx = aAy - a^2B$ so that at least if $A \neq 0$ we have z, y determined. (In fact one has $By^* - z^*C = a(1 - A)$ from the above so if z is determined then so is y if B is not zero etc). Thus $\mathbb{C}[\mathbb{C}\mathbb{P}^1]$ is viewed inside $\mathbb{C}[\mathbb{C}\mathbb{P}^3]$ in this patch as

$$(14) \quad \left(\begin{array}{ccc} a & x & * \\ x^* & A - a & * \\ & * & * \end{array} \right), \quad x^*x = a(A - a),$$

where the unspecified entries are determined as above using the relations in terms of x and a .

Similar analysis holds in the other coordinate patches, although we shall not check this here as this is a well-known classical result. In other patches we would see the various copies of $\mathbb{C}[\mathbb{C}\mathbb{P}^1]$ appearing elsewhere in the above matrix. \square

This situation now provides us with yet another way to view the instanton bundle. Let \mathcal{M} be a finite rank projective $\mathbb{C}[\mathbb{C}\mathbb{M}^\#]$ -module. Then J induces a module map $J : \mathcal{M} \rightarrow \mathcal{M}$ whose fixed point submodule is a finite rank projective $\mathbb{C}[S^4]$ -module. In particular, if we take \mathcal{M} to be the $\mathbb{C}[\mathbb{C}\mathbb{M}^\#]$ -module given by the defining tautological projector (7), then as explained above as well as in that section, the fixed point submodule is precisely the tautological bundle $\mathcal{E} = \mathbb{C}[S^4]^4 e$ of Proposition 3.2 which defines the instanton bundle over S^4 .

Now the map $\eta : \mathbb{C}[S^4] \rightarrow \mathbb{C}^-[\mathbb{C}\mathbb{P}^3]$ induces the 'push-out' of the $\mathbb{C}[S^4]$ -module \mathcal{E} along η to obtain an 'auxiliary' $\mathbb{C}^-[\mathbb{C}\mathbb{P}^3]$ -module $\tilde{\mathcal{E}}$, given by viewing the projector $e \in M_4(\mathbb{C}[S^4])$ as a projector $\tilde{e} \in M_4(\mathbb{C}^-[\mathbb{C}\mathbb{P}^3])$, so that $\tilde{\mathcal{E}} := \mathbb{C}^-[\mathbb{C}\mathbb{P}^3]^4 \tilde{e}$, giving a

bundle over twistor space. Explicitly, we have

$$\begin{aligned} \tilde{e} &= \begin{pmatrix} \eta(a) & 0 & \eta(z) & \eta(-w^*) \\ 0 & \eta(a) & \eta(w) & \eta(z^*) \\ \eta(z^*) & \eta(w^*) & \eta(1-a) & 0 \\ \eta(-w) & \eta(z) & 0 & \eta(1-a) \end{pmatrix} \\ &= \begin{pmatrix} a+b & 0 & y+v^* & z-w^* \\ 0 & a+b & w-z^* & y^*+v \\ y^*+v & w^*-z & 1-(a+b) & 0 \\ z^*-w & y+v^* & 0 & 1-(a+b) \end{pmatrix} \in M_4(\mathbb{C}^-[CP^3]). \end{aligned}$$

Moreover, if one sets $a+b = A$, $y+v^* = B$, $w-z^* = C$ for fixed real A and complex B, C as in Proposition 3.13, then we have

$$\tilde{e} = \begin{pmatrix} A & 0 & B & -C^* \\ 0 & A & C & B^* \\ B^* & C^* & 1-A & 0 \\ -C & B & 0 & 1-A \end{pmatrix},$$

a constant projector of rank two. Then viewing the fibre $\mathbb{C}[CP^1]$ as a subset of $\mathbb{C}[CP^3]$ as in (14), it is easily seen that $\mathbb{C}[CP^1]^4 \tilde{e}$ is a free $\mathbb{C}[CP^1]$ -module of rank two. This is just the coordinate algebra version of saying that for all $x = (A, B, C) \in S^4$ the instanton bundle pulled back from S^4 to CP^3 is trivial upon restriction to each fibre $\hat{x} = CP^1$, and we may thus see the instanton bundle \mathcal{E} over $\mathbb{C}[S^4]$ as coming from the bundle $\tilde{\mathcal{E}}$ over twistor space. This is an easy example of the *Penrose-Ward transform*, which we shall discuss in more detail later.

4. THE QUANTUM CONFORMAL GROUP

The advantage of writing space-time and twistor space as homogeneous spaces in the language of coordinate functions is that we are now free to apply the standard theory of quantisation by a cocycle twist.

To this end, we recall that if H is a Hopf algebra with coproduct $\Delta : H \rightarrow H \otimes H$, counit $\epsilon : H \rightarrow \mathbb{C}$ and antipode $S : H \rightarrow H$, then a two-cocycle F on H means $F : H \otimes H \rightarrow \mathbb{C}$ which is convolution invertible and unital (i.e. a 2-cochain) in the sense

$$F(h_{(1)}, g_{(1)})F^{-1}(h_{(2)}, g_{(2)}) = F^{-1}(h_{(1)}, g_{(1)})F(h_{(2)}, g_{(2)}) = \epsilon(h)\epsilon(g)$$

(for some map F^{-1}) and obeys $\partial F = 1$ in the sense

$$F(g_{(1)}, f_{(1)})F(h_{(1)}, g_{(1)}f_{(2)})F^{-1}(h_{(2)}g_{(3)}, f_{(3)})F^{-1}(h_{(3)}, g_{(4)}) = \epsilon(f)\epsilon(h)\epsilon(g).$$

We have used Sweedler notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$ and suppressed the summation. In this case there is a ‘cotwisted’ Hopf algebra H_F with the same coalgebra structure

and counit as H but with modified product \bullet and antipode S_F [11]

$$(15) \quad h \bullet g = F(h_{(1)} \otimes g_{(1)}) h_{(2)} g_{(2)} F^{-1}(h_{(3)} \otimes g_{(3)})$$

$$S_F(h) = U(h_{(1)}) S h_{(2)} U^{-1}(h_{(2)}), \quad U(h) = F(h_{(1)}, S h_{(2)})$$

for $h, g \in H$, where we use the product and antipode of H on the right hand sides and $U^{-1}(h_{(1)}) U(h_{(2)}) = \epsilon(h) = U(h_{(1)}) U^{-1}(h_{(2)})$ defines the inverse functional. If H is a coquasitriangular Hopf algebra then so is H_F . In particular, if H is commutative then H_F is cotriangular with ‘universal R-matrix’ and induced (symmetric) braiding given by

$$R(h, g) = F(g_{(1)}, h_{(1)}) F^{-1}(h_{(2)}, g_{(2)}), \quad \Psi_{V,W}(v \otimes w) = R(w_{(1)}, v_{(1)}) w_{(2)} \otimes v_{(2)}$$

for any two left comodules V, W . We use the Sweedler notation for the left coactions as well. In the cotriangular case one has $\Psi^2 = \text{id}$, so every object on the category of H_F -comodules inherits nontrivial statistics in which transposition is replaced by this non-standard transposition.

The nice property of this construction is that the category of H -comodules is actually equivalent to that of H_F -comodules, so there is an invertible functor which ‘functorially quantises’ any construction in the first category (any H -covariant construction) to give an H_F -covariant one. So not only is the classical Hopf algebra H quantised but also any H -covariant construction as well. This is a particularly easy example of the ‘braid statistics approach’ to quantisation, whereby deformation is achieved by deforming the category of vector spaces to a braided one [11].

In particular, if A is a left H -comodule algebra, we automatically obtain a left H_F -comodule algebra A_F which as a vector space is the same as A , but has the modified product

$$(16) \quad a \bullet b = F(a_{(1)}, b_{(1)}) a_{(2)} b_{(2)},$$

for $a, b \in A$, where we have again used the Sweedler notation $\Delta_L(a) = a_{(1)} \otimes a_{(2)}$ for the coaction $\Delta_L : A \rightarrow H \otimes A$. The same applies to any other covariant algebra. For example if $\Omega(A)$ is an H -covariant differential calculus (see later) then this functorially quantises as $\Omega(A_F) := \Omega(A)_F$ by this same construction.

Finally, if $H' \rightarrow H$ is a homomorphism of Hopf algebras then any cocycle F on H pulls back to one on H' and as a result one has a homomorphism $H'_F \rightarrow H_F$. In what follows we take $H = \mathbb{C}[\mathbb{C}^4]$ (the translation group of \mathbb{C}^4) and H' variously the coordinate algebras of $\tilde{K}, \tilde{H}, \text{GL}_4$.

In particular, since the group GL_4 acts on the quadric $\widetilde{\mathbb{C}\text{M}^\#}$, we have (as in section 1.1) a coaction Δ_L of the coordinate ring $\mathbb{C}[\text{GL}_4]$ on $\mathbb{C}[\widetilde{\mathbb{C}\text{M}^\#}]$, and we shall first deform this picture. In order to do this we note first that the conformal transformations of $\mathbb{C}\text{M}^\#$ break down into compositions of translations, rotations,

dilations and inversions. Written with respect to the aforementioned double null coordinates, GL_4 decomposes into 2×2 blocks

$$(17) \quad \begin{pmatrix} \gamma & \tau \\ \sigma & \tilde{\gamma} \end{pmatrix}$$

with overall non-zero determinant, where the entries of τ constitute the translations and the entries of σ contain the inversions. The diagonal blocks $\gamma \times \tilde{\gamma}$ constitute the space-time rotations as well as the dilations. Writing $M_2 := M_2(\mathbb{C})$, GL_4 decomposes as the subset of nonzero determinant

$$\mathrm{GL}_4 \subset \mathbb{C}^4 \rtimes (M_2 \times M_2) \ltimes \mathbb{C}^4$$

where the outer factors denote σ, τ and $\gamma \times \tilde{\gamma} \in M_2 \times M_2$. In practice it is convenient to work in a ‘patch’ GL_4^- where γ is assumed invertible. Then by factorising the matrix we deduce that

$$\det \begin{pmatrix} \gamma & \tau \\ \sigma & \tilde{\gamma} \end{pmatrix} = \det(\gamma) \det(\tilde{\gamma} - \sigma\gamma^{-1}\tau)$$

which is actually a part of a universal formula for determinants of matrices with entries in a noncommutative algebra (here the algebra is M_2 and we compose with the determinant map on this algebra). We see that as a set, GL_4^- is $\mathbb{C}^4 \times \mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathbb{C}^4$, where the two copies of GL_2 refer to γ and $\tilde{\gamma} - \sigma\gamma^{-1}\tau$. There is of course another patch GL_4^+ where we similarly assume $\tilde{\gamma}$ invertible.

In terms of coordinate functions for $\mathbb{C}[\mathrm{GL}_4]$ we therefore have four matrix generators $\tau, \sigma, \gamma, \tilde{\gamma}$ organised as above. These together have a matrix form of coproduct

$$\Delta \begin{pmatrix} \gamma & \tau \\ \sigma & \tilde{\gamma} \end{pmatrix} = \begin{pmatrix} \gamma & \tau \\ \sigma & \tilde{\gamma} \end{pmatrix} \otimes \begin{pmatrix} \gamma & \tau \\ \sigma & \tilde{\gamma} \end{pmatrix}.$$

In the classical case the generators commute and an invertible element D obeying $D = \det a$ is adjoined. For $\mathbb{C}[\mathrm{GL}_4^-]$ we instead adjoin inverses to $d = \det(\gamma)$ and $\tilde{d} = \det(\tilde{\gamma} - \sigma\gamma^{-1}\tau)$.

We focus next on the translation sector $H = \mathbb{C}[\mathbb{C}^4]$ generated by some $t_{A'}^A$, where $A' \in \{3, 4\}$ and $A \in \{1, 2\}$ to line up with our conventions for GL_4 . These generators have a standard additive coproduct. We let $\partial_A^{A'}$ be the Lie algebra of translation generators dual to this, so

$$\langle \partial_A^{A'}, t_{B'}^B \rangle = \delta_B^{A'} \delta_A^B$$

which extends to the action on products of the $t_{A'}^A$ by differentiation and evaluation at zero (hence the notation). In this notation the we use cocycle

$$F(h, g) = \langle \exp(\frac{\iota}{2} \theta_{A'B}^{AB} \partial_A^{A'} \otimes \partial_B^{B'}), h \otimes g \rangle.$$

Cotwisting here does not change H itself, $H = H_F$, because its coproduct is cocommutative (the group \mathbb{C}^4 is Abelian) but it twists $A = \mathbb{C}[\mathbb{C}^4]$ as a comodule algebra into the Moyal plane. This is by now well-known both in the module form and the

above comodule form. We now pull this cocycle back to $\mathbb{C}[\mathrm{GL}_4]$, where it takes the same form as above on the generators $\tau_{A'}^A$ (which project onto $t_{A'}^A$). The pairing extends as zero on the other generators. One can view the $\partial_A^{A'}$ in the Lie algebra of GL_4 as the nilpotent 4×4 matrices with entry 1 in the A, A' position for some $A = 1, 2, A' = 3, 4$ and zeros elsewhere, extending the above picture. Either way, one computes

$$\begin{aligned} F_{\nu\beta}^{\mu\alpha} = F(a_\nu^\mu, a_\beta^\alpha) &= \langle \exp(\frac{\imath}{2}\theta_{A'B'}^{AB}\partial_A^{A'} \otimes \partial_B^{B'}), a_\nu^\mu \otimes a_\beta^\alpha \rangle \\ &= \delta_\nu^\mu \delta_\beta^\alpha + \frac{\imath}{2}\theta_{A'B'}^{AB}\delta_A^\mu \delta_B^\alpha \delta_\nu^{A'} \delta_\beta^{B'} \\ &= \delta_\nu^\mu \delta_\beta^\alpha + \frac{\imath}{2}\theta_{\nu\beta}^{\mu\alpha}, \end{aligned}$$

where it is understood that $\theta_{\nu\beta}^{\mu\alpha}$ is zero when $\{\mu, \alpha\} \neq \{1, 2\}, \{\nu, \beta\} \neq \{3, 4\}$. We also compute

$$U(a_\nu^\mu) = F(a_a^\mu, S a_\nu^a) = \langle \exp(-\frac{\imath}{2}\theta_{A'B'}^{AB}\partial_A^{A'} \otimes \partial_B^{B'}), a_\alpha^\mu \otimes a_\nu^\alpha \rangle = \delta_\nu^\mu - \frac{\imath}{2}\theta_{\alpha\beta}^{\mu\alpha} = \delta_\nu^\mu.$$

Then following equations (15) the deformed coordinate algebra $\mathbb{C}_F[\mathrm{GL}_4]$ has undeformed antipode on the generators and deformed product

$$a_\nu^\mu \bullet a_\beta^\alpha = F_{mn}^{\mu\alpha} a_p^m a_q^n F_{\nu\beta}^{-1pq}$$

where $a_\nu^\mu \in \mathbb{C}[\mathrm{GL}_4]$ are the generators of the classical algebra. The commutation relations can be written in R-matrix form (as for any matrix coquasitriangular Hopf algebra) as

$$R_{\alpha\beta}^{\mu\nu} a_\gamma^\alpha \bullet a_\delta^\beta = a_\beta^\nu \bullet a_\alpha^\mu R_{\gamma\delta}^{\alpha\beta}, \quad R_{\alpha\beta}^{\mu\nu} = F_{\delta\gamma}^{\nu\mu} F_{\alpha\beta}^{-1\gamma\delta}$$

where in our particular case

$$R_{\nu\beta}^{\mu\alpha} = \delta_\nu^\mu \delta_\beta^\alpha - \imath \theta_{\nu\beta}^{-\mu\alpha}, \quad \theta_{\nu\beta}^{-\mu\alpha} = \frac{1}{2}(\theta_{\nu\beta}^{\mu\alpha} - \theta_{\beta\nu}^{\alpha\mu})$$

has the same form but now with only the antisymmetric part of θ in the sense shown.

We give the resulting relations explicitly in the $\gamma, \tilde{\gamma}, \sigma, \tau$ block form (17). These are in fact all 2×2 matrix relations with indices A, A' etc. as explained but when no confusion can arise we write the indices in an apparently GL_4 form. For example, in writing γ_ν^μ it is implicit that $\mu, \nu \in \{1, 2\}$, whereas for σ_ν^μ it is understood that $\mu \in \{3, 4\}$ and $\nu \in \{1, 2\}$.

Theorem 4.1. *The quantum group coordinate algebra $\mathbb{C}_F[\mathrm{GL}_4]$ has deformed product*

$$\begin{aligned} \tau_\nu^\mu \bullet \tau_\beta^\alpha &= \tau_\nu^\mu \tau_\beta^\alpha + \frac{\imath}{2}\theta_{cd}^{\mu\alpha} \tilde{\gamma}_c^\alpha \tilde{\gamma}_\beta^d - \frac{\imath}{2}\gamma_c^\mu \gamma_\beta^\alpha \theta_{\nu\beta}^{cd} + \frac{1}{4}\theta_{ab}^{\mu\alpha} \sigma_c^a \sigma_\beta^b \theta_{\nu\beta}^{cd}, \\ \gamma_\nu^\mu \bullet \tau_\beta^\alpha &= \gamma_\nu^\mu \tau_\beta^\alpha + \frac{\imath}{2}\theta_{cd}^{\mu\alpha} \sigma_\nu^c \tilde{\gamma}_\beta^d, & \tau_\nu^\mu \bullet \gamma_\beta^\alpha &= \tau_\nu^\mu \gamma_\beta^\alpha + \frac{\imath}{2}\theta_{cd}^{\mu\alpha} \tilde{\gamma}_\nu^c \sigma_\beta^d, \\ \tilde{\gamma}_\nu^\mu \bullet \tau_\beta^\alpha &= \tilde{\gamma}_\nu^\mu \tau_\beta^\alpha - \frac{\imath}{2}\sigma_c^\mu \gamma_\beta^\alpha \theta_{\nu\beta}^{cd}, & \tau_\nu^\mu \bullet \tilde{\gamma}_\beta^\alpha &= \tau_\nu^\mu \tilde{\gamma}_\beta^\alpha - \frac{\imath}{2}\gamma_c^\mu \sigma_\beta^c \theta_{\nu\beta}^{cd}, \\ \gamma_\nu^\mu \bullet \gamma_\beta^\alpha &= \gamma_\nu^\mu \gamma_\beta^\alpha + \frac{\imath}{2}\theta_{cd}^{\mu\alpha} \sigma_\nu^c \sigma_\beta^d, & \tilde{\gamma}_\nu^\mu \bullet \tilde{\gamma}_\beta^\alpha &= \tilde{\gamma}_\nu^\mu \tilde{\gamma}_\beta^\alpha - \frac{\imath}{2}\sigma_c^\mu \sigma_\beta^c \theta_{\nu\beta}^{cd} \end{aligned}$$

with the remaining relations, antipode and coproduct undeformed on the generators. The quantum group is generated by matrices $\gamma, \tilde{\gamma}, \tau, \sigma$ of generators with commutation relations

$$\begin{aligned} [\gamma_\nu^\mu, \gamma_\beta^\alpha]_\bullet &= \imath \theta_{cd}^{-\mu\alpha} \sigma_\nu^c \bullet \sigma_\beta^d, & [\tilde{\gamma}_\nu^\mu, \tilde{\gamma}_\beta^\alpha]_\bullet &= -\imath \sigma_d^\alpha \bullet \sigma_c^\mu \theta_{\nu\beta}^{-cd} \\ [\gamma_\nu^\mu, \tau_\beta^\alpha]_\bullet &= \imath \theta_{cd}^{-\mu\alpha} \sigma_\nu^c \bullet \tilde{\gamma}_\beta^d, & [\tilde{\gamma}_\nu^\mu, \tau_\beta^\alpha]_\bullet &= -\imath \gamma_d^\alpha \bullet \sigma_c^\mu \theta_{\nu\beta}^{-cd} \\ [\tau_\nu^\mu, \tau_\beta^\alpha]_\bullet &= \imath \theta_{cd}^{-\mu\alpha} \tilde{\gamma}_\nu^c \bullet \tilde{\gamma}_\beta^d - \imath \gamma_d^\alpha \bullet \gamma_c^\mu \theta_{\nu\beta}^{-cd} \end{aligned}$$

and a certain determinant inverted.

Proof. Finishing the computations above with the explicit form of F we have

$$a_\nu^\mu \bullet a_\beta^\alpha = a_\nu^\mu a_\beta^\alpha + \frac{\imath}{2} \theta_{cd}^{\mu\alpha} a_\nu^c a_\beta^d - \frac{\imath}{2} a_c^\mu a_d^\alpha \theta_{\nu\beta}^{cd} + \frac{1}{4} \theta_{ab}^{\mu\alpha} a_c^a a_d^b \theta_{\nu\beta}^{cd}.$$

Noting that $\theta_{\nu\beta}^{\mu\alpha} = 0$ unless $\mu, \alpha \in \{1, 2\}$ and $\nu, \beta \in \{3, 4\}$ we can write these for the 2×2 blocks as shown. For the commutation relations we have similarly

$$[a_\nu^\mu, a_\beta^\alpha]_\bullet = \imath \theta_{cd}^{-\mu\alpha} a_\nu^c \bullet a_\beta^d - \imath a_d^\alpha \bullet a_c^\mu \theta_{\nu\beta}^{-cd}$$

which we similarly decompose as stated. Note that different terms here drop out due to the range of the indices for nonzero θ_- , which are same as for θ . There is in principle a formula also for the determinant written in terms of the \bullet product. It can be obtained via braided ‘antisymmetric tensors’ from the R -matrix and will necessarily be product of 2×2 determinants in the ‘patches’ where γ or $\tilde{\gamma}$ are invertible in the noncommutative algebra. \square

One may proceed to compute these more explicitly, for example

$$[\gamma_\nu^\mu, \gamma_\beta^\alpha]_\bullet = \imath \theta_{34}^{-\mu\alpha} \sigma_\nu^3 \sigma_\beta^4 + \imath \theta_{43}^{-\mu\alpha} \sigma_\beta^3 \sigma_\nu^4 + \imath \theta_{33}^{-\mu\alpha} \sigma_\nu^3 \sigma_\beta^3 + \imath \theta_{44}^{-\mu\alpha} \sigma_\nu^4 \sigma_\beta^4$$

and so forth.

We similarly calculate the resulting products on the coordinate algebras of the deformed homogeneous spaces. Indeed, using equation (16), we have the following results.

Proposition 4.2. *The covariantly twisted algebra $\mathbb{C}_F[\widetilde{\text{CM}}^\#]$ has the deformed product*

$$\begin{aligned} x^{\mu\nu} \bullet x^{\alpha\beta} &= x^{\mu\nu} x^{\alpha\beta} + \frac{\imath}{2} (\theta_{ad}^{\mu\beta} x^{a\nu} x^{\alpha d} + \theta_{ac}^{\mu\alpha} x^{a\nu} x^{c\beta} + \theta_{bd}^{\nu\beta} x^{\mu b} x^{\alpha d} + \theta_{bc}^{\nu\alpha} x^{\mu b} x^{c\beta}) \\ &\quad - \frac{1}{4} (\theta_{bc}^{\nu\alpha} \theta_{ad}^{\mu\beta} + \theta_{bd}^{\nu\beta} \theta_{ac}^{\mu\alpha}) x^{ab} x^{cd} \end{aligned}$$

and is isomorphic to the subalgebra

$$\mathbb{C}_F[\text{GL}_4]^{\mathbb{C}_F[\tilde{H}]}$$

where F is pulled back to $\mathbb{C}[\tilde{H}]$. Products of generators with $t = x^{34}$ are undeformed.

Proof. The isomorphism $\mathbb{C}_F[\widetilde{\mathbb{C}\mathbb{M}^\#}] \cong \mathbb{C}_F[\mathrm{GL}_4]^{\mathbb{C}_F[\tilde{H}]}$ is a consequence of the functoriality of the cocycle twist. The deformed product is simply a matter of calculating the twisted product on $\mathbb{C}_F[\widetilde{\mathbb{C}\mathbb{M}^\#}]$. The coaction $\Delta_L(x^{\mu\nu}) = a_a^\mu a_b^\nu \otimes x^{ab}$ combined with the formula (16) yields

$$x^{\mu\nu} \bullet x^{\alpha\beta} = F(a_a^\mu a_b^\nu, a_c^\alpha a_d^\beta) x^{ab} x^{cd} = F_{mn}^{\mu\alpha} F_{ap}^{m\beta} F_{qc}^{\nu n} F_{bd}^{qp} x^{ab} x^{cd},$$

using that F in our particular case is multiplicative (a Hopf algebra bicharacter on $\mathbb{C}[\mathbb{C}^4]$ and hence when pulled back to $\mathbb{C}[\mathrm{GL}_4]$). Alternatively, one may compute it directly from the original definition as an exponentiated operator, going out to $\partial\partial\otimes\partial\partial$ terms before evaluating at zero in \mathbb{C}^4 . Either way we have the result stated when we recall that $\theta_{bd}^{\mu\alpha}$ is understood to be zero unless $\{\mu, \alpha\} = \{1, 2\}$, $\{b, d\} = \{3, 4\}$. We note also the commutation relations

$$x^{\alpha\beta} \bullet x^{\mu\nu} = R_{mn}^{\mu\alpha} R_{ap}^{m\beta} R_{qc}^{\nu n} R_{bd}^{qp} x^{ab} \bullet x^{cd}$$

following from $b \bullet a = R(a_{(1)}, b_{(1)}) a_{(2)} \bullet b_{(2)}$ computed in the same way as above but now with R in place of F . Since $R^{-1} = R_{21}$ these relations may be written in a ‘reflection’ form on regarding x as a matrix. Finally, since θ is zero when $\{\mu, \alpha\} \neq \{1, 2\}$ we see that the \bullet product of the generator $t = x^{34}$ with any other generator is undeformed, which also implies that t is central in the deformed algebra. \square

Examining the resulting relations associated to the twisted product more closely, one finds

$$\begin{aligned} [z, \tilde{z}]_\bullet &= [-x^{23}, x^{14}]_\bullet = -\frac{\imath}{2} \theta_{43}^{21} x^{43} x^{34} + \frac{\imath}{2} \theta_{34}^{12} x^{34} x^{43} = \imath \theta^{-21}_{43} t^2, \\ [w, \tilde{w}]_\bullet &= [-x^{13}, x^{24}]_\bullet = -\frac{\imath}{2} \theta_{43}^{12} x^{43} x^{34} + \frac{\imath}{2} \theta_{34}^{21} x^{34} x^{43} = \imath \theta^{-12}_{43} t^2, \end{aligned}$$

with the remaining commutators amongst these affine Minkowski space generators undeformed. Since products with t are undeformed, the above can be viewed as the commutation relations among the affine generators $t, w, \tilde{w}, z, \tilde{z}$ of $\mathbb{C}_F[\widetilde{\mathbb{C}\mathbb{M}^\#}]$. The commutation relations for the s generator are

$$\begin{aligned} [s, z]_\bullet &= [x^{12}, -x^{23}]_\bullet = -\frac{\imath}{2} \theta_{ac}^{12} x^{a2} x^{c3} + \frac{\imath}{2} \theta_{ac}^{21} x^{a3} x^{c2} \\ &= -\frac{\imath}{2} \theta_{34}^{12} x^{32} x^{43} - \frac{\imath}{2} \theta_{44}^{12} x^{42} x^{43} + \frac{\imath}{2} \theta_{43}^{21} x^{43} x^{32} + \frac{\imath}{2} \theta_{44}^{21} x^{43} x^{42} \\ &= \imath \theta^{-12}_{34} zt + \imath \theta^{-21}_{44} \tilde{w}t, \end{aligned}$$

for example, as well as

$$\begin{aligned} [s, \tilde{z}]_\bullet &= \imath \theta^{-12}_{33} wt + \imath \theta^{-21}_{43} \tilde{z}t, \\ [s, w]_\bullet &= \imath \theta^{-12}_{43} wt + \imath \theta^{-21}_{44} \tilde{z}t, \\ [s, \tilde{w}]_\bullet &= \imath \theta^{-12}_{33} zt + \imath \theta^{-21}_{34} \tilde{w}t. \end{aligned}$$

Again, we may equally well use the \bullet product on the right hand side of each equation.

Proposition 4.3. *The twisted algebra $\mathbb{C}_F[\tilde{T}]$ has deformed product*

$$Z^\mu \bullet Z^\nu = Z^\mu Z^\nu + \frac{i}{2} \theta_{ab}^{\mu\nu} Z^a Z^b.$$

It is isomorphic to the subalgebra

$$\mathbb{C}_F[\mathrm{GL}_4]^{\mathbb{C}_F[\tilde{K}]}$$

where F is pulled back to $\mathbb{C}[\tilde{K}]$. Products of generators with Z^3, Z^4 are undeformed.

Proof. The isomorphism $\mathbb{C}_F[\tilde{T}] \cong \mathbb{C}_F[\mathrm{GL}_4]^{\mathbb{C}_F[\tilde{K}]}$ is again a consequence of the theory of cocycle twisting. An application of equation (16) gives the new product

$$Z^\nu \bullet Z^\mu = F_{ab}^{\nu\mu} Z^a Z^b = F_{ab}^{\nu\mu} F^{-1ba}_{cd} Z^c \bullet Z^d = R_{cd}^{\mu\nu} Z^c \bullet Z^d$$

and we compute the first of these explicitly. The remaining relations tell us that this is a ‘braided vector space’ associated to the R -matrix, see [11, Ch. 10]. As before, the form of θ implies that products with Z^3, Z^4 are undeformed. Hence these are central. \square

We conclude that the only nontrivial commutation relation is the Z^1 - Z^2 one, which we compute explicitly as

$$[Z^1, Z^2]_\bullet = -i\theta_{ab}^{-21} Z^a \bullet Z^b = i(\theta_{34}^{-12} + \theta_{43}^{-12}) Z^3 Z^4 + i\theta_{33}^{-12} Z^3 Z^3 + i\theta_{44}^{-12} Z^4 Z^4.$$

where we could as well use the \bullet on the right.

We remark that although our deformation of the conformal group is different from and more general than that previously obtained in [8] (which insists on only first order terms in the deformation parameter), it is of note that the deformed commutation relations associated to twistor space and conformal space-time are in agreement with those proposed in recent literature [15, 9, 8, 4] *provided* one supposes that the four parameters

$$\theta_{33}^{-12} = \theta_{44}^{-12} = \theta_{34}^{-11} = \theta_{34}^{-22} = 0.$$

This says that $\theta^{-AB}_{A'B'}$, as a 4×4 matrix with rows AA' and columns BB' (the usual presentation) has the form

$$\theta^- = \begin{pmatrix} 0 & 0 & 0 & \theta_1 \\ 0 & 0 & -\theta_2 & 0 \\ 0 & \theta_2 & 0 & 0 \\ -\theta_1 & 0 & 0 & 0 \end{pmatrix}, \quad \theta_1 = \theta_{34}^{-12}, \quad \theta_2 = \theta_{34}^{-21}.$$

The quantum group deformation we propose then agrees with [8] on the generators $\gamma_\nu^\mu, \sigma_\beta^\alpha$ and this in fact is the reason that the space-time and twistor algebras then agree, since their generators may be viewed as living in the subalgebra generated by the first two columns of a *via* the isomorphisms given in propositions 4.2 and 4.3.

Finally we give the commutation relations in the twisted coordinate algebra $\mathbb{C}_F[\tilde{\mathcal{F}}]$ of the correspondence space, which may be computed either by viewing it as a twisted comodule algebra for $\mathbb{C}_F[\mathrm{GL}_4]$ or by identification with the appropriate subalgebra of $\mathbb{C}_F[\mathrm{GL}_4]$ and calculating there. Either way, one obtains

$$(18) \quad \begin{aligned} [s, Z^1]_{\bullet} &= -\imath\theta^{-21}_{33}wZ^3 - \imath\theta^{-21}_{34}wZ^4 + \imath\theta^{-21}_{43}\tilde{z}Z^3 + \imath\theta^{-21}_{44}\tilde{z}Z^4 + \imath\theta^{-11}_{34}tZ^2, \\ [s, Z^2]_{\bullet} &= \imath\theta^{-12}_{33}zZ^3 + \imath\theta^{-12}_{34}zZ^4 - \imath\theta^{-12}_{43}\tilde{w}Z^3 - \imath\theta^{-12}_{44}\tilde{w}Z^4 + \imath\theta^{-22}_{34}tZ^1, \\ [z, Z^1]_{\bullet} &= \imath\theta^{-21}_{43}tZ^3 + \imath\theta^{-21}_{44}tZ^4, & [z, Z^2]_{\bullet} &= \imath\theta^{-22}_{43}tZ^3, \\ [\tilde{z}, Z^1]_{\bullet} &= \imath\theta^{-11}_{34}tZ^4, & [\tilde{z}, Z^2]_{\bullet} &= \imath\theta^{-12}_{33}tZ^3 + \imath\theta^{-12}_{34}tZ^4, \\ [w, Z^1]_{\bullet} &= \imath\theta^{-11}_{43}tZ^3, & [w, Z^2]_{\bullet} &= \imath\theta^{-12}_{43}tZ^3 + \imath\theta^{-12}_{44}tZ^4, \\ [\tilde{w}, Z^1]_{\bullet} &= \imath\theta^{-21}_{33}tZ^3 + \imath\theta^{-21}_{34}tZ^4, & [\tilde{w}, Z^2]_{\bullet} &= \imath\theta^{-22}_{34}tZ^4, \end{aligned}$$

where we may equally write \bullet on the right hand side of each relation. The generators t, Z^3, Z^4 are of course central. The relations (1) twist by replacing the old product by \bullet . In terms of the old product they become

$$(19) \quad \begin{aligned} \tilde{z}Z^3 + wZ^4 - tZ^1 &= 0, \\ \tilde{w}Z^3 + zZ^4 - tZ^2 &= 0, \\ sZ^3 + wZ^2 + -zZ^1 + \frac{\imath}{2}((\theta_{43}^{12} - \theta_{43}^{21})tZ^3 + \theta_{44}^{-12}tZ^4) &= 0, \\ sZ^4 - \tilde{z}Z^2 + \tilde{w}Z^1 + \frac{\imath}{2}(\theta_{33}^{-21}tZ^3 + (\theta_{34}^{21} - \theta_{34}^{12})tZ^4) &= 0. \end{aligned}$$

5. QUANTUM DIFFERENTIAL CALCULI ON $\mathbb{C}_F[\mathrm{GL}_4]$, $\mathbb{C}_F[\widetilde{\mathrm{CM}}^{\#}]$ AND $\mathbb{C}_F[\tilde{T}]$

We recall that a *differential calculus* of an algebra A consists of an A - A -bimodule $\Omega^1 A$ and a map $d : A \rightarrow \Omega^1 A$ obeying the Leibniz rule such that $\Omega^1 A$ is spanned by elements of the form adb . Every unital algebra has a universal calculus $\Omega_{un}^1 = \ker \mu$ where μ is the product map of A . The differential is $d_{un}(a) = 1 \otimes a - a \otimes 1$. Any other calculus is a quotient of Ω_{un}^1 by a sub-bimodule N_A .

When A is a Hopf algebra, it coacts on itself by left and right translation *via* the coproduct Δ : we say a calculus on A is *left covariant* if this coaction extends to a left coaction $\Delta_L : \Omega^1 A \rightarrow A \otimes \Omega^1 A$ such that d is an intertwiner and Δ_L is a bimodule map, that is

$$\begin{aligned} \Delta_L(da) &= (\mathrm{id} \otimes d)\Delta(a), \\ a \cdot \Delta_L(\omega) &= \Delta_L(a \cdot \omega), \quad \Delta_L(\omega) \cdot b = \Delta_L(\omega \cdot b) \end{aligned}$$

for all $a, b \in A, \omega \in \Omega^1 A$, where A acts on $A \otimes \Omega^1 A$ in the tensor product representation. Equivalently, $\Delta_L(a \cdot \omega) = (\Delta a) \cdot \Delta_L(\omega)$ etc., with the second product as an $A \otimes A$ -module. We then say that a one-form $\omega \in \Omega^1 A$ is *left invariant* if it is invariant under left translation by Δ_L . Of course, similar definitions may be made with ‘left’ replaced by ‘right’. It is bicovariant if both definitions hold and the left and right coactions commute. We similarly have the notion of the calculus on an H -comodule algebra A being H -covariant, namely that the coaction extends to the

calculus such that it commutes with d and is multiplicative with respect to the bimodule product.

We now quantise the differential structures on our spaces and groups by the same covariant twist method. For groups the important thing to know is that the classical exterior algebra of differential forms $\Omega(\mathrm{GL}_4)$ (in our case) is a super-Hopf algebra where the coproduct on degree zero elements is that of $\mathbb{C}[\mathrm{GL}_4]$ while on degree one it is $\Delta_L + \Delta_R$ for the classical coactions induced by left and right translation (so $\Delta_L da = a_{(1)} \otimes da_{(2)}$ etc.) We view F as a cocycle on this super-Hopf algebra by extending it as zero, and make a cotwist in the super-algebra version of the cotwist of $\mathbb{C}[\mathrm{GL}_4]$. Then $\Omega(\mathbb{C}_F[\mathrm{GL}_4])$ has the bimodule and wedge products

$$\begin{aligned} a_\nu^\mu \bullet da_\beta^\alpha &= F_{mn}^{\mu\alpha} a_p^m da_q^n F^{-1pq}_{\nu\beta}, & da_\nu^\mu \bullet a_\beta^\alpha &= F_{mn}^{\mu\alpha} (da_p^m) a_q^n F^{-1pq}_{\nu\beta}, \\ da_\nu^\mu \bullet da_\beta^\alpha &= F_{mn}^{\mu\alpha} da_p^m \wedge da_q^n F^{-1pq}_{\nu\beta}, \end{aligned}$$

while d itself is not deformed. The commutation relations are

$$R_{ab}^{\mu\alpha} a_\nu^a \bullet da_\beta^b = da_\beta^a \bullet a_b^\mu R_{\nu\beta}^{ab}, \quad R_{ab}^{\mu\alpha} da_\nu^a \bullet da_\beta^b = -da_\beta^a \bullet da_b^\mu R_{\nu\beta}^{ab}.$$

In terms of the decomposition (17) the deformed products come out as

$$\begin{aligned} \gamma_\nu^\mu \bullet d\gamma_\beta^\alpha &= \gamma_\nu^\mu d\gamma_\beta^\alpha + \frac{\imath}{2} \theta_{ab}^{\mu\alpha} \sigma_\nu^a d\sigma_\beta^b, & d\gamma_\nu^\mu \bullet \gamma_\beta^\alpha &= (d\gamma_\nu^\mu) \gamma_\beta^\alpha + \frac{\imath}{2} \theta_{ab}^{\mu\alpha} (d\sigma_\nu^a) \sigma_\beta^b, \\ d\gamma_\nu^\mu \bullet d\gamma_\beta^\alpha &= d\gamma_\nu^\mu \wedge d\gamma_\beta^\alpha + \frac{\imath}{2} \theta_{ab}^{\mu\alpha} d\sigma_\nu^a \wedge d\sigma_\beta^b, \\ \tilde{\gamma}_\nu^\mu \bullet d\tilde{\gamma}_\beta^\alpha &= \tilde{\gamma}_\nu^\mu d\tilde{\gamma}_\beta^\alpha - \frac{\imath}{2} \sigma_c^\alpha d\sigma_d^c \theta_{\nu\beta}^{cd}, & d\tilde{\gamma}_\nu^\mu \bullet \tilde{\gamma}_\beta^\alpha &= (d\tilde{\gamma}_\nu^\mu) \tilde{\gamma}_\beta^\alpha - \frac{\imath}{2} (d\sigma_c^\alpha) \sigma_d^c \theta_{\nu\beta}^{cd}, \\ d\tilde{\gamma}_\nu^\mu \bullet d\tilde{\gamma}_\beta^\alpha &= d\tilde{\gamma}_\nu^\mu \wedge d\tilde{\gamma}_\beta^\alpha - \frac{\imath}{2} d\sigma_c^\alpha \wedge d\sigma_d^c \theta_{\nu\beta}^{cd}, \\ \gamma_\nu^\mu \bullet d\tau_\beta^\alpha &= \gamma_\nu^\mu d\tau_\beta^\alpha + \frac{\imath}{2} \theta_{ab}^{\mu\alpha} \sigma_\nu^a d\tilde{\gamma}_\beta^b, & d\gamma_\nu^\mu \bullet \tau_\beta^\alpha &= (d\gamma_\nu^\mu) \tau_\beta^\alpha + \frac{\imath}{2} \theta_{ab}^{\mu\alpha} (d\sigma_\nu^a) \tilde{\gamma}_\beta^b, \\ d\gamma_\nu^\mu \bullet d\tau_\beta^\alpha &= d\gamma_\nu^\mu \wedge d\tau_\beta^\alpha + \frac{\imath}{2} \theta_{ab}^{\mu\alpha} d\sigma_\nu^a \wedge d\tilde{\gamma}_\beta^b, \\ \tau_\nu^\mu \bullet d\gamma_\beta^\alpha &= \tau_\nu^\mu d\gamma_\beta^\alpha + \frac{\imath}{2} \theta_{ab}^{\mu\alpha} \tilde{\gamma}_\nu^a d\sigma_\beta^b, & d\tau_\nu^\mu \bullet \gamma_\beta^\alpha &= (d\tau_\nu^\mu) (\gamma_\beta^\alpha) + \frac{\imath}{2} \theta_{ab}^{\mu\alpha} (d\tilde{\gamma}_\nu^a) \sigma_\beta^b, \\ d\tau_\nu^\mu \bullet d\gamma_\beta^\alpha &= d\tau_\nu^\mu \wedge d\gamma_\beta^\alpha + \frac{\imath}{2} \theta_{ab}^{\mu\alpha} d\tilde{\gamma}_\nu^a \wedge d\sigma_\beta^b, \\ \tilde{\gamma}_\nu^\mu \bullet d\tau_\beta^\alpha &= \tilde{\gamma}_\nu^\mu d\tau_\beta^\alpha - \frac{\imath}{2} \sigma_c^\mu \gamma_d^\alpha \theta_{\nu\beta}^{cd}, & d\tilde{\gamma}_\nu^\mu \bullet \tau_\beta^\alpha &= (d\tilde{\gamma}_\nu^\mu) \tau_\beta^\alpha - \frac{\imath}{2} (d\sigma_c^\mu) \gamma_d^\alpha \theta_{\nu\beta}^{cd}, \\ d\tilde{\gamma}_\nu^\mu \bullet d\tau_\beta^\alpha &= d\tilde{\gamma}_\nu^\mu \wedge d\tau_\beta^\alpha - \frac{\imath}{2} d\sigma_c^\mu \wedge \gamma_d^\alpha \theta_{\nu\beta}^{cd}, \\ \tau_\nu^\mu \bullet d\tilde{\gamma}_\beta^\alpha &= \tau_\nu^\mu d\tilde{\gamma}_\beta^\alpha - \frac{\imath}{2} \gamma_c^\mu d\sigma_d^c \theta_{\nu\beta}^{cd}, & d\tau_\nu^\mu \bullet \tilde{\gamma}_\beta^\alpha &= (d\tau_\nu^\mu) \tilde{\gamma}_\beta^\alpha - \frac{\imath}{2} (d\gamma_c^\mu) \sigma_d^c \theta_{\nu\beta}^{cd}, \\ d\tau_\nu^\mu \bullet d\tilde{\gamma}_\beta^\alpha &= d\tau_\nu^\mu \wedge d\tilde{\gamma}_\beta^\alpha - \frac{\imath}{2} d\gamma_c^\mu \wedge d\sigma_d^c \theta_{\nu\beta}^{cd}, \\ \tau_\nu^\mu \bullet d\tau_\beta^\alpha &= \tau_\nu^\mu d\tau_\beta^\alpha + \frac{\imath}{2} \theta_{ab}^{\mu\alpha} \tilde{\gamma}_\nu^a d\tilde{\gamma}_\beta^b - \frac{\imath}{2} \gamma_c^\mu d\gamma_d^\alpha \theta_{\nu\beta}^{cd} + \frac{1}{4} \theta_{ab}^{\mu\alpha} \sigma_c^a d\sigma_d^b \theta_{\nu\beta}^{cd}, \\ d\tau_\nu^\mu \bullet \tau_\beta^\alpha &= (d\tau_\nu^\mu) \tau_\beta^\alpha + \frac{\imath}{2} \theta_{ab}^{\mu\alpha} (d\tilde{\gamma}_\nu^a) \tilde{\gamma}_\beta^b - \frac{\imath}{2} (d\gamma_c^\mu) \gamma_d^\alpha \theta_{\nu\beta}^{cd} + \frac{1}{4} \theta_{ab}^{\mu\alpha} (d\sigma_c^a) \sigma_d^b \theta_{\nu\beta}^{cd}, \\ d\tau_\nu^\mu \bullet d\tau_\beta^\alpha &= d\tau_\nu^\mu \wedge d\tau_\beta^\alpha + \frac{\imath}{2} \theta_{ab}^{\mu\alpha} d\tilde{\gamma}_\nu^a \wedge d\tilde{\gamma}_\beta^b - \frac{\imath}{2} d\gamma_c^\mu \wedge d\gamma_d^\alpha \theta_{\nu\beta}^{cd} + \frac{1}{4} \theta_{ab}^{\mu\alpha} d\sigma_c^a \wedge d\sigma_d^b \theta_{\nu\beta}^{cd}, \end{aligned}$$

with remaining relations undeformed. As before we adopt the convention that in each set of equations, the indices α, β, μ, ν lie in the appropriate ranges for each 2×2 block. One may also calculate the explicit commutation relations in closed form; they will be similar to the above but with θ^- in place of θ .

Similarly, since the classical differential structures on $\widetilde{\mathbb{C}\mathbb{M}^\#}, \widetilde{T}$ are covariant under GL_4 , we have coactions on their classical exterior algebras induced from the coactions on the spaces themselves, such that d is equivariant. We can hence covariantly twist these in the same way as the algebras themselves. Thus $\Omega(\mathbb{C}_F[\widetilde{T}])$ has structure

$$(20) \quad \begin{aligned} Z^\nu \bullet dZ^\mu &= F_{ab}^{\nu\mu} Z^a dZ^b, & dZ^\nu \bullet Z^\mu &= F_{ab}^{\nu\mu} (dZ^a) Z^b \\ dZ^\nu \bullet dZ^\mu &= F_{ab}^{\nu\mu} dZ^a \wedge dZ^b \end{aligned}$$

The commutation relations are similarly

$$Z^\nu \bullet dZ^\mu = R_{ab}^{\mu\nu} dZ^a \bullet Z^b, \quad dZ^\nu \bullet dZ^\mu = -R_{ab}^{\mu\nu} dZ^a \bullet dZ^b.$$

These formulae are essentially as for the coordinate algebra, but now with d inserted, and are (in some form) standard for braided linear spaces define by an R -matrix. More explicitly,

$$\begin{aligned} Z^\mu \bullet dZ^\nu &= Z^\mu dZ^\nu + \frac{\imath}{2} \theta_{ab}^{\mu\nu} Z^a dZ^b, & dZ^\mu \bullet Z^\nu &= (dZ^\mu) Z^\nu + \frac{\imath}{2} \theta_{ab}^{\mu\nu} dZ^a Z^b \\ dZ^\mu \bullet dZ^\nu &= dZ^\mu \wedge dZ^\nu + \frac{\imath}{2} \theta_{ab}^{\mu\nu} dZ^a \wedge dZ^b \end{aligned}$$

so that the Z^3, Z^4, dZ^3, dZ^4 products are undeformed. In terms of commutation relations

$$[Z^\mu, dZ^\nu]_\bullet = \imath \theta_{ab}^{-\mu\nu} Z^a dZ^b, \quad \{dZ^\mu, dZ^\nu\}_\bullet = \imath \theta_{ab}^{-\mu\nu} dZ^a \wedge dZ^b$$

where the right hand sides are for $a, b \in \{3, 4\}$ and could be written with the bullet product equally well. The only nonclassical commutation relations here are those with $\mu, \nu \in \{1, 2\}$.

Similarly, $\Omega(\mathbb{C}_F[\widetilde{\mathbb{C}\mathbb{M}^\#}])$ has structure

$$\begin{aligned} x^{\mu\nu} \bullet dx^{\alpha\beta} &= F_{mn}^{\mu\alpha} F_{ap}^{m\beta} F_{qc}^{\nu n} F_{bd}^{qy} x^{ab} dx^{cd}, & dx^{\mu\nu} \bullet x^{\alpha\beta} &= F_{mn}^{\mu\alpha} F_{ap}^{m\beta} F_{qc}^{\nu n} F_{bd}^{qy} (dx^{ab}) x^{cd} \\ dx^{\mu\nu} \bullet dx^{\alpha\beta} &= F_{mn}^{\mu\alpha} F_{ap}^{m\beta} F_{qc}^{\nu n} F_{bd}^{qy} dx^{ab} \wedge dx^{cd}. \end{aligned}$$

On the affine Minkowski generators and t we find explicitly:

$$(21) \quad \begin{aligned} z \bullet d\tilde{z} &= z d\tilde{z} + \frac{\imath}{2} \theta_{43}^{21} t dt, & dz \bullet \tilde{z} &= (dz) \tilde{z} + \frac{\imath}{2} \theta_{43}^{21} (dt) t, \\ \tilde{z} \bullet dz &= \tilde{z} dz + \frac{\imath}{2} \theta_{34}^{12} t dt, & d\tilde{z} \bullet z &= (d\tilde{z}) z + \frac{\imath}{2} \theta_{34}^{12} (dt) t, \\ dz \bullet d\tilde{z} &= dz \wedge d\tilde{z} + \frac{\imath}{2} \theta_{43}^{21} dt \wedge dt = dz \wedge d\tilde{z}, \\ w \bullet d\tilde{w} &= w d\tilde{w} + \frac{\imath}{2} \theta_{43}^{12} t dt, & dw \bullet \tilde{w} &= (dw) \tilde{w} + \frac{\imath}{2} \theta_{43}^{12} (dt) t, \\ \tilde{w} \bullet dw &= \tilde{w} dw + \frac{\imath}{2} \theta_{34}^{21} t dt, & d\tilde{w} \bullet w &= (d\tilde{w}) w + \frac{\imath}{2} \theta_{34}^{21} (dt) t, \end{aligned}$$

$$dw \bullet d\tilde{w} = dw \wedge d\tilde{w} + \frac{\imath}{2}\theta_{43}^{12}dt \wedge dt = dw \wedge d\tilde{w},$$

with other relations amongst these generators undeformed, as are the relations involving dt , whence we may equally use the \bullet product in terms which involve t, dt . The relations in the calculus involving s, ds are more complicated. We write just the final commutation relations for these:

$$\begin{aligned} [s, dz]_{\bullet} &= -\frac{\imath}{2}\theta_{34}^{12}zdt + \frac{\imath}{2}\theta_{43}^{21}tdz - \frac{\imath}{2}\theta_{44}^{12}\tilde{w}dt + \frac{\imath}{2}\theta_{44}^{21}td\tilde{w}; \\ [ds, z]_{\bullet} &= -\frac{\imath}{2}\theta_{34}^{12}(dz)t + \frac{\imath}{2}\theta_{43}^{21}(dt)z - \frac{\imath}{2}\theta_{44}^{12}(d\tilde{w})t + \frac{\imath}{2}\theta_{44}^{21}(dt)\tilde{w}; \\ [ds, dz]_{\bullet} &= -\frac{\imath}{2}\theta_{34}^{12}dz \wedge dt + \frac{\imath}{2}\theta_{43}^{21}dt \wedge dz - \frac{\imath}{2}\theta_{44}^{12}d\tilde{w} \wedge dt + \frac{\imath}{2}\theta_{44}^{21}dt \wedge d\tilde{w}; \\ [s, d\tilde{z}]_{\bullet} &= -\frac{\imath}{2}\theta_{33}^{21}wdt + \frac{\imath}{2}\theta_{33}^{12}tdw + \frac{\imath}{2}\theta_{43}^{12}\tilde{z}dt - \frac{\imath}{2}\theta_{34}^{12}td\tilde{z}; \\ [ds, \tilde{z}]_{\bullet} &= -\frac{\imath}{2}\theta_{33}^{21}(dw)t + \frac{\imath}{2}\theta_{33}^{12}(dt)w + \frac{\imath}{2}\theta_{43}^{12}(d\tilde{z})t - \frac{\imath}{2}\theta_{34}^{12}(dt)\tilde{z}; \\ [ds, d\tilde{z}]_{\bullet} &= -\frac{\imath}{2}\theta_{33}^{21}dw \wedge dt + \frac{\imath}{2}\theta_{33}^{12}dt \wedge dw + \frac{\imath}{2}\theta_{43}^{12}d\tilde{z} \wedge dt - \frac{\imath}{2}\theta_{34}^{12}dt \wedge d\tilde{z}; \\ [s, dw]_{\bullet} &= -\frac{\imath}{2}\theta_{34}^{21}wdt + \frac{\imath}{2}\theta_{43}^{12}tdw + \frac{\imath}{2}\theta_{44}^{21}\tilde{z}dt - \frac{\imath}{2}\theta_{44}^{12}td\tilde{z}; \\ [ds, w]_{\bullet} &= -\frac{\imath}{2}\theta_{34}^{21}(dw)t + \frac{\imath}{2}\theta_{43}^{12}(dt)w + \frac{\imath}{2}\theta_{44}^{21}(d\tilde{z})t - \frac{\imath}{2}\theta_{44}^{12}(dt)\tilde{z}; \\ [ds, dw]_{\bullet} &= -\frac{\imath}{2}\theta_{34}^{21}dw \wedge dt + \frac{\imath}{2}\theta_{43}^{12}dt \wedge dw + \frac{\imath}{2}\theta_{44}^{21}d\tilde{z} \wedge dt - \frac{\imath}{2}\theta_{44}^{12}dt \wedge d\tilde{z}; \\ [s, d\tilde{w}]_{\bullet} &= \frac{\imath}{2}\theta_{33}^{12}zdt - \frac{\imath}{2}\theta_{33}^{21}tdz - \frac{\imath}{2}\theta_{43}^{12}\tilde{w}dt + \frac{\imath}{2}\theta_{34}^{21}td\tilde{w}; \\ [ds, \tilde{w}]_{\bullet} &= \frac{\imath}{2}\theta_{33}^{12}(dz)t - \frac{\imath}{2}\theta_{33}^{21}(dt)z - \frac{\imath}{2}\theta_{43}^{12}(d\tilde{w})t + \frac{\imath}{2}\theta_{34}^{21}(dt)\tilde{w}; \\ [ds, d\tilde{w}]_{\bullet} &= \frac{\imath}{2}\theta_{33}^{12}dz \wedge dt - \frac{\imath}{2}\theta_{33}^{21}dt \wedge dz - \frac{\imath}{2}\theta_{43}^{12}d\tilde{w} \wedge dt + \frac{\imath}{2}\theta_{34}^{21}dt \wedge d\tilde{w}. \end{aligned}$$

The calculus of the correspondence space algebra $\mathbb{C}_F[\tilde{\mathcal{F}}]$ is generated by dZ^μ , $ds, dt, dz, d\tilde{z}, dw, d\tilde{w}$ with twisted relations given by (18) with d inserted where appropriate, as well as the relations given by differentiating (19) using the Leibniz rule. Explicitly we have

$$\begin{aligned} (22) \quad [z, dZ^1]_{\bullet} &= \imath\theta^{-21}_{43}tdZ^3 + \imath\theta^{-21}_{44}tdZ^4, & [z, dZ^2]_{\bullet} &= \imath\theta^{-22}_{43}tdZ^3, \\ [dz, Z^1]_{\bullet} &= \imath\theta^{-21}_{43}(dt)Z^3 + \imath\theta^{-21}_{44}(dt)Z^4, & [dz, Z^2]_{\bullet} &= \imath\theta^{-22}_{43}(dt)Z^3 \\ [dz, dZ^1]_{\bullet} &= \imath\theta^{-21}_{43}dt \wedge dZ^3 + \imath\theta^{-21}_{44}dt \wedge dZ^4, & [dz, dZ^2]_{\bullet} &= \imath\theta^{-22}_{43}dt \wedge dZ^3, \end{aligned}$$

for example, where we may equally use the \bullet product on the right-hand sides. Moreover,

$$\begin{aligned}
& (d\tilde{z})Z^3 + (dw)Z^4 - (dt)Z^1 + \tilde{z}dZ^3 + wdZ^4 - tdZ^1 = 0, \\
& (d\tilde{w})Z^3 + (dz)Z^4 - (dt)Z^2 + \tilde{w}dZ^3 + zdZ^4 - tdZ^2 = 0, \\
& (ds)Z^3 + (dw)Z^2 + -(dz)Z^1 + \frac{i}{2}((\theta_{43}^{12} - \theta_{43}^{21})(dt)Z^3 + \theta_{44}^{-12}(dt)Z^4) \\
& \quad + sdZ^3 + wdZ^2 + -zdZ^1 + \frac{i}{2}((\theta_{43}^{12} - \theta_{43}^{21})tdZ^3 + \theta_{44}^{-12}tdZ^4) = 0, \\
& (ds)Z^4 - (d\tilde{z})Z^2 + (d\tilde{w})Z^1 + \frac{i}{2}(\theta_{33}^{-21}(dt)Z^3 + (\theta_{34}^{21} - \theta_{34}^{12})(dt)Z^4) \\
& \quad + sdZ^4 - \tilde{z}dZ^2 + \tilde{w}dZ^1 + \frac{i}{2}(\theta_{33}^{-21}tdZ^3 + (\theta_{34}^{21} - \theta_{34}^{12})tdZ^4) = 0.
\end{aligned}$$

Of course, these relations may be written much more compactly using the \bullet product (as explained, the twisted relations are the same as the classical case, save for replacing the old product by the twisted one). The other relations are obtained similarly, hence we refrain from writing them out explicitly, although we stress that dZ^3, dZ^4, dt are central in the calculus.

6. QUANTISATION BY COTWISTS IN THE SU_n $*$ -ALGEBRA VERSION

Our second setting is to work with twistor space and space-time as real manifolds in a $*$ -algebra context. To this end we gave a ‘projector’ description of our spaces $\mathbb{CP}^3, \mathbb{CM}^\#, \mathcal{F}$ as well as their realisation as SU_4 homogeneous spaces. We also show that this setting too quantises nicely by a cochain twist. This approach is directly compatible with C^* -algebra methods, although we shall not perform the C^* -algebra completions here. We shall however simultaneously quantise all real manifolds defined by $n \times n$ -matrices of generators with SU_n -covariant conditions, a class which (as we have seen) includes all partial flag varieties based on \mathbb{C}^n , with $n = 4$ the case relevant for the paper.

The main new ingredient is that in the general theory of cotwisting, we should add that H is a Hopf $*$ -algebra in the sense that Δ is a $*$ -algebra map and $(S \circ *)^2 = \text{id}$, and that the cocycle is real in the sense [11]

$$\overline{F(h, g)} = F((S^2g)^*, (S^2h)^*).$$

In this case the twisted Hopf algebra acquires a new $*$ -structure

$$h^{*F} = \sum \overline{V^{-1}(S^{-1}h_{(1)})(h_{(2)})^*V(S^{-1}h_{(3)})}, \quad V(h) = U^{-1}(h_{(1)})U(S^{-1}h_{(2)}).$$

(We note the small correction to the formula stated in [11]). Also, if A is a left comodule algebra and a $*$ -algebra, we require the coaction Δ_L to be a $*$ -algebra map. Then A_F has a new $*$ -structure

$$a^{*F} = \overline{V^{-1}(S^{-1}a_{(1)})(a_{(2)})^*}.$$

The twisting subgroup in our application will be $S(U(1) \times U(1) \times U(1) \times U(1))$ since this is contained in all relevant subgroups of SU_4 , or rather the larger group SU_n with twisting subgroup appearing in the Hopf $*$ -algebra picture as

$$H = \mathbb{C}[S(U(1)^n)] = \mathbb{C}[t_\mu, t_\mu^{-1}; \mu = 1, \dots, n] / \langle t_1 \cdots t_n = 1 \rangle$$

$$t_\mu^* = t_\mu^{-1}, \quad \Delta t_\mu = t_\mu \otimes t_\mu \quad \epsilon t_\mu = 1, \quad S t_\mu = t_\mu^{-1}.$$

This is also the group algebra of the Abelian group $\mathbb{Z}^n / \mathbb{Z}(1, 1, \dots, 1)$ (where the vector here has n entries, all equal to 1). We define a basis of H by

$$t^{\vec{a}} = t_1^{a_1} \cdots t_n^{a_n}, \quad \vec{a} \in \mathbb{Z}^n / \mathbb{Z}(1, 1, \dots, 1).$$

Note that we could of course eliminate one of the $U(1)$ factors and identify the group with $U(1)^{n-1}$ and the dual group with \mathbb{Z}^{n-1} , and this would be entirely equivalent in what follows but not canonical. We prefer to keep manifest the natural inclusion on the diagonal of SU_n . This inclusion appears now as the $*$ -Hopf algebra surjection

$$\pi : \mathbb{C}[SU_n] \rightarrow H, \quad \pi(a_\nu^\mu) = \delta_\nu^\mu t_\mu.$$

Next, we define a cocycle $F : H \otimes H \rightarrow \mathbb{C}$ by

$$F(t^{\vec{a}}, t^{\vec{b}}) = e^{i\vec{a} \cdot \theta \cdot \vec{b}}$$

where $\theta \in M_n(\mathbb{C})$ is any matrix for which every row and every column adds up to zero (so that $(1, 1, \dots, 1)$ is in the null space from either side). Such a matrix is fully determined by an arbitrary choice of (say) lower $(n-1) \times (n-1)$ diagonal block. Thus the data here is an arbitrary $(n-1) \times (n-1)$ matrix just as we would have if we had eliminated t_1 in the first place. The reality condition and the functional V work out as

$$\theta^\dagger = -\theta, \quad U(t^{\vec{a}}) = e^{-i\vec{a}\theta\vec{a}}, \quad V(t^{\vec{a}}) = 1.$$

The latter means that the $*$ -structures do not deform.

We now pull back this cocycle under π to a cocycle on $\mathbb{C}[SU_n]$ with matrix

$$\mathbb{F} = F(a_\nu^\mu, a_\beta^\alpha) = \delta_\nu^\mu \delta_\beta^\alpha F(t_\nu, t_\beta) = \delta_\nu^\mu \delta_\beta^\alpha e^{i\theta_{\nu\beta}}.$$

We then use these new F -matrices in place of those in Sections 4,5 since the general formulae in terms of F -matrices are identical, being determined by the coactions and coproducts, with the additional twist of the $*$ -operation computed in the same way. Thus we find easily that $\mathbb{C}_F[SU_n]$ has the deformed product and antipode (and undeformed $*$ -structure):

$$a_\nu^\mu \bullet a_\beta^\alpha = e^{i(\theta_{\mu\alpha} - \theta_{\nu\beta})} a_\nu^\mu a_\beta^\alpha = e^{i(\theta_{\mu\alpha} - \theta_{\nu\beta} - \theta_{\alpha\mu} + \theta_{\beta\nu})} a_\beta^\alpha \bullet a_\nu^\mu, \quad S_F a_\nu^\mu = e^{-i(\theta_{\mu\mu} - \theta_{\nu\nu})} S a_\nu^\mu$$

which means that $\mathbb{C}_F[SU_n]$ has a compact form if we use new generators

$$\hat{a}_\nu^\mu = e^{\frac{i}{2}(\theta_{\mu\mu} - \theta_{\nu\nu})} a_\nu^\mu$$

in the sense

$$\Delta \hat{a}_\nu^\mu = \hat{a}_\alpha^\mu \otimes \hat{a}_\nu^\alpha, \quad S_F \hat{a}_\alpha^\mu \bullet \hat{a}_\nu^\alpha = \delta_\nu^\mu = \hat{a}_\alpha^\mu \bullet S_F \hat{a}_\nu^\alpha, \quad \hat{a}_\nu^{\mu*} = S_F \hat{a}_\mu^\nu.$$

They have the same form of commutation relations as the a_ν^μ with respect to the \bullet product. Note that for a C^* -algebra treatment we will certainly want the commutation relations in ‘Weyl form’ with a purely phase factor and hence θ to be real-valued, hence antisymmetric and hence with zeros on the diagonal. So in this case natural case there will be no difference between the \hat{a} and the a generators.

Similarly we find by comodule cotwist that

$$Z^\mu \bullet Z^\nu = e^{i\theta_{\mu\nu}} Z^\mu Z^\nu, \quad Z^{\mu*} \bullet Z^\nu = e^{-i\theta_{\mu\nu}} Z^{\mu*} Z^\nu, \quad Z^\mu \bullet Z^{\nu*} = e^{-i\theta_{\mu\nu}} Z^\mu Z^{\nu*},$$

or directly from the unitary transformation of any projectors

$$P_\nu^\mu \bullet P_\beta^\alpha = F(a_a^\mu S a_\nu^c, a_b^\alpha a_\beta^d) P_c^\alpha P_d^b = F(t_\mu t_\nu^{-1}, t_\alpha t_\beta^{-1}) P_\nu^\mu P_\beta^\alpha = e^{i(\theta_{\mu\alpha} - \theta_{\nu\alpha} + \theta_{\nu\beta} - \theta_{\mu\beta})} P_\nu^\mu P_\beta^\alpha.$$

The commutation relations for the entries of Z, P respectively have the same form as the deformation relations but with $\theta_{\mu\nu}$ replaced by $2\theta_{\mu\nu}^- \equiv \theta_{\mu\nu} - \theta_{\nu\mu}$.

The P_ν^μ are no longer projectors with respect to the \bullet product but the new generators

$$\hat{P}_\nu^\mu = e^{-i\theta_{\mu\nu} + \frac{i}{2}(\theta_{\mu\mu} + \theta_{\nu\nu})} P_\nu^\mu$$

are. They enjoy the same commutation relations as the P_ν^μ with respect to the bullet product. Moreover

$$\text{Tr } \hat{P} = \text{Tr } P, \quad \hat{P}_\nu^{\mu*} = \hat{P}_\mu^\nu.$$

Thus we see for example that $\mathbb{C}_F[\mathbb{CP}^3]$ has quantised commutation relations for the matrix entries of generators \hat{Q} with further relations $\text{Tr } \hat{Q} = 1$ and $*$ -structure $\hat{Q}^\dagger = \hat{Q}$, i.e. exactly the same form for the matrix- \bullet relations as in the classical case. Applying these computations but now with $\text{Tr } P = 2$ for the matrix generator P , we obtain $\mathbb{C}_F[\mathbb{CM}^\#]$ in exactly the same way but with $\text{Tr } \hat{P} = 2$, so that in the projector picture we cover both cases at the same time but with different values for the trace.

For $\mathbb{C}_F[\mathcal{F}]$ we have projectors \hat{Q} of trace 1 and \hat{P} of trace 2. Their products are deformed in the same manner as the P - P relations, leading to

$$\hat{P}_\nu^\mu \bullet \hat{Q}_\beta^\alpha = e^{2i(\theta_{\mu\alpha}^- - \theta_{\nu\alpha}^- + \theta_{\nu\beta}^- - \theta_{\mu\beta}^-)} \hat{Q}_\beta^\alpha \bullet \hat{P}_\nu^\mu$$

for the quantised commutation relations between entries. Moreover, $\hat{P} \bullet \hat{Q} = \hat{Q} = \hat{Q} \bullet \hat{P}$ by a similar computation as for $\hat{P}^2 = \hat{P}$. In particular, we see that $\hat{Q} \in M_4(\mathbb{C}_F[\mathbb{CP}^3])$ and $\hat{P} \in M_4(\mathbb{C}_F[\mathbb{CM}^\#])$ are projectors which define tautological quantum vector bundles over these quantum spaces and their pull-backs to $\mathbb{C}_F[\mathcal{F}]$.

Rather than proving all these facts for each algebra, let us prove them for the quantisation of any real manifold $X \subset M_n(\mathbb{C})^r$ defined as the set of r -tuples of matrices P_1, \dots, P_r obeying relations defined by the operations of: (a) matrix product; (b) trace; (c) the $(\)^\dagger$ operation of Hermitian conjugation. We say that X is

defined by ‘matrix relations’. Clearly any such X has on it an action of SU_n acting by conjugation. We define $\mathbb{C}[X]$ to be the (possibly $*$ -) algebra defined by treating the matrix entries $P_{i\nu}^\mu$ as polynomial generators, the matrix relations as relations in the algebra, and P_i^\dagger (when specified) as a definition of $P_{i\mu}^{\nu*}$. The coaction of $\mathbb{C}[SU_n]$ is

$$\Delta_L P_{i\nu}^\mu = a_a^\mu S a_\nu^c \otimes P_{i_c}^a.$$

We have already seen several examples of such coordinate algebras with matrix relations.

Proposition 6.1. *Let $\mathbb{C}[X]$ be a $*$ -algebra defined by ‘matrix relations’ among matrices of generators P_i , $i = 1, \dots, r$. Its quantisation $\mathbb{C}_F[X]$ by cocycle cotwist using the cocycle above is the free associative algebra with matrices of generators \hat{P}_i modulo the commutation relations*

$$\hat{P}_{i\nu}^\mu \bullet \hat{P}_{j\beta}^\alpha = e^{2i(\theta_{\mu\alpha}^- - \theta_{\nu\alpha}^- + \theta_{\nu\beta}^- - \theta_{\mu\beta}^-)} \hat{P}_{j\beta}^\alpha \bullet \hat{P}_{i\nu}^\mu$$

and the matrix relations of $\mathbb{C}[X]$ with P_i replaced by \hat{P}_i .

Proof. All the P_i have the same coaction, hence for the deformed product for any $P, Q \in \{P_1, \dots, P_r\}$ we have

$$P_\nu^\mu \bullet Q_\beta^\alpha = e^{i(\theta_{\mu\alpha} - \theta_{\nu\alpha} + \theta_{\nu\beta} - \theta_{\mu\beta})} P_\nu^\mu Q_\beta^\alpha$$

by the same computation as for the P - P relations above. This implies the commutation relations stated for the entries of P_i and hence of the \hat{P}_i . Again motivated by the example we define

$$\hat{P}_{i\nu}^\mu = e^{-i\theta_{\mu\nu} + \frac{i}{2}(\theta_{\mu\mu} + \theta_{\nu\nu})} P_{i\nu}^\mu$$

and verify that

$$\hat{P} \bullet \hat{Q} = \widehat{(PQ)}, \quad (\hat{P})^\dagger = \widehat{(P^\dagger)}, \quad \text{Tr } \hat{P} = \text{Tr } P$$

for any P, Q taken from our collection. Thus

$$\begin{aligned} (\hat{P} \bullet \hat{Q})_\nu^\mu &= \hat{P}_\alpha^\mu \bullet \hat{P}_\nu^\alpha = e^{-i\theta_{\mu\alpha} + \frac{i}{2}(\theta_{\mu\mu} + \theta_{\alpha\alpha}) - i\theta_{\alpha\nu} + \frac{i}{2}(\theta_{\alpha\alpha} + \theta_{\nu\nu})} P_\alpha^\mu \bullet P_\nu^\alpha \\ &= e^{-i\theta_{\mu\alpha} + \frac{i}{2}(\theta_{\mu\mu} + \theta_{\alpha\alpha}) - i\theta_{\alpha\nu} + \frac{i}{2}(\theta_{\alpha\alpha} + \theta_{\nu\nu})} e^{i(\theta_{\mu\alpha} - \theta_{\alpha\alpha} + \theta_{\alpha\nu} - \theta_{\mu\nu})} P_\alpha^\mu Q_\nu^\alpha \\ &= e^{-i\theta_{\mu\nu} + \frac{i}{2}(\theta_{\mu\mu} + \theta_{\nu\nu})} P_\alpha^\mu Q_\nu^\alpha = \widehat{(PQ)}_\nu^\mu \\ \hat{P}_\mu^{\nu*} &= e^{i\overline{\theta_{\nu\mu}} - \frac{i}{2}(\overline{\theta_{\nu\nu}} + \overline{\theta_{\mu\mu}})} P_{\mu}^{\nu*} = e^{-i\theta_{\mu\nu} + \frac{i}{2}(\theta_{\mu\mu} + \theta_{\nu\nu})} P_{\nu}^{\dagger\mu} = \widehat{(P^\dagger)}_\nu^\mu. \end{aligned}$$

The proof for the trace is immediate from the definition. We also note that $\hat{\delta}_\nu^\mu = \delta_\nu^\mu$ as the quantisation of the constant identity projector (the identity for matrix multiplication). \square

For example, we now have the quantisation of all flag varieties $\mathbb{C}_F[\mathcal{F}_{k_1, \dots, k_r}(\mathbb{C}^n)]$ with projectors \hat{P}_i having this new form of commutation relations for their matrix entries, but with matrix- \bullet products having the same form as in the classical case

given in Section 3. Again, the \hat{P}_i define r tautological projectors with values in the quantum algebra and hence r tautological classes in the noncommutative K -theory, strictly quantising the commutative situation.

Let us also make some immediate observations from the relations in the proposition. We see that diagonal elements $\hat{P}_{i\mu}^\mu$ (no sum) are central. We also see that $\hat{P}_{i\nu}^\mu$ and $\hat{P}_{i\mu}^\nu = (\hat{P}_{i\nu}^\mu)^*$ commute (so all matrix entry generators are normal in the $*$ -algebra sense). On the other hand nontrivial commutation relations arise when three of the four indices are different and that if we take the adjoint of a generator on both sides of a commutation relation, we should also invert the commutation factor. This means that elements of the form $(\hat{P}_{i\nu}^\mu)^* \hat{P}_{i\nu}^\mu$ (no summation) are always central.

These observations mean that $\mathbb{C}_F[\mathbb{CP}^1]$ is necessarily undeformed in the new generators. For a nontrivial deformation the smallest example is then $\mathbb{C}_F[\mathbb{CP}^2]$. Writing its matrix generator as

$$\hat{Q} = \begin{pmatrix} a & x & y \\ x^* & b & z \\ y^* & z^* & c \end{pmatrix}$$

one has a, b, c are self-adjoint and central with $a + b + c = 1$ and

$$xy = e^{i\theta}yx, \quad yz = e^{i\theta}zy, \quad xz = e^{i\theta}zx, \quad \theta = 2(\theta_{12}^- + \theta_{23}^- + \theta_{31}^-)$$

and the projection relations exactly as stated in Proposition 3.4 (whose statement and proof assumed only that $a, b, c, x^*x, y^*y, z^*z$ are central; no other commutation relations were actually needed). Also note that since a, b, c are central it is natural to set them to constants even in the quantum case. In the quotient $a = b = c = \frac{1}{3}$ we can define $U = 3x, W = 3y, V = 3z$. Then we have the algebra

$$UV = W, \quad U^* = U^{-1}, \quad V^* = V^{-1}, \quad W^* = W^{-1}$$

$$UW = e^{i\theta}WU, \quad WV = e^{i\theta}VW, \quad UV = e^{i\theta}VU.$$

Actually this is just the usual noncommutative torus $\mathbb{C}_\theta[S^1 \times S^1]$ with W defined by the above relations and no additional constraints. Similarly in general, for any actual values with $b, c > 0$ and $b + c < 1$ we will have the same result but with different rescaling factors for U, V, W , i.e. again noncommutative tori as quantum versions of a family of inclusions $S^1 \times S^1 \subset \mathbb{CP}^2$. We can consider this family as a quantum analogue of $\mathbb{C}^* \times \mathbb{C}^* \subset \mathbb{CP}^2$. This conforms to our expectation of $\mathbb{C}_F[\mathbb{CP}^2]$ as a ‘quantum toric variety’. Moreover, by arguments analogous to the classical case in Section 3.2 we can view the localisation $\mathbb{C}_F[\mathbb{CP}^2][a^{-1}]$ as a punctured quantum S^4 with generators x, y, a, a^{-1} and the relations

$$xy = e^{i\theta}yx, \quad x^*x + y^*y = a(1 - a),$$

with a central.

One can also check that the cotriangular Hopf $*$ -algebra $\mathbb{C}_F[\mathrm{SU}_n]$ coacts on $\mathbb{C}_F[X]$ now as the quantum version of our classical coactions, as is required by the general theory, namely that these quantised spaces may be realised as quantum homogeneous spaces if one wishes. Again from general theory, these spaces are Ψ -commutative with respect to the induced involutive braiding built from θ^- and appearing in the commutation relations for the matrix entries.

Finally, the differential calculi are twisted by the same methods as in Section 5. The formulae are similar to the deformation of the coordinate algebras, with the insertion of d just as before. Thus $\Omega(\mathbb{C}_F[\mathrm{SU}_n])$ has structure

$$\begin{aligned} a_\nu^\mu \bullet da_\beta^\alpha &= e^{i(\theta_{\mu\alpha} - \theta_{\nu\beta})} a_\nu^\mu da_\beta^\alpha, & da_\nu^\mu \bullet a_\beta^\alpha &= e^{i(\theta_{\mu\alpha} - \theta_{\nu\beta})} da_\nu^\mu a_\beta^\alpha, \\ da_\nu^\mu \bullet da_\beta^\alpha &= e^{i(\theta_{\mu\alpha} - \theta_{\nu\beta})} da_\nu^\mu \wedge da_\beta^\alpha \end{aligned}$$

and commutation relations

$$\hat{a}_\nu^\mu \bullet d\hat{a}_\beta^\alpha = e^{2i(\theta_{\mu\alpha}^- - \theta_{\nu\beta}^-)} \hat{a}_\nu^\mu \bullet d\hat{a}_\beta^\alpha, \quad d\hat{a}_\nu^\mu \bullet d\hat{a}_\beta^\alpha = -e^{2i(\theta_{\mu\alpha}^- - \theta_{\nu\beta}^-)} d\hat{a}_\nu^\mu \bullet d\hat{a}_\beta^\alpha.$$

Similarly, for the quantisation $\mathbb{C}_F[X]$ above of an algebra with matrix relations, $\Omega(\mathbb{C}_F[X])$ has commutation relations

$$\begin{aligned} \hat{P}_{i\nu}^\mu \bullet d\hat{P}_{j\beta}^\alpha &= e^{2i(\theta_{\mu\alpha}^- - \theta_{\nu\alpha}^- + \theta_{\nu\beta}^- - \theta_{\mu\beta}^-)} d\hat{P}_{j\beta}^\alpha \bullet \hat{P}_{i\nu}^\mu, \\ d\hat{P}_{i\nu}^\mu \bullet d\hat{P}_{j\beta}^\alpha &= -e^{2i(\theta_{\mu\alpha}^- - \theta_{\nu\alpha}^- + \theta_{\nu\beta}^- - \theta_{\mu\beta}^-)} d\hat{P}_{j\beta}^\alpha \bullet d\hat{P}_{i\nu}^\mu. \end{aligned}$$

6.1. Quantum $\mathbb{C}_F[\mathrm{CM}^\#]$, $\mathbb{C}_F[S^4]$ and the quantum instanton. In this section we specialise the above general theory to $\mathbb{C}[\mathrm{CM}^\#]$ and its cocycle twist quantisation. We also find a natural one-parameter family of the $\theta_{\mu\nu}$ -parameters for which $\mathbb{C}_F[\mathrm{CM}^\#]$ has a $*$ -algebra quotient $\mathbb{C}_F[S^4]$. We find that this recovers the S_θ^4 previously introduced by Connes and Landi [5] and that the quantum tautological bundle (as a projective module) on $\mathbb{C}_F[\mathrm{CM}^\#]$ pulls back in this case to a bundle with Grassmann connection equal to the noncommutative instanton found in [10]. This is very different from the approach in [10].

We start with some notations. Since here $n = 4$, $\theta_{\mu\nu}$ is a 4×4 matrix with all rows and columns summing to zero. For convenience we limit ourselves to the case where θ is real and hence (see above) antisymmetric (only the antisymmetric part enters into the commutation relations so this is no real loss). As a result it is equivalent to giving a 3×3 real antisymmetric matrix, i.e. it has within it only three independent parameters.

Lemma 6.2.

$$\theta_A = \theta_{12} + \theta_{23} + \theta_{31}, \quad \theta_B = \theta_{23} + \theta_{34} + \theta_{42}, \quad \theta = \theta_{13} - \theta_{23} + \theta_{24} - \theta_{14}$$

determine any antisymmetric theta completely.

Proof. We write out the three independent equations $\sum_j \theta_{ij} = 0$ for $j = 1, 2, 3$ as

$$\theta_{12} + \theta_{13} + \theta_{14} = 0, \quad -\theta_{12} + \theta_{23} + \theta_{24} = 0, \quad -\theta_{13} - \theta_{23} + \theta_{34} = 0$$

and sum the first two, and sum the last two to give:

$$\theta_{13} + \theta_{23} + \theta_{14} + \theta_{24} = 0, \quad \theta_{24} + \theta_{34} - \theta_{12} - \theta_{13} = 0.$$

Adding the first to θ equations tells us that $\theta = 2(\theta_{13} + \theta_{24})$, while adding the second to $\theta_A - \theta_B$ tells us that $\theta_A - \theta_B = 2(\theta_{24} - \theta_{13})$. Hence, knowing $(\theta, \theta_A - \theta_B)$ is equivalent to knowing θ_{13}, θ_{24} . Finally,

$$\theta_A + \theta_B = \theta_{12} + 2\theta_{23} + \theta_{34} - \theta_{13} - \theta_{24} = \theta_{12} + 3\theta_{23} - \theta_{24} = 4(\theta_{12} - \theta_{24})$$

using the third of our original three equations to identify θ_{23} and then the second of our original three equations to replace it. Hence knowing $\theta, \theta_A - \theta_B$, we see that knowing $\theta_A + \theta_B$ is equivalent to knowing θ_{12} . The remaining $\theta_{14}, \theta_{23}, \theta_{34}$ are determined from our original three equations. This completes the proof, which also provides the explicit formulae:

$$\begin{aligned} \theta_{24} &= \frac{1}{4}(\theta + \theta_A - \theta_B), & \theta_{13} &= \frac{1}{4}(\theta - \theta_A + \theta_B), & \theta_{12} &= \frac{1}{4}\theta + \frac{1}{2}\theta_A \\ \theta_{14} &= -\frac{1}{2}\theta - \frac{1}{4}\theta_A - \frac{1}{4}\theta_B, & \theta_{23} &= \frac{1}{4}\theta_A + \frac{1}{4}\theta_B, & \theta_{34} &= \frac{1}{4}\theta + \frac{1}{2}\theta_B \end{aligned}$$

□

We are now ready to compute the commutation relations between the entries of

$$\hat{P} = \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{B}^\dagger & \hat{D} \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} z & \tilde{w} \\ w & \tilde{z} \end{pmatrix}$$

as in (7), except that now the matrix entry generators are for the quantum algebra $\mathbb{C}_F[\mathbb{CM}^\#]$ (we omit their hats). From the general remarks after Proposition 6.1 we know that the generators along the diagonal, i.e. a, α_3, δ_3 , are central (and self-conjugate under $*$). Also from general remarks we know that all matrix entry generators \hat{P} are normal (they commute with their own conjugate under $*$). Moreover, it is easy to see that if $xy = \lambda yx$ is a commutation relation between any two matrix entries then so is $xy^* = \bar{\lambda} y^* x$, again due to the form of the factors in Proposition 6.1.

Proposition 6.3. *The nontrivial commutation relations of $\mathbb{C}_F[\mathbb{CM}^\#]$ are*

$$\begin{aligned} \alpha\delta &= e^{2i\theta}\delta\alpha, \\ \alpha z &= e^{2i\theta_A}z\alpha, & \alpha\tilde{w} &= e^{2i(\theta+\theta_A)}\tilde{w}\alpha, & \alpha w &= e^{2i\theta_A}w\alpha, & \alpha\tilde{z} &= e^{2i(\theta+\theta_A)}\tilde{z}\alpha, \\ \delta z &= e^{-2i(\theta+\theta_B)}z\delta, & \delta w &= e^{-2i\theta_B}w\delta, & \delta\tilde{w} &= e^{-2i(\theta+\theta_B)}\tilde{w}\delta, & \delta\tilde{z} &= e^{-2i\theta_B}\tilde{z}\delta, \\ z\tilde{w} &= e^{2i(\theta+\theta_B)}\tilde{w}z, & zw &= e^{2i\theta_A}wz, & z\tilde{z} &= e^{2i(\theta+\theta_A+\theta_B)}\tilde{z}z \\ \tilde{w}w &= e^{2i(\theta_A-\theta_B)}w\tilde{w}, & \tilde{w}\tilde{z} &= e^{2i(\theta+\theta_A)}\tilde{z}\tilde{w}, & w\tilde{z} &= e^{2i\theta_B}\tilde{z}w, \end{aligned}$$

and similar relations with inverse coefficient when a generator is replaced by its conjugate under $*$. The further (projector) relations of $\mathbb{C}_F[\mathbb{C}\mathbb{M}^\#]$ are exactly the same as in stated in Corollary 3.3 except for the last two auxiliary relations:

$$\begin{aligned} (\alpha_3 + \delta_3) \begin{pmatrix} z \\ \tilde{z} \end{pmatrix} &= \begin{pmatrix} -\alpha & -e^{-2i(\theta+\theta_B)}\delta^* \\ e^{2i\theta_B}\delta & \alpha^* \end{pmatrix} \begin{pmatrix} w \\ \tilde{w} \end{pmatrix}, \\ (\alpha_3 - \delta_3) \begin{pmatrix} w \\ \tilde{w} \end{pmatrix} &= \begin{pmatrix} -\alpha^* & -e^{-2i\theta_B}\delta^* \\ e^{2i(\theta+\theta_B)}\delta & \alpha \end{pmatrix} \begin{pmatrix} z \\ \tilde{z} \end{pmatrix}. \end{aligned}$$

Proof. Here the product is the twisted \bullet product which we do not denote explicitly. We use the commutation relations in Proposition 6.1, computing the various instances of

$$\theta_{ijkl} = \theta_{ik} - \theta_{jk} + \theta_{jl} - \theta_{il}$$

in terms of the combinations in Lemma 6.2. This gives

$$\begin{aligned} \alpha \hat{B}^i_1 &= e^{2i\theta_A} \hat{B}^i_1 \alpha, & \alpha \hat{B}^i_2 &= e^{2i(\theta+\theta_A)} \hat{B}^i_2 \alpha, & i &= 1, 2 \\ \delta \hat{B}^1_i &= e^{-2i(\theta+\theta_B)} \hat{B}^1_i \delta, & \delta \hat{B}^2_i &= e^{-2i\theta_B} \hat{B}^2_i \delta, & i &= 1, 2 \\ \hat{B}^1_1 \hat{B}^1_2 &= e^{2i(\theta+\theta_B)} \hat{B}^1_2 \hat{B}^1_1, & \hat{B}^1_1 \hat{B}^2_1 &= e^{2i\theta_A} \hat{B}^2_1 \hat{B}^1_1, \\ \hat{B}^1_1 \hat{B}^2_2 &= e^{2i(\theta+\theta_A+\theta_B)} \hat{B}^2_2 \hat{B}^1_1, & \hat{B}^1_2 \hat{B}^2_1 &= e^{2i(\theta_A-\theta_B)} \hat{B}^2_1 \hat{B}^1_2, \\ \hat{B}^1_2 \hat{B}^2_2 &= e^{2i(\theta+\theta_A)} \hat{B}^2_2 \hat{B}^1_2, & \hat{B}^2_1 \hat{B}^2_2 &= e^{2i\theta_B} \hat{B}^2_2 \hat{B}^2_1, \end{aligned}$$

which we write out more explicitly as stated. As explained above, the diagonal elements of A, D are central and for general reasons the conjugate relations are as stated. Finally, we explicitly recompute the content of the noncommutative versions of (3)-(4) to find the relations required for \hat{P} to be a projector (this is equivalent to computing the bullet product from the classical relations). The a, α, α^* form a commutative subalgebra, as do a, δ, δ^* , so the calculations of $A(1-A)$ and $(1-D)D$ are not affected. We can compute BB^\dagger without any commutativity assumptions, and in fact we stated all results from (3) in Corollary 3.3 carefully so as to still be correct without such assumptions. Being similarly careful for (4) gives the remaining two auxiliary equations (without any commutativity assumptions) as

$$(\alpha_3 + \delta_3) \begin{pmatrix} z \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} -\tilde{w}\delta^* - \alpha w \\ w\delta + \alpha^* \tilde{w} \end{pmatrix}, \quad (\alpha_3 - \delta_3) \begin{pmatrix} w \\ \tilde{w} \end{pmatrix} = \begin{pmatrix} \tilde{z}\delta^* + \alpha^* z \\ -z\delta - \alpha \tilde{z} \end{pmatrix}$$

which we write in ‘matrix’ form using the above deformed commutation relations. \square

Note that the ‘Cartesian’ decomposition $\hat{B} = t + ix \cdot \sigma$ may also be computed but it involves sin and cos factors, whereas the ‘twistor’ coordinates, where we work with \hat{B}^i_j directly as generators, have simple phase factors as above. Next, we look at the possible cases where $\mathbb{C}_F[\mathbb{C}\mathbb{M}^\#]$ has a quotient analogous to $\mathbb{C}[S^4]$ in the

classical case. We saw in the classical case that $\alpha = \delta = 0$ and t, x are Hermitian, or equivalently that

$$(23) \quad z^* = \tilde{z}, \quad w^* = -\tilde{w}.$$

Now in the quantum case the $*$ -operation on the entries of \hat{P} are given by a multiple of the undeformed $*$ -operation (as shown in the proof of Proposition 6.1). Hence the analogous relations in $\mathbb{C}_F[S^4]$, if it exists as a $*$ -algebra quotient, will have the same form as (23) but with some twisting factors.

Proposition 6.4. *The twisting quantisation $\mathbb{C}_F[\text{CM}^\#]$ is compatible with the $*$ -algebra quotient $\mathbb{C}[\text{CM}^\#] \rightarrow \mathbb{C}[S^4]$ if and only if $\theta_A = \theta_B = -\frac{1}{2}\theta$. In this case*

$$z^* = e^{\frac{i}{2}\theta} \tilde{z}, \quad w^* = -e^{-\frac{i}{2}\theta} \tilde{w}$$

Proof. If the twisting quantisation is compatible with the $*$ -algebra quotient, we have

$$\hat{B}^1_1 = e^{-i\theta_{13}} B^1_1, \quad \hat{B}^1_2 = e^{-i\theta_{14}} B^1_2, \quad \hat{B}^2_1 = e^{-i\theta_{23}} B^2_1, \quad \hat{B}^2_2 = e^{-i\theta_{24}} B^2_2$$

from which we deduce the required $*$ -operations for the quotient. For example, $\hat{B}^1_1^* = e^{i\theta_{13} + i\theta_{24}} \hat{B}^2_2$ and use the above lemma to identify the factor here as $e^{i\theta/2}$. Therefore we obtain the formulae as stated for the $*$ -structure necessarily in the quotient. Next, working out $\mathbb{C}_F[\text{CM}^\#]$ using Proposition 6.3 we have on the one hand

$$z\tilde{w}^* = e^{-2i(\theta + \theta_B)} \tilde{w}^* z$$

and on the other hand

$$zw = e^{2i\theta_A} wz.$$

For these to coincide as needed by any relation of the form (23) (independently of any deformation factors there) we need $-(\theta + \theta_B) = \theta_A$. Similarly for compatibility of the z^*w relation with the $\tilde{z}w$ relation, we need $\theta_A = \theta_B$. This determines θ_A, θ_B as stated for the required quotient to be a $*$ -algebra quotient. These are also sufficient as far as the commutation relations are concerned. The precise form of $*$ -structure stated allows one to verify the other relations in the quotient as well. \square

We see that while $\mathbb{C}_F[\text{CM}^\#]$ has a 3-parameter deformation, there is only a 1-parameter deformation that pulls back to $\mathbb{C}_F[S^4]$. The latter has only a, z, w, z^*, w^* as generators with relations

$$[a, z] = [a, w] = 0, \quad zw = e^{-i\theta} wz, \quad zw^* = e^{i\theta} w^* z, \quad z^*z + w^*w = a(1 - a),$$

which after a minor change of variables is exactly the S^4_θ in [5]. The ‘pull-back’ of the projector \hat{P} to $\mathbb{C}_F[S^4]$ is

$$\hat{e} = \begin{pmatrix} a & \hat{B} \\ \hat{B}^\dagger & 1 - a \end{pmatrix}, \quad a^* = a, \quad \hat{B} = \begin{pmatrix} z & -e^{\frac{i\theta}{2}} w^* \\ w & e^{-\frac{i\theta}{2}} z^* \end{pmatrix}$$

which up to the change of notations is the ‘defining projector’ in the Connes-Landi approach to S^4 . Whereas it is obtained in [5] from considerations of cyclic cohomology, we obtain it by a straightforward twisting-quantisation. In view of Proposition 3.2 we define the noncommutative 1-instanton to be the Grassmann connection for the projector \hat{e} on $\mathcal{E} = \mathbb{C}_F[S^4]^4 \hat{e}$. This should not come as any surprise since the whole point in [5] was to define the noncommutative S^4 by a projector generating the K-theory as the 1-instanton bundle does classically. However, we now obtain \hat{e} not by this requirement but by twisting-quantisation and as a ‘pull-back’ of the tautological bundle on $\mathbb{C}_F[\mathbb{C}\mathbb{M}^\#]$.

Finally, our approach also canonically constructs $\Omega(\mathbb{C}_F[\mathbb{C}\mathbb{M}^\#])$ and one may check that this quotients in the one-parameter case to $\Omega(\mathbb{C}_F[S^4])$, coinciding with the calculus used in [5]. As explained above, the classical (anti)commutation relations are modified by the same phase factors as in the commutation relations above. One may then obtain explicit formulae for the instanton connection and for the Grassmann connection on $\mathbb{C}_F[\mathbb{C}\mathbb{M}^\#]^4 \hat{P}$.

6.2. Quantum twistor space $\mathbb{C}_F[\mathbb{C}\mathbb{P}^3]$. In our $*$ -algebra approach the classical algebra $\mathbb{C}[\mathbb{C}\mathbb{P}^3]$ has a matrix of generators $Q^\mu{}_\nu$ with exactly the same form as for $\mathbb{C}[\mathbb{C}\mathbb{M}^\#]$, with the only difference being now $\text{Tr } Q = 1$, which significantly affects the content of the ‘projector’ relations of the $*$ -algebra. However, the commutation relations in the quantum case $\mathbb{C}_F[\mathbb{C}\mathbb{P}^3]$ according to Proposition 6.1 have exactly the same form as $\mathbb{C}_F[\mathbb{C}\mathbb{M}^\#]$ if we use the same cocycle F . Hence the commutation relations between different matrix entries in the quantum case can be read off from Proposition 6.3. We describe them in the special 1-parameter case found in the previous section where $\theta_A = \theta_B = -\frac{1}{2}\theta$.

Using the same notations as in Section 3.2 but now with potentially noncommutative generators, $\mathbb{C}_F[\mathbb{C}\mathbb{P}^3]$ has a matrix of generators

$$\hat{Q} = \begin{pmatrix} a & x & y & z \\ x^* & b & w & v \\ y^* & w^* & c & u \\ z^* & v^* & u^* & d \end{pmatrix}, \quad a^* = a, b^* = b, c^* = c, d^* = d, a + b + c + d = 1$$

(we omit hats on the generators). We will use the same shorthand $X = x^*x$ etc as before. As we know on general grounds above, any twisting quantisation $\mathbb{C}_F[\mathbb{C}\mathbb{P}^3]$ has all entries of the quantum matrix \hat{Q} normal (commuting with their adjoints), and the diagonal elements and quantum versions of X, Y, Z, U, V, W central. Moreover, all proofs and statements in Section 3.2 were given whilst being careful *not* to assume that x, y, z, u, v, w mutually commute, only that these elements are normal and X, Y, Z, U, V, W central. Hence the relations stated there are also exactly the projector relations for this algebra:

Proposition 6.5. *For the 1-parameter family of cocycles $\theta_A = \theta_B = -\frac{1}{2}\theta$ the quantisations $\mathbb{C}_F[\mathbb{CP}^3]$ and $\mathbb{C}_F^-(\mathbb{CP}^3)$ have exactly the projection relations as in Propositions 3.5 and 3.7 but now with the commutation relations*

$$xz = e^{i\theta}zx, \quad yx = e^{i\theta}xy, \quad yz = e^{i\theta}zy,$$

and the auxiliary commutation relations

$$uv = e^{i\theta}vu, \quad uw = e^{i\theta}wu, \quad vw = e^{i\theta}wv,$$

$$x(u, v, w) = (e^{2i\theta}u, e^{i\theta}v, e^{-i\theta}w)x,$$

$$y(u, v, w) = (e^{i\theta}u, v, e^{-i\theta}w)y, \quad z(u, v, w) = (e^{i\theta}u, e^{i\theta}v, w)z,$$

and similar relations with inverse factor if any generator in a relation is replaced by its adjoint under $*$.

Proof. As explained, the commutation relations are the same as for the entries of \hat{P} in $\mathbb{C}_F[\mathbb{CM}^\#]$ with a different notation of the matrix entries. We read them off and specialise to the 1-parameter case of interest. The ‘auxiliary’ set are deduced from those among the x, y, z if $a \neq 0$, since in this case u, v, w are given in terms of these and their adjoints. \square

If we localise by inverting a , then by analogous arguments to the classical case, the resulting ‘patch’ of $\mathbb{C}_F^-(\mathbb{CP}^3)$ becomes a quantum punctured S^6 with complex generators x, y, z , invertible central self-adjoint generator a and commutation relations as above, and the relation $x^*x + y^*y + z^*z = a(1 - a)$. Also by the same arguments as in the classical case, if we set a, b, c, d to actual fixed numbers (which still makes sense since they are central) then

$$\mathbb{C}_F^-(\mathbb{CP}^3) \Big|_{\substack{b, c, d > 0 \\ b+c+d=1}} = \mathbb{C}_\theta[S^1 \times S^1 \times S^1]$$

where the right hand side has relations as above for three circles but now with commutation relations between the x, y, z circle generators as stated in the proposition. For each set of values of b, c, d we have a quantum analogue of $S^1 \times S^1 \times S^1 \subset \mathbb{CP}^3$ and if we leave them undetermined then in some sense a quantum version of $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* \subset \mathbb{CP}^3$, i.e. $\mathbb{C}_F^-(\mathbb{CP}^3)$ is in this sense a ‘quantum toric variety’. We have seen the same pattern of results already for $\mathbb{C}_F[\mathbb{CP}^2]$ and $\mathbb{C}[\mathbb{CP}^1]$.

Having established the quantum versions of $\mathbb{C}[S^4]$ and twistor space $\mathbb{C}^-(\mathbb{CP}^3)$, we now investigate the quantum version of the fibration $\mathbb{CP}^3 \rightarrow S^4$. In terms of coordinate algebras one has an antilinear involution $J : \mathbb{C}_F^-(\mathbb{CP}^3) \rightarrow \mathbb{C}_F^-(\mathbb{CP}^3)$ analogous to Lemma 3.9. The form of J , however, has to be modified by some phase factors to fit the commutation relations of Proposition 6.5, and is now given by

$$\begin{aligned} J(y) &= e^{i\theta}v^*, & J(y^*) &= e^{-i\theta}v, & J(v) &= e^{-i\theta}y^*, & J(v^*) &= e^{i\theta}y, \\ J(w) &= z^*, & J(w^*) &= -z, & J(z) &= -w^*, & J(z^*) &= -w, \\ J(x) &= -x, & J(u) &= -u, & J(a) &= b, & J(b) &= a. \end{aligned}$$

The map J then extends to $\mathbb{C}_F[\text{CM}^\#]$ and by arguments analogous to those given in the previous section and in Section 3.2, the fixed point subalgebra under J is once again precisely $\mathbb{C}_F[S^4]$. We arrive in this way at the analogous main conclusion, which we verify directly:

Proposition 6.6. *There is an algebra inclusion*

$$\eta : \mathbb{C}_F[S^4] \hookrightarrow \mathbb{C}_F^-[\mathbb{CP}^3]$$

given by

$$\eta(a) = a + b, \quad \eta(z) = e^{i\theta}y + v^*, \quad \eta(w) = w - z^*.$$

Proof. Once again the main relation to investigate is the image of the sphere relation $zz^* + ww^* = a(1 - a)$. Applying η to the left hand side, we obtain

$$\eta(zz^* + ww^*) = yy^* + e^{i\theta}yv + e^{-i\theta}v^*y^* + v^*v + ww^* - wz - z^*w^* + z^*z.$$

We now compute that

$$ayv = yav = yx^*z = e^{-i\theta}x^*yz = e^{-i\theta}awz,$$

where the first equality uses centrality of a , the second uses the projector relation $av = x^*z$ and the third uses Proposition 6.5. Similarly one obtains that $byv = e^{-i\theta}bwz$, $cyv = e^{-i\theta}cwz$, $dyv = e^{-i\theta}dwz$, so that adding these four relations now reveals that $yv = e^{-i\theta}wz$ in $\mathbb{C}_F^-[\mathbb{CP}^3]$. Finally using the relations in Proposition 3.8 (which are still valid as the projector relations in our noncommutative case) we see that

$$\eta(zz^* + ww^*) = Y + V + W + Z = (a + b)(c + d) = (a + b)(1 - (a + b)) = \eta(a(1 - a)).$$

To verify the preservation of the algebra structure of $\mathbb{C}_F[S^4]$ under η we also have to check the commutation relations, of which the nontrivial one is $zw = e^{-i\theta}wz$. Indeed, $\eta(zw) = (e^{i\theta}y + v^*)(w - z^*) = e^{-i\theta}(w - z^*)(e^{i\theta}y + v^*) = \eta(e^{-i\theta}wz)$ using the commutation relations in Proposition 6.5. \square

Just as in Section 3.2 we may compute the ‘push-out’ of the quantum instanton bundle along η , given by viewing the tautological projector \hat{e} as an element $\tilde{e} \in M_4(\mathbb{C}_F^-[\mathbb{CP}^3])$. Explicitly, we have (following the method of Section 3.2)

$$\tilde{e} = \begin{pmatrix} a + b & \hat{M} \\ \hat{M}^\dagger & 1 - (a + b) \end{pmatrix} \in M_4(\mathbb{C}_F^-[\mathbb{CP}^3]), \quad \hat{M} = \begin{pmatrix} y + v^* & -e^{\frac{i\theta}{2}}(w^* - z) \\ w - z^* & e^{-\frac{i\theta}{2}}(y^* + v) \end{pmatrix},$$

and the auxiliary bundle over twistor space is then $\tilde{\mathcal{E}} = \mathbb{C}_F^-[\mathbb{CP}^3]^4 \tilde{e}$. In this way the quantum instanton may be thought of as coming from a bundle over quantum twistor space, just as in the classical case.

7. THE PENROSE-WARD TRANSFORM

The main application of the double fibration (2) is to study relationships between vector bundles over twistor space and space-time, and between their associated geometric data. We begin by discussing how differential forms on these spaces are related, before considering more general vector bundles. Although we restrict our attention to the special case of (2), our remarks are not specific to this example and one should keep in mind the picture of a pair of fibrations of homogeneous spaces

$$(24) \quad \begin{array}{ccc} & G/R & \\ p \swarrow & & \searrow q \\ G/H & & G/K, \end{array}$$

where G is, say, a complex Lie group with parabolic subgroups H, K, R such that $R = H \cap K$. In general, the differential calculi of the coordinate algebras of these spaces need not be compatible in any sensible way, however when there is some form of compatibility (namely a certain transversality condition on the fibrations at the level of the correspondence space G/R) then the double fibration has some nice properties. Indeed, as we shall see, such a compatibility between calculi allows one to ‘transform’ geometric data from G/H to G/K and *vice versa*.

This is the motivation behind the *Penrose-Ward transform*, which we shall describe in this section. The idea is that certain classes of vector bundles over a given subset of G/H correspond to vector bundles over an appropriate subset of G/K equipped with a connection possessing anti-self-dual curvature, the correspondence given by pull-back along p followed by direct image along q . In fact, as discussed, we have already seen this transform in action, albeit in the simplified case where $G/H = \mathbb{CP}^3$ and the given subset of $G/K = \mathbb{CM}^\#$ is S^4 , so that the double fibration collapses to a single fibration. In Section 3.2 we gave a coordinate algebra description of this fibration, with the analogous quantum version computed in Section 6.2. In this section we outline how the transform works in a different situation, namely between the affine piece $\mathbb{CM} \subset \mathbb{CM}^\#$ of space-time and the corresponding patch of twistor space (so that we are utilising the full geometry of the double fibration): we also outline how it quantises under the cotwist given in Section 4.

7.1. Localised coordinate algebras. We note that classically the driving force behind the construction that we shall describe is the theory of holomorphic functions and holomorphic sections of vector bundles. Of course, if one works globally then one cannot expect to be able to say very much at all (in general, one cannot expect to find enough functions). Hence here we resort to working *locally*, on some open set U of space-time $\mathbb{CM}^\#$ having nice topological properties¹. As remarked earlier, of particular interest are the principal open sets U_f , for which one can explicitly write

¹Namely that the intersection of each α -plane \hat{Z} with U is either empty or connected and simply connected, so that if we write $W := q^{-1}(U)$ and $Y := p(W)$ then the fibres of the maps

down the coordinate algebra $\mathcal{O}_{\mathbb{C}\mathbb{M}^\#}(U_f)$ in our language and expect the construction to go through. In what follows, however, we choose to be explicit and restrict our attention to the case of

$$\mathcal{O}_{\mathbb{C}\mathbb{M}^\#}(U_t) = \widetilde{\mathbb{C}[\mathbb{C}\mathbb{M}^\#]}(t^{-1})_0,$$

the ‘affine’ piece of space-time, for which we have the inclusion of algebras $\mathbb{C}[\mathbb{C}\mathbb{M}] \hookrightarrow \widetilde{\mathbb{C}[\mathbb{C}\mathbb{M}^\#]}(t^{-1})_0$ given by

$$x_1 \rightarrow t^{-1}z, \quad x_2 \rightarrow t^{-1}\tilde{z}, \quad x_3 \rightarrow t^{-1}w, \quad x_4 \rightarrow t^{-1}\tilde{w}.$$

In fact we also choose to write $\tilde{s} := t^{-1}s$, so that the quadric relation in $\widetilde{\mathbb{C}[\mathbb{C}\mathbb{M}^\#]}$ becomes $\tilde{s} = x_1x_2 - x_3x_4$. Thus in the algebra $\widetilde{\mathbb{C}[\mathbb{C}\mathbb{M}^\#]}(t^{-1})_0$ the generator \tilde{s} is redundant and we may as well work with $\mathbb{C}[\mathbb{C}\mathbb{M}]$.

Since the points in $\widetilde{\mathbb{C}\mathbb{M}^\#}$ for which $t = 0$ correspond to the points in \tilde{T} where $Z^3 = Z^4 = 0$, the homogeneous twistor space of affine space-time $\mathbb{C}[\mathbb{C}\mathbb{M}]$ has coordinate algebra $\mathbb{C}[\tilde{T}]$ with the extra conditions that Z^3, Z^4 are not both zero. The twistor space T_t of $\mathbb{C}\mathbb{M}$ is thus covered by two coordinate patches, where $Z^3 \neq 0$ and where $Z^4 \neq 0$. These patches have coordinate algebras

$$\mathbb{C}[T_{Z^3}] := \mathbb{C}[\tilde{T}]/((Z^3)^{-1})_0, \quad \mathbb{C}[T_{Z^4}] := \mathbb{C}[\tilde{T}]/((Z^4)^{-1})_0$$

respectively. Note that when both Z^3 and Z^4 are non-zero the two algebras are isomorphic (even in the twisted case, since Z^3, Z^4 remain central in the algebra), with ‘transition functions’ $Z^\mu(Z^3)^{-1} \mapsto Z^\mu(Z^4)^{-1}$. This isomorphism simply says that both coordinate patches look locally like \mathbb{C}^3 , in agreement that our expectation that twistor space is a complex 3-manifold.

In passing from $\mathbb{C}\mathbb{M}^\#$ to $\mathbb{C}\mathbb{M}$ we delete the ‘region at infinity’ where $t = 0$, and similarly we obtain the corresponding twistor space T_t by deleting the region where $Z^3 = Z^4 = 0$. At the homogeneous level this region has coordinate algebra

$$\mathbb{C}[\tilde{T}]/\langle Z^3 = Z^4 = 0 \rangle \cong \mathbb{C}[Z^1, Z^2],$$

describing a line $\mathbb{C}\mathbb{P}^1$ at the projective level. In the twisted framework, the generators Z^3, Z^4 are central so the quotient still makes sense and we have

$$\mathbb{C}_F[\tilde{T}]/\langle Z^3 = Z^4 = 0 \rangle \cong \mathbb{C}_F[Z^1, Z^2],$$

describing a noncommutative $\mathbb{C}\mathbb{P}^1$.

At the level of the homogeneous correspondence space we also have t^{-1} adjoined and Z^3, Z^4 not both zero, and we write $\mathbb{C}[\tilde{\mathcal{F}}_t] := \mathbb{C}[\tilde{\mathcal{F}}](t^{-1})_0$ for the corresponding coordinate algebra. We note that in terms of the $\mathbb{C}[\mathbb{C}\mathbb{M}]$ generators, the relations (1) now read

$$(25) \quad Z^1 = x_2Z^3 + x_3Z^4, \quad Z^2 = x_4Z^3 + x_1Z^4.$$

$p: W \rightarrow Y$ and $q: W \rightarrow U$ are connected, so that operations such as pull-back and direct image are well-behaved.

At the projective level, the correspondence space is also covered by two coordinate patches. Thus when $Z^3 \neq 0$ and when $Z^4 \neq 0$ we respectively mean

$$\mathbb{C}[\mathcal{F}_{Z^3}] := \mathbb{C}[\tilde{\mathcal{F}}_t]((Z^3)^{-1})_0, \quad \mathbb{C}[\mathcal{F}_{Z^4}] := \mathbb{C}[\tilde{\mathcal{F}}_t]((Z^4)^{-1})_0.$$

Again when Z^3 and Z^4 are both non-zero these algebras are seen to be isomorphic under appropriate transition functions. Thus at the homogeneous level, the coordinate algebra $\mathbb{C}[\tilde{\mathcal{F}}_t]$ describes a local trivialisation of the correspondence space in the form $\tilde{\mathcal{F}}_t \cong \mathbb{C}^2 \times \mathbb{C}\mathbb{M}$. The two coordinate patches \mathcal{F}_{Z^3} and \mathcal{F}_{Z^4} together give a trivialisation of the projective correspondence space in the form $\mathcal{F}_t = \mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{M}$.

Regarding the differential calculus of $\mathbb{C}[\widetilde{\mathbb{C}\mathbb{M}^\#}](t^{-1})_0$, it is easy to see that since $d(t^{-1}t) = 0$, from the Leibniz rule we have $d(t^{-1}) = -t^{-2}dt$. Since dt is central even in the twisted calculus, adjoining this extra generator t^{-1} causes no problems, and it remains to check that the calculus is well-defined in the degree zero subalgebra $\mathbb{C}[\widetilde{\mathbb{C}\mathbb{M}^\#}](t^{-1})_0$. Indeed, we see that for example

$$dx_1 = d(t^{-1}z) = t^{-1}dz - (t^{-1}z)(t^{-1}dt),$$

which is again of overall degree zero. Similar statements hold regarding the differential calculi of $\mathbb{C}[T_{Z^3}]$ and $\mathbb{C}[T_{Z^4}]$ as well as those of $\mathbb{C}[\mathcal{F}_{Z^3}]$ and $\mathbb{C}[\mathcal{F}_{Z^4}]$.

7.2. Differential aspects of the double fibration. We now consider the pull-back and direct image of one-forms on our algebras. Indeed, we shall examine how the differential calculi occurring in the fibration (2) are related and derive the promised ‘transversality condition’ required in order to transfer data from one side of the fibration to the other. For now, we consider only the classical (i.e. untwisted) situation.

Initially we work at the homogeneous level, with $\mathbb{C}[\tilde{T}]$ and $\mathbb{C}[\tilde{\mathcal{F}}_t]$, later passing to local coordinates by adjoining an inverse for either Z^3 or Z^4 , as described above. We define

$$(26) \quad \Omega_p^1 := \Omega^1\mathbb{C}[\tilde{\mathcal{F}}_t]/p^*\Omega^1\mathbb{C}[\tilde{T}]$$

to be the set of *relative* one-forms (the one-forms which are dual to those vectors which are tangent to the fibres of p). Note that Ω_p^1 is just the sub-bimodule of $\Omega^1\mathbb{C}[\tilde{\mathcal{F}}_t]$ spanned by $d\tilde{s}$, dx_1 , dx_2 , dx_3 , dx_4 , so that

$$(27) \quad \Omega^1\mathbb{C}[\tilde{\mathcal{F}}_t] = p^*\Omega^1\mathbb{C}[\tilde{T}] \oplus \Omega_p^1.$$

There is of course an associated projection

$$\pi_p : \Omega^1\mathbb{C}[\tilde{\mathcal{F}}_t] \rightarrow \Omega_p^1$$

and hence an associated relative exterior derivative $d_p : \mathbb{C}[\tilde{\mathcal{F}}_t] \rightarrow \Omega_p^1$ given by composition of d with this projection,

$$(28) \quad d_p = \pi_p \circ d : \mathbb{C}[\tilde{\mathcal{F}}_t] \rightarrow \Omega_p^1.$$

Similarly, we define the relative two-forms by

$$\Omega_p^2 := \Omega^2\mathbb{C}[\tilde{\mathcal{F}}_t]/(p^*\Omega^1\mathbb{C}[\tilde{T}] \wedge \Omega^1\mathbb{C}[\tilde{\mathcal{F}}_t])$$

so that d_p extends to a map

$$d_p : \Omega_p^1 \rightarrow \Omega_p^2$$

by composing $d : \Omega^1\mathbb{C}[\tilde{\mathcal{F}}_t] \rightarrow \Omega^2\mathbb{C}[\tilde{\mathcal{F}}_t]$ with the projection $\Omega^1\mathbb{C}[\tilde{\mathcal{F}}_t] \rightarrow \Omega_p^1$. We see that d_p obeys the relative Leibniz rule,

$$(29) \quad d_p(fg) = (d_p f)g + f(d_p g), \quad f, g \in \mathbb{C}[\tilde{\mathcal{F}}_t].$$

It is clear by construction that the kernel of d_p consists precisely of the functions in $\mathbb{C}[\tilde{\mathcal{F}}_t]$ which are constant on the fibres of p , whence we recover $\mathbb{C}[\tilde{T}]$ by means of the functions in $\mathbb{C}[\tilde{\mathcal{F}}_t]$ which are covariantly constant with respect to d_p (since functions in $\mathbb{C}[\tilde{T}]$ may be identified with those functions in $\mathbb{C}[\tilde{\mathcal{F}}_t]$ which are constant on the fibres of p). Moreover, the derivative d_p is *relatively flat*, i.e. its curvature d_p^2 is zero.

The next stage is to consider the direct image of the one-forms Ω_p^1 along q . In the usual theory the direct image [20] of a vector bundle $\pi : E' \rightarrow \mathcal{F}_t$ over \mathcal{F}_t along the fibration $q : \mathcal{F}_t \rightarrow \mathbb{CM}$ is by definition the bundle $E := q_* E' \rightarrow \mathbb{CM}$ whose fibre over $x \in \mathbb{CM}$ is $H^0(q^{-1}(x), E')$, the space of global sections of the restriction of E' to $q^{-1}(x)$. Of course, this definition does not in general result in a well-defined vector bundle. For this, we assume that each $q^{-1}(x)$ is compact and connected so that $H^0(q^{-1}(x), E')$ is finite dimensional, and that $H^0(q^{-1}(x), E')$ is of constant dimension as $x \in \mathbb{CM}$ varies. In our situation these criteria are clearly satisfied.

We recall that in the given local trivialisation, the correspondence space looks like $\mathcal{F}_t = \mathbb{CP}^1 \times \mathbb{CM}$ with coordinate functions $\zeta, x_1, x_2, x_3, x_4$, where $\zeta = Z^3(Z^4)^{-1}$ or $Z^4(Z^3)^{-1}$ (depending on which coordinate patch we are in). Writing \mathcal{E}' for the space of sections of the bundle $E' \rightarrow \mathcal{F}_t$, the space of sections \mathcal{E} of the direct image bundle E is in general obtained by computing \mathcal{E}' as a $\mathbb{C}[\mathbb{CM}]$ -module. So although any section $\xi \in \mathcal{E}'$ of E' is in general a function of the twistor coordinate ζ as well as the space-time coordinates x_j , its restriction to any fibre $q^{-1}(x)$ is by Liouville's theorem independent of ζ (i.e. ζ is constant on $q^{-1}(x)$). Of course, when we restrict the section ξ to each $q^{-1}(x)$, its dependence on ζ varies as $x \in \mathbb{CM}$ varies (in that although ζ is constant on each $q^{-1}(x)$, it possibly takes different values as x varies), and this means that in taking the direct image we may write ζ as a function of the space-time coordinates $x_j, j = 1, \dots, 4$. One could also argue by noting that \mathbb{CM} is topologically trivial so has no cohomology, whence one has for example

$$H^0(\mathbb{CP}^1 \times \mathbb{CM}, \mathbb{C}) = H^0(\mathbb{CP}^1, \mathbb{C}) \otimes H^0(\mathbb{CM}, \mathbb{C}) \cong H^0(\mathbb{CM}, \mathbb{C}).$$

At the homogeneous level, although the fibres of the map $q : \tilde{\mathcal{F}}_t \rightarrow \mathbb{CM}$ are not compact, the effect of the above argument may be achieved by regarding the twistor coordinates Z^3, Z^4 as functions of the space-time coordinates under the

direct image. The upshot of this argument is that to compute the direct image of a $\mathbb{C}[\tilde{\mathcal{F}}_t]$ -module \mathcal{E}' , we ‘remove’ the dependence of \mathcal{E}' on the twistor coordinates Z^i , hence obtaining a $\mathbb{C}[\mathbb{C}\mathbb{M}]$ -module \mathcal{E} . In what follows, this will be our naive definition of direct image, by analogy with the classical case. Since this argument really belongs in the language of cohomology, we satisfy ourselves with this definition for now, deferring a more precise treatment to a sequel.

Proposition 7.1. *There is an isomorphism $q_*q^*\Omega^1\mathbb{C}[\mathbb{C}\mathbb{M}] \cong \Omega^1\mathbb{C}[\mathbb{C}\mathbb{M}]$.*

Proof. The generators $dx_j, j = 1, \dots, 4$ of $\Omega^1\mathbb{C}[\mathbb{C}\mathbb{M}]$ pull back to their counterparts $dx_i, i = 1, \dots, 4$ in $\mathbb{C}[\tilde{\mathcal{F}}_t]$ and these span $q^*\Omega^1\mathbb{C}[\mathbb{C}\mathbb{M}]$ as a $\mathbb{C}[\tilde{\mathcal{F}}_t]$ -bimodule. Taking the direct image involves computing $q^*\Omega^1\mathbb{C}[\mathbb{C}\mathbb{M}]$ as a $\mathbb{C}[\mathbb{C}\mathbb{M}]$ -bimodule, and as such it is spanned by elements of the form $d\tilde{s}, dx_i, Z^i d\tilde{s}, Z^i dx_j$, for $i, j = 1, \dots, 4$. As already observed, the generator \tilde{s} is essentially redundant, hence so are the generators involving $d\tilde{s}$. Moreover, the relations (25) allow us to write Z^1 and Z^2 in terms of Z^3, Z^4 , whence we are left with elements of the form $dx_j, Z^3 dx_j$ and $Z^4 dx_j$ for $j = 1, \dots, 4$.

As explained above, the direct image is given by writing the Z -coordinates as functions of the space-time coordinates, and it is clear that the resulting differential calculus $q_*q^*\Omega^1\mathbb{C}[\mathbb{C}\mathbb{M}]$ of $\mathbb{C}[\mathbb{C}\mathbb{M}]$ is isomorphic to the one we first thought of. We remark that we have used the same symbol d to denote the exterior derivative in the different calculi $\Omega^1\mathbb{C}[\mathbb{C}\mathbb{M}]$, $q^*\Omega^1\mathbb{C}[\mathbb{C}\mathbb{M}]$ and $q_*q^*\Omega^1\mathbb{C}[\mathbb{C}\mathbb{M}]$. Although they are in principle different, our notation causes no confusion here. \square

Proposition 7.2. *For $s = 1, 2$ the direct images $q_*\Omega_p^s$ are given by:*

- $q_*\Omega_p^1 \cong \Omega^1\mathbb{C}[\mathbb{C}\mathbb{M}];$
- $q_*\Omega_p^2 \cong \Omega_+^2\mathbb{C}[\mathbb{C}\mathbb{M}],$

where $\Omega_+^2\mathbb{C}[\mathbb{C}\mathbb{M}]$ denotes the space of two-forms in $\Omega^2\mathbb{C}[\mathbb{C}\mathbb{M}]$ which are self-dual with respect to the Hodge $*$ -operator defined by the metric $\eta = 2(dx_1 dx_2 - dx_3 dx_4)$.

Proof. We first consider the $\mathbb{C}[\tilde{\mathcal{F}}_t]$ -bimodule Ω_p^1 , and write $q_*\Omega_p^1$ for the same vector space considered as a $\mathbb{C}[\mathbb{C}\mathbb{M}]$ -bimodule. Quotienting $\Omega^1\mathbb{C}[\tilde{\mathcal{F}}_t]$ by the one-forms pulled back from $\mathbb{C}[\tilde{T}]$ means that Ω_p^1 is spanned as a $\mathbb{C}[\tilde{\mathcal{F}}_t]$ -bimodule by $d\tilde{s}$ and $dx_i, i = 1, \dots, 4$.

The direct image is computed just as before, so $q_*\Omega_p^1$ is as a vector space the same calculus Ω_p^1 but now considered as a $\mathbb{C}[\mathbb{C}\mathbb{M}]$ -bimodule. In what follows we shall write d for the exterior derivatives in the calculi $\Omega^1\mathbb{C}[\mathbb{C}\mathbb{M}]$ and $\Omega^1\mathbb{C}[\tilde{\mathcal{F}}_t]$ as they are the usual operators (the ones we which we wrote down and quantised in section 5). However, in calculating the direct image $q_*\Omega_p^1$ of the calculus Ω_p^1 , we must introduce different notation for the image of the operator d_p under q_* . To this end, we write $q_*d_p := \underline{d}$, so that as a $\mathbb{C}[\mathbb{C}\mathbb{M}]$ -bimodule the calculus $q_*\Omega_p^1$ is spanned by elements of the form $\underline{d}\tilde{s}$ and $\underline{d}x_j, i, j = 1, \dots, 4$. As already observed, the generator \tilde{s} is essentially redundant, hence so is the generator $\underline{d}\tilde{s}$.

The identity (29) becomes a Leibniz rule for \underline{d} upon taking the direct image q_* , whence $(q_*\Omega_p^1, \underline{d})$ is a first order differential calculus of $\mathbb{C}[\text{CM}]$. We must investigate its relationship with the calculus $\Omega^1\mathbb{C}[\text{CM}]$.

Differentiating the relations (25) and quotienting by generators dZ^i yields the relations

$$(30) \quad Z^3 d_p x_2 + Z^4 d_p x_3 = 0, \quad Z^3 d_p x_4 + Z^4 d_p x_1 = 0$$

in the relative calculus Ω_p^1 . Thus as a $\mathbb{C}[\tilde{\mathcal{F}}_t]$ -bimodule, Ω_p^1 has rank two, since the basis elements $d_p x_j$ are not independent. However, in the direct image $q_*\Omega_p^1$, which is just Ω_p^1 considered as a $\mathbb{C}[\text{CM}]$ -bimodule, these basis elements (now written $\underline{d}x_j$) are independent. Thus it is clear that the calculus $(q_*\Omega_p^1, \underline{d})$ is isomorphic to $(\Omega^1\mathbb{C}[\text{CM}], d)$ in the sense that as bimodules they are isomorphic, and that this isomorphism is an intertwiner for the derivatives \underline{d} and d .

Using the relations (30) it is easy to see that in the direct image bimodule $q_*\Omega_p^2 := \Lambda^2(q_*\Omega_p^1)$ we have

$$\underline{d}x_1 \wedge \underline{d}x_4 = \underline{d}x_2 \wedge \underline{d}x_3 = \underline{d}x_1 \wedge \underline{d}x_2 + \underline{d}x_3 \wedge \underline{d}x_4 = 0,$$

which we recognise (since we are in double null coordinates) as the anti-self-dual two-forms, whence it is the self-dual two-forms which survive under the direct image. It is evident that these arguments are valid upon passing to either coordinate patch at the projective level, i.e. upon adjoining either $(Z^3)^{-1}$ or $(Z^4)^{-1}$ and taking the degree zero part of the resulting calculus. \square

It is clear that the composition of maps

$$q^*\Omega^1\mathbb{C}[\text{CM}] \rightarrow \Omega^1\mathbb{C}[\tilde{\mathcal{F}}_t] \rightarrow \Omega_p^1$$

determines the isomorphism

$$\Omega^1\mathbb{C}[\text{CM}] \rightarrow q_*\Omega_p^1$$

by taking direct images² (this is also true in each of the coordinate patches). The sequence

$$(31) \quad \mathbb{C}[\tilde{\mathcal{F}}_t] \rightarrow \Omega_p^1 \rightarrow \Omega_p^2,$$

where the two maps are just d_p , becomes the sequence

$$\mathbb{C}[\text{CM}] \rightarrow \Omega^1\mathbb{C}[\text{CM}] \rightarrow \Omega_+^2\mathbb{C}[\text{CM}]$$

upon taking direct images (the maps become \underline{d}). The condition that $d_p^2 = 0$ is then equivalent to the statement that the curvature \underline{d}^2 is annihilated by the map

$$\Omega^2\mathbb{C}[\text{CM}] \rightarrow q_*\Omega_p^2 = \Omega_+^2,$$

²We note that for the general double fibration (24), a similar analysis will go through provided one has the transversality condition that $q_*\Omega_p^1 \cong \Omega^1\mathbb{C}[G/H]$.

i.e. that the curvature \underline{d}^2 is anti-self-dual. So although the connection d_p is flat, its image under q_* is not. Although the derivatives d and \underline{d} agree as maps $\mathbb{C}[\mathbb{CM}] \rightarrow \Omega^1\mathbb{C}[\mathbb{CM}]$, they do not agree beyond the one-forms. The image of $d : \Omega^1\mathbb{C}[\mathbb{CM}] \rightarrow \Omega^2\mathbb{C}[\mathbb{CM}]$ consists of all holomorphic two-forms, whereas \underline{d} maps $\Omega^1\mathbb{C}[\mathbb{CM}] \cong q_*\Omega_p^1$ onto the anti-self-dual two-forms. At the level of the correspondence space, the reason for this is that in the sequence (31), the calculus Ω_p^1 comes equipped with relations (30), whereas the pull-back $q^*\Omega^1\mathbb{C}[\mathbb{CM}]$ has no such relations.

We are now in a position to investigate how this construction behaves under the twisting discussed in Sections 4 and 5. Of course, we need only check that the various steps of the procedure remain valid under the quantisation functor. To this end, our first observation is that although the relations in the first order differential calculus of $\mathbb{C}[\mathbb{CM}]$ are deformed, equations (21) show that the two-forms are undeformed, as is the metric ν , hence the Hodge $*$ -operator is also undeformed in this case. It follows that the notion of anti-self-duality of two-forms is the same as in the classical case.

Furthermore, the definition of relative one-forms (26) still makes sense since the decomposition (27) is clearly unaffected by the twisting. Since we are working with the affine piece of (now noncommutative) space-time, the relevant relations in the correspondence space algebra $\mathbb{C}_F[\tilde{\mathcal{F}}_t]$ are given by (25), which are unchanged under twisting since t, Z^3, Z^4 remain central in the algebra.

Finally we observe that the proofs of Propositions 7.2 and 7.1 go through unchanged. The key steps use the fact that when the generators Z^3, Z^4 and t are invertible, one may adjoin their inverses to the coordinate algebras and differential calculi and take the degree zero parts. Since these generators remain central under twisting, this argument remains valid and we have the following twisted consequence of Proposition 7.2.

Proposition 7.3. *Let $d_p : \mathbb{C}_F[\tilde{\mathcal{F}}_t] \rightarrow \Omega_p^1$ be the differential operator defined by the composition of maps $d_p = \pi \circ d$,*

$$d_p : \mathbb{C}_F[\tilde{\mathcal{F}}_t] \rightarrow \Omega^1\mathbb{C}_F[\tilde{F}_t] \rightarrow \Omega_p^1.$$

Then the direct image $\underline{d} : \mathbb{C}_F[\mathbb{CM}] \rightarrow q_\Omega_p^1 \cong \Omega^1\mathbb{C}_F[\mathbb{CM}]$ is a differential operator whose curvature \underline{d}^2 takes values in the anti-self-dual two-forms $\Omega_-^2\mathbb{C}_F[\mathbb{CM}]$.*

We remark that this is no coincidence: the transform between one-forms on T_t and the operator \underline{d} on the corresponding affine patch \mathbb{CM} of space-time goes through to this noncommutative picture precisely because of the choice of cocycle made in Section 4. Indeed, we reiterate that any construction which is covariant under a chosen symmetry group will also be covariant after applying the quantisation functor. In this case, the symmetry group is the subgroup generated by conformal translations (see Section 4), which clearly acts covariantly on affine space-time \mathbb{CM} . By the very nature of the twistor double fibration, this translation group also acts

covariantly on the corresponding subsets \tilde{T} and $\tilde{\mathcal{F}}_t$ of (homogeneous) twistor space and the correspondence space respectively, and it is therefore no surprise that the transform outlined above works in the quantum case as well.

As discussed, it is true classically that one can expect such a transform between subsets of $\mathbb{C}\mathbb{M}^\#$ and the corresponding subsets of twistor space T provided the required topological properties (such as connectedness and simple-connectedness of the fibres) are met. We now see, however, that the same is not necessarily true in the quantum case. For a given open subset U of $\mathbb{C}\mathbb{M}^\#$, we expect the transform between U and its twistor counterpart $\hat{U} = p(q^{-1}(U)) \subset T$ to carry over to the quantum case provided the twisting group of symmetries is chosen in a way so as to preserve U .

7.3. Outline of the Penrose-Ward transform for vector bundles. The previous section described how the one-forms on twistor space give rise to a differential operator on forms over space-time having anti-self-dual curvature. The main feature of this relationship is that bundle data on twistor space correspond to differential data on space-time. The idea of the full Penrose-Ward transform is to generalise this construction from differential forms to sections of more general vector bundles.

We begin with a finite rank $\mathbb{C}[\tilde{T}]$ -module³ $\tilde{\mathcal{E}}$ describing the holomorphic sections of a (trivial) holomorphic vector bundle \tilde{E} over homogeneous twistor space \tilde{T} . The pull-back $p^*\mathcal{E}$ is the $\mathbb{C}[\tilde{\mathcal{F}}_t]$ -module

$$(32) \quad \tilde{\mathcal{E}}' := p^*\tilde{\mathcal{E}} = \mathbb{C}[\tilde{\mathcal{F}}_t] \otimes_{\mathbb{C}[\tilde{T}]} \tilde{\mathcal{E}}.$$

The key observation is then that there is a relative connection ∇_p on $p^*\tilde{\mathcal{E}}$ defined by

$$\nabla_p = d_p \otimes 1$$

with respect to the decomposition (32). Again there is a relative Leibniz rule

$$(33) \quad \nabla_p(f\xi) = f(\nabla_p\xi) + (d_p f) \otimes \xi$$

for $f \in \mathbb{C}[\tilde{\mathcal{F}}_t]$ and $\xi \in p^*\tilde{\mathcal{E}}$. Moreover, ∇_p extends to $\mathbb{C}[\tilde{\mathcal{F}}_t]$ -valued k -forms by defining

$$\Omega_p^k \tilde{\mathcal{E}}' := \Omega_p^k \otimes_{\mathbb{C}[\tilde{\mathcal{F}}_t]} \tilde{\mathcal{E}}'$$

for $k \geq 0$ and extending $\nabla_p = d_p \otimes 1$ with respect to this decomposition. It is clear that the curvature satisfies $\nabla_p^2 = 0$ and we say that ∇_p is *relatively flat*.

Conversely, if the fibres of p are connected and simply connected (as they clearly are in our situation; we assume this in the general case of (24)), then if $\tilde{\mathcal{E}}'$ is a finite rank $\mathbb{C}[\tilde{\mathcal{F}}_t]$ -module admitting a flat relative connection ∇' (that is, a complex-linear

³In this section we shall for convenience work with left modules over the algebras in question.

map $\nabla' : \tilde{\mathcal{E}}' \rightarrow \Omega_p^1 \otimes_{\mathbb{C}[\tilde{\mathcal{F}}_t]} \tilde{\mathcal{E}}'$ satisfying (33) and $(\nabla')^2 = 0$, we may recover a finite rank $\mathbb{C}[\tilde{T}]$ -module $\tilde{\mathcal{E}}$ by means of the covariantly constant sections, namely

$$\tilde{\mathcal{E}} := \{\xi \in \tilde{\mathcal{E}}' \mid \nabla' \xi = 0\}.$$

This argument gives rise to the following result.

Proposition 7.4. *There is a one-to-one correspondence between finite rank $\mathbb{C}[\tilde{T}]$ -modules $\tilde{\mathcal{E}}$ and finite rank $\mathbb{C}[\tilde{\mathcal{F}}_t]$ -modules $\tilde{\mathcal{E}}'$ admitting a flat relative connection ∇_p ,*

$$\nabla_p(f\xi) = f(\nabla_p \xi) + (d_p f) \otimes \xi, \quad \nabla_p^2 = 0$$

for all $f \in \mathbb{C}[\tilde{\mathcal{F}}_t]$ and $\xi \in \tilde{\mathcal{E}}'$.

We remark that here we do not see the non-trivial structure of the bundles involved (all of our modules describing vector bundles are free) as in our local picture all bundles are trivial. However, given these local formulae it will be possible at a later stage to patch together what happens at the global level.

The Penrose-Ward transform arises by considering what happens to a relative connection ∇_p under direct image along q . We here impose the additional assumption that $\tilde{\mathcal{E}}'$ is also the pull-back of a bundle on space-time, so $\tilde{\mathcal{E}}' = p^* \tilde{\mathcal{E}} = q^* \mathcal{E}$ for some finite rank $\mathbb{C}[\mathbb{CM}]$ -module \mathcal{E} . This is equivalent to assuming that the bundle $\tilde{\mathcal{E}}'$ is trivial upon restriction to each of the fibres of the map $q : \tilde{\mathcal{F}}_t \rightarrow \mathbb{CM}$. The direct image is computed exactly as described in the previous section.

Thus the direct image of $\tilde{\mathcal{E}}'$ is $q_* \tilde{\mathcal{E}}' = q_* q^* \mathcal{E} \cong \mathcal{E}$ and, just as in Proposition 7.2, equation (33) for ∇_p becomes a Leibniz rule for $\underline{\nabla} := q_* \nabla$, whence ∇_p maps onto a genuine connection on \mathcal{E} . The sequence

$$\tilde{\mathcal{E}}' \rightarrow \Omega_p^1 \otimes_{\mathbb{C}[\tilde{\mathcal{F}}_t]} \tilde{\mathcal{E}}' \rightarrow \Omega_p^2 \otimes_{\mathbb{C}[\tilde{\mathcal{F}}_t]} \tilde{\mathcal{E}}',$$

where the two maps are ∇_p , becomes the sequence

$$\mathcal{E} \rightarrow \Omega^1 \mathbb{C}[\mathbb{CM}] \otimes_{\mathbb{C}[\mathbb{CM}]} \mathcal{E} \rightarrow \Omega_+^2 \otimes_{\mathbb{C}[\mathbb{CM}]} \mathcal{E},$$

where the maps here are $\underline{\nabla}$. Moreover, just as in the previous section it follows that under direct image the condition that $\nabla_p^2 = 0$ is equivalent to the condition that the curvature $\underline{\nabla}^2$ is annihilated by the mapping

$$\Omega^2 \mathbb{C}[\mathbb{CM}] \otimes_{\mathbb{C}[\mathbb{CM}]} \mathcal{E} \rightarrow q_* \Omega_p^2 \otimes_{\mathbb{C}[\mathbb{CM}]} \mathcal{E} = \Omega_+^2 \otimes_{\mathbb{C}[\mathbb{CM}]} \mathcal{E},$$

so that $\underline{\nabla}$ has anti-self-dual curvature.

7.4. Tautological bundle on \mathbb{CM} and its Ward transform. We remark that in the previous section we began with a bundle over homogeneous twistor space \tilde{T} , whereas it is usual to work with bundles over the projective version T . As such, we implicitly assume that in doing so we obtain from $\tilde{\mathcal{E}}$ corresponding $\mathbb{C}[T_{Z^3}]$ - and $\mathbb{C}[T_{Z^4}]$ -modules which are compatible in the patch where Z^3 and Z^4 are both non-zero, as was the case for the coordinate algebras $\mathbb{C}[T_{Z^3}]$ and $\mathbb{C}[T_{Z^4}]$, $\mathbb{C}[\mathcal{F}_{Z^3}]$ and

$\mathbb{C}[\mathcal{F}_{Z^4}]$ via the transition functions $Z^\mu(Z^3)^{-1} \mapsto Z^\mu(Z^4)^{-1}$. We similarly assume this for the corresponding calculi on these coordinate patches.

In order to neatly capture these issues of coordinate patching, the general construction really belongs in the language of cohomology: as explained, the details will be addressed elsewhere. For the time being we give an illustration of the transform in the coordinate algebra framework, as well as an indication of what happens under twisting, through the tautological example introduced in Section 3.1.

We recall the identification of conformal space-time and the correspondence space as flag varieties $\mathbb{CM}^\# = \mathcal{F}_2(\mathbb{C}^4)$ and $\mathcal{F} = \mathcal{F}_{1,2}(\mathbb{C}^4)$ respectively, and the resulting fibration

$$q : \mathcal{F}_{1,2}(\mathbb{C}^4) \rightarrow \mathcal{F}_2(\mathbb{C}^4),$$

where the fibre over a point $x \in \mathcal{F}_2(\mathbb{C}^4)$ is the set of all one-dimensional subspaces of \mathbb{C}^4 contained in the two-plane $x \subset \mathbb{C}^4$, so is topologically a projective line $\mathbb{C}\mathbb{P}^1$. We also have a fibration at the homogeneous level,

$$\tilde{\mathcal{F}} \rightarrow \mathcal{F}_2(\mathbb{C}^4),$$

where this time the fibre over $x \in \mathcal{F}_2(\mathbb{C}^4)$ is the set of all vectors which lie in the two-plane x . As explained, there are no non-constant global holomorphic sections of this bundle, so as before we pass to the affine piece of space-time \mathbb{CM} in order to avoid this trivial case.

We identify \mathbb{C}^4 with its dual and take the basis (Z^1, Z^2, Z^3, Z^4) . In the patch $\tilde{\mathcal{F}}_t$ the relations in $\mathbb{C}[\tilde{\mathcal{F}}_t]$ are

$$(34) \quad Z^1 = x_2 Z^3 + x_3 Z^4, \quad Z^2 = x_4 Z^3 + x_1 Z^4,$$

which may be seen as giving a trivialisation of the tautological bundle over $\mathbb{CM}^\#$ in this coordinate patch. In this trivialisation the space \mathcal{E} of sections of the bundle is just the free module over $\mathbb{C}[\mathbb{CM}]$ of rank two, spanned by Z^3 and Z^4 , i.e. $\mathcal{E} \cong \mathbb{C}[\mathbb{CM}] \otimes \mathbb{C}^2$. We equip this module with the anti-self-dual connection $\underline{d} \otimes 1$ constructed in Section 7.2.

It is now easy to see that the pull-back $\tilde{\mathcal{E}}'$ of \mathcal{E} along q is just the free $\mathbb{C}[\tilde{\mathcal{F}}_t]$ -module of rank two. By construction, the connection \underline{d} on \mathcal{E} pulls back to the relative connection $d_p \otimes 1$ on $\tilde{\mathcal{E}}' = \mathbb{C}[\tilde{\mathcal{F}}_t] \otimes \mathbb{C}^2$. As discussed earlier the corresponding $\mathbb{C}[\tilde{T}]$ -module $\tilde{\mathcal{E}}$ is obtained as the kernel of the partial connection ∇_p , which is precisely the free rank two $\mathbb{C}[\tilde{T}]$ -module $\tilde{\mathcal{E}} = \mathbb{C}[\tilde{T}] \otimes \mathbb{C}^2$. It is also clear that these modules satisfy the condition that $\tilde{\mathcal{E}}' = p^* \tilde{\mathcal{E}} = q^* \mathcal{E}$ (the corresponding vector bundles are trivial in this case, whence they are automatically trivial when restricted to each fibre of p and of q).

Lastly, it is obvious that the bundle over \tilde{T} described by the module $\tilde{\mathcal{E}}$ descends to a (trivial) bundle over the twistor space T_t of \mathbb{CM} . On the coordinate patches where Z^3 and Z^4 are non-zero one obtains respectively a free rank two $\mathbb{C}[T_{Z^3}]$ -module $\tilde{\mathcal{E}}_{Z^3}$ by inverting Z^3 and a $\mathbb{C}[T_{Z^4}]$ -module $\tilde{\mathcal{E}}_{Z^4}$ by inverting Z^4 , and since

the algebras $\mathbb{C}[T_{Z^3}]$ and $\mathbb{C}[T_{Z^4}]$ agree when Z^3, Z^4 are both non-zero, the same is true of the sections in the localised modules $\tilde{\mathcal{E}}_{Z^3}$ and $\tilde{\mathcal{E}}_{Z^4}$. The construction in this example remains valid under the twisting described in Section 4.1 due to our earlier observation that the relations (34) are unchanged.

Explicitly then, the tautological bundle of $\mathbb{C}\mathbb{M}^\#$ in the affine patch $\mathbb{C}\mathbb{M}$ is simply the free rank two $\mathbb{C}[\mathbb{C}\mathbb{M}]$ -module $\mathcal{E} = \mathbb{C}[\mathbb{C}\mathbb{M}]^2 = \mathbb{C}[\mathbb{C}\mathbb{M}] \otimes \mathbb{C}^2$, which we equip with the anti-self-dual connection $\underline{d} \otimes 1$ constructed in Section 7.2 (note that one reserves the term *instanton* specifically for anti-self-dual connections over S^4). Its Penrose-Ward transform is the trivial rank two holomorphic bundle over the twistor space T_t of $\mathbb{C}\mathbb{M}$. Since these spaces are topologically trivial, we see that the Penrose-Ward transform is here very different in flavour to the transform given in Section 3.2. It is however clear that this example of the transform quantises in exactly the same way as the rank one case of Section 7.2.

8. THE ADHM CONSTRUCTION

8.1. The classical ADHM construction. We begin this section with a brief summary of the ADHM construction for connections with anti-self-dual curvature on vector bundles over Minkowski space $\mathbb{C}\mathbb{M}$ [1, 14], with a view to dualising and then twisting the construction.

A monad over $\tilde{T} = \mathbb{C}^4$ is a sequence of linear maps

$$A \xrightarrow{\rho_Z} B \xrightarrow{\tau_Z} C,$$

between complex vector spaces A, B, C of dimensions $k, 2k+n, k$ respectively, such that for all $Z \in \mathbb{C}^4$, $\tau_Z \rho_Z : A \rightarrow C$ is zero and for all $Z \in \mathbb{C}^4$, ρ_Z is injective and τ_Z is surjective. Moreover, we insist that ρ_Z, τ_Z each depend linearly on $Z \in \tilde{T}$. The spaces A, B, C should be thought of as typical fibres of trivial vector bundles over $\mathbb{C}\mathbb{P}^3$ of ranks $k, 2k+n, k$, respectively.

A monad determines a rank- n holomorphic vector bundle on $T = \mathbb{C}\mathbb{P}^3$ whose fibre at $[Z]$ is $\text{Ker } \tau_Z / \text{Im } \rho_Z$ (where $[Z]$ denotes the projective equivalence class of $Z \in \mathbb{C}^4$). Moreover, any holomorphic vector bundle on $\mathbb{C}\mathbb{P}^3$ trivial on each projective line comes from such a monad, unique up to the action of $\text{GL}(A) \times \text{GL}(B) \times \text{GL}(C)$. For a proof of this we refer to [16], although we note that the condition $\tau_Z \rho_Z = 0$ implies $\text{Im } \rho_Z \subset \text{Ker } \tau_Z$ for all $z \in \mathbb{C}^4$ so the cohomology makes sense, and the fact that ρ_Z, τ_Z have maximal rank at every Z implies that each fibre has dimension n . The idea behind the ADHM construction is to use the same monad data to construct a rank n vector bundle over $\mathbb{C}\mathbb{M}^\#$ with second Chern class $c_2 = k$ (in the physics literature this is usually called the *topological charge* of the bundle).

For each $W, Z \in \tilde{T}$ we write $x = \widetilde{W \wedge Z}$ for the corresponding element x of (homogeneous) conformal space-time $\mathbb{CM}^\# \subset \Lambda^2 \tilde{T}$. Then define

$$E_x = \text{Ker } \tau_z \cap \text{Ker } \tau_w, \quad F_x = (\rho_z A) \cap (\rho_w A), \quad \Delta_x = \tau_z \rho_w.$$

Proposition 8.1. [14] *The vector spaces E_x, F_x and the map Δ_x depend on x , rather than on Z, W individually.*

Proof. We first consider Δ_x and suppose that $x = Z \wedge W = Z \wedge W'$ for some $W' = W + \lambda Z, \lambda \in \mathbb{C}$. Then we have

$$\tau_z \rho_{W'} - \tau_z \rho_W = \tau_z \rho_{W' - W} = \tau_z \rho_{\lambda Z} = \lambda \tau_z \rho_Z = 0.$$

Now writing $Z' = Z + \lambda W'$ we see that

$$\tau_{Z'} \rho_{W'} - \tau_z \rho_W = (\tau_z + \lambda \tau_{W'}) \rho_{W'} - \tau_z \rho_W = \lambda \tau_z \rho_{W'} - \tau_z \rho_W = 0$$

by the first calculation, proving the claim for Δ_x .

Next we consider $b \in \text{Ker } \tau_z \cap \text{Ker } \tau_w$ and suppose $x = Z \wedge W = Z \wedge W'$ where $W' = W + \lambda Z$. Then

$$\tau_{W'} b = (\tau_w + \lambda \tau_z) b = 0,$$

so that $\text{Ker } \tau_z \cap \text{Ker } \tau_w = \text{Ker } \tau_z \cap \text{Ker } \tau_{W'}$. Moreover, if $Z' = Z + \lambda W'$ we see that

$$\tau_{Z'} b = (\tau_z + \lambda \tau_{W'}) b = 0,$$

so that $\text{Ker } \tau_z \cap \text{Ker } \tau_w = \text{Ker } \tau_{Z'} \cap \text{Ker } \tau_{W'}$, establishing the second claim. The third follows similarly. \square

As in [14], we write U for the set of $x \in \mathbb{CM}^\#$ on which Δ_x is invertible.

Proposition 8.2. *For all $x \in U$ we have the decomposition,*

$$(35) \quad B = E_x \oplus \text{Im } \rho_z \oplus \text{Im } \rho_w.$$

In particular, for all $x \in \mathbb{CM}^\#$ we have $F_x = 0$.

Proof. For $x \in U$, define

$$P_x = 1 - \rho_w \Delta_x^{-1} \tau_z + \rho_z \Delta_x^{-1} \tau_w : B \rightarrow B.$$

Now P_x is linear in x and is dependent only on x and not on Z, W individually. It is easily shown that $P_x^2 = P_x$ and $P_x B = E_x$, whence P_x is the projection onto E_x . Moreover,

$$-\rho_w \Delta_x^{-1} \tau_z, \quad \rho_z \Delta_x^{-1} \tau_w,$$

are the projections onto the second and third summands respectively. Hence we have proven the claim provided we can show that $F_x = 0$, since the sum $\text{Im } \rho_z \oplus \text{Im } \rho_w$ is then direct.

Suppose that $F_x = (\rho_z A) \cap (\rho_w A)$ is not zero for some $x \in \mathbb{CM}^\#$, so there exists a non-zero $b \in \rho_{W'} A$ for all $W' \in \tilde{T}$ such that $x = Z \wedge W = Z \wedge W'$. Then in particular

$a := \rho_{W'}^{-1}b$ is non-zero and defines a holomorphic section of a vector bundle over the two-dimensional subspace of \tilde{T} spanned by all such W' (whose typical fibre is just A), and hence (at the projective level) a non-zero holomorphic section of the bundle $\mathcal{O}(-1) \otimes A$ over $\hat{x} = \mathbb{CP}^1$, where $\mathcal{O}(-1)$ denotes the tautological line bundle over \mathbb{CP}^1 . It is however well-known that this bundle has no non-zero global sections, whence we must in fact have $F_x = 0$. \square

This procedure has thus constructed a rank n vector bundle over U whose fibre over $x \in U$ is E_x (again noting that the construction is independent of the scaling of $x \in \mathbb{CM}^\#$). The bundle E is obtained as a sub-bundle of the trivial bundle $U \times B$: the projection P_x identifies the fibre of E at each $x \in \mathbb{CM}^\#$ as well as defining a connection on E by orthogonal projection of the trivial connection on $U \times B$.

8.2. ADHM in the *-algebra picture. In this section we mention how the ADHM construction ought to operate in our SU_4 *-algebra framework. In passing to the affine description of our manifolds, we encode them as real rather than complex manifolds, obtaining a global coordinate algebra description. Thus we expect that the ADHM construction ought to go through at some global level in our *-algebra picture. For now we suppress the underlying holomorphic structure of the bundles we construct, with the complex structure to be added elsewhere.

Indeed, we observe that the key ingredient in the ADHM construction is the decomposition (35),

$$B = E_x \oplus \text{Im } \rho_z \oplus \text{Im } \rho_w,$$

which identifies the required bundle over space-time as a sub-bundle of the trivial bundle with fibre B . We wish to give a version of this decomposition labelled by points at the projective level, that is in terms of points of $T = \mathbb{CP}^3$ and $\mathbb{CM}^\# = \mathcal{F}_2(\mathbb{C}^4)$, rather than in terms of the homogeneous representatives $Z \in \tilde{T}$ and $x \in \mathbb{CM}^\#$ used above. We shall do this as before by identifying points of \mathbb{CP}^3 with rank one projectors Q in $M_4(\mathbb{C})$, similarly points of $\mathbb{CM}^\#$ with rank two projectors P . Points of the correspondence space are identified with pairs of such projectors (Q, P) such that $QP = Q = PQ$.

Recall that in the previous description, given $x \in \Lambda^2 \tilde{T}$ and $Z \in \hat{x}$ there are many W such that $x = Z \wedge W$. In the alternative description, given a projection $P \in \mathbb{CM}^\#$, the corresponding picture is that there are $Q, Q' \in T$ with $QP = Q = PQ$ and $Q'P = Q' = PQ'$ such that $\text{Im } P = \text{Im } Q \oplus \text{Im } Q'$. Given $P \in \mathbb{CM}^\#$, $Q \in T$ with $(Q, P) \in \mathcal{F}$, there is a canonical choice for Q' , namely $Q' := P - Q$. Indeed, P identifies a two-dimensional subspace of \mathbb{C}^4 and Q picks out a one-dimensional subspace of this plane. The projector $Q' = P - Q$ picks out a line in \mathbb{C}^4 in the orthogonal complement of the line determined by Q .

As in the monad description of the construction outlined earlier, the idea is to begin with a trivial bundle of rank $2k + n$ with typical fibre $B = \mathbb{C}^{2k+n}$ and to

present sufficient information to canonically identify a decomposition of B in the form (35). In the monad description this was done by assuming that $\tau_Z \rho_Z = 0$ and that the maps τ, ρ were linearly dependent on $Z \in \tilde{T}$. Of course, this makes full use of the additive structure on \tilde{T} , a property which we do not have in the projector version: here we suggest an alternative approach.

In order to obtain such a decomposition it is necessary to determine the reason for each assumption in the monad construction and to then translate this assumption into the projector picture. The first observation is that the effect of the map ρ is to identify a k -dimensional subspace of B for each point in twistor space (for each $Q \in T$ the map ρ_Q is simply the associated embedding of A into B). We note that this may be achieved directly by specifying for each $Q \in T$ a rank k projection $\rho_Q : B \rightarrow B$.

The construction then requires us to decompose each point of space-time in terms of a pair of twistors. As observed above, given $P \in \mathbb{CM}^\#$ and any $Q \in T$ such that $PQ = Q = QP$ we have $P = Q + Q'$, where $Q' = P - Q$ (it is easy to check that Q' is indeed another projection of rank one). The claim is then that the corresponding k -dimensional subspaces of B have zero intersection and that their direct sum is independent of the choice of decomposition of P . In the monad construction this was obtained using the assumed linear dependence of ρ on points $Z \in \tilde{T}$, although in terms of projectors this is instead achieved by assuming that the projections $\rho_Q, \rho_{Q'}$ are orthogonal whenever Q, Q' are orthogonal (clearly if $Q + Q' = P$ then Q, Q' are orthogonal), i.e.

$$\rho_Q \rho_{Q'} = \rho_{Q'} \rho_Q = 0$$

for all Q, Q' such that $QQ' = Q'Q = 0$. Moreover, we impose that $\rho_Q + \rho_{Q'}$ depends only on $Q + Q'$, so that the direct sum of the images of these projections depends only on the sum of the projections.

Then for each $P \in \mathbb{CM}^\#$ we define a subspace B_P of B of dimension n ,

$$B = B_P \oplus \text{Im } \rho_Q \oplus \text{Im } \rho_{Q'}$$

by constructing the projection

$$e_P := 1 - \rho_Q - \rho_{Q'}$$

on \mathbb{C}^4 , which is well-defined since the assumptions we have made on the family ρ_Q imply that the projectors e_P, ρ_Q and $\rho_{Q'}$ are pairwise orthogonal. This constructs a rank n vector bundle over $\mathbb{CM}^\#$ whose fibre over $P \in \mathbb{CM}^\#$ is $\text{Im } e_P = B_P$.

8.3. The tautological bundle on $\mathbb{CM}^\#$ and its corresponding monad. We illustrate these ideas by constructing a specific example in the $*$ -algebra picture. We construct a ‘tautological monad’ which appears extremely naturally in the projector version and turns out to correspond to the 1-instanton bundle of Section 3.2.

We once again recall that compactified space-time may be identified with the flag variety $\mathcal{F}_2(\mathbb{C}^4)$ of two-planes in \mathbb{C}^4 and this space has its associated tautological bundle whose fibre at $x \in \mathcal{F}_2(\mathbb{C}^4)$ is the two-plane in \mathbb{C}^4 which defines x (it is precisely this observation which gave rise to the projector description of space-time in the first place). Then we take $B = \mathbb{C}^4$ in the ADHM construction: note that we expect to take $n = 2, k = 1$, which agrees with the fact that $\dim B = 2k + n = 4$.

Then for each point $P \in \mathbb{C}\mathbb{M}^\#$ we are required to decompose it as the sum of a pair Q, Q' of rank one projectors (each representing a twistor). Here this is easy to do: we simply choose any one-dimensional subspace of the image of P and take Q to be the rank one projector whose image is this line. As discussed, the canonical choice for Q' is just $Q' := P - Q$.

In doing so, we have tautologically specified the one-dimensional subspace of $B = \mathbb{C}^4$ associated to $Q \in T$ (recalling that $k = 1$ here), simply defining $\rho_Q := Q$. We now check that these data satisfy the conditions outlined in the previous section. It is tautologically clear that for all $Q_1, Q_2 \in T$ we have that ρ_{Q_1} and ρ_{Q_2} are orthogonal projections if and only if Q_1 and Q_2 are orthogonal. Moreover, if we fix $P \in \mathbb{C}\mathbb{M}^\#, Q \in T$ such that $PQ = Q = QP$ and take $Q' = P - Q$ then

$$\rho_Q + \rho_{Q'} = Q + Q' = P,$$

which of course depends only on $Q + Q'$ rather than on Q, Q' individually.

Thus we construct the subspace B_P as the complement of the direct sum of the images of ρ_Q and $\rho_{Q'}$. As explained, this is done by constructing the projection

$$e_P = 1 - \rho_Q - \rho_{Q'} = 1 - P.$$

Thus as P varies we get a rank two vector bundle over $\mathbb{C}\mathbb{M}^\#$, which is easily seen to be the complement in \mathbb{C}^4 of the tautological vector bundle over $\mathbb{C}\mathbb{M}^\#$. We equip this bundle with the Grassmann connection obtained by orthogonal projection of the trivial connection on the trivial bundle $\mathbb{C}\mathbb{M}^\# \times \mathbb{C}^4$, as discussed earlier.

This gives a monad description of the tautological bundle over $\mathbb{C}\mathbb{M}^\# = \mathcal{F}_2(\mathbb{C}^4)$. As explained, the instanton bundle over S^4 is obtained by restriction of this bundle to the two-planes $x \in \mathcal{F}_2(\mathbb{C}^4)$ which are invariant under the map J defined in Section 3.2. Hence we consider this construction only for $x \in S^4$, and by Proposition 3.11 we have that $Q' = J(Q)$ in the above. We now wish to give the monad version of the corresponding bundle over twistor space, for which we need the map τ_Q . We recall that τ is meant to satisfy $\tau_Q \rho_Q = 0$ for all $Q \in \mathbb{C}\mathbb{P}^3$, and we use this property to construct τ by putting $\tau_Q := \rho_{J(Q)} = J(Q)$, so that in this tautological example we have

$$\tau_Q \rho_Q = \rho_{J(Q)} \rho_Q = J(Q)Q = 0.$$

The bundle over twistor space corresponding to the instanton then appears as the vector bundle whose fibre over $Q \in \mathbb{CP}^3$ is the cohomology

$$\tilde{\mathcal{E}} = \text{Ker } \tau_Q / \text{Im } \rho_Q = \text{Ker } \rho_{J(Q)} / \text{Im } \rho_Q,$$

which in the case of the 1-instanton is the rank two bundle $\tilde{E} = \text{Ker } J(Q) / \text{Im } Q$, and one may easily check that this bundle over twistor space agrees with the one computed in Section 3.2 using the Penrose-Ward transform. Indeed, the crucial property is that it is trivial upon restriction to $\hat{P} = p(q^{-1}(P))$ for all $P \in S^4$, which is straightforward to see through the observation that as Q varies with P fixed, Q and $J(Q)$ always span the same plane (the one defined by P), and the fibre of \tilde{E} over all such Q is precisely the orthogonal complement to this plane.

Acknowledgements. We would like to thank G. Landi and S.P. Smith for helpful comments on versions of the manuscript during their respective visits to the Noncommutative Geometry Programme at the Newton Institute.

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