

# Master-slave synchronization and invariant manifolds for coupled stochastic systems\*

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## Abstract

We deal with abstract systems of two coupled nonlinear stochastic (infinite dimensional) equations subjected to additive white noise type process. This kind of systems may describe various interaction phenomena in a continuum random medium. Under suitable conditions we prove the existence of an exponentially attracting random invariant manifold for the coupled system and show that this system can be reduced to a single equation with modified nonlinearity. This result means that under some conditions we observe (nonlinear) synchronization phenomena in the coupled system. Our applications include stochastic systems consisting of (i) parabolic and hyperbolic equations, (ii) two hyperbolic equations, and (iii) Klein-Gordon and Schrödinger equations. We also show that the random manifold constructed converges to its deterministic counterpart when the intensity of noise tends to zero.

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# 1 Introduction

Let  $X_1$  and  $X_2$  be (infinite dimensional separable) Hilbert spaces. The main object of our work is the following system of differential equations

$$U_t + A_1U = F_1(U, V) + B_1\dot{N}_1(t, \omega), \quad t > 0, \quad \text{in } X_1, \quad (1)$$

and

$$V_t + A_2V = F_2(U, V) + B_2\dot{N}_2(t, \omega), \quad t > 0, \quad \text{in } X_2, \quad (2)$$

where  $A_1$  and  $A_2$  are generators of  $C_0$ -semigroups,  $F_1$  and  $F_2$  are continuous (nonlinear) mappings,

$$F_1 : X_1 \times X_2 \mapsto X_1, \quad F_2 : X_1 \times X_2 \mapsto X_2.$$

Here above  $B_1\dot{N}_1(t, \omega)$  and  $B_2\dot{N}_2(t, \omega)$  are white noise processes in  $X_1$  and  $X_2$  which will be specified later for a random parameter  $\omega \in \Omega$ .

Our main goal is to apply the theory of random invariant manifolds to study synchronization phenomena of the stochastic problem (1), (2).

Recently the subject of synchronization of coupled (identical or not) systems has received considerable attention. There are now several monographs [30, 36, 38] in this field, which contain extensive lists of references. In the case of infinite dimensional systems the synchronization problem has been studied in [10, 33] for the case of coupled (deterministic) parabolic systems. The synchronization of stochastic stationary solutions (i.e. single valued random attractors) of finite dimensional stochastic systems has been considered in [9] (see also [1, 24] for similar results in deterministic nonautonomous systems). The synchronization of the dynamics of parabolic stochastic systems in two thin layers at the level of global pullback attractors has been studied in [8]. From mathematical point of view the synchronization phenomena can be treated as the existence of an invariant manifold of a special type in the phase space of the coupled system. For instance, if the problems (1) and (2) have the same phase space ( $X_1 = X_2$ ), then the possibility of synchronized regimes means that the set

$$M = \{(U, V) \in X_1 \times X_2 : U = V \in X_1 = X_2\}$$

is invariant with respect to the flow generated by the coupled system. If this invariant set is globally asymptotically stable, then given any solution of the first equation and any solution of the second equation, the difference

between two solutions becomes small as  $t \rightarrow +\infty$ . In this case we observe full (asymptotic) synchronization of systems (1) and (2). From this point of view it is natural (see [16, 17] for the deterministic case) to consider the question on the existence of a random invariant manifold of a more general form. In particular, we are looking for a random manifold given by

$$M_t(\omega) = \{(U, V) \in X_1 \times X_2 : U = \Phi_t(\omega, V) \in X_1, V \in X_2\}, \quad (3)$$

where  $\Phi_t(\omega, \cdot) : X_2 \mapsto X_1$  is a random Lipschitz mapping, adopting the following definition.

**Definition 1.1** *System (1) is said to be (asymptotically) synchronized with system (2), if there exists a random Lipschitz mapping  $\Phi_t : X_2 \mapsto X_1$  such that*

$$\lim_{t \rightarrow +\infty} \|U(t, \omega) - \Phi_t(\omega, V(t, \omega))\|_{X_1} = 0 \quad \text{for all } \omega \in \Omega$$

*for any solution  $(U(t, \omega), V(t, \omega))$  to problem (1) and (2). In this case (2) is called master system and (1) is slave system.*

The theory of invariant and inertial manifolds for various classes of infinite dimensional dynamical systems has been developed by many authors, see, e.g., monographs [13, 19, 23, 37] for the deterministic case and papers [5, 12, 22, 14, 18, 21, 27, 35] for the stochastic case and also the references therein. There are two approaches to construction of invariant manifolds: Hadamard graph transform method (see, e.g., [19] and also [4, 22, 21, 34]) and Lyapunov-Perron method. In this work we follow the idea of the Lyapunov-Perron method in the form presented in [29] for the deterministic case which was also used in [7] for the case of stochastic hyperbolic-parabolic problem.

Our main objective is to establish possibility of synchronization and to prove a reduction principle for the random dynamical system generated by the problem (1) and (2) which allows us to rewrite our coupled system as a single stochastic equation in  $X_2$  with a conveniently modified nonlinear term. To be more precise, we prove that under some conditions the random dynamical system generated by (1) and (2) has an invariant exponentially attracting random manifold of the type (3) where  $\Phi_t : \Omega \times X_2 \mapsto X_1$  is a Lipschitz mapping for each  $\omega \in \Omega$  and a stationary process with respect to  $t$ . The existence of this manifold  $M$  makes it possible to prove that the long-time behavior of the system (1) and (2) can be described by the reduced problem

$$V_t + A_2 V = F_2(\Phi_t(\omega, V), V) + B_2 \dot{N}_2(t, \omega), \quad \text{in } X_2.$$

For a similar result in the deterministic framework we refer to [26, 16] for parabolic/hyperbolic systems and to [17] for general case. The same result in the deterministic case can be also derived from Theorem 1.1 and Corollary 1.4 in [25].

The paper is organized as follows. In the preliminary Section 2 we formulate our main hypotheses and represent the problem as a first order stochastic differential equation. For the reader's convenience, we recall basic definitions from the theory of random dynamical systems, and collect several results on stochastic convolutions of a form adapted to our situation. Then we introduce several kinds of noise processes which serve for the random excitation of our system. In this section we also provide a result on the existence and uniqueness of mild solutions to problem (1) and (2) and show that this problem generates a filtered random dynamical system (RDS). Section 3 contains our main result which is a type of reduction principle (see Theorem 3.1). We show the existence of a fixed point of the random Lyapunov–Perron method. This fixed point provides us an invariant manifold. In addition, we obtain the tracking property showing that the manifold is exponentially attracting. In Section 4 we estimate the distance between  $M(\omega)$  and its deterministic counterpart  $M^{det}$  in terms of the covariance operators of  $N_1$  and  $N_2$  (see Theorem 4.1). In particular, we prove that  $M(\omega)$  converges to  $M^{det}$  when the intensity of the noise tends to zero which means the persistence of synchronization in the zero noise limit (in contrast with phenomena which takes place for some classes of parabolic systems; see, e.g., [8]). Then in Section 5 we consider applications to coupled (i) parabolic and hyperbolic equations, (ii) parabolic PDE and ODE, (iii) two hyperbolic equations, and (iv) Klein-Gordon and Schrödinger equations.

## 2 Preliminaries

Our main goal in this section is to describe rigorously the model given by the system of stochastic differential equations (1) and (2).

### 2.1 Assumptions on the nonlinear evolution equation

We assume that

- (A1)** Let  $X_1$  and  $X_2$  be two separable Hilbert spaces. Let  $A_1$  be the generator of a linear  $C_0$ -semigroup  $S^1(t) = e^{-A_1 t}$  on  $X_1$  which satisfies the

estimate

$$\|S^1(t)\|_{\mathcal{L}(X_1)} \leq M_1 \exp\{-\gamma_1 t\}, \quad t \geq 0.^1 \quad (4)$$

for some positive constants  $M_1, \gamma_1$ . Similarly, let  $A_2$  be the generator of a linear  $C_0$ -group  $S^2$  on  $X_2$  satisfying the estimates

$$\|S^2(t)\|_{\mathcal{L}(X_2)} \leq M_2 \exp\{-\gamma_2 t\}, \quad t \leq 0, \quad (5)$$

for some constant  $M_2 \geq 1, \gamma_2 \geq 0$ .

**(A2)**  $F_1$  and  $F_2$  are nonlinear mappings,

$$F_1 : X_1 \times X_2 \mapsto X_1, \quad F_2 : X_1 \times X_2 \mapsto X_2,$$

and there exist constants  $L_1$  and  $L_2$  such that

$$\|F_1(U_1, V_1) - F_1(U_2, V_2)\|_{X_1} \leq L_1 (\|U_1 - U_2\|_{X_1}^2 + \|V_1 - V_2\|_{X_2}^2)^{1/2} \quad (6)$$

and

$$\|F_2(U_1, V_1) - F_2(U_2, V_2)\|_{X_2} \leq L_2 (\|U_1 - U_2\|_{X_1}^2 + \|V_1 - V_2\|_{X_2}^2)^{1/2}. \quad (7)$$

Below we consider the space  $X = X_1 \times X_2$  equipped with the norm

$$\|Y\|_X = (\|U\|_{X_1}^2 + \|V\|_{X_2}^2)^{1/2}, \quad Y = (U, V),$$

and denote by  $Q$  and  $P$  the orthoprojectors on  $X$  onto the first and second components, i.e.

$$Q(U, V) = (U, 0) \simeq U \in X_1 \quad \text{and} \quad P(U, V) = (0, V) \simeq V \in X_2 \quad (8)$$

for  $(U, V) \in X$ .

**Remark 2.1** If  $\gamma_1 > \gamma_2$  then the properties (4) and (5) ensure the so-called *dichotomy* estimates for the linear  $C_0$ -semigroup  $S$  on  $X$  given by

$$S(t) = \begin{pmatrix} S^1(t) & 0 \\ 0 & S^2(t) \end{pmatrix}, \quad t \geq 0, \quad (9)$$

in the space  $X$  with respect to the pair of projectors  $Q$  and  $P$  given by (8); see, e.g., [13, Chapter 6].

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<sup>1</sup>Here and below we denote by  $\|\cdot\|_{\mathcal{L}(X)}$  the operator norm of linear operators on  $X$ .

Since  $S^2$  is  $C_0$ -(semi)group we can guarantee (see [31, p. 4]) the existence of constants  $\tilde{M}_2 \geq 1$  and  $\tilde{\gamma}_2 \geq 0$  such that

$$\|S^2(t)\| \leq \tilde{M}_2 \exp\{\tilde{\gamma}_2 t\}, \quad t \geq 0, \quad (10)$$

These constants  $\tilde{M}_2$  and  $\tilde{\gamma}_2$  play some auxiliary rôle and do not enter in our main results.

As we see below the assumption that  $\gamma_2 \geq 0$  can be also relaxed. However, it seems the case when  $\gamma_2 < 0$  has no substantial physical meaning, but the corresponding analysis requires some special considerations. This is why we assume that  $\gamma_2 \geq 0$  from the very beginning.

With these assumptions **(A1)** and **(A2)** we can rewrite system (1) and (2) as a single first order stochastic equation in the space  $X = X_1 \times X_2$  on the interval  $[s, \infty)$

$$Y_t + AY = F(Y) + B\dot{N}(t), \quad t > s, \quad Y(s) = Y_0 \in X, \quad (11)$$

where  $s \in \mathbb{R}$ ,  $Y(t) = (U(t), V(t))$ .  $\dot{N} = (\dot{N}_1, \dot{N}_2)$  is a noise vector on the time set  $\mathbb{R}$  over an appropriate probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and  $B_1, B_2$  are linear operators which will be introduced below, and

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad F(Y) = \begin{pmatrix} F_1(U, V) \\ F_2(U, V) \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}.$$

Obviously, the operator  $A$  is the generator of the linear  $C_0$ -semigroup (9) on  $X$ .

## 2.2 Random dynamical systems and Ornstein–Uhlenbeck processes

We first recall concepts from the theory of random dynamical systems, see Arnold [3] for more details.

**Definition 2.2** Let  $X$  be a topological space. A random dynamical system (RDS) with time  $\mathbb{R}_+$  and state space  $X$  is a pair  $(\theta, \phi)$  consisting of the following two objects:

1. A metric dynamical system (MDS)  $\theta \equiv (\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t, t \in \mathbb{R}\})$ , i.e. a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with a random flow  $\theta : \mathbb{R} \times \Omega \rightarrow \Omega$ :

- (a)  $\theta_0 = \text{id}_\Omega$ ,  $\theta_t \circ \theta_s = \theta_{t+s}$  for all  $t, s \in \mathbb{R}$ ;
- (b) the map  $(t, \omega) \mapsto \theta_t \omega$  is measurable and  $\theta_t \mathbb{P} = \mathbb{P}$  for all  $t \in \mathbb{R}$ .

2. A cocycle  $\phi$  over  $\theta$  is a measurable mapping

$$\phi : \mathbb{R}_+ \times \Omega \times X \mapsto X, \quad (t, \omega, x) \mapsto \phi(t, \omega)x$$

such that the cocycle property

$$\phi(0, \omega) = \text{id}_X, \quad \phi(t + s, \omega) = \phi(t, \theta_s \omega) \circ \phi(s, \omega)$$

holds for all  $t, s \geq 0$  and  $\omega \in \Omega$ .

Over a metric dynamical system we introduce a special class of random variables.

**Definition 2.3** *A nonnegative random variable  $R \geq 0$  is called tempered if*

$$\lim_{t \rightarrow \pm\infty} \frac{1}{|t|} \log^+ R(\theta_t \omega) = 0 \quad \text{almost surely.} \quad (12)$$

In the case that such a random variable is not tempered we have under the additional assumption that the measure  $\mathbb{P}$  is ergodic the only alternative:

$$\limsup_{t \rightarrow \pm\infty} \frac{1}{|t|} \log^+ R(\theta_t \omega) = +\infty \quad \text{almost surely.} \quad (13)$$

Hence if the growth of  $t \mapsto R(\theta_t \omega)$  is exponentially bounded almost surely such that we can exclude (13) we just know that  $R$  is tempered. On the other hand for a tempered random variable  $R$  we can find a modification of the metric dynamical system such that (12) holds for every  $\omega \in \Omega$ .

We now introduce random fields which we will call *Ornstein-Uhlenbeck processes*. These fields are introduced by the stochastic convolutions of our semigroups  $S^1$ ,  $S^2$  and a noise vector  $N(t) = (N_1(t), N_2(t)) \in H_0 \times H_0$ , where  $H_0$  is some separable Hilbert space. Let  $\Pi = \{(s, t) \in \mathbb{R}^2 : s \leq t\}$ . Especially for  $(s, t) \in \Pi$  the Ornstein-Uhlenbeck processes  $\eta_1(t, s)$ ,  $\eta_2(t, s)$  are versions the stochastic convolutions are defined by

$$\eta_1(t, s) = \int_s^t S^1(t - \tau) B_1 dN_1(\tau), \quad \eta_2(t, s) = \int_s^t S^2(t - \tau) B_2 dN_2(\tau). \quad (14)$$

Here  $B_i$  are linear bounded operators from  $H_0$  into  $X_i$ . Particular hypotheses on these random fields will be formulated below. Later we will give

three examples of noises generating Ornstein-Uhlenbeck processes satisfying these hypotheses. We also note that  $\eta_1$  and  $\eta_2$  are solutions of particular stochastic linear differential equations.

Since we do not assume that  $S^2$  is exponentially stable, it is convenient to consider the group  $S^{2,a}$  generated by  $A_2 + a \text{id}$  on  $X_2$ ,  $S^{2,a}(t) \equiv S^2(t)e^{-at}$ . Because  $S^2$  satisfies estimate (10) for  $a = \tilde{\gamma}_2 + 1$  we have the inequality

$$\|S^{2,a}(t)\| \leq \tilde{M}_2 \exp\{-t\}, \quad t \geq 0.$$

Similar to the second equation of (14) we introduce by  $\eta_{2,a}(t, s)$  as a version of the stochastic convolution with respect to the group  $S^{2,a}$ :

$$\eta_{2,a}(t, s) = \int_s^t S^{2,a}(t - \tau) B_2 dN_2(\tau). \quad (15)$$

Supposing that  $\eta_{2,a}(t, s, \omega)$  is defined for  $\Pi \times \Omega$  then  $\eta_2, \eta_{2,a}$  are connected by the following equality

$$\eta_2(t, s) = \eta_{2,a}(t, s) + a \int_s^t S^2(t - \tau) \eta_{2,a}(\tau, s) d\tau \quad \text{for } (s, t) \in \Pi, \omega \in \Omega, \quad (16)$$

which follows easily from the structure of  $\eta_2, \eta_{2,a}$ .

We now formulate some assumptions about the random fields  $\eta_1, \eta_2^a$ .

**(R1)** There exist measurable mappings

$$\eta_1 : \Pi \times \Omega \mapsto X_1 \quad \text{and} \quad \eta_{2,a} : \Pi \times \Omega \mapsto X_2$$

such that the mappings

$$\Pi \ni (s, t) \mapsto \eta_1(t, s, \omega) \in X_1, \quad \Pi \ni (s, t) \mapsto \eta_{2,a}(t, s, \omega) \in X_2$$

are càdlàg depending on the noise paths on  $[s, \infty)$  for every  $s \in \mathbb{R}$ ,  $\omega \in \Omega$  and on  $(-\infty, t]$  for every  $t \in \mathbb{R}$ ,  $\omega \in \Omega$ . In case that the noise is continuous we can assume that these mappings are continuous.

**(R2)** For every  $(s, t) \in \Pi$  the mapping

$$\Omega \ni \omega \mapsto \eta_1(t, s, \omega) \in X_1 \text{ is } \mathcal{F}, \mathcal{B}(X_1) \text{ - measurable}$$

and the mapping

$$\Omega \ni \omega \mapsto \eta_{2,a}(t, s, \omega) \in X_2 \text{ is } \mathcal{F}, \mathcal{B}(X_2) \text{ - measurable.}$$

**(R3)** For every  $\tau \in \mathbb{R}$ ,  $(s, t) \in \Pi$  and  $\omega \in \Omega$  we have

$$\begin{aligned}\eta_1(t, s, \omega) &= \eta_1(t + \tau, s + \tau, \theta_{-\tau}\omega), \\ \eta_{2,a}(t, s, \omega) &= \eta_{2,a}(t + \tau, s + \tau, \theta_{-\tau}\omega).\end{aligned}$$

**(R4)** We have for  $-\infty \leq \tau < s \leq t$ ,  $\omega \in \Omega$

$$\eta_1(t, s, \omega) = \eta_1(t, \tau, \omega) - S^1(t - s)\eta_1(t, s, \omega)$$

and similar for  $\eta_{2,a}$  with respect to  $S^{2,a}$ .

**(R5)** For  $\omega \in \Omega$  and  $t \in \mathbb{R}$  the limits

$$\lim_{s \rightarrow -\infty} \eta_1(t, s, \omega) \equiv \eta_1(t, -\infty, \omega), \quad \eta_1(\omega) \equiv \eta_1(0, -\infty, \omega)$$

exist. In addition, we have for  $t \in \mathbb{R}$ ,  $\omega \in \Omega$

$$\eta_1(t, -\infty, \omega) = \eta_1(\theta_t \omega)$$

and  $\|\eta_1(\omega)\|_{X_1}$  is a tempered random variable. The mapping  $\mathbb{R} \ni t \mapsto \eta_1(\theta_t \omega) \in X_1$  is continuous or càdlàg depending if the noise is continuous or càdlàg and  $\eta_1$  (which follows from the definition) is  $\mathcal{F}, \mathcal{B}(X_1)$ -measurable.

**(R6)** We assume that the random function

$$\tau \mapsto \Sigma(s, \tau, \omega) \equiv (\eta_1(\tau), \omega), -S^2(\tau - s)\eta_2(s, \tau, \omega)), \quad \tau \leq s$$

satisfies the following integrability condition

$$\int_{-\infty}^s e^{2\mu(\tau-s)} \|\Sigma(s, \tau, \omega)\|_{X_1}^2 d\tau < \infty$$

for every  $s \in \mathbb{R}$  and  $\omega \in \Omega$ , where  $\mu \in (\gamma_2, \gamma_1)$  will be chosen later.

**Remark 2.4** (1) We note that using relation (16) we can also construct a version of  $\eta_2(t, s, \omega)$  such that **(R1)–(R4)** hold for every  $(s, t) \in \Pi$ ,  $\omega \in \Omega$ .

(2) Instead **(R6)** we can assume that hypotheses **(R6s)** holds:

$$\begin{aligned}r_1(s, \omega) &= \sup_{t < s} \left\{ e^{2\mu(t-s)} \|\eta_1(\theta_t \omega)\|_{X_1}^2 \right\} \\ r_2(s, \omega) &= \sup_{t < s} \left\{ e^{2\mu(t-s)} \|S^2(t - s)\eta_2(s, t, \omega)\|_{X_2}^2 \right\}\end{aligned}$$

are finite random variables for every  $s \in \mathbb{R}$  and  $\mu > \gamma_2$ . We also note that the finiteness of  $r_1(s, \omega)$  follows from the temperedness property in **(R5)**.

We now formulate examples such that the hypotheses **(R1)–(R6)** are satisfied.

### 2.2.1 The Brownian Motion

Let  $p_K = (\Omega', \mathcal{G}, \mathbb{P}_K)$  be the probability space  $(C_0(H_0), \mathcal{B}(C_0(H_0)), \mathbb{P}_K)$ , where  $C_0(H_0)$  is the set of continuous functions on  $\mathbb{R}$  with values in  $H_0$  which are 0 at 0 equipped with the compact open topology,  $\mathcal{B}(C_0(H_0))$  is the Borel- $\sigma$ -algebra for this space and  $\mathbb{P}_K$  is the Wiener measure related to the covariance operator  $K$  which means that  $K$  is a symmetric positive operator of trace class. This probability space is called canonical twosided Wiener process. For two covariance operators  $K_1, K_2$  we define now the product space  $p_{K_1} \times p_{K_2}$  with  $\Omega = \Omega' \times \Omega', \mathcal{F} = \mathcal{G} \otimes \mathcal{G}$  and  $\mathbb{P} = \mathbb{P}_{K_1} \times \mathbb{P}_{K_2}$ . The two factors of this product define two independent twosided Wiener processes  $N_1, N_2$  with covariance  $K_1, K_2$ . In addition,  $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ , where  $\theta$  is given by the flow called Wiener shift

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \text{for } \omega \in \Omega, t \in \mathbb{R},$$

defines a metric dynamical system. For details see Arnold, [3, Appendix].

Let  $\bar{\mathcal{F}}$  be the completion of  $\mathcal{F}$  with respect to  $\mathbb{P}$ . Then let  $(\mathcal{F}_{s,t})_{t \in \mathbb{R}}$  containing the zero measure sets of  $\bar{\mathcal{F}}$  which is assumed to be right continuous. We can suppose that  $N_1, N_2$  are also independent Wiener processes with respect to the filtered probability space  $(\Omega, \bar{\mathcal{F}}, (\mathcal{F}_{s,t})_{s < t}, \bar{\mathbb{P}})$ .

We then suppose that  $B_i, i = 1, 2$  are linear bounded operators from  $H_0$  to  $X_i$ . Let us assume that for  $(s, t) \in \Pi$

$$\|B_i\|_{L(H_0, X_i)}^2 \text{tr } K_i \int_s^t \|S^i(t - \tau)\|_{L(X_i, X_i)}^2 d\tau < \infty, \quad (17)$$

where  $\|\cdot\|_{L(X, Y)}$  denotes the operator norm for operators from  $X$  into  $Y$ . Then we know that the random convolutions (14) are well defined, see DaPrato and Zabczyk [20], Chapter 4. In particular, these random variables are  $(\mathcal{F}_{s,t})_{t \geq s}$ -measurable for every  $s \in \mathbb{R}$  and continuous, almost surely. One can also prove (see, e.g., [20]) that

$$\mathbb{E} \|\eta_i(t, s)\|_{X_i}^2 \leq \|B_i\|_{L(H_0, X_i)}^2 \text{tr } K_i \int_s^t \|S^i(t - \tau)\|_{L(X_i, X_i)}^2 d\tau.$$

We now have

**Lemma 2.5** *Under the assumptions (17) there are versions to the random fields introduced in (14) and (15) such that **(R1)–(R5)** and **(R6s)** are satisfied. In particular, for these versions (16) is satisfied.*

For the proof we refer to Chueshov and Scheutzow [18, Proposition 3.1].

### 2.2.2 The fractional Brownian motion

Now we consider  $N_1, N_2$  to be two independent infinite dimensional twosided fractional Brownian motions. A one dimensional twosided fractional Brownian motion with Hurst parameter  $H \in (0, 1)$  is a Gaussian process  $\beta$  with mean zero and covariance

$$R(s, t) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}) \quad \text{for } t, s \in \mathbb{R}.$$

Note that such a process is neither a semimartingale nor a Markov process for  $H \neq 1/2$ . But on the other hand, a fractional Brownian motion has stationary increments.

We now introduce an infinite dimensional fractional motion. Let  $K$  be a linear bounded symmetric positive operator on  $H_0$  of trace class. The spectrum and the associated eigenelements of  $K$  are denoted by  $\{\mu_i, e_i : i \in \mathbb{N}\}$ .  $\{e_i : i \in \mathbb{N}\}$  forms an complete orthonormal system in  $H_0$ .

**Definition 2.6** *Let  $\{\beta_i : i \in \mathbb{N}\}$  be a sequence of independent one dimensional twosided fractional Brownian motions. Then a  $H_0$ -valued continuous process with covariance  $K$  defined by*

$$N(t) = \sum_{i=1}^{\infty} \sqrt{\mu_i} \beta_i(t) e_i, \quad t \in \mathbb{R}$$

*is called an infinite dimensional twosided fractional Brownian motion on  $H_0$ .*

Note that for the fractional Brownian motion one can construct in the same way as for the Brownian motion a metric dynamical system  $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$  which is ergodic with  $\mathbb{P} = \mathbb{P}_{K_1} \times \mathbb{P}_{K_2}$ , where  $K_1, K_2$  are the corresponding covariances. In particular, there exist two independent twosided fractional Brownian motions on  $H_0$ .

We now construct random convolutions  $\eta_1, \eta_{2,a}$  and  $\eta_2$  in  $H_0$  and  $B_1 = B_2 = \text{id}$ . For the applications we have in mind this setting is sufficient. We can conclude from Schmalfuß and Maslowski [28]:

**Lemma 2.7** (1) *For every  $x \in H_0$  there exists a continuous version of*

$$S^1(t)x + \int_0^t S^1(t - \tau) dN_1(\tau), \quad S^{2,a}(t)x + \int_0^t S^{2,a}(t - \tau) dN_2(\tau).$$

*This version is denoted by  $\psi_1(t, \omega)x, \psi_2(t, \omega)x$ .*

(2) There exist random variables  $\eta_1, \eta_{2,a}$  with values in  $H_0$  such that for  $t \geq 0$

$$\psi_1(t, \omega)\eta_1(\omega) = \eta_1(\theta_t\omega), \quad \psi_2(t, \omega)\eta_{2,a}(\omega) = \eta_{2,a}(\theta_t\omega).$$

From these facts we can derive

**Lemma 2.8** *Suppose that  $N_1, N_2$  are two independent infinite dimensional twosided fractional Brownian motions such that the above properties are satisfied. Then the hypotheses **(R1)**–**(R5)** and **(R6s)** hold.*

**Proof.**  $\eta_1$  from **(R5)** is just given in Lemma 2.7. In particular, from Lemma 2.7, 2.8 follows directly that

$$t \rightarrow \eta_1(\theta_t\omega) = \eta_1(-\infty, t, \omega)$$

is continuous. Then the continuity and measurability properties or **(R1)**, **(R2)** and **(R3)** follows directly from

$$\eta_1(s, t) \equiv \eta_1(\theta_t\omega) - S^1(t-s)\eta_1(\theta_s\omega) \quad (18)$$

and similarly for  $\eta_{2,a}$ . In a similar way we can proof **(R4)**. The temperedness conclusion follows from the proof Maslowski and Schmalfuß [28], Theorem 3.2, where it has been used that with respect to some Hölder space  $C^\alpha$  with appropriate Hölder exponent  $\alpha$ ,

$$\|N_i(\cdot, \theta_j\omega)\|_{C^\alpha([0,1];H_0)}$$

has a subexponential growth for  $j \rightarrow \pm\infty$ . □

### 2.2.3 $\alpha$ -stable Lévy processes

Let  $(\lambda_i^j)_{i=1, \dots, n_j}$ ,  $j = 1, 2$  be two finite sequences of  $\alpha$ -stable one dimensional twosided Lévy processes which are mutually independent. We can assume that these processes are càdlàg. For definition we refer to Applebaum [2]. Then if  $(e_i^1)_{i=1, \dots, n_1}$ ,  $(e_i^2)_{i=1, \dots, n_2}$  be two finite subsets of an orthonormal base of  $H_0$ . Set

$$N_1(t) = \sum_{i=1}^{n_1} \lambda_i^1(t)e_i^1, \quad N_2(t) = \sum_{i=1}^{n_2} \lambda_i^2(t)e_i^2.$$

Since  $N_1, N_2$  are càdlàg processes with independent increments we can describe these processes by a metric dynamical system  $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ . For details we refer to Arnold [3, Appendix A].

We choose versions  $\eta_1, \eta_{2,a}$  of

$$\eta_1(\omega) = \int_s^t S^1(t-\tau)B_1dN_1(\tau), \quad \eta_{2,a}(\omega) = \int_s^t S^{2,a}(t-\tau)B_2dN_2(\tau)$$

such that **(R1)**–**(R4)** hold. Since  $N_1, N_2$  are semimartingals by the integration by parts formula we can choose versions for the integrals in the last formula as

$$A_1 \int_{-\infty}^0 S^1(\tau)B_1N_1(\tau)d\tau$$

which then allows to conclude that  $\mathbb{R} \ni t \rightarrow \eta_1(\theta_t\omega)$  is càdlàg a such that **(R5)** follows. **(R1)**–**(R4)** then follows similar to (18). In particular, the random fields  $\eta_1, \eta_2, \eta_{2,a}$  depend càdlàg on  $s$  and on  $t$ . To prove **(R6)** we refer to Pruitt [32], where the limit behavior of the  $\alpha$ -stable Lévy processes  $\lambda_i^j$

$$\lim_{t \rightarrow \infty} |t|^{-\eta} \sup_{\tau \in [0,t]} |\lambda_i^j(\tau)| = 0, \quad \lim_{t \rightarrow \infty} |t|^{-\eta} \sup_{\tau \in [-t,0]} |\lambda_i^j(\tau)| = 0$$

for  $\eta > \alpha$  has been established (which implies the temperedness of  $\|\eta_1\|$  and  $\|\eta_{2,a}\|$ ) and to [2] for the existence of the corresponding stationary processes. Recently the same observation was used in Liu et al. [39].

### 2.3 Mild solutions and generation of an RDS

We denote by  $D([a, b]; X)$  the space of strongly càdlàg functions on the interval  $[a, b]$  with values in  $X$  equipped with the norm of uniform convergence. Later when we consider random problems driven by a Brownian or fractional Brownian motion we can replace  $D([a, b]; X)$  by the space  $C([a, b]; X)$  of continuous functions on  $[a, b]$  with values in  $X$ .

Below we assume that **(A1)**, **(A2)**, **(R1)**–**(R6)** hold true.

**Definition 2.9** Let  $s \in \mathbb{R}, T > s, Y_0 \in X$ . A process  $Y(t) \equiv Y(t, s, \omega, V_0)$  which, for each  $\omega \in \Omega$ , belongs to the space  $D([s, T]; X)$  (or  $C([0, T]; X)$ ) is said to be a *mild solution* to problem (11) on the interval  $[s, T]$  if  $Y(s) = Y_0 \in X$  and

$$Y(t) = \mathcal{E}[Y](t) \equiv S(t-s)Y_0 + \int_s^t S(t-\tau)F(Y(\tau))d\tau + \eta(t, s) \quad (19)$$

for all  $t \in [s, T]$  and  $\omega \in \Omega$ , where  $\eta(t, s) = (\eta_1(t, s), \eta_2(t, s))$ .

We have the following result about existence and uniqueness of mild solutions to (11).

**Theorem 2.10** *For every  $Y_0 \in X$  and  $T > s$  problem (11) has a unique mild solution  $Y(t)$  on the interval  $[s, T]$ . Furthermore, the process  $t \mapsto Y(t, s, Y_0)$  is adapted to the corresponding filtration  $\{\mathcal{F}_{s,t}\}_{t \geq s}$ .*

*Define the map  $\phi : \mathbb{R}_+ \times \Omega \times X \mapsto X$  by the formula  $\phi(t, \omega)Y_0 \equiv Y(t, 0, \omega, Y_0)$ . Then*

- (i)  $\phi$  is a cocycle of a random dynamical system on  $X$ , and
- (ii)  $Y(t, s, \omega, Y_0) = \phi(t - s, \theta_s \omega)Y_0$  solves (19) for every  $s \in \mathbb{R}$ , every  $t > s$ , and every  $\omega \in \Omega$ .

**Proof.** By the standard fix point argument one can prove the existence of a unique mild solution on any interval  $[s, T]$ . We note this argument is applied in the space  $D([s, T]; X)$  because the uniform limit of a sequence càdlàg functions is itself a càdlàg function. The same remains true if we replace càdlàg functions by continuous functions.

Now using by the relations **(R3)** and (16) we deduce from the uniqueness of the mild solutions that

$$Y(t, s, \omega, Y_0) = Y(t + \tau, s + \tau, \theta_{-\tau} \omega, Y_0), \quad s \leq t, \tau \in \mathbb{R}, \quad (20)$$

for all  $Y_0 \in X$ ,  $\omega \in \Omega$ , as well as

$$Y(t, 0, \omega, Y_0) = Y(t, s, \omega, Y(s, 0, \omega, Y_0)), \quad 0 \leq s \leq t, Y_0 \in X, \omega \in \Omega.$$

Therefore

$$\begin{aligned} \phi(t + s, \omega)Y_0 &= Y(t + s, 0, \omega, Y_0) = Y(t + s, s, \omega; Y(s, 0, \omega, Y_0)) \\ &= Y(t, 0, \theta_s \omega, Y(s, 0, \omega, Y_0)) = \phi(t, \theta_s \omega)\phi(s, \omega)Y_0, \end{aligned}$$

for  $t, s \geq 0$ , i.e.  $\phi$  satisfies the cocycle property. The càdlàg/continuity and measurability properties of  $\phi$  follow from those of  $Y$ . It also follows from (20) that

$$\phi(t - s, \theta_s \omega)Y_0 = Y(t - s, 0, \theta_s \omega, Y_0) = Y(t, s, \omega, Y_0),$$

which completes the proof.  $\square$

### 3 Existence of an invariant manifold

Now we can prove the main result of this paper.

**Theorem 3.1** *Assume that (A1)–(A2), (R1)–(R6) hold and*

$$\gamma_1 - \gamma_2 > \left( \sqrt{M_2 L_2} + \sqrt{M_1 L_1} \right)^2. \quad (21)$$

*Then, there exists a random mapping  $\Phi(\cdot, \cdot) : \Omega \times X_2 \mapsto X_1$  such that*

$$\|\Phi(\omega, V_1) - \Phi(\omega, V_2)\|_{X_1} \leq C \|V_1 - V_2\|_{X_2}, \quad (22)$$

*for all  $V_1, V_2 \in X_2, \omega \in \Omega$ , where  $C$  is a constant independent of the arguments. Moreover, the random manifold*

$$\mathbf{M}(\omega) = \{(\Phi(\omega, V), V) : V \in X_2\} \subset X,$$

*is strictly invariant with respect to the cocycle  $\phi$ :  $\phi(t, \omega)\mathbf{M}(\omega) = \mathbf{M}(\theta_t \omega)$ .*

*This manifold  $\mathbf{M}$  is exponentially attracting in the following sense. Let*

$$\mu = \frac{\sqrt{M_2 L_2} \gamma_1 + \sqrt{M_1 L_1} \gamma_2}{\sqrt{M_2 L_2} + \sqrt{M_1 L_1}}. \quad (23)$$

*Then there exist a random variable  $R_1 > 0$  and a constant  $C_1 > 0$  such that for any  $Y_0 \in X$  to (11) there exists  $Y^* \in \mathbf{M}(\omega)$ :*

$$\left[ \int_0^\infty e^{2\mu t} \|\phi(t, \omega)Y_0 - \phi(t, \omega)Y^*\|_X^2 dt \right]^{1/2} \leq R_1(\omega) + C_1 \|Y_0\|_X. \quad (24)$$

*If we assume that the stronger requirement of (R6s) holds, then we have a random variable  $R_2$  and a constant  $C_2$  such that*

$$\|\phi(t, \omega)Y_0 - \phi(t, \omega)Y^*\|_X \leq e^{-\mu t} (R_2(\omega) + C_2 \|Y_0\|_X), \quad t > 0. \quad (25)$$

**Remark 3.2** It follows from (25) and from the invariance property of  $\mathbf{M}(\omega)$  with respect to the cocycle  $\phi$  that for every bounded set  $B$  there exists a  $C_B > 0$

$$\sup \{ \text{dist}_X(\phi(t, \omega)Y_0, \mathbf{M}(\theta_t \omega)) : Y_0 \in B \} \leq C_B(\omega) e^{-\mu t}, \quad \omega \in \Omega.$$

If  $R_2(\omega)$  is tempered, then relation (25) also implies that

$$\lim_{t \rightarrow \infty} \sup \{ e^{\tilde{\mu} t} \text{dist}_X(\phi(t, \theta_{-t} \omega)Y_0, \mathbf{M}(\omega)) : Y_0 \in B \} = 0, \quad \omega \in \Omega,$$

for any  $\tilde{\mu} < \mu$ . Thus, in the case of tempered  $R_2(\omega)$  the manifold  $M(\omega)$  is uniformly exponentially attracting in the both *forward* and *pullback* sense. By the Lipschitz continuity of  $\Phi(\omega, \cdot)$  we obtain that  $M$  is a random closed set, i.e.

$$\omega \mapsto \text{dist}_X(y, M(\omega))$$

is measurable for any  $y \in X$

In the following we prove Theorem 3.1. We proceed in several steps.

### 3.1 Construction of the inertial manifold

We apply the Lyapunov-Perron procedure (see, e.g., [11, 13, 29]) but modified for stochastic systems (see [12, 14, 18, 15]).

Following [29] for each fixed  $s \in \mathbb{R}$ , we consider the spaces

$$\mathcal{X}_s = \left\{ Y(\cdot) : e^{\mu(\cdot-s)}Y(\cdot) \in L_2(-\infty, s, X) \right\},$$

where  $\mu \in (\gamma_2, \gamma_1)$  is given by (23). On this space we introduce the norm

$$\|Y\|_{\mathcal{X}_s} \equiv \left( \int_{-\infty}^s e^{2\mu(t-s)} \|Y(t)\|_X^2 dt \right)^{1/2}.$$

In order to construct an invariant manifold we should first solve the integral equation

$$Y = \mathcal{T}_{V_0}[Y, \omega](\cdot, s) \quad \text{on } \mathcal{X}_s \quad (26)$$

for every  $s \in \mathbb{R}$ , where  $V_0 \in PX$  and  $\mathcal{T}_{V_0}[Y, \omega] \equiv \mathcal{L}_{V_0}[F(Y), \omega]$ . Here  $\mathcal{L}_{V_0}[Y, \omega]$  is defined on  $\mathcal{X}_s$  by

$$\begin{aligned} \mathcal{L}_{V_0}[Y, \omega](\sigma, s) &= S^2(\sigma - s)V_0 - \int_{\sigma}^s S^2(\sigma - \tau)PY(\tau, s)d\tau \\ &+ \int_{-\infty}^{\sigma} S^1(\sigma - \tau)QY(\tau, s)d\tau + \Sigma(s, \sigma, \omega) \end{aligned} \quad (27)$$

with

$$\Sigma(s, \sigma, \omega) \equiv (\eta_1(\theta_{\sigma}\omega), -S^2(\sigma - s)\eta_2(s, \sigma, \omega)) \quad (28)$$

for every  $\sigma \in (-\infty, s]$ . A solutions of (26) is denoted by  $Y = Y_{V_0}(\cdot, s, \omega)$ . We first point out some properties of the stochastic term in (27), which is useful for our considerations. It is easy to see from **(R3)** that

$$\Sigma(s, \sigma + s, \omega) = \Sigma(0, \sigma, \theta_s\omega) \quad \text{for all } \sigma \leq 0, s \in \mathbb{R}, \omega \in \Omega.$$

Therefore a simple calculation gives us the following relation between the solutions to the problem (26) for different values of  $s$ :

$$Y_{V_0}(\tau + s, s, \omega) = Y_{V_0}(\tau, 0, \theta_s \omega) \quad \text{for all } \tau \leq 0, s \in \mathbb{R}, \omega \in \Omega. \quad (29)$$

Similar to the deterministic case considered in [16] we can prove:

**Proposition 3.3** *Let  $s \in \mathbb{R}$  and  $\gamma_2 < \mu < \gamma_1$ . Then, for every  $V_0 \in PX$  and  $\omega \in \Omega$  the operator  $\mathcal{T}_{V_0}[\cdot, \omega]$  is from  $\mathcal{X}_s$  into itself and*

$$|\mathcal{T}_{V_{01}}[Y_1, \omega] - \mathcal{T}_{V_{02}}[Y_2, \omega]|_{\mathcal{X}_s} \leq \|V_{01} - V_{02}\|_{X_1} + \kappa(\mu) \cdot |Y_1 - Y_2|_{\mathcal{X}_s}, \quad \omega \in \Omega,$$

for every  $V_{01}, V_{02} \in X_2$  and  $Y_1, Y_2 \in \mathcal{X}_s$ , where

$$\kappa(\mu) = \frac{M_2 L_2}{\mu - \gamma_2} + \frac{M_1 L_1}{\gamma_1 - \mu}. \quad (30)$$

We need for this Lemma 3.4 and Proposition 3.5.

**Lemma 3.4** *Let  $f \in L_2(\mathbb{R})$  and  $\delta > 0$ . Then*

$$I_1(f)(t) = \int_t^\infty e^{\delta(t-\tau)} f(\tau) d\tau \in L_2(\mathbb{R}),$$

$$I_2(f)(t) = \int_{-\infty}^t e^{-\delta(t-\tau)} f(\tau) d\tau \in L_2(\mathbb{R}),$$

and

$$\int_{\mathbb{R}} |I_i(f)(t)|^2 dt \leq \frac{1}{\delta^2} \int_{\mathbb{R}} |f(t)|^2 dt, \quad i = 1, 2. \quad (31)$$

**Proof.** On can see that  $I_2(f)(t) = I_1(f_-)(-t)$ , where  $f_-(t) = f(-t)$ . Therefore it is sufficient to deal with  $I_1$  only. After the Fourier transform relation (31) easily follows from the Plancherel formula. Details can be found in [16].  $\square$

**Proposition 3.5** *For every  $V_0 \in X_1$  the operator  $\mathcal{L}_{V_0}^{det}$  given by*

$$\begin{aligned} \mathcal{L}_{V_0}^{det}[Y](\sigma, s) &= S^2(\sigma - s)V_0 - \int_\sigma^s S^2(\sigma - \tau)PY(\tau, s)d\tau \\ &+ \int_{-\infty}^\sigma S^1(\sigma - \tau)QY(\tau, s)d\tau \end{aligned} \quad (32)$$

is a continuous mapping from  $\mathcal{X}_s$  into itself and for any  $Y_1, Y_2 \in \mathcal{X}_s$  we have that

$$|\mathcal{L}_{V_0}^{det}[PY_1] - \mathcal{L}_{V_0}^{det}[PY_2]|_{\mathcal{X}_s} \leq \frac{M_2}{\mu - \gamma_2} \cdot |PY_1 - PY_2|_{\mathcal{X}_s} \quad (33)$$

and

$$|\mathcal{L}_{V_0}^{det}[QY_1] - \mathcal{L}_{V_0}^{det}[QY_2]|_{\mathcal{X}_s} \leq \frac{M_1}{\gamma_1 - \mu} \cdot |QY_1 - QY_2|_{\mathcal{X}_s}. \quad (34)$$

**Proof.** Since  $\mathcal{L}_{V_0}^{det}[Y_1] - \mathcal{L}_{V_0}^{det}[Y_2] = \mathcal{L}_0^{det}[Y_1 - Y_2]$ , to obtain (33) and (34) we need only to estimate  $|\mathcal{L}_0^{det}[PY]|_{\mathcal{X}_s}$  and  $|\mathcal{L}_0^{det}[QY]|_{\mathcal{X}_s}$  for any  $Y \in \mathcal{X}_s$ . One can see that

$$e^{\mu(\sigma-s)} \|\mathcal{L}_0^{det}[PY](\sigma)\|_X \leq M_2 \int_{\sigma}^s e^{(\mu-\gamma_2)(\sigma-\tau)} \cdot e^{\mu(\tau-s)} \|PY(\tau)\|_{X_1} d\tau, \quad \sigma \leq s.$$

Therefore, applying this estimate given for  $I_1$  in Lemma 3.4 with  $\delta = \mu - \gamma_2$  and  $f(t)$  defined by the relation:  $f(t) = e^{\mu(t-s)} \|Y(t)\|_X$  for  $t \leq s$  and  $f(t) = 0$  for  $t > s$ , we obtain that

$$|\mathcal{L}_0^{det}[PY]|_{\mathcal{X}_s} \leq \frac{M_2}{\mu - \gamma_2} \cdot |PY|_{\mathcal{X}_s} \quad \text{for any } Y \in \mathcal{X}_s. \quad (35)$$

In a similar way, Lemma 3.4 for  $I_2$  yields that

$$|\mathcal{L}_0^{det}[QY]|_{\mathcal{X}_s} \leq \frac{M_1}{\gamma_1 - \mu} \cdot |QY|_{\mathcal{X}_s} \quad \text{for any } Y \in \mathcal{X}_s. \quad (36)$$

Relations (35) and (36) imply (33) and (34). The continuity of the mapping  $\mathcal{L}_{V_0}^{det}$  follows from (33) and (34) and from the relation

$$|\mathcal{L}_{V_0}^{det}[Y_1] - \mathcal{L}_{V_0}^{det}[Y_2]|_{\mathcal{X}_s}^2 = |\mathcal{L}_{V_0}^{det}[PY_1] - \mathcal{L}_{V_0}^{det}[PY_2]|_{\mathcal{X}_s}^2 + |\mathcal{L}_{V_0}^{det}[QY_1] - \mathcal{L}_{V_0}^{det}[QY_2]|_{\mathcal{X}_s}^2.$$

□

Let  $\mu$  be given by (23) which minimizes (30). In this case

$$\kappa(\mu) = \frac{(\sqrt{M_2 L_2} + \sqrt{M_1 L_1})^2}{\gamma_1 - \gamma_2}$$

and we have that  $\kappa(\mu) < 1$  under condition (21). Thus  $\mathcal{T}_{V_0}[\cdot, \omega]$  is a contraction in  $\mathcal{X}_s$  and hence (26) has a unique solution  $Y(\cdot, s) \equiv Y_{V_0}(\cdot, s, \omega)$  in the space  $\mathcal{X}_s$  for each  $\omega \in \Omega$ . Using the same (standard) argument as in the deterministic case (see [16]) one can show that this solution  $Y(\cdot, s)$  possesses the properties

$$Y(\cdot) \equiv Y(\cdot, s) \in D((-\infty, s], X),$$

and

$$\sup_{t \leq s} \left\{ e^{\mu(t-s)} \|Y_{V_{01}}(t, s, \omega) - Y_{V_{02}}(t, s, \omega)\|_X \right\} \leq C \|V_{01} - V_{02}\|_X \quad (37)$$

for any  $V_{01}, V_{02} \in PX$  and  $\omega \in \Omega$ , where  $C$  is a positive constant.

For every  $s \in \mathbb{R}$  we define  $\Phi_s : \Omega \times X_2 \rightarrow X_1$  as

$$\Phi_s(\omega, V_0) \equiv \int_{-\infty}^s S^1(s - \tau) F_1(Y_{V_0}(\tau, s, \omega)) d\tau + \eta_1(\theta_s \omega) = Q\mathcal{T}_{V_0}[Y, \omega](s, s). \quad (38)$$

Now we prove that  $M$  is forward invariant, i.e.  $\phi(t, \omega)M(\omega) \subseteq M(\omega)$ . To see this we note that if  $s < t$ , then

$$\tilde{Y}(\sigma, t, \omega) \equiv \begin{cases} Y_{V_0}(\sigma, s, \omega) & : \quad \sigma \leq s \\ Y(\sigma, s, \omega, V_0 + \Phi_s(\omega, V_0)) & : \quad \sigma \in [s, t] \end{cases}$$

satisfies

$$\begin{aligned} \tilde{Y}(\sigma, t, \omega) &= S^2(\sigma - t)PY(t, s, \omega, Y_{V_0}(s, s, \omega)) \\ &\quad + \int_t^\sigma S^2(\sigma - \tau)F_2(\tilde{Y}(\tau, t, \omega))d\tau \\ &\quad + \int_{-\infty}^\sigma S^1(\sigma - \tau)F_1(\tilde{Y}(\tau, t, \omega))d\tau \\ &\quad - S^2(\sigma - t)\eta_2(t, \sigma, \omega) + \eta_1(\theta_\sigma \omega) \end{aligned}$$

for both  $\sigma \leq s$  and  $\sigma \in (s, t]$ . Hence  $\tilde{Y}$  is a fixed point of

$$\mathcal{T}_{PY(t, s, \omega, V_0 + \Phi_s(\omega, V_0))}[\cdot, \omega](\cdot, t)$$

and

$$\Phi_t(\omega, PY(t, s, \omega, V_0 + \Phi_s(\omega, V_0))) = QY(t, s, \omega, V_0 + \Phi_s(\omega, V_0))$$

which means forward invariance of  $M$ . The strict invariance will be proved later, in Section 3.3.

It is easy to see from (29) that  $\Phi_s(\omega, V_0) = \Phi_0(\theta_s \omega, V_0) \equiv \Phi(\theta_s \omega, V_0)$ , i.e.  $s \mapsto \Phi_s(\omega, V_0)$  is a stationary process. Moreover, the relation (37) implies the Lipschitz property (22).

### 3.2 Tracking properties

We will use the method developed in [29] for the proof of the tracking property for inertial manifolds in the deterministic case.

Let  $Y_0 = (U_0, V_0) \in X$ . We consider the following space

$$\mathcal{Z} = \left\{ Z(\cdot) : \mathbb{R} \rightarrow X : |Z|_{\mathcal{Z}}^2 \equiv \int_{-\infty}^{\infty} e^{2\mu t} \|Z(t)\|_X^2 dt < \infty \right\}$$

for  $\mu$  given in (23) and define the random function

$$Z_0(t, \omega) = \begin{cases} -Y_0 + \mathcal{T}_{PY_0}[Y, \omega](t, 0), & \text{for } t \leq 0; \\ S(t) [-Y_0 + \mathcal{T}_{PY_0}[Y, \omega](0, 0)], & \text{for } t > 0, \end{cases}$$

where  $\mathcal{T}_{PY_0}$  is defined in (26). Below we need the following properties of the random function  $Z_0(t, \omega)$ .

**Lemma 3.6** *For every  $\omega \in \Omega$  the random function  $Z_0(t, \omega)$  belongs to  $\mathcal{Z}$ . Moreover, there exist a deterministic constant  $C_1$  and a scalar random variable  $\bar{R}_1(\omega)$  such that*

$$|Z_0|_{\mathcal{Z}} \leq \bar{R}_1(\omega) + C_1 \|Y_0\|_X \quad (39)$$

*If we assume in addition **(R6s)**, then there also exist a  $C_2 > 0$  and a scalar random variable  $\bar{R}_2(\omega)$  such that*

$$\sup_{t \in \mathbb{R}} \{e^{\mu t} \|Z_0(t)\|_X\} \leq \bar{R}_2(\omega) + C \|Y_0\|_X. \quad (40)$$

**Proof.** We split  $Z_0(t, \omega)$  into a deterministic and a stochastic part,

$$Z_0(t, \omega) = Z_0^{det}(t) + Z_0^{st}(t, \omega),$$

where

$$Z_0^{det}(t) = \begin{cases} -Y_0 + \mathcal{L}_{PY_0}^{det}[F(Y)](t, 0), & \text{for } t \leq 0; \\ S(t) [-Y_0 + \mathcal{L}_{PY_0}^{det}[F(Y)](0, 0)], & \text{for } t > 0, \end{cases}$$

and

$$Z_0^{st}(t, \omega) = \begin{cases} \Sigma(0, t, \omega), & \text{for } t \leq 0; \\ (S^1(t)\eta_1(\omega), 0), & \text{for } t > 0, \end{cases}$$

Since

$$\begin{aligned} R_1^*(\omega) &\equiv |Z_0^{st}(\omega)|_{\mathcal{Z}}^2 \leq \int_{-\infty}^0 e^{2\mu t} [\|S^2(t)\eta_2(0, t, \omega)\|_{X_2}^2 + \|\eta_1(\theta_t \omega)\|_{X_1}^2] dt \\ &\quad + \frac{M_1}{\gamma_1} \|\eta_1(\omega)\|_{X_1}^2 < \infty \end{aligned}$$

and if **(R6s)** holds

$$\begin{aligned} R_2^*(\omega) &\equiv \sup_{t \in \mathbb{R}} \{e^{\mu t} \|Z_0^{st}(t, \omega)\|_X\} \\ &\leq c_0 \sup_{t \in \mathbb{R}_-} \{e^{\mu t} [\|S^2(t)\eta_2(0, t, \omega)\|_{X_2} + \|\eta_1(\theta_t \omega)\|_{X_1}]\} < \infty, \end{aligned}$$

it follows from hypotheses imposed that  $R_1^*(\omega)$  and  $R_2^*(\omega)$  are finite for every  $\omega$ . Therefore, estimating the deterministic part  $Z_0^{det}(t)$  by the standard method we obtain the estimates (39) and (40) with  $\bar{R}_i(\omega) = D_1 + D_2 R_i^*(\omega)$ , where  $D_1$  and  $D_2$  are constants, independent of  $\omega$ .  $\square$

Let now  $Y(t) = Y(t, 0, \omega, Y_0)$ ,  $t \in \mathbb{R}$ , be the solution to (11) for  $t \geq 0$  and  $Y_0 \in X$  for  $t \leq 0$ . We define an integral operator  $\mathcal{R} : \mathcal{Z} \mapsto \mathcal{Z}$  by the formula

$$\begin{aligned} \mathcal{R}[Z](t) &= Z_0(t) + \int_{-\infty}^t S^1(t-\tau)Q [F(Z(\tau) + Y(\tau)) - F(Y(\tau))] d\tau \\ &\quad - \int_t^{\infty} S^2(t-\tau)P [F(Z(\tau) + Y(\tau)) - F(Y(\tau))] d\tau. \end{aligned}$$

Let us prove that  $\mathcal{R}$  is a contraction in  $\mathcal{Z}$ .

By (7) and (5) we have that

$$\begin{aligned} &e^{\mu t} \|P(\mathcal{R}[Z_1](t) - \mathcal{R}[Z_2](t))\|_X \\ &\leq M_2 L_2 \int_t^{\infty} e^{(\mu-\gamma_2)(t-\tau)} e^{\mu\tau} \|Z_1(\tau) - Z_2(\tau)\|_X d\tau \end{aligned} \quad (41)$$

By Lemma 3.4 with  $\delta = \mu - \gamma_2$  and  $f(t) = M_2 L_2 e^{\mu t} \|Z_1(t) - Z_2(t)\|_X$  we obtain that

$$\|P(\mathcal{R}[Z_1] - \mathcal{R}[Z_2])\|_{\mathcal{Z}} \leq \frac{M_2 L_2}{\mu - \gamma_2} \cdot \|Z_1 - Z_2\|_{\mathcal{Z}}.$$

Similarly, (6) and (4) yields

$$\begin{aligned} &e^{\mu t} \|Q(\mathcal{R}[Z_1](t) - \mathcal{R}[Z_2](t))\|_X \\ &\leq M_1 L_1 \int_{-\infty}^t e^{(\mu-\gamma_1)(t-\tau)} e^{\mu\tau} \|Z_1(\tau) - Z_2(\tau)\|_X d\tau \end{aligned}$$

and thus applying Lemma 3.4 again we have that

$$\|Q(\mathcal{R}[Z_1] - \mathcal{R}[Z_2])\|_{\mathcal{Z}} \leq \frac{M_1 L_1}{\gamma_1 - \mu} \cdot \|Z_1 - Z_2\|_{\mathcal{Z}}.$$

If  $\mu$  is given by (23), we can write

$$\|\mathcal{R}[Z_1] - \mathcal{R}[Z_2]\|_{\mathcal{Z}} \leq \kappa(\mu) \cdot \|Z_1 - Z_2\|_{\mathcal{Z}} \quad \text{for every } Z_1, Z_2 \in \mathcal{Z} \quad (42)$$

where  $\kappa(\mu) < 1$ . Thus by the contraction principle there exists a unique solution  $Z \in \mathcal{Z}$  to the equation  $Z = \mathcal{R}[Z]$  in  $\mathcal{Z}$ .

Now using the same calculation as in [16] and [29] we can conclude that the function  $\tilde{Y}(t) = Z(t) + Y(t)$ , where  $Z \in \mathcal{Z}$  solves the equation  $Z = \mathcal{R}[Z]$ , satisfies the relation

$$\tilde{Y}(t) = \begin{cases} \mathcal{T}_{P\tilde{Y}(0)}[\tilde{Y}, \omega](t, 0), & \text{if } t \leq 0; \\ \phi(t, \omega)\tilde{Y}(0), & \text{if } t > 0. \end{cases}$$

In particular,  $\tilde{Y}(0) = \mathcal{T}_{P\tilde{Y}(0)}[\tilde{Y}, \omega](0, 0)$  and, therefore, by the definition of the operator  $\mathcal{T}_{P\tilde{Y}(0)}$  we obtain that

$$\tilde{Y}(0) = P\tilde{Y}(0) + \int_{-\infty}^0 S(-\tau)QF(\tilde{Y}(\tau))d\tau + Q\eta(0, -\infty).$$

By (38) this implies that  $\tilde{Y}(0) = \left(\Phi(\omega, P\tilde{Y}(0)), P\tilde{Y}(0)\right)$ . Therefore  $\tilde{Y}(t) = \phi(t, \omega)\tilde{Y}(0) \in \mathbf{M}(\theta_t\omega)$  for  $t \geq 0$ . Thus to complete the proof of the tracking property in (24) and (25) we only need to establish appropriate estimates for  $Z(t)$ .

Since

$$Z(t) = \mathcal{R}[Z](t) = Z_0(t) + \mathcal{R}[Z](t) - \mathcal{R}[0](t), \quad (43)$$

from (39) and (42) we obtain the relation

$$|Z|_{\mathcal{Z}} \leq (1 - \kappa(\mu))^{-1} \cdot |Z_0|_{\mathcal{Z}} \leq (1 - \kappa(\mu))^{-1} \cdot (\bar{R}_1(\omega) + C\|Y_0\|_X), \quad (44)$$

which implies (24).

Now we prove (25). From (41) we have that for  $t \in \mathbb{R}$

$$\begin{aligned} e^{\mu t} \|P(\mathcal{R}[Z](t) - \mathcal{R}[0](t))\|_X &\leq M_2 L_2 \int_t^{\infty} e^{(\mu - \gamma_2)(t - \tau)} \cdot e^{\mu \tau} \|Z(\tau)\|_X d\tau \\ &\leq M_2 L_2 \left[ \int_t^{\infty} e^{2(\mu - \gamma_2)(t - \tau)} d\tau \right]^{1/2} \cdot |Z|_{\mathcal{Z}} = \frac{M_2 L_2}{\sqrt{2(\mu - \gamma_2)}} \cdot |Z|_{\mathcal{Z}}. \end{aligned}$$

Thus

$$\sup_{t \in \mathbb{R}} \{e^{\mu t} \|P(\mathcal{R}[Z](t) - \mathcal{R}[0](t))\|_X\} \leq \frac{M_2 L_2}{\sqrt{2(\mu - \gamma_2)}} \cdot |Z|_{\mathcal{Z}}. \quad (45)$$

Similarly, we have that

$$\begin{aligned} e^{\mu t} \|Q(\mathcal{R}[Z](t) - \mathcal{R}[0](t))\|_X \\ \leq M_1 L_1 \int_{-\infty}^t e^{-\gamma_1(t - \tau)} e^{\mu \tau} \|Z(\tau)\|_X d\tau \leq \frac{M_1 L_1}{\sqrt{2(\gamma_1 - \mu)}} \cdot |Z|_{\mathcal{Z}}. \end{aligned} \quad (46)$$

Consequently, using relations (43), (45) and (40) we obtain that for appropriate  $c_1 > 0$ ,  $c_2 > 0$

$$\sup_{t \in \mathbb{R}} \{e^{\mu t} \|Z(t)\|_X\} \leq c_1 \bar{R}_2(\omega) + c_2 |Z|_{\mathcal{Z}}.$$

Thus by (44) we have

$$\sup_{t \in \mathbb{R}} \{e^{\mu t} \|Z(t)\|_X\} \leq c_3(\bar{R}_1(\omega) + \bar{R}_2(\omega)) + c_4 \|Y_0\|_X.$$

for appropriate (deterministic) constants  $c_3$  and  $c_4$ . This implies (25) and completes the proof of Theorem 3.1.

### 3.3 The reduced system and the strict invariance

Assume the hypotheses of Theorem 3.1 hold and let  $\Phi \equiv \Phi_0$  be given by (38) with  $s = 0$ . Consider the problem

$$\begin{aligned} V_t + A_2 V &= F_2(\Phi(\theta_t \omega, V), V) + B_2 \dot{N}_2, \quad t > s, \quad \text{in } X_2, \\ V(s) &= V_0, \end{aligned} \quad (47)$$

and define its *mild solution* on the interval  $[s, T]$  as a random function

$$V(t) \equiv V(t, s, \omega, V_0) \in D([s, T], X_2)$$

such that

$$V(t) = S^2(t-s)V_0 + \int_s^t S^2(t-\tau) F_2(\Phi(\theta_t \omega, V(\tau)), V(\tau)) d\tau + \eta_2(t, s) \quad (48)$$

for almost all  $t \in [s, T]$  and  $\omega \in \Omega$ .

**Proposition 3.7** *Let  $V_0 \in X_2$ . Then under the conditions of Theorem 3.1 problem (47) has a mild solution on any interval  $[s, T]$ . This solution is unique and any mild solution  $V$  to problem (47) generates a mild solution to problem (1) and (2) with initial condition  $(\Phi(\theta_s \omega, V_0), V_0)$  by the formula*

$$Y(t) = (U(t), V(t)) = (\Phi(\theta_t \omega, V(t)), V(t)). \quad (49)$$

*Moreover, in this case the manifold  $\mathbf{M}$  is strictly invariant with respect to the cocycle  $\phi$  generated by (1) and (2).*

**Proof.** The existence of a solution to (47) follows by the Lipschitz continuity of  $\Phi$ . Since  $S^2$  is a group we can solve (48) backwards in time and, hence, one can prove that  $\mathbf{M}$  is strictly invariant with respect to the cocycle  $\phi(t, \omega)$ .  $\square$

Observe now that Theorem 3.1 implies that for any mild solution  $Y$  to problem (1) and (2) with initial data  $Y_0 \in X$ , there exists a mild solution  $V(t)$  to reduced problem (47) such that

$$\|V(t) - QY(t)\|_{X_2}^2 + \|\Phi(\omega, V(t)) - PY(t)\|_{X_1}^2 \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

exponentially fast (in the sense of (24) and (25)). Thus under the conditions of Theorem 3.1, the long-time behaviour of solutions to (1) and (2) can be described completely by solutions to problem (47). Moreover, due to relation (49), every limiting regime of the reduced system (47) is realized in the coupled system (1) and (2).

## 4 Distance between random and deterministic manifolds

Theorem 3.1 can be also applied to the deterministic version of problem (1) and (2):

$$\begin{aligned} U_t + A_1 U &= F_1(U, V), \quad t > 0, \quad \text{in } X_1, \\ V_t + A_2 V &= F_2(U, V), \quad t > 0, \quad \text{in } X_2. \end{aligned} \quad (50)$$

In this case Theorem 3.1 gives us the existence of (deterministic) invariant exponentially attracting manifold  $M^{det}$  of the form

$$M^{det} = \left\{ (\Phi^{det}(V), V) : V \in X_2 \right\} \subset X,$$

where  $\Phi^{det} : X_2 \mapsto X_1$  is a globally Lipschitz mapping.

Our goal in this section is to estimate the mean value distance between the deterministic ( $M^{det}$ ) and random ( $M(\omega)$ ) manifolds.

**Theorem 4.1** *The following estimate holds,*

$$\|\Phi(\omega, V_0) - \Phi^{det}(V_0)\|_{X_1}^2 \leq 2\|\eta_1(0, -\infty, \omega)\|_{X_1}^2 + b_1|\Sigma(0, \cdot, \omega)|_{X_0}^2, \quad (51)$$

where  $b_1 > 0$  is a constant. In particular, in the case of white noises (see Section 2.2.1) there exist a positive constant  $C$  such that

$$\mathbb{E} \left\{ \sup_{V_0 \in X_2} \|\Phi(\cdot, V_0) - \Phi^{det}(V_0)\|_{X_1}^2 \right\} \leq C(\text{tr } K_1 + \text{tr } K_2). \quad (52)$$

Thus, in the latter case the random manifold  $M(\omega)$  is close to its deterministic counterpart when  $\text{tr } K_1 + \text{tr } K_2$  becomes small.

**Proof.** It follows from the definition (see (38)) of the functions  $\Phi$  and  $\Phi^{det}$  that

$$\begin{aligned} & \Phi(\omega, V_0) - \Phi^{det}(V_0) \\ &= \int_{-\infty}^0 S^1(\tau) \left[ F_1(Y^{st}(\tau)) - F_1(Y^{det}(\tau)) \right] d\tau + \eta_1(0, -\infty), \end{aligned}$$

where  $Y^{st}(t)$  and  $Y^{det}(t)$  are defined on the semi-axis  $(-\infty, 0]$  and solve the equations

$$Y^{st}(t) = \mathcal{L}_{V_0}[F(Y^{st}), \omega](t, 0) \text{ and } Y^{det}(t) = \mathcal{L}_{V_0}^{det}[F(Y^{det})](t, 0), \quad (53)$$

where  $\mathcal{L}_{V_0}$  and  $\mathcal{L}_{V_0}^{det}$  are defined as in (27) and (32).

Using the same method as in the proof of relation (46) we can conclude that

$$\|\Phi(\cdot, V_0) - \Phi^{det}(V_0)\|_{X_1} \leq \|\eta_1(0, -\infty)\|_{X_1} + a_1|Y^{st} - Y^{det}|_{x_0} \quad (54)$$

where  $a_1$  is a deterministic constant. By (53) we have that

$$|Y^{st} - Y^{det}|_{x_0} \leq |\mathcal{T}_{V_0}[Y^{st}, \omega](\cdot, 0) - \mathcal{T}_{V_0}[Y^{det}, \omega](\cdot, 0)|_{x_0} + |\Sigma(0, \cdot)|_{x_0},$$

where  $\mathcal{T}_{V_0}[V, \omega](t, 0)$  is the same as in (26) and  $\Sigma(s, t)$  is given by (28). Thus by Proposition 3.3 we have that

$$|Y^{st} - Y^{det}|_{x_0} \leq (1 - q)^{-1}|\Sigma(0, \cdot)|_{x_0},$$

where  $q < 1$ . Therefore, using (54) we obtain the estimate (51).

It easily follows from the definition of  $\Sigma(s, t)$  (see (28)) in the white noise case that

$$\mathbb{E}\|\eta_1(0, -\infty)\|_{X_1}^2 \leq C_1 \cdot \text{tr } K_1 \quad \text{and} \quad \mathbb{E}|\Sigma(0, \cdot)|_{x_0}^2 \leq C_2 (\text{tr } K_1 + \text{tr } K_2). \quad (55)$$

Therefore (52) follows from (51) and (55).  $\square$

**Remark 4.2** If we consider the scaled noises  $\varepsilon N_1$  and  $\varepsilon N_2$  instead of  $N_1$  and  $N_2$ , then (51) implies that

$$\lim_{\varepsilon \rightarrow 0} \|\Phi_\varepsilon(\omega, V_0) - \Phi^{det}(V_0)\|_{X_1} = 0 \quad \text{for every } \omega \in \Omega.$$

## 5 Applications

In this section we consider several applications of the Theorem 3.1.

## 5.1 Coupled parabolic-hyperbolic system

Let  $D$  be a bounded domain in  $\mathbb{R}^d$ ,  $\Gamma \equiv \partial D$  a  $C^1$ -manifold. Let  $\{a_{ij}\}_{i,j=1}^d$  and  $\{b_{ij}\}_{i,j=1}^d$  be symmetric matrices of measurable functions such that

$$\begin{aligned} c_0|\xi|^2 &\leq \sum_{i,j=1}^d a_{ij}(x)\xi_j\xi_i \leq c_1|\xi|^2 \\ c_0|\xi|^2 &\leq \sum_{i,j=1}^d b_{ij}(x)\xi_j\xi_i \leq c_1|\xi|^2, \quad \xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d, \end{aligned}$$

for some positive constants  $c_0, c_1$  and  $x \in D$ . Let  $\Gamma = \overline{\Gamma_0 \cup \Gamma_1}$ , where  $\Gamma_0$  and  $\Gamma_1$  are (relatively) open subsets of  $\Gamma$  such that  $\Gamma_0 \cap \Gamma_1 = \emptyset$ . ( $\Gamma_0 = \emptyset$  or  $\Gamma_0 = \Gamma$  are allowed). Let  $a_0, b_0$  be nonnegative parameters and  $a_0^\Gamma$  is a positive function in  $L_\infty(\Gamma_1)$ . We consider the following coupled system consisting of the parabolic-hyperbolic problem

$$u_t - \sum_{i,j=1}^d \partial_i [a_{ij}(x)\partial_j u] + a_0 u = f_1(u, v, v_t) + \dot{N}_1, \quad (56)$$

$$u = 0 \text{ on } \Gamma_0, \quad \sum_{i,j=1}^d n_i a_{ij} \partial_j u + a_0^\Gamma(x)u = 0 \text{ on } \Gamma_1 \quad (57)$$

$$v_t - \sum_{i,j=1}^d \partial_j [b_{ij}(x)\partial_j v] + b_0 v = f_2(u, v, v_t) + \dot{N}_2, \quad (58)$$

$$v = 0 \text{ on } \Gamma. \quad (59)$$

where  $n = (n_1, \dots, n_d)$  is the outer normal vector of  $\Gamma$ .

The functions

$$f_1 : \mathbb{R}^3 \mapsto \mathbb{R} \quad \text{and} \quad f_2 : \mathbb{R}^3 \mapsto \mathbb{R}$$

possess the properties:

$$|f_1(w) - f_1(w^*)| \leq l_1 |w - w^*|_{\mathbb{R}^3}. \quad (60)$$

and

$$|f_2(w) - f_2(w^*)| \leq l_2 |w - w^*|_{\mathbb{R}^3}. \quad (61)$$

for all  $w, w^* \in \mathbb{R}^3$ .

Let  $A_1$  be a positive self-adjoint operator on  $X_1 = L_2(D)$  generated by the bilinear form

$$(A_1 u, u^*) = \sum_{i,j=1}^d \int_{\Omega} a_{ij} \partial_j u \partial_i u^* dx + a_0 \int_{\Omega} u u^* dx + \int_{\Gamma_1} a_0^{\Gamma} u u^* d\Gamma$$

This operator has a compact inverse and generates a  $C_0$ -semigroup  $S^1(t) = e^{-tA_1}$ . We have that  $\|S^1(t)\|_{X_1} \leq e^{-\lambda_{A_1} t}$  for  $t \geq 0$ , where  $\lambda_{A_1} \equiv \inf \text{spec}(A_1)$ . We note that  $\lambda_{A_1} > 0$  provided either  $\Gamma_0 \neq \emptyset$ , or  $a_0 > 0$ , or  $a_0^{\Gamma} > 0$ .

Let us write (58) as a system

$$\begin{pmatrix} v_{0t} \\ v_{1t} \end{pmatrix} + \begin{pmatrix} 0 & -I \\ B & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} + \begin{pmatrix} 0 \\ f_2(u, v_0, v_1) \end{pmatrix} = \begin{pmatrix} 0 \\ \dot{N}_2 \end{pmatrix}, \quad (62)$$

where  $B$  is a positive self-adjoint operator defined by

$$Bv = - \sum_{i,j=1}^d \partial_j [b_{ij}(x) \partial_j v] + b_0 v, \quad v \in \mathcal{D}(B) \equiv H^2(D) \cap H_0^1(D).$$

We set  $\lambda_B \equiv \inf \text{spec} B > 0$ . We denote by  $A_2$  the generator of a unitary  $C_0$ - (semi)group corresponded to the linear part of (62) on  $X_2 = H_0^1(D) \times L_2(D)$ . We equip the space  $X_2$  with the energy type norm

$$\|(w, w_1)\|_{X_2}^2 = \int_D (|B^{1/2} w_0|^2 + |w_1|^2) dx, \quad V = (v_0, v_1). \quad (63)$$

We have  $\|S^2(t)\|_{X_2} \leq e^{\gamma_2 t}$  for  $t \in \mathbb{R}$  with  $\gamma_2 \equiv 0$ . We also can define

$$F_1(U, V)[x] = f_1(u(x), v(x), v_t(x)), \quad U = u(\cdot), \quad V = (v(\cdot), v_t(\cdot)),$$

and

$$F_2(U, V)[x] = (0, f_2(u(x), v(x), v_t(x))), \quad U = u(\cdot), \quad V = (v(\cdot), v_t(\cdot)),$$

for  $x \in D$  giving us Lipschitz continuous operators from  $X = X_1 \times X_2$  into  $X_i$  for  $i = 1, 2$  resp. By the particular form of the noise in (62) it is appropriate to consider  $\eta_2 \in L_2(D) = X_1$  such that with respect to Subsection 2.2.2 we can set  $X_1 = H_0$ .

Under all these assumption we have **(A1)**, **(A2)**, **(R1)**– **(R6)**. Thus we obtain that the mild solution of (56)–(59) generates a random dynamical system. In addition, we have

$$M_1 = M_2 = 1, \quad \gamma_1 = \lambda_{A_1} \geq a_0, \quad \gamma_2 = 0$$

and also for the Lipschitz constants of  $F_1, F_2$ :

$$L_1 = l_1 \max \left\{ 1, \frac{1}{\sqrt{\lambda_B}} \right\}, \quad L_2 = l_2 \max \left\{ 1, \frac{1}{\sqrt{\lambda_B}} \right\}.$$

Thus under the condition

$$\lambda_{A_1} > \left( \sqrt{l_1} + \sqrt{l_2} \right)^2 \max \left\{ 1, \frac{1}{\sqrt{\lambda_B}} \right\}$$

(58) synchronizes (56) by Theorem 3.1.

**Remark 5.1** (i) We note that it is not important that  $D$  is a bounded domain and Dirichlet boundary conditions for  $v$  hold. The only facts which we use in the proof are (i)  $B$  is a self adjoint operator with  $\inf \text{spec}(B) > 0$ , and (ii)  $A_1$  generates exponentially stable  $C_0$ -semigroup. Thus we can consider unbounded domains and equip the corresponding differential operation with other (self-adjoint) boundary conditions. We will use this observation in our subsequent applications.

(ii) Coupled models like (56) and (58) arises in the study of wave phenomena which are heat generating or temperature related (see, e.g., [16, 26] for the deterministic case and [7] for a stochastic thermoelastic problem and the references therein).

## 5.2 Coupled parabolic PDE and ODE systems

Let  $f_i : \mathbb{R}^{1+m} \mapsto \mathbb{R}$ ,  $i = 1, 2$ , be a globally Lipschitz functions:

$$|f_i(w) - f_i(w^*)| \leq l_i |w - w^*|_{\mathbb{R}^{1+m}} \quad \text{for all } w, w^* \in \mathbb{R}^{1+m}.$$

In a bounded domain  $D \subset \mathbb{R}^d$  we consider the following parabolic equation

$$u_t - \Delta u + f_1(u, v) = \dot{N}_1, \quad u|_{\partial D} = 0, \quad (64)$$

coupled with the ordinary differential equation in  $\mathbb{R}^m$ :

$$v_t + f_2(u, v) = \dot{N}_2. \quad (65)$$

In (65)  $t \mapsto v(t, \cdot)$  is a function with values in  $[L^2(D)]^m$  which satisfies ODE with respect to  $t$  (the variable  $x$  is present as a parameter). So for the fixed  $u \in L^2(D)$  (and  $x \in D$ ) we can solve (65) as an equation in  $[L^2(D)]^m$ . This kind of coupled problems arises in biology. For instance, the well-known

Hodgkin–Huxley system belongs to this class (see, e.g., [13, 23] and the references therein).

The problem (64) and (65) can be embedded in our framework with the spaces  $X_1 = [L^2(D)]^m$  and  $X_2 = [L^2(D)]^m$  and operators  $A_1 = -\Delta$  in  $X_1$  with the domain  $(H^2 \cap H_0^1)(D)$  and  $A_2 \equiv 0$  in  $X_2$ . The semigroup  $S^1$  generated by  $A_1$  is the same as in the previous example and  $S^2 \equiv id$ . It is clear that the dichotomy properties holds with  $\gamma_1 = \lambda_{A_1} \equiv \inf \text{spec}(A_1) > 0$  and  $\gamma_2 = 0$ . We also have that  $M_1 = M_2 = 1$ . Thus under the condition

$$\lambda_{A_1} > \left( \sqrt{l_1} + \sqrt{l_2} \right)^2$$

we observe the master–slave synchronization phenomenon.

### 5.3 Two coupled hyperbolic systems

In a smooth domain  $D \subseteq \mathbb{R}^d$  we consider two coupled wave equations for scalar functions  $u$  and  $v$ :

$$\begin{aligned} u_{tt} + \nu u_t - \sum_{i,j=1}^d \partial_i [a_{ij}(x) \partial_j u] + a_0 u + f_1(u, v, v_t) &= \dot{N}_1, \\ u &= 0 \quad \text{on } \Gamma \\ v_{tt} - \sum_{i,j=1}^d \partial_j [b_{ij}(x) \partial_j v] + b_0 v + f_2(u, v, v_t) &= \dot{N}_2, \\ v &= 0 \quad \text{on } \Gamma \end{aligned} \tag{66}$$

In the same way as in Subsection 5.1 the linear part of the second equation generates a unitary  $C_0$ –group  $S^2$  on  $X_2 = H_0^1(D) \times L_2(D)$  with norm (63). Let us rewrite the first equation of (66) as

$$\begin{pmatrix} u_{0t} \\ u_{1t} \end{pmatrix} + \begin{pmatrix} 0 & -I \\ A & \nu \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} + \begin{pmatrix} 0 \\ f_1(u_0, v_0, v_1) \end{pmatrix} = \begin{pmatrix} 0 \\ \dot{N}_1 \end{pmatrix}, \tag{67}$$

where  $A$  is a positive self-adjoint operator defined by

$$Av = - \sum_{i,j=1}^d \partial_j [a_{ij}(x) \partial_j v] + a_0 v, \quad v \in \mathcal{D}(A) \equiv H^2(D) \cap H_0^1(D).$$

Then the linear part of equation (67) generates a  $C_0$ –semigroup  $S^1$  on the phase space

$$X_1 = D(A^{1/2}) \times L_2(D), \quad \|U\|_{X_1}^2 = \|A^{1/2} u_0\|^2 + \|u_1 + 2\gamma u_0\|^2, \tag{68}$$

where  $U = (u_0, u_1)$  and  $\|\cdot\|$  is the norm of  $L_2(D)$ . The parameter  $\gamma > 0$  will be chosen below. This choice of the norm is motivated by the following assertion.

**Lemma 5.2** *Let  $U(t) = (u(t), u_t(t))$  be a solution to*

$$u_{tt} + \nu u_t + Au = 0, \quad u = 0 \quad \text{on } \Gamma.$$

*If we choose  $\gamma \equiv \min\left\{\frac{\nu}{8}, \frac{\lambda_A}{4\nu}\right\}$  in definition (68) of the norm in  $X_1$ , then*

$$\|U(t)\|_{X_1} \leq e^{-\gamma t} \|U(0)\|_{X_1} \quad \text{for every } t > 0. \quad (69)$$

**Proof.** We refer to Proposition 1.2 in [37, Chap.IV]. □

Let the Lipschitz continuous mappings  $F_1, F_2$  on  $X_1 \times X_2$  be defined by mappings  $f_1, f_2$  given by (60), (61) with Lipschitz constants  $l_1, l_2$ .

Obviously we have that

$$M_1, M_2 \equiv 1, \quad \gamma_2 \equiv 0, \quad \gamma_1 \equiv \gamma.$$

Simple calculations shows that

$$L_1 = l_1 \max\left\{1, \frac{1}{\sqrt{\lambda_A}}, \frac{1}{\sqrt{\lambda_B}}\right\}, \quad L_2 = l_2 \max\left\{1, \frac{1}{\sqrt{\lambda_A}}, \frac{1}{\sqrt{\lambda_B}}\right\}.$$

Thus under the condition

$$\min\left\{\frac{\nu}{8}, \frac{\lambda_A}{4\nu}\right\} > \left(\sqrt{l_1} + \sqrt{l_2}\right)^2 \max\left\{1, \frac{1}{\sqrt{\lambda_A}}, \frac{1}{\sqrt{\lambda_B}}\right\}$$

there exists an exponentially attracting invariant manifold. In particular, by Theorem 3.1 the equation (67) synchronizes the dynamics governed by the first equation in (66).

We note that we can also include in  $f_i$  dependence on  $u_t$  and obtain conditions for synchronization. However in this case, due to the structure of the norm (68) the calculation of  $L_1$  and  $L_2$  is not so direct and this Lipschitz constants may depend on  $\gamma$ . We do not give these calculations here.

## 5.4 Coupled Klein-Gordon-Schrödinger system

The following coupled model arises in quantum physics (see, e.g., [6] and the references therein):

$$u_{tt} + \nu u_t - \Delta u + m^2 u + f_1(u, v) = \dot{N}_1 \quad \text{in } \mathbb{R}^d, \quad (70a)$$

$$iv_t + \Delta v + f_2(u, v) = \dot{N}_2 \quad \text{in } \mathbb{R}^d, \quad (70b)$$

where  $\nu, m > 0$ . Here  $u$  is real and  $v$  is complex functions. In contrast with previous examples here we concentrate on the case when  $D = \mathbb{R}^d$ . In the case when  $D$  is a domain in  $\mathbb{R}^d$  we need to impose some (self-adjoint) boundary conditions.

We assume that the functions

$$f_1 : \mathbb{R} \times \mathbb{C} \mapsto \mathbb{R}, \quad \text{and} \quad f_2 : \mathbb{R} \times \mathbb{C} \mapsto \mathbb{C}$$

are globally Lipschitz, i.e.,

$$\begin{aligned} |f_1(w) - f_1(w^*)| &\leq l_1 |w - w^*|_{\mathbb{R} \times \mathbb{C}}, \\ |f_2(w) - f_2(w^*)| &\leq l_2 |w - w^*|_{\mathbb{R} \times \mathbb{C}}, \quad w, w^* \in \mathbb{R} \times \mathbb{C}. \end{aligned}$$

To apply Theorem 3.1 we rewrite (70) as (1) and (2) with  $U = (u(\cdot), u_t(\cdot))$  and  $V = v(\cdot)$ , The corresponding phase spaces

$$X_1 = H^1(\mathbb{R}^d) \times L_2(\mathbb{R}^d), \quad X_2 = L_2^{\mathbb{C}}(\mathbb{R}^d),$$

where  $L_2^{\mathbb{C}}(\mathbb{R}^d)$  is the space of square integrable *complex* functions.

We consider in  $L_2(\mathbb{R}^d)$  the operator  $A = -\Delta + m^2$  with the domain  $\mathcal{D}(A) = H^2(\mathbb{R}^d)$ . It is clear from the Fourier analysis that  $A$  is a positive self-adjoint operator with  $\lambda_A \equiv \inf \text{spec}(A) = m^2$ . We equip  $X_1$  with the norm given in (68) with this operator  $A$ . Thus by Lemma 5.2 the linear part of (70a) generates  $C_0$ -semigroup  $S^1$  for which we have  $M_1 = 1$  and  $\gamma = \min \left\{ \frac{\nu}{8}, \frac{m^2}{4\nu} \right\}$ . Since the linear part of (70b) generates the unitary group (this follows from the Fourier analysis again), we also have that  $M_2 = 1$  and  $\gamma_2 = 0$ . A calculation as in the previous examples gives us that

$$L_1 = l_1 \max \left\{ 1, \frac{1}{m} \right\}, \quad L_2 = l_2 \max \left\{ 1, \frac{1}{m} \right\}.$$

Thus under the condition

$$\min \left\{ \frac{\nu}{2}, \frac{m^2}{\nu} \right\} \geq 4 \left( \sqrt{l_1} + \sqrt{l_2} \right)^2 \max \{ 1, m^{-1} \}$$

system (70b) synchronizes (70a).

**Remark 5.3** In conclusion we note that we are not able to apply our main result (Theorem 3.1) in the case of two coupled parabolic equation. The main reason is that the backward time estimate in (4) cannot be obtained in the case when the master equation is parabolic. In the purely parabolic case the approach to synchronization relies on the construction of appropriate Lyapunov type functions [10, 33]. Our approach is alternative in some sense and covers another, in comparison with [10, 33], kind of problems.

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