Of Copulas, Quantiles, Ranks and Spectra an L_1-approach to spectral analysis

Holger Dette[a*], Marc Hallin[b*†], Tobias Kley[a*‡] and Stanislav Volgushev[a*]

[a] Ruhr-Universität Bochum

[b] ECARES, Université Libre de Bruxelles, and ORFE, Princeton University

Abstract

In this paper we present an alternative method for the spectral analysis of a strictly stationary time series $\{Y_t\}_{t\in\mathbb{Z}}$. We define a "new" spectrum as the Fourier transform of the differences between copulas of the pairs (Y_t, Y_{t-k}) and the independence copula. This object is called *copula spectral density kernel* and allows to separate marginal and serial aspects of a time series. We show that it is intrinsically related to the concept of quantile regression. Like in quantile regression, which provides more information about the conditional distribution than the classical location-scale model, the copula spectral density kernel is more informative than the spectral density obtained from the autocovariances. In particular the approach provides a complete description of the distributions of all pairs (Y_t, Y_{t-k}). Moreover, it inherits the robustness properties of classical quantile regression, because it does not require any distributional assumptions such as the existence of finite moments. In order to estimate the copula spectral density kernel we introduce rank-based Laplace periodograms which are calculated as bilinear forms of weighted L_1-projections of the ranks of the observed time series onto a harmonic regression model. We establish the asymptotic distribution of those periodograms, and the consistency of adequately smoothed versions. The finite-sample properties of the new methodology, and its potential for applications are briefly investigated by simulations and a short empirical example.

AMS 1980 subject classification : 62M15, 62G35.

Key words and phrases : Time series, Spectral analysis, Periodogram, Quantile regression, Copulas, Ranks, Time reversibility .

1 Introduction.

1.1 The location-scale paradigm.

Whether linear or not, most traditional time series models are of the conditional location/scale type: conditional on past values $Y_{t-1}, Y_{t-2}, \ldots$, the random variable Y_t satisfies an equation of the form

$$Y_t = \psi(Y_{t-1}, Y_{t-2}, \dots) + \sigma(Y_{t-1}, Y_{t-2}, \dots)\varepsilon_t \qquad t \in \mathbb{Z}, \tag{1.1}$$

*Supported by the Sonderforschungsbereich "Statistical modelling of nonlinear dynamic processes" (SFB 823) of the Deutsche Forschungsgemeinschaft.

†Académie Royale de Belgique, CentER, Tilburg University, and ECORE. Supported by a Discovery Grant of the Australian Research Council.

‡Supported by a PhD Grant of the Ruhr-Universität Bochum and by the Ruhr-University Research School funded by Germany's Excellence Initiative [DFG GSC 98/1].

where $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is white noise, and ε_t is independent of $Y_{t-1}, Y_{t-2}, \dots$. The $(Y_{t-1}, Y_{t-2}, \dots)$-measurable functions ψ and σ are (conditional) location and scale functions, possibly parametrized by some $\boldsymbol{\vartheta}$. Equation (1.1) may characterize a data-generating process – in which case "=" is to be considered as "almost sure equality" – or, more generally, it simply describes Y_t's conditional (on $Y_{t-1}, Y_{t-2}, \dots$) distribution – and "=" is to be interpreted as "equality in (conditional) distribution". Such distinction is, however, irrelevant from a statistical point of view, as it has no impact on likelihoods.

In model (1.1), the distribution of Y_t conditional on $Y_{t-1}, Y_{t-2}, \dots$ is nothing but the distribution of ε_t, rescaled by the conditional scale parameter $\sigma(Y_{t-1}, Y_{t-2}, \dots)$ and shifted by the conditional location parameter $\psi(Y_{t-1}, Y_{t-2}, \dots)$. Sophisticated as they may be, the mappings

$$(Y_{t-1}, Y_{t-2}, \dots) \mapsto (\psi(Y_{t-1}, Y_{t-2}, \dots),\ \sigma(Y_{t-1}, Y_{t-2}, \dots))$$

only can account for a very limited type of the dynamics of the process $\{Y_t\}_{t\in\mathbb{Z}}$. The dynamics of volatility provided by this model are quite poor, being of a pure rescaling nature. In particular no impact of past values on skewness, kurtosis, tails, can be reflected. All standardized conditional distributions strictly coincide with that of ε, and all conditional τ-quantiles, hence all values at risk, follow, irrespective of τ, from those of ε via one single linear transformation.

Note that the interpretation of ψ and σ depends on the identification constraints on ε: if ε is assumed to have mean zero and variance one, then ψ and σ are a conditional mean and a conditional standard error, respectively. In this case models of the form (1.1) clearly belong to the L_2-Gaussian legacy. If ε is assumed to have median zero and expected absolute deviation or median absolute deviation one, ψ and σ are a conditional median and a conditional expected or median absolute deviation.

On the basis of these "remarks", the following questions naturally arise: Can we do better? Can we go beyond that (conditional) "location-scale paradigm"? Can we model richer dynamics under which the conditional quantiles of Y are not just a shifted and rescaled version of those of ε and under which the whole conditional distribution of Y_t, not just its location and scale, can be affected by the past? And, can we achieve this in a statistically tractable way?

1.2 Marginal and serial features.

Another drawback of models of the form (1.1) is their sensitivity to nonlinear marginal transformations. If two statisticians observe the same time series, but measure it on different scales, Y_t and Y_t^3 or e^{Y_t}, for instance, and both adjust a model of the form (1.1) to their measurements, they will end up with drastically different analyses and predictions. The only way to avoid this problem consists in disentangling the marginal (viz., related to the scale of measurement) aspects of the series under study from its serial aspects, that is, basing the description of serial dependence features on quantities such as the $F_Y(Y_t)$'s, where F_Y is Y_t's marginal distribution function. Those quantities do not depend on the measurement scale since they are invariant under continuous monotone increasing transformations.

This point of view is closely related to the concept of copulas (see Nelsen [2006] or Genest and Favre [2007]). Consider, for instance, a strictly stationary Markovian process $\{Y_t\}_{t\in\mathbb{Z}}$ of order one. This process is fully characterized by the joint distribution of (Y_t, Y_{t-1}) or, equivalently, by the marginal distribution function F_Y (or the quantile function F_Y^{-1}) of Y_t, along with the joint distribution of $(U_t, U_{t-1}) := (F_Y(Y_t),\ F_Y(Y_{t-1}))$, a "serial copula of order one". In such a description, the marginal features of the process $\{Y_t\}_{t\in\mathbb{Z}}$ are entirely described by F_Y, independently of the serial features, that are accounted for by the serial copula. Two statisticians observing the same phenomenon but recording Y_t and e^{Y_t}, respectively, would use distinct quantile functions, but they would agree on serial features.

In more general cases, serial copulas of order one are not sufficient, and higher-order or multiple copulas may be needed. Note that the description of the model in this context is clearly "in distribution": U_t is not related to U_{t-1} through any direct interpretable "almost sure relation" reflecting some "physical" data-generating mechanism.

1.3 A new nonparametric approach.

The objective of this paper is to show how to overcome the limitations of conditional location-scale modelling described in Sections 1.1 and 1.2, and to provide statistical tools for a new approach to time series modelling. Not surprisingly, those tools are essentially related to copulas, quantiles and ranks. The traditional nonparametric techniques, such as spectral analysis (in its usual L_2-form), which only account for second-order serial features, cannot handle such objects, and we therefore propose and develop an original, flexible and fully nonparametric L_1-spectral analysis method.

While classical spectral densities are obtained as Fourier transforms of classical covariance functions, we rather define spectral density *kernels*, associated with covariance *kernels* of the form

$$\gamma_k(x_1, x_2) := \text{Cov}(I\{Y_t \leq x_1\}, I\{Y_{t-k} \leq x_2\}) \tag{1.2}$$

(Laplace cross-covariance kernels) or

$$\gamma_k^U(\tau_1, \tau_2) := \text{Cov}(I\{U_t \leq \tau_1\}, I\{U_{t-k} \leq \tau_2\}) \tag{1.3}$$

(copula cross-covariance kernels), where $U_t := F_Y(Y_t)$ and F_Y denotes the marginal distribution of the strictly stationary process $\{Y_t\}_{t\in\mathbb{Z}}$. Contrary to covariance functions, the *kernels* $\{\gamma_k(x_1, x_2)|x_1, x_2 \in \mathbb{R}\}$ and $\{\gamma_k^U(\tau_1, \tau_2)|\tau_1, \tau_2 \in [0, 1]\}$ allow for a complete description of arbitrary bivariate distributions for the couples (Y_t, Y_{t-k}) and arbitrary bivariate copulas of the pairs (U_t, U_{t-k}), respectively, and thus escape the conditional location-scale paradigm discussed in Section 1.1. They are able to account for sophisticated dependence features that covariance-based methods are unable to detect, such as time-irreversibility, tail dependence, varying conditional skewness or kurtosis, etc. And, in view of the desired separation between marginal and serial features expressed in Section 1.2, special virtues, such as invariance/equivariance (with respect to continuous order-preserving marginal transformations), can be expected from the copula covariance kernels defined in (1.3).

Classical nonparametric spectral-based inference methods have proven quite effective [Granger, 1964, Bloomfield, 1976], essentially in a Gaussian context, where dependencies are fully characterized by autocovariance functions. Therefore, it can be anticipated that similar methods, based on estimated versions of Laplace or copula spectral kernels (associated with Laplace and copula covariance kernels, respectively) would be quite useful in the study of series exhibiting those features that classical covariance-related spectra cannot account for.

Estimation of Laplace and copula spectral kernels, however, requires a substitute for the *ordinary periodogram* concept considered in the classical approach. We therefore introduce Laplace and copula *periodogram kernels*. While ordinary periodograms are defined via least squares regression of the observations on the sines and cosines of the harmonic basis, our periodogram kernels are obtained via quantile regression in the Koenker and Bassett [1978] sense. A study of their asymptotic properties shows that, just as ordinary periodograms, they produce asymptotically unbiased estimates (more precisely, the mean of their asymptotic distribution is 2π times the corresponding spectrum), and we therefore also consider smoothed versions that yield consistency. Asymptotic results show that copula periodograms, as anticipated, are preferable to the Laplace ones, as their asymptotic behavior only depends on the bivariate copulas of the pairs (U_t, U_{t-k}), not on the (in general unknown) marginal distribution F_Y of the Y_t's.

Unfortunately, copula periodogram kernels are not statistics, since their definition involves the transformation of Y_t into U_t, hence the knowledge of the marginal distribution function F_Y. We therefore introduce a third periodogram kernel, based on the empirical version of F_Y, that is, on the *ranks* of the random variables $Y_1, \ldots, Y_n$, and establish, under mild assumptions, the asymptotic equivalence of that rank-based Laplace periodogram with the copula one. Smoothed rank-based Laplace periodogram kernels, accordingly, seem to be the adequate tools in this context. We conclude with a brief numerical illustration – simulations and an empirical application – of their potential use in practical problems.

1.4 Review of related literature

Quantities of the form (1.2) and (1.3) arise naturally when the clipped series $(I\{Y_t \leq x\})_{t\in\mathbb{Z}}$ and $(I\{U_t \leq \tau\})_{t\in\mathbb{Z}}$ are investigated. Processes of this type have been considered earlier in the literature (see, for instance, Kedem [1980]). In the field of signal processing the idea to replace the quadratic by other loss functions has been discussed by Katkovnik [1998] who proposed to use L_p-distances and analyzed the properties of these *M-periodograms*. Hong [2000] used the Laplace covariances corresponding to positive lags to construct a test for serial dependence. Linton and Whang [2007] considered sequences of Laplace cross-correlations $\gamma_k(\tau,\tau)/\gamma_0(\tau,\tau)$ (called *quantilogram* by these authors) in order to test for directional predictability. Li [2008] suggested least absolute deviation estimators in a harmonic regression model assuming that the median of the random variables Y_t vanishes. The focus of this author was on the quantity

$$f_{0,0}(\omega) = \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \gamma_k(0,0) \exp(\mathrm{i}k\omega),$$

which he called Laplace spectrum. He constructed an asymptotically unbiased estimator for a quantity which differs from $f_{0,0}(\omega_j)$ by a factor involving $1/(F_Y'(0))^2$ (here ω_j denotes the jth Fourier frequency) and extended these results to arbitrary quantiles (see Li [2011]). An important drawback of this method consists in the fact that it requires estimates of the quantity $F_Y'(0)$ in order to obtain an estimate of the Laplace spectrum. Moreover, Li [2008, 2011] does not prove consistency of a smoothed version of his estimates. Very recently, Hagemann [2011] proposed an alternative method to estimate the Laplace spectrum (called *quantile spectrum* by this author), which is based on an estimate of a linearization of Li [2008]'s statistic. This approach does not suffer from the drawbacks of Li [2008]'s method and yields consistent estimates avoiding estimation of the marginal density. On the other hand this method does not allow a direct interpretation in terms of (weighted) absolute deviation estimates.
In order to obtain a complete description of the two-dimensional distributions at lag k, Hong [1999] introduced the *generalized spectrum* defined as the covariance $\mathrm{Cov}(e^{\mathrm{i}x_1 Y_t}, e^{\mathrm{i}x_2 Y_{t+k}})$, and this approach was used by Chung and Hong [2007] to test for directional pedictability. Recently, Lee and Rao [2011] considered a Fourier transform of the differences between the joint density of the pairs (Y_t, Y_{t-k}) and the product of the two marginal densities to investigate serial dependence.

Unlike ours, these methods are not invariant with respect to transformations of the marginal distributions.

Finally, there exist some recent proposals to use pair-copula constructions to describe dependencies in the time-domain. Domma et al. [2009] assumed a first-order Markov process, such that only distributions of pairs (Y_t, Y_{t+1}) at lag $k = 1$ need to be considered. Smith et al. [2010] decomposed the distribution at a point in time, conditional upon the past, into the product of a sequence of bivariate copula densities and the marginal density, known as D-vine by Bedford and Cooke [2002].

The approach presented in this paper differs from these references in many important aspects. Essentially, it combines their attractive features while avoiding some of their drawbacks. It shares the quantile-based flavor of Kedem [1980], Linton and Whang [2007], Li [2008, 2011] and Hagemann [2011]. In contrast to the latter, however, we do not focus on a particular quantile, and consider copula *cross*-covariances $\gamma_k^U(\tau_1, \tau_2)$ for *all* (not necessarily equal) values of τ_1, τ_2, while Li [2008, 2011] and Hagemann [2011] restrict to the case $\tau_1 = \tau_2$. As a consequence, we obtain, as in the characteristic function approach of Hong [1999], a complete characterization of the dependencies among the pairs (Y_t, Y_{t-k}). This allows to address such important features as time reversibility [see Proposition 2.1] or tail dependence in general. By replacing the original observations with their ranks, we furthermore achieve an attractive invariance property with respect to modifications of marginal distributions, which is not satisfied in the case of Hong [1999]'s method. Moreover, we also avoid the scaling problem of Li's estimates and establish the consistency of a smoothed version of periodograms. Finally, because our method is related to the concept of copulas it allows to separate marginal and serial aspects of a time series, which makes it attractive for practitioners.

1.5 Outline of the paper.

The paper is organized as follows. In Section 2.1, we introduce the concepts of *Laplace* and *copula cross-covariance kernels* which, in this quantile-based approach, are to replace the ordinary autocovariance function. The corresponding spectra and periodograms are introduced in Sections 2.2 and 2.3, respectively. Section 3 deals with the asymptotic properties of the Laplace, copula, and rank-based Laplace periodograms. In Section 4, smoothed periodograms are considered, and the smoothed rank-based Laplace periodogram kernel is shown to be a consistent estimator of the copula spectral density. Some numerical illustration is provided in Section 5, and most of the technical details are concentrated in an appendix.

2 An L_1-approach to spectral analysis.

2.1 The Laplace and copula cross-covariance kernels.

Covariances clearly are not sufficient for describing a serial copula. We therefore introduce the following concept, which will be convenient for that purpose. Let $\{Y_t\}_{t\in\mathbb{Z}}$ be a strictly stationary process and define the *copula cross-covariance kernel* of lag $k \in \mathbb{Z}$ of $\{Y_t\}_{t\in\mathbb{Z}}$ as

$$\gamma_k^U := \left\{\gamma_k^U(\tau_1, \tau_2) \mid (\tau_1, \tau_2) \in (0,1)^2\right\}$$

where $\gamma_k^U(\tau_1, \tau_2)$ is defined in (1.3). Similarly, define the *Laplace cross-covariance kernel* of lag $k \in \mathbb{Z}$ of $\{Y_t\}_{t\in\mathbb{Z}}$ as

$$\gamma_k := \Big\{\gamma_k(x_1, x_2) \mid (x_1, x_2) \in \mathbb{R}^2\Big\},$$

where $\gamma_k(x_1, x_2)$ is defined in (1.2). Contrary to traditional cross-covariances, copula and Laplace cross-covariance kernels exist for all k (no finite variance assumption needed). The words "covariance" and "cross-covariance" are used out of time series classical terminology; but we only consider covariances of indicators, which provide a canonical description of their joint distributions. The copula cross-covariance kernel of order k indeed entirely characterizes the joint distribution of (U_t, U_{t-k}), and conversely, without requiring any information on the distribution function F_Y of Y_t. Along with F_Y, the copula cross-covariance kernel of order k entirely characterizes the Laplace cross-covariance kernel of order k and the joint distribution of (Y_t, Y_{t-k}), and conversely. If $\int x^2 \mathrm{d}F_Y < \infty$, the distribution function F_Y of Y_t and the collection of copula cross-covariance kernels of all orders jointly characterize the autocovariance function of $\{Y_t\}_{t\in\mathbb{Z}}$.

2.2 The Laplace and copula spectral density kernels.

Assume that the Laplace cross-covariance kernels γ_k (equivalently, the copula cross-covariance kernels γ_k^U), $k \in \mathbb{Z}$ are absolutely summable, that is, they satisfy $\sum_{k=-\infty}^{\infty} |\gamma_k(x_1, x_2)| < \infty$ for all $(x_1, x_2) \in \mathbb{R}^2$. Then, γ_k admits the representation

$$\gamma_k(x_1, x_2) = \int_{-\pi}^{\pi} e^{\mathrm{i}k\omega} \mathfrak{f}_{x_1,x_2}(\omega) \mathrm{d}\omega, \quad (x_1, x_2) \in \mathbb{R}^2$$

with

$$\mathfrak{f}_{x_1,x_2}(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_k(x_1, x_2) e^{-\mathrm{i}k\omega}, \quad (x_1, x_2) \in \mathbb{R}^2. \tag{2.1}$$

The collection $\{\omega \mapsto \mathfrak{f}_{x_1,x_2}(\omega) | (x_1, x_2) \in \mathbb{R}^2\}$, called the *Laplace spectral density kernel*, is such that each mapping $\omega \in (-\pi, \pi] \mapsto \mathfrak{f}_{x_1,x_2}(\omega)$, $(x_1, x_2) \in \mathbb{R}^2$, is continuous and satisfies (writing $\bar{z}$ for the complex conjugate of $z \in \mathbb{C}$)

$$\mathfrak{f}_{x_1,x_2}(-\omega) = \mathfrak{f}_{x_2,x_1}(\omega) = \overline{\mathfrak{f}_{x_1,x_2}(\omega)}. \tag{2.2}$$

Similarly define the *copula spectral density kernel* as

$$\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_k^U(\tau_1, \tau_2) e^{-\mathrm{i}k\omega}, \quad (\tau_1, \tau_2) \in (0, 1)^2. \tag{2.3}$$

where $q_{\tau_i} := F_Y^{-1}(\tau_i)$ $(i = 1, 2)$. Note that $\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}$ is the Fourier transform of the differences between copulas of the pairs (Y_t, Y_{t-k}) and the independence copula. Clearly, the same identity (2.2) holds for $\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega)$ as for $\mathfrak{f}_{x_1,x_2}(\omega)$.

Throughout this paper we denote by $\stackrel{d}{=}$ equality in distribution and define $\Im z$ and $\Re z$ as the imaginary and real part of $z \in \mathbb{C}$, respectively. Obviously we have $\Im \mathfrak{f}_{x_1,x_2}(\omega) = 0$ for all ω if and only if $\gamma_k(x_1, x_2) = \gamma_{-k}(x_1, x_2)$ for all k and we obtain the following result.

Proposition 2.1 The following statements are equivalent:

(1) $(Y_t, Y_{t+k}) \stackrel{d}{=} (Y_t, Y_{t-k})$ for all k (time-reversibility).

(2) $\Im \mathfrak{f}_{x_1,x_2}(\omega) = 0$ for all ω and x_1, x_2 .

(3) $\Im \mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega) = 0$ for all ω and τ_1, τ_2 .

2.3 The Laplace, copula, and rank-based Laplace periodogram kernels.

The Laplace (resp., the copula) cross-covariance kernels describe the behaviour of Y_t (resp., of $U_t = F_Y(Y_t)$) via a characterization of their conditional quantiles.

If a quantile-based approach is to be considered, it seems intuitively reasonable that the traditional L_2-tools, which are closely related with the concepts of mean and variance, be abandoned in favor of quantile-specific ones. In particular, traditional L_2-projections should be replaced with (weighted) L_1-ones. Recall that, in traditional spectral analysis, estimation is usually based on the *ordinary periodogram*

$$I_n(\omega_{j,n}) := \frac{1}{n}\Big|\sum_{t=1}^{n} Y_t e^{\mathrm{i}t\omega_{j,n}}\Big|^2,$$

where $\omega_{j,n} = 2\pi j/n \in \mathcal{F}_n := \{2\pi j/n |\ j = \frac{1}{2}, \ldots, \lfloor\frac{n-1}{2}\rfloor, \lfloor\frac{n}{2}\rfloor\}$ denote the positive *Fourier frequencies*. A straightforward calculation shows that this can be expressed as

$$I_n(\omega_{j,n}) = \frac{n}{4}\|\hat{\mathbf{b}}_{n,\mathrm{OLS}}(\omega_{j,n})\|^2 := \frac{n}{4}\hat{\mathbf{b}}'_{n,\mathrm{OLS}}(\omega_{j,n}) \begin{pmatrix} 1 & -\mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} \hat{\mathbf{b}}_{n,\mathrm{OLS}}(\omega_{j,n}),$$

where

$$(\hat{a}_{n,\mathrm{OLS}}(\omega_{j,n}), \hat{\mathbf{b}}'_{n,\mathrm{OLS}}(\omega_{j,n})) := \mathrm{Argmin}_{(a,\mathbf{b}')\in\mathbb{R}^3} \sum_{t=1}^{n} \big(Y_t - (a, \mathbf{b}')\mathbf{c}_t(\omega_{j,n})\big)^2 \tag{2.4}$$

is the ordinary least squares estimator in the linear model with regression constants $\mathbf{c}_t(\omega_{j,n}) := (1, \cos(t\omega_{j,n}), \sin(t\omega_{j,n}))'$, corresponding to an L_2-projection of the observed series onto the harmonic basis.

If, instead of a representation of Y_t itself, we are interested in a representation, in terms of the harmonic basis, of Y_t's quantile of order τ, it may seem natural to replace that ordinary periodogram $I_n(\omega_{j,n})$ with

$$\hat{L}_{n,\tau}(\omega_{j,n}) := \frac{n}{4}\|\hat{\mathbf{b}}_{n,\tau}(\omega_{j,n})\|^2 := \frac{n}{4}\hat{\mathbf{b}}'_{n,\tau}(\omega_{j,n}) \begin{pmatrix} 1 & -\mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} \hat{\mathbf{b}}_{n,\tau}(\omega_{j,n}),$$

where

$$(\hat{a}_{n,\tau}(\omega_{j,n}), \hat{\mathbf{b}}_{n,\tau}(\omega_{j,n})) := \mathrm{Argmin}_{(a,\mathbf{b}')\in\mathbb{R}^3} \sum_{t=1}^{n} \rho_\tau\left(Y_t - (a, \mathbf{b}')\mathbf{c}_t(\omega_{j,n})\right), \tag{2.5}$$

and

$$\rho_\tau(x) := x(\tau - I\{x \le 0\}) = (1-\tau)|x|I\{x \le 0\} + \tau|x|I\{x > 0\}, \quad \tau \in (0,1),$$

is the so-called *check function* (see Koenker [2005]). In definition (2.5), the L_2-loss function, which yields the classical definition (2.4), is thus replaced by Koenker and Bassett's weighted L_1-loss which produces quantile regression estimates [Koenker and Bassett, 1978]. That this indeed is a sensible definition will follow from the asymptotic results of Section 3.

This L_1-approach has been taken by Li [2008] for the particular case $\tau = 1/2$, leading to a least absolute deviations (LAD) regression coefficient $\hat{\mathbf{b}}_{n,0.5}$ and later by Li [2011] for arbitrary $\tau \in (0,1)$.

More generally, for a given series $Y_1, \ldots, Y_n$, define the *Laplace periodogram kernel* as the collection

$$\hat{L}_{n,\tau_1,\tau_2}(\omega_{j,n}) := \frac{n}{4}\hat{\mathbf{b}}'_{n,\tau_1}(\omega_{j,n}) \begin{pmatrix} 1 & -\mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} \hat{\mathbf{b}}_{n,\tau_2}(\omega_{j,n}),\ (\tau_1,\tau_2) \in (0,1)^2,\ \omega_{j,n} \in \mathcal{F}_n. \tag{2.6}$$

For any $(\tau_1, \tau_2, \omega_{j,n})$, computation of $\hat{L}_{n,\tau_1,\tau_2}(\omega_{j,n})$ is immediate via the simplex algorithm (as in classical Koenker-Bassett quantile regression, see Koenker [2005]).

Similarly, define the *copula periodogram kernel* as the Laplace periodogram kernel $\hat{L}^U_{n,\tau_1,\tau_2}(\omega_{j,n})$ associated with the series $U_1, \ldots, U_n$. This means that $\hat{L}^U_{n,\tau_1,\tau_2}(\omega_{j,n})$ is obtained from (2.6) by replacing the estimate $\hat{\mathbf{b}}_{n,\tau}$ by the second and third component of the vector

$$(\hat{a}, (\hat{\mathbf{b}}^U)') := \mathrm{Argmin}_{(a,\mathbf{b}')\in\mathbb{R}^3} \sum_{t=1}^{n} \rho_\tau \left(U_t - (a, \mathbf{b}')\mathbf{c}_t(\omega_{j,n})\right).$$

Finally, because the marginal distribution function F_Y required for the calculation of the data $U_t = F_Y(Y_t)$ is not known, we introduce the *empirical* or *rank-based Laplace periodogram kernel* as the Laplace periodogram kernel $\underset{\sim}{\hat{L}}_{n,\tau_1,\tau_2}(\omega_{j,n})$ associated with the series $n^{-1}R_1^{(n)}, \ldots, n^{-1}R_n^{(n)}$, where $R_t^{(n)}$ denotes the rank of Y_t among $Y_1, \ldots, Y_n$. In other words, $\underset{\sim}{\hat{L}}_{n,\tau_1,\tau_2}(\omega_{j,n})$ is obtained from (2.6) by replacing the estimate $\hat{\mathbf{b}}_{n,\tau}$ by the second and third component of the vector

$$(\hat{a},\ \underset{\sim}{\hat{\mathbf{b}}'}\ ) := \mathrm{Argmin}_{(a,\mathbf{b}')\in\mathbb{R}^3} \sum_{t=1}^{n} \rho_\tau \left(n^{-1}R_t^{(n)} - (a, \mathbf{b}')\mathbf{c}_t(\omega_{j,n})\right).$$

Before we continue, a few remarks regarding the philosophy of the notations used in this paper might be appropriate. With $\hat{T}$ we usually denote a statistic obtained from the original time series $Y_1, \ldots, Y_n$ such as $\hat{L}_{n,\tau_1,\tau_2}$. The notation $\hat{T}^U$ means the T has been calculated from the "data" $U_1, \ldots, U_n$ – a typical example is $\hat{L}^U_{n,\tau_1,\tau_2}$ (note that $U_t = F_Y(Y_t)$). Finally, the notation $\underset{\sim}{\hat{T}}$ reflects the fact that $\hat{T}$ has been calculated from the normalized ranks $n^{-1}R_1^{(n)}, \ldots, n^{-1}R_n^{(n)}$ such as the rank-based Laplace periodogram kernel $\underset{\sim}{\hat{L}}_{n,\tau_1,\tau_2}$.

3 Asymptotic properties.

3.1 Asymptotics of Laplace and copula periodogram kernels.

We now proceed to deriving the asymptotic distributions of the Laplace and rank-based Laplace periodogram kernels, which, as we shall see, establishes their relation to the spectral density kernels defined in (2.1) and (2.3). Throughout the rest of the paper we make the following basic assumptions.

Assumption (A1) The process $\{Y_t\}_{t\in\mathbb{Z}}$ is strictly stationary and m_n-*decomposable* [Chanda et al., 1990], i.e. it admits a representation

$$Y_t = X_{t,n} + D_{t,n}, \qquad t \in \mathbb{Z},\ n \in \mathbb{N}, \tag{3.1}$$

where $\{X_{t,n}\}$ and $\{D_{t,n}\}$ are strictly stationary and $\{X_{t,n}\}$ is m_n-dependent with $m_n = O(n^{1/4-a})$ for some $a > 0$ and, for some nonnegative sequences (η_n) and (κ_n) converging to 0 as $n \to \infty$,

$$\mathrm{P}(|D_{t,n}| > \eta_n) \leq \kappa_n. \tag{3.2}$$

ASSUMPTION (A2) The distribution function $F_{n,X}$ of $X_{t,n}$ and, for any t_1, t_2, the joint distribution functions of $(X_{t_1,n}, X_{t_2,n})$ are twice continuously differentiable, with uniformly (with respect to n and all their arguments) bounded derivatives. Moreover we assume that there exist $d > 0$ and $n_0 < \infty$ such that

$$\inf_{n \geq n_0} \inf_{|x - q_{n,\tau}| \leq d} f_{n,X}(x) > 0,$$

where $f_{n,X}$ and $q_{n,\tau} := F^{-1}_{n,X}(\tau)$ denote the density and τ-quantile corresponding to the distribution $F_{n,X}$.

ASSUMPTION (A3) For any $\tau \in (0,1)$,

$$\sup_n \sum_{k=-n}^{n} |\mathrm{Cov}(I\{X_{1,n} \leq q_{n,\tau}\}, I\{X_{k,n} \leq q_{n,\tau}\})| < \infty.$$

ASSUMPTION (A4) $\qquad |f_{n,X}(q_{n,\tau}) - f_Y(q_\tau)| = o((\log n)^{-1})$.

Let $q_\tau := F_Y^{-1}(\tau)$ denote the τ-quantile of F_Y and consider a ν-tuple $\Omega := \{\omega_1, \ldots, \omega_\nu\} \subset (0,\pi)$ of distinct frequencies. Denote by $\hat{L}_{n,\tau_1,\tau_2}$ and $\hat{L}^U_{n,\tau_1,\tau_2}$, respectively, the Laplace and copula periodogram kernels associated with a realization of length n. For each $(\tau_1,\tau_2) \in (0,1)^2$ and $\omega \in (0,\pi)$, write

$$\overset{\circ}{\mathfrak{f}}_{\tau_1,\tau_2}(\omega) := \mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega)/(f_Y(q_{\tau_1})f_Y(q_{\tau_2})) \tag{3.3}$$

for the *scaled* version of the spectral density kernel $\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega)$ defined in (2.3). In the following two statements $\xrightarrow{\mathcal{L}}$ denotes convergence in distribution, and χ^2_k denotes a chi-square distribution with k degrees of freedom. We also introduce the piecewise constant function (defined on the interval $(0,\pi)$)

$$g_n(\omega) := \omega_{j,n}, \tag{3.4}$$

where $\omega_{j,n}$ is the Fourier frequency closest to ω such that $\omega \in (\omega_{j,n} - \frac{2\pi}{n}, \omega_{j,n} + \frac{2\pi}{n}]$. The following result is the key for understanding the asymptotic properties of the Laplace periodogram kernel.

Theorem 3.1 Let $\Omega := \{\omega_1, \ldots, \omega_\nu\} \subset (0,\pi)$ denote distinct frequencies and $T := \{\tau_1, \ldots, \tau_p\} \subset (0,1)$ distinct quantile orders. Let Assumptions (A1)–(A4) be satisfied with (A2) and (A3) holding for every $\tau \in T$. Also assume $\kappa_n + \eta_n = o(n^{-1})$. Then

$$\sqrt{n}\Big(\hat{\mathbf{b}}_{n,\tau}(g_n(\omega))\Big)_{\tau \in T, \omega \in \Omega} \xrightarrow[n \to \infty]{\mathcal{L}} \Big(N_\tau(\omega)\Big)_{\tau \in T, \omega \in \Omega},$$

where $(N_\tau(\omega))_{\tau \in T, \omega \in \Omega}$ denotes a vector of centered normal distributed random variables with

$$M(\tau_{k_1}, \tau_{k_2}) := \mathrm{Cov}(N_{\tau_1}(\omega_1), N_{\tau_2}(\omega_2)) = \begin{cases} 2\pi \begin{pmatrix} \Re \overset{\circ}{\mathfrak{f}}_{\tau_1,\tau_1}(\omega) & -\Im \overset{\circ}{\mathfrak{f}}_{\tau_1,\tau_2}(\omega) \\ \Im \overset{\circ}{\mathfrak{f}}_{\tau_1,\tau_2}(\omega) & \Re \overset{\circ}{\mathfrak{f}}_{\tau_2,\tau_2}(\omega) \end{pmatrix} & \text{if } \omega_1 = \omega_2 =: \omega \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{if } \omega_1 \neq \omega_2. \end{cases} \tag{3.5}$$

Proof. The proof consists of two basic steps which will be sketched in this section. The technical details for the arguments presented here can be found in Appendix A.

Step 1: The first step consists of a linearization of the estimate $\hat{\mathbf{b}}_{n,\tau}(\omega_{j,n})$ defined in (2.5). To be precise we define for any $\tau \in (0,1)$, $\omega \in (0,\pi)$ and $\boldsymbol{\delta} \in \mathbb{R}^3$ the functions

$$\hat{Z}_{n,\tau,\omega}(\boldsymbol{\delta}) := \sum_{t=1}^{n} \big(\rho_\tau(Y_t - q_\tau - n^{-1/2}\mathbf{c}_t'(\omega)\boldsymbol{\delta}) - \rho_\tau(Y_t - q_\tau)\big), \tag{3.6}$$

$$\hat{Z}^X_{n,\tau,\omega}(\boldsymbol{\delta}) := \sum_{t=1}^{n} \big(\rho_\tau(X_{t,n} - q_{n,\tau} - n^{-1/2}\mathbf{c}_t'(\omega)\boldsymbol{\delta}) - \rho_\tau(X_{t,n} - q_{n,\tau})\big), \tag{3.7}$$

where $\mathbf{c}_t(\omega) := (1, \cos(\omega t), \sin(\omega t))'$, and q_τ and $q_{n,\tau}$ denote the τ-quantiles of F_Y and $F_{n,X}$, respectively. Further define

$$Z^X_{n,\tau,\omega}(\boldsymbol{\delta}) := -\boldsymbol{\delta}'\boldsymbol{\zeta}^X_{n,\tau,\omega} + \frac{1}{2}\boldsymbol{\delta}'\mathbf{Q}^X_{n,\omega}\boldsymbol{\delta},$$

where

$$\boldsymbol{\zeta}^X_{n,\tau,\omega} := n^{-1/2}\sum_{t=1}^{n} \mathbf{c}_t(\omega)(\tau - I\{X_{t,n} \le q_{n,\tau}\}) \tag{3.8}$$

$$\mathbf{Q}^X_{n,\tau,\omega} := f_{n,X}(q_{n,\tau})\, n^{-1}\sum_{t=1}^{n} \mathbf{c}_t(\omega)\mathbf{c}_t'(\omega). \tag{3.9}$$

We first show that the minimizers

$$\hat{\boldsymbol{\delta}}_{n,\tau,\omega} := \arg\min_{\boldsymbol{\delta}} \hat{Z}_{n,\tau,\omega}(\boldsymbol{\delta}) \quad \text{and} \quad \boldsymbol{\delta}^X_{n,\tau,\omega} := \arg\min_{\boldsymbol{\delta}} Z^X_{n,\tau,\omega}(\boldsymbol{\delta}) = (\mathbf{Q}^X_{n,\tau,\omega})^{-1}\boldsymbol{\zeta}^X_{n,\tau,\omega} \tag{3.10}$$

are close in probability (uniformly with respect to $\omega \in \mathcal{F}_n$). Note that from the definition in (2.5) it follows that the random variable $\sqrt{n}\hat{\mathbf{b}}_{n,\tau}(\omega_{j,n})$ coincides with the second and third component of the vector $\hat{\boldsymbol{\delta}}_{n,\tau,\omega}$. Moreover, for $\omega_{j,n} = 2\pi j/n$ we have

$$\mathbf{Q}^X_{n,\tau,\omega_{j,n}} = f_{n,X}(q_{n,\tau})\ \mathrm{diag}(1, 1/2, 1/2), \tag{3.11}$$

where $\mathrm{diag}(a_1, \ldots, a_k)$ denotes the diagonal matrix with diagonal elements $a_1, \ldots, a_k$.

More precisely, we establish the following estimate

$$\sup_{\omega \in \mathcal{F}_n} \|\hat{\boldsymbol{\delta}}_{n,\tau,\omega} - \boldsymbol{\delta}^X_{n,\tau,\omega}\| = o_{\mathrm{P}}\big((n^{-1/8} \vee (n^{-1/6}m_n^{1/3}))(\log n)^{3/2}\big). \tag{3.12}$$

This result is obtained from the following arguments, for which the details are provided in Section 6.1. Roughly speaking, estimates of the type (3.12) can be obtained by showing that the corresponding functions $\hat{Z}_{n,\tau,\omega}$ and $Z^X_{n,\tau,\omega}$ are uniformly close in probability. A precise statement is given in Lemma 6.1 (see Section 6.1.1), where we show that (3.12) follows if the estimate

$$\sup_{\omega \in \mathcal{F}_n} \sup_{\|\boldsymbol{\delta} - \boldsymbol{\delta}^X_{n,\tau,\omega}\| \le \epsilon} |\hat{Z}_{n,\tau,\omega}(\boldsymbol{\delta}) - Z^X_{n,\tau,\omega}(\boldsymbol{\delta})| = o_{\mathrm{P}}\big((n^{-1/4} \vee (n^{-1/3}m_n^{2/3}))(\log n)^3\big) \tag{3.13}$$

can be proved for some $\epsilon > 0$. In order to prove this estimate we show in Section 6.1.2 that

$$\sup_{\omega \in \mathcal{F}_n} \sup_{\|\boldsymbol{\delta} - \boldsymbol{\delta}^X_{n,\tau,\omega}\| \le \epsilon} |\hat{Z}_{n,\tau,\omega}(\boldsymbol{\delta}) - \hat{Z}^X_{n,\tau,\omega}(\boldsymbol{\delta})| = o_{\mathrm{P}}\big((n^{-1/4} \vee (n^{-1/3}m_n^{2/3}))(\log n)^3\big), \tag{3.14}$$

where the function $\hat{Z}^X_{n,\tau,\omega}$ is defined in (3.7). It remains to prove that the functions $\hat{Z}^X_{n,\tau,\omega}$ and $Z^X_{n,\tau,\omega}$ are uniformly close in probability, which is done in two steps. First we apply Lemma 6.2 (see Section 6.1.1), which yields that there exists a finite constant A such that, with probability tending to one,

$$\sup_{\omega\in\mathcal{F}_n}\sup_{\|\boldsymbol{\delta}-\boldsymbol{\delta}^X_{n,\tau,\omega}\|\leq\epsilon}|\hat{Z}^X_{n,\tau,\omega}(\boldsymbol{\delta})-Z^X_{n,\tau,\omega}(\boldsymbol{\delta})| \;\leq\; \sup_{\omega\in\mathcal{F}_n}\sup_{\|\boldsymbol{\delta}\|\leq\epsilon+A\sqrt{\log n}}|\hat{Z}^X_{n,\tau,\omega}(\boldsymbol{\delta})-Z^X_{n,\tau,\omega}(\boldsymbol{\delta})|.$$

Secondly, we show in Section 6.1.3 that this result entails

$$\sup_{\omega\in\mathcal{F}_n}\sup_{\|\boldsymbol{\delta}-\boldsymbol{\delta}^X_{n,\tau,\omega}\|\leq\epsilon}|\hat{Z}^X_{n,\tau,\omega}(\boldsymbol{\delta})-Z^X_{n,\tau,\omega}(\boldsymbol{\delta})| = o_{\mathrm{P}}\big((n^{-1/4}\vee(n^{-1/3}m_n^{2/3}))(\log n)^3\big). \tag{3.15}$$

Combining the estimates (3.14) and (3.15) yields (3.13) and as a consequence (by an application of Lemma 6.1) we obtain the estimate (3.12).

Step 2: As we have discussed at the beginning of the first step, the asymptotic properties of $\sqrt{n}\hat{\mathbf{b}}_{n,\tau}(\omega_{j,n})$ can be obtained from those of the random variables $\boldsymbol{\delta}^X_{n,\tau,\omega}$ for which an explicit expression is available. More precisely, for given sets $\Omega := \{\omega_1,\ldots,\omega_\nu\}\subset(0,\pi)$ of Fourier frequencies and $T := \{\tau_1,\ldots,\tau_p\}\subset(0,1)$ consider the linear combination with coefficients $\boldsymbol{\lambda}_{ik}\in\mathbb{R}^2$, $i=1,\ldots,\nu$, $k=1,\ldots,p$

$$\begin{aligned}\sum_{k=1}^p\sum_{i=1}^\nu\boldsymbol{\lambda}'_{ik}\sqrt{n}\hat{\mathbf{b}}_{n,\tau_k}(g_n(\omega_i)) &= \sum_{k=1}^p\frac{2}{f_{n,X}(q_{n,\tau_k})}\sum_{i=1}^\nu\boldsymbol{\lambda}'_{ik}\sum_{t=1}^n\frac{\mathbf{v}_{tn}(\omega_i)}{\sqrt{n}}(\tau_k-I\{X_{t,n}\leq q_{n,\tau_k}\})+o_{\mathrm{P}}(1)\\ &= \sum_{k=1}^p\frac{2}{f_Y(q_{\tau_k})}\sum_{i=1}^\nu\boldsymbol{\lambda}'_{ik}\sum_{t=1}^n\frac{\mathbf{v}_{tn}(\omega_i)}{\sqrt{n}}(\tau_k-I\{X_{t,n}\leq q_{n,\tau_k}\})+o_{\mathrm{P}}(1),\end{aligned} \tag{3.16}$$

where $\mathbf{v}_{tn}(\omega) := (\cos(g_n(\omega)t),\sin(g_n(\omega)t))'$. The first equality is a consequence of (3.10), (3.11) and (3.12). For the second one, observe that the sum with respect to t, that is

$$S(i,k) := \sum_{t=1}^n\frac{\mathbf{v}_{tn}(\omega_i)}{\sqrt{n}}(\tau_k-I\{X_{t,n}\leq q_{n,\tau_k}\}) \tag{3.17}$$

converges in distribution for every $k\in\{1,...,p\}, i\in\{1,...,\nu\}$ as we will show below. This implies that each of those sums is stochastically bounded, and together with assumption (A4) this yields the second equality. To prove the convergence in distribution, note that the summands in the sum with index t in the last line are uniformly bounded and form a triangular array of centered, m_n-dependent random vectors. For the covariance we obtain

$$\begin{aligned}&\mathrm{Cov}\Big(\sum_{t_1=1}^n\frac{1}{\sqrt{n}}\mathbf{v}_{t_1,n}(\omega_{i_1})(\tau_{k_1}-I\{X_{t_1,n}\leq q_{\tau_{k_1},n}\}),\sum_{t_2=1}^n\frac{1}{\sqrt{n}}\mathbf{v}_{t_2,n}(\omega_{i_2})(\tau_{k_2}-I\{X_{t_2,n}\leq q_{\tau_{k_2},n}\})\Big)\\ &=\frac{1}{n}\sum_{t_1=1}^n\sum_{t_2=1}^n\mathbf{v}_{t_1,n}(\omega_{i_1})\mathbf{v}'_{t_2,n}(\omega_{i_2})\,\mathrm{Cov}(I\{X_{t_1,n}\leq q_{\tau_{k_1},n}\},I\{X_{t_2,n}\leq q_{\tau_{k_2},n}\})\\ &=\frac{1}{n}\sum_{t_1=1}^n\sum_{t_2=1}^n\mathbf{v}_{t_1,n}(\omega_{i_1})\mathbf{v}'_{t_2,n}(\omega_{i_2})\,\mathrm{Cov}(I\{Y_{t_1}\leq q_{\tau_{k_1}}\},I\{Y_{t_2}\leq q_{\tau_{k_2}}\})+o(1),\end{aligned}$$

where the last equality is due to the fact that

$$\Big\|\frac{1}{n}\sum_{t_1=1}^{n}\sum_{t_2=1}^{n}\mathbf{v}_{t_1,n}(\omega_{i_1})\mathbf{v}'_{t_2,n}(\omega_{i_2})\Big(\operatorname{Cov}(I\{X_{t_1,n}\le q_{n,\tau_{k_1}}\},I\{X_{t_2,n}\le q_{n,\tau_{k_2}}\}) - \operatorname{Cov}(I\{Y_{t_1}\le q_{\tau_{k_1}}\},I\{Y_{t_2}\le q_{\tau_{k_2}}\})\Big)\Big\|_\infty$$

$$\le \frac{1}{n}\sum_{t_1=1}^{n}\sum_{t_2=1}^{n}\Big(\mathrm{E}\big|I\{X_{t_1,n}\le q_{n,\tau_{k_1}}\}I\{X_{t_2,n}\le q_{n,\tau_{k_2}}\} - I\{Y_{t_1,n}\le q_{\tau_{k_1}}\}I\{X_{t_2,n}\le q_{n,\tau_{k_2}}\}\big| + \mathrm{E}\big|I\{Y_{t_1,n}\le q_{\tau_{k_1}}\}I\{X_{t_2,n}\le q_{n,\tau_{k_2}}\} - I\{Y_{t_1}\le q_{\tau_{k_1}}\}I\{Y_{t_2}\le q_{\tau_{k_2}}\}\big|\Big)$$

$$\le \frac{1}{n}\sum_{t_1=1}^{n}\sum_{t_2=1}^{n}\Big(\mathrm{P}\big(|X_{t_1,n}-q_{n,\tau_{k_1}}|\le|D_{t_1,n}|+|q_{n,\tau_{k_1}}-q_{\tau_{k_1}}|\big) + \mathrm{P}\big(|X_{t_2,n}-q_{n,\tau_{k_2}}|\le|D_{t_2,n}|+|q_{n,\tau_{k_2}}-q_{\tau_{k_2}}|\big)\Big) = nO(\kappa_n+\eta_n)$$

and the assumption that $\kappa_n+\eta_n = o(n^{-1})$. Along the same lines as in the Proof of Theorem 2 of Li [2008], we then obtain

$$\lim_{n\to\infty}\frac{4}{n}\sum_{t_1=1}^{n}\sum_{t_2=1}^{n}\mathbf{v}_{t_1,n}(\omega_{i_1})\mathbf{v}'_{t_2,n}(\omega_{i_2})\frac{\operatorname{Cov}(I\{Y_{t_1}\le q_{\tau_{k_1}}\},I\{Y_{t_2}\le q_{\tau_{k_2}}\})}{f_Y(q_{\tau_{k_1}})f_Y(q_{\tau_{k_2}})} = M(\tau_{k_1},\tau_{k_2})$$

where $M(\tau_{k_1},\tau_{k_2})$ is defined in (3.5). By a Central Limit Theorem for triangular arrays of m_n-dependent random variables (Lemma 6.6), this proves the weak convergence of the sum $S(k,i)$ defined in (3.17). Next, apply Lemma 6.6 to the first term in the last line of (3.16) to obtain

$$\sum_{k=1}^{p}\frac{2}{f_Y(q_{\tau_k})}\sum_{i=1}^{\nu}\boldsymbol{\lambda}'_{ik}\sum_{t=1}^{n}\frac{\mathbf{v}_{tn}(\omega_i)}{\sqrt{n}}(\tau_k - I\{X_{tn}\le q_{n,\tau_k}\}) \xrightarrow{\mathcal{L}} \mathcal{N}\Big(0,\operatorname{Var}(\sum_{k=1}^{p}\sum_{i=1}^{\nu}\boldsymbol{\lambda}'_{ik}N_{\tau_k}(\omega_i))\Big),$$

where $(N_\tau(\omega))_{\tau\in T,\,\omega\in\Omega}$ denotes a vector of centered normal distributed random variables with $\operatorname{Cov}(N_{\tau_1}(\omega_1),N_{\tau_2}(\omega_2)) = M(\tau_{k_1},\tau_{k_2})$ defined in (3.5). Because of (3.16), the quantity

$$\sqrt{n}\sum_{k=1}^{p}\sum_{i=1}^{\nu}\boldsymbol{\lambda}'_{ik}\hat{\mathbf{b}}_{\tau_k}(g_n(\omega_i))$$

converges to the same limit distribution. Thus it follows by an application of the traditional Cramér-Wold device that

$$\Big(\sqrt{n}\hat{\mathbf{b}}_{n,\tau}(g_n(\omega))\Big)_{\tau\in T,\,\omega\in\Omega} \xrightarrow[n\to\infty]{\mathcal{L}} \Big(N_\tau(\omega)\Big)_{\tau\in T,\,\omega\in\Omega}.$$

□

As an immediate consequence of the above result the Continuous Mapping Theorem yields the asymptotic distribution of a collection of Laplace periodogram kernels.

Theorem 3.2 *Under the assumptions of Theorem 3.1*

$$(\hat{L}_{n,\tau_1,\tau_2}(g_n(\omega_1)),\ldots,\hat{L}_{n,\tau_1,\tau_2}(g_n(\omega_\nu))) \xrightarrow{\mathcal{L}} (L_{\tau_1,\tau_2}(\omega_1),\ldots,L_{\tau_1,\tau_2}(\omega_\nu)), \tag{3.18}$$

where the random variables L_{τ_1,τ_2} associated with distinct frequencies are mutually independent. Moreover,

$$L_{\tau_1,\tau_2}(\omega) \sim \pi \overset{\circ}{\mathfrak{f}}_{\tau_1,\tau_2}(\omega)\chi_2^2 \quad \textit{if} \;\; \tau_1 = \tau_2, \tag{3.19}$$

and

$$L_{\tau_1,\tau_2}(\omega) \overset{d}{=} \frac{1}{4}(Z_{11}, Z_{12}) \begin{pmatrix} 1 & -\mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} \begin{pmatrix} Z_{21} \\ Z_{22} \end{pmatrix} \quad \textit{if} \;\; \tau_1 \neq \tau_2,$$

where $(Z_{11}, Z_{12}, Z_{21}, Z_{22})' \sim \mathcal{N}(0, \mathbf{\Sigma}_4(\omega))$ with covariance matrix

$$\mathbf{\Sigma}_4(\omega) := 2\pi \begin{pmatrix} \overset{\circ}{\mathfrak{f}}_{\tau_1\tau_1}(\omega) & 0 & \Re\overset{\circ}{\mathfrak{f}}_{\tau_1\tau_2}(\omega) & -\Im\overset{\circ}{\mathfrak{f}}_{\tau_1\tau_2}(\omega) \\ 0 & \overset{\circ}{\mathfrak{f}}_{\tau_1\tau_2}(\omega) & \Im\overset{\circ}{\mathfrak{f}}_{\tau_1\tau_2}(\omega) & \Re\overset{\circ}{\mathfrak{f}}_{\tau_1\tau_2}(\omega) \\ \Re\overset{\circ}{\mathfrak{f}}_{\tau_1\tau_2}(\omega) & \Im\overset{\circ}{\mathfrak{f}}_{\tau_1\tau_2}(\omega) & \overset{\circ}{\mathfrak{f}}_{\tau_1\tau_2}(\omega) & 0 \\ -\Im\overset{\circ}{\mathfrak{f}}_{\tau_1\tau_2}(\omega) & \Re\overset{\circ}{\mathfrak{f}}_{\tau_1\tau_2}(\omega) & 0 & \overset{\circ}{\mathfrak{f}}_{\tau_1\tau_2}(\omega) \end{pmatrix}. \tag{3.20}$$

It follows from Theorem 3.2 that, for all $(\tau_1, \tau_2) \in (0,1)^2$ and $\omega \in (0, \pi)$,

$$\mathrm{E}[L_{\tau_1,\tau_2}(\omega)] = 2\pi \overset{\circ}{\mathfrak{f}}_{\tau_1,\tau_2}(\omega),$$

which indicates that an estimator of the scaled spectral density $2\pi \overset{\circ}{\mathfrak{f}}_{\tau_1,\tau_2}(\omega)$ defined in (3.3) could be based on an average of quantities of the form $\hat{L}_{n,\tau_1,\tau_2}(\omega)$. Moreover, the following result is an immediate consequence of Theorem 3.2 and yields the asymptotic distribution of the copula periodogram kernel.

Corollary 3.1 *Let $\Omega := \{\omega_1, \dots, \omega_\nu\} \subset (0, \pi)$ denote distinct frequencies and $(\tau_1, \tau_2) \in (0,1)^2$. If Assumptions (A1)–(A4) hold for every $\tau \in \{\tau_1, \tau_2\}$, with $\kappa_n + \eta_n = o(n^{-1})$, then*

$$(\hat{L}^U_{n,\tau_1,\tau_2}(g_n(\omega_1)), \dots, \hat{L}^U_{n,\tau_1,\tau_2}(g_n(\omega_\nu))) \xrightarrow{\mathcal{L}} (L^U_{\tau_1,\tau_2}(\omega_1), \dots, L^U_{\tau_1,\tau_2}(\omega_\nu)), \tag{3.21}$$

where $g_n(\omega)$ is defined in (3.4)*. The random variables $L^U_{\tau_1,\tau_2}$ in* (3.24) *associated with distinct frequencies are mutually independent,*

$$L^U_{\tau_1,\tau_2}(\omega) \sim \pi\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega)\chi_2^2 \quad \textit{if} \;\; \tau_1 = \tau_2, \tag{3.22}$$

$$L^U_{\tau_1,\tau_2}(\omega) \overset{d}{=} \frac{1}{4}(Z_{11}, Z_{12}) \begin{pmatrix} 1 & -\mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} \begin{pmatrix} Z_{21} \\ Z_{22} \end{pmatrix} \quad \textit{if} \;\; \tau_1 \neq \tau_2,$$

and $(Z_{11}, Z_{12}, Z_{21}, Z_{22})' \sim \mathcal{N}(0, \mathbf{\Sigma}_4(\omega))$ with covariance matrix

$$\mathbf{\Sigma}_4(\omega) := 2\pi \begin{pmatrix} \mathfrak{f}_{q_{\tau_1}q_{\tau_1}}(\omega) & 0 & \Re\mathfrak{f}_{q_{\tau_1}q_{\tau_2}}(\omega) & -\Im\mathfrak{f}_{q_{\tau_1}q_{\tau_2}}(\omega) \\ 0 & \mathfrak{f}_{q_{\tau_1}q_{\tau_2}}(\omega) & \Im\mathfrak{f}_{q_{\tau_1}q_{\tau_2}}(\omega) & \Re\mathfrak{f}_{q_{\tau_1}q_{\tau_2}}(\omega) \\ \Re\mathfrak{f}_{q_{\tau_1}q_{\tau_2}}(\omega) & \Im\mathfrak{f}_{q_{\tau_1}q_{\tau_2}}(\omega) & \mathfrak{f}_{q_{\tau_1}q_{\tau_2}}(\omega) & 0 \\ -\Im\mathfrak{f}_{q_{\tau_1}q_{\tau_2}}(\omega) & \Re\mathfrak{f}_{q_{\tau_1}q_{\tau_2}}(\omega) & 0 & \mathfrak{f}_{q_{\tau_1}q_{\tau_2}}(\omega) \end{pmatrix}. \tag{3.23}$$

In particular,

$$\mathrm{E}[L^U_{\tau_1,\tau_2}(\omega)] = 2\pi\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega).$$

This means that copula periodogram kernels $\hat{L}^U_{n,\tau_1,\tau_2}$, rather than the Laplace ones $\hat{L}_{n,\tau_1,\tau_2}$, thus seem to be the appropriate tools for statistical analysis regarding $\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}$. Unfortunately, they are not statistics, since they involve the unknown marginal distribution F_Y which in practice is unspecified. This problem will be solved in the following section.

3.2 Asymptotics of rank-based Laplace periodogram kernels.

The final result of this section establishes the asymptotic equivalence of the copula and rank-based Laplace periodogram kernels $\hat{L}^U_{n,\tau_1\tau_2}(\omega)$ and $\underset{\sim}{\hat{L}}_{n,\tau_1\tau_2}(\omega)$, where the latter do not involve F_Y and can be computed from the data. In particular the following results show that $\underset{\sim}{\hat{\mathbf{b}}}_{n,\tau}$, $\underset{\sim}{\hat{L}}_{n,\tau_1,\tau_2}(\omega)$ are asymptotically distribution-free in the sense that their asymptotic distributions only depend on the process $\{U_t\}_{t\in\mathbb{Z}}$.

Theorem 3.3 Let $\Omega := \{\omega_1,\dots,\omega_\nu\} \subset (0,\pi)$ denote distinct frequencies and $T := \{\tau_1,\dots,\tau_p\} \subset (0,1)$ distinct quantile orders. Let Assumptions (A1)–(A3) be satisfied with (A2) and (A3) holding for every $\tau \in T$. Also assume $\kappa_n + \eta_n = o(n^{-1})$. Then

$$\Big(\underset{\sim}{\hat{\mathbf{b}}}_{n,\tau}(g_n(\omega))\Big)_{\tau\in T,\,\omega\in\Omega} \xrightarrow[n\to\infty]{\mathcal{L}} \Big(N^U_{\tau,\omega}\Big)_{\tau\in T,\,\omega\in\Omega}$$

where $(N^U_{\tau,\omega})_{\tau\in T,\,\omega\in\Omega}$ denotes a vector of centered normal distributed random variables with

$$\mathrm{Cov}(N^U_{\tau_1,\omega_1}, N^U_{\tau_2,\omega_2}) = \begin{cases} \begin{pmatrix} \Re\mathfrak{f}_{q_{\tau_1},q_{\tau_1}}(\omega) & -\Im\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega) \\ \Im\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega) & \Re\mathfrak{f}_{q_{\tau_2},q_{\tau_2}}(\omega) \end{pmatrix} & \text{if } \omega_1 = \omega_2 =: \omega, \text{ and} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{if } \omega_1 \neq \omega_2. \end{cases}$$

Together with the above result, the Continuous Mapping Theorem yields

Theorem 3.4 *Under the assumptions of Theorem 3.2*

$$(\underset{\sim}{\hat{L}}_{n,\tau_1,\tau_2}(g_n(\omega_1)),\dots,\underset{\sim}{\hat{L}}_{n,\tau_1,\tau_2}(g_n(\omega_\nu))) \xrightarrow{\mathcal{L}} (L^U_{\tau_1,\tau_2}(\omega_1),\dots,L^U_{\tau_1,\tau_2}(\omega_\nu)), \tag{3.24}$$

where $g_n(\omega)$ and the distribution of the random variables $L^U_{\tau_1,\tau_2}$ are defined in (3.4) *and Corollary 3.1, respectively.*

Proof of Theorem 3.3. Let $\hat{F}_{n,Y}$ denote the empirical distribution function of $Y_1,\dots,Y_n$, $\mathbf{e}_1 := (1,0,0)'$, $\boldsymbol{\delta} = (\delta_1,\delta_2,\delta_3)'$, and $U_{t,n} := F_{n,X}(X_{t,n})$. We introduce the functions

$$\begin{aligned}
\underset{\sim}{\hat{Z}}_{n,\tau,\omega}(\boldsymbol{\delta}) &:= \sum_{t=1}^n \big(\rho_\tau(\hat{F}_{n,Y}(Y_t) - \tau - n^{-1/2}\mathbf{c}_t'(\omega)\boldsymbol{\delta}) - \rho_\tau(\hat{F}_{n,Y}(Y_t) - \tau)\big) \\
\underset{\sim}{\hat{Z}}^X_{n,\tau,\omega}(\boldsymbol{\delta}) &:= \sum_{t=1}^n \big(\rho_\tau(\hat{F}_{n,Y}(X_{t,n}) - \tau - n^{-1/2}\mathbf{c}_t'(\omega)\boldsymbol{\delta}) - \rho_\tau(\hat{F}_{n,Y}(X_{t,n}) - \tau)\big) \\
\hat{Z}^U_{n,\tau,\omega}(\boldsymbol{\delta}) &:= \sum_{t=1}^n \big(\rho_\tau(U_{t,n} - \tau - n^{-1/2}\mathbf{c}_t'(\omega)\boldsymbol{\delta}) - \rho_\tau(U_{t,n} - \tau)\big) - \delta_1\sqrt{n}(F_{n,X}(\hat{F}^{-1}_{n,Y}(\tau)) - \tau) \\
Z^U_{n,\tau,\omega}(\boldsymbol{\delta}) &:= -\boldsymbol{\delta}'\big(\boldsymbol{\zeta}^U_{n,\tau,\omega} + \mathbf{e}_1'\sqrt{n}(F_{n,X}(\hat{F}^{-1}_{n,Y}(\tau)) - \tau)\big) + \frac{1}{2}\boldsymbol{\delta}'\mathbf{Q}^U_{n,\omega}\boldsymbol{\delta}
\end{aligned}$$

where

$$\mathbf{Q}^U_{n,\omega} := \frac{1}{n}\sum_{t=1}^n \mathbf{c}_t(\omega)\mathbf{c}_t'(\omega) \text{ and } \boldsymbol{\zeta}^U_{n,\tau,\omega} := n^{-1/2}\sum_{t=1}^n \mathbf{c}_t(\omega)\big(\tau - I\{U_{t,n} \le \tau\}\big).$$

If we can show that the difference $\underset{\sim}{\hat{Z}}_{n,\tau,\omega}(\boldsymbol{\delta}) - Z^U_{n,\tau,\omega}(\boldsymbol{\delta})$ is uniformly small in probability, a slight modification of the arguments presented in the proof Theorem 3.2 yield a uniform linearization of $\underset{\sim}{\hat{\boldsymbol{\delta}}}_{n,\tau,\omega} := \arg\min_{\boldsymbol{\delta}} \underset{\sim}{\hat{Z}}_{n,\tau,\omega}(\boldsymbol{\delta})$. More precisely, we show

$$\sup_{\omega\in\mathcal{F}_n} \|\underset{\sim}{\hat{\boldsymbol{\delta}}}_{n,\tau,\omega} - \boldsymbol{\delta}^U_{n,\tau,\omega}\| = o_{\mathrm{P}}\big(n^{-1/8} m_n^{1/4} \log n\big) \tag{3.25}$$

where

$$\boldsymbol{\delta}^U_{n,\tau,\omega} := \arg\min_{\boldsymbol{\delta}} Z^U_{n,\tau,\omega}(\boldsymbol{\delta}) = (\mathbf{Q}^U_{n,\omega})^{-1}\big(\boldsymbol{\zeta}^U_{n,\tau,\omega} + \mathbf{e}_1\sqrt{n}(F_{n,X}(\hat{F}^{-1}_{n,Y}(\tau)) - \tau)\big).$$

The asymptotic normality of the linearization $\boldsymbol{\delta}^U_{n,\tau,\omega}$ then follows by similar arguments as given in Step (2) of the proof of Theorem 3.2 and the details are omitted for the sake of brevity.

In order to prove (3.25) we note that Lemma 6.1 in the Appendix also holds with $\hat{Z}_{n,\tau,\omega}(\boldsymbol{\delta})$, $Z^X_{n,\tau,\omega}(\boldsymbol{\delta})$, $\boldsymbol{\delta}^X_{n,\tau,\omega}$ and $\hat{\boldsymbol{\delta}}_{n,\tau,\omega}$ replaced by $\underset{\sim}{\hat{Z}}_{n,\tau,\omega}(\boldsymbol{\delta})$, $Z^U_{n,\tau,\omega}(\boldsymbol{\delta})$, $\boldsymbol{\delta}^U_{n,\tau,\omega}$ and $\underset{\sim}{\hat{\boldsymbol{\delta}}}_{n,\tau,\omega}$, respectively. Therefore, it suffices to establish that, for some $\epsilon > 0$,

$$\sup_{\omega\in\mathcal{F}_n} \sup_{\|\boldsymbol{\delta}-\boldsymbol{\delta}^U_{n,\tau,\omega}\|\le\epsilon} |\underset{\sim}{\hat{Z}}_{n,\tau,\omega}(\boldsymbol{\delta}) - Z^U_{n,\tau,\omega}(\boldsymbol{\delta})| = o_{\mathrm{P}}\big(n^{-1/4} m_n^{1/2} (\log n)^2\big). \tag{3.26}$$

Note that $\boldsymbol{\delta}^U_{n,\tau,\omega}$ decomposes into a term containing $\boldsymbol{\zeta}^U_{n,\tau,\omega}$, to which Lemma 6.2 applies, and a term involving $\sqrt{n}(F_{n,X}(\hat{F}^{-1}_{n,Y}(\tau)) - \tau)$, which, for every τ, converges in distribution, so that $\mathrm{P}(\sqrt{n}(F_{n,X}(\hat{F}^{-1}_{n,Y}(\tau)) - \tau) > A\sqrt{\log n}) \to 0$ for any $A > 0$. Therefore, there exists a finite constant A such that

$$\lim_{n\to\infty} \mathrm{P}\big(\sup_{\omega\in\mathcal{F}_n} \|\boldsymbol{\delta}^U_{n,\tau,\omega}\|_\infty > A\sqrt{\log n}\big) = 0.$$

It follows that, in order to establish (3.26), we may restrict to a supremum with respect to the set $\|\boldsymbol{\delta}\| \le 2A\sqrt{\log n}$. Knight's identity (Knight [1998]; see p. 121 of Koenker [2005]) yields

$$\underset{\sim}{\hat{Z}}^X_{n,\tau,\omega}(\boldsymbol{\delta}) = \underset{\sim}{\hat{Z}}^X_{n,\tau,\omega,1}(\boldsymbol{\delta}) + \underset{\sim}{\hat{Z}}^X_{n,\tau,\omega,2}(\boldsymbol{\delta}),$$

where

$$\begin{aligned}
\underset{\sim}{\hat{Z}}^X_{n,\tau,\omega,1}(\boldsymbol{\delta}) &= -\boldsymbol{\delta}' n^{-1/2} \sum_{t=1}^n \mathbf{c}_t(\omega)\big(\tau - I\{U_{t,n} \le F_{n,X}(\hat{F}^{-1}_{n,Y}(\tau))\}\big),\\
\underset{\sim}{\hat{Z}}^X_{n,\tau,\omega,2}(\boldsymbol{\delta}) &= \sum_{t=1}^n \int_0^{n^{-1/2}\mathbf{c}_t'(\omega)\boldsymbol{\delta}} \big(I\{U_{t,n} \le F_{n,X}(\hat{F}^{-1}_{n,Y}(s+\tau))\} - I\{U_{t,n} \le F_{n,X}(\hat{F}^{-1}_{n,Y}(\tau))\}\big)\mathrm{d}s.
\end{aligned}$$

A similar representation holds for $Z^U_{n,\tau,\omega}(\boldsymbol{\delta})$. Now the proof of (3.26) is a consequence of the following three auxiliary results, which are proved in Sections 6.2.1–6.2.3:

$$\sup_{\omega\in\mathcal{F}_n} \sup_{\|\boldsymbol{\delta}\|\le A\sqrt{\log n}} \big|\underset{\sim}{\hat{Z}}_{n,\tau,\omega}(\boldsymbol{\delta}) - \underset{\sim}{\hat{Z}}^X_{n,\tau,\omega}(\boldsymbol{\delta})\big| = O_{\mathrm{P}}\big((\kappa_n + \eta_n + r_n(2\eta_n, m_n))\sqrt{n\log n}\big) \tag{3.27}$$

where $r_n(2\eta_n, m_n)) = o((n\log n)^{-1/2})$ (for a precise definition see equation 6.8 in Appendix B),

$$\begin{aligned}
\sup_{\omega\in\mathcal{F}_n} \sup_{\|\boldsymbol{\delta}\|\le A\sqrt{\log n}} \Big|\underset{\sim}{\hat{Z}}^X_{n,\tau,\omega,1}(\boldsymbol{\delta}) - \boldsymbol{\delta}' n^{-1/2} \sum_{t=1}^n \mathbf{c}_t(\omega)(\tau - I\{U_{t,n} \le \tau\}) \\
- \delta_1\sqrt{n}(F_{n,X}(\hat{F}^{-1}_{n,Y}(\tau)) - \tau)\Big| = O_{\mathrm{P}}\big(n^{-1/4} m_n^{1/2} (\log n)^{3/2}\big).
\end{aligned} \tag{3.28}$$

$$\sup_{\omega\in\mathcal{F}_n}\sup_{\|\boldsymbol{\delta}\|\le A\sqrt{\log n}}\Big|\hat{\underset{\sim}{Z}}{}^X_{n,\tau,\omega,2}(\boldsymbol{\delta})-\sum_{t=1}^n\int_0^{n^{-1/2}\mathbf{c}_t'(\omega)\boldsymbol{\delta}}\big(I\{U_{t,n}\le s+\tau\}-I\{U_{t,n}\le\tau\}\big)\mathrm{d}s\Big|$$
$$=O_{\mathrm{P}}\big(n^{-1/4}m_n^{1/2}(\log n)^{3/2}+\sqrt{n\log n}\big(\kappa_n+\eta_n+r_n(\eta_n,m_n)\big)\big). \quad (3.29)$$

Note that the combination of (3.28) and (3.29) implies that $\hat{\underset{\sim}{Z}}{}^X_{n,\tau,\omega}$ and $\hat{Z}^U_{n,\tau,\omega}$ are uniformly close in probability. Finally, we obtain from (3.15)

$$\sup_{\omega\in\mathcal{F}_n}\sup_{\|\boldsymbol{\delta}\|\le A\sqrt{\log n}}|\hat{Z}^U_{n,\tau,\omega}(\boldsymbol{\delta})-Z^U_{n,\tau,\omega}(\boldsymbol{\delta})|=o_{\mathrm{P}}((n^{-1/4}\vee n^{-1/3}m_n^{2/3})(\log n)^3), \quad (3.30)$$

where we may replace $\hat{Z}^X_{n,\tau,\omega}(\boldsymbol{\delta})$ with $\hat{Z}^U_{n,\tau,\omega}(\boldsymbol{\delta})$ and $Z^X_{n,\tau,\omega}(\boldsymbol{\delta})$ with $Z^U_{n,\tau,\omega}(\boldsymbol{\delta})$, since $U_{1,n},\dots,U_{n,n}$ are m_n-dependent as required and the additional term $\delta_1\sqrt{n}(F_{n,X}(\hat{F}^{-1}_{n,Y}(\tau))-\tau)$ appears in both $\hat{Z}^U_{n,\tau,\omega}(\boldsymbol{\delta})$ and $Z^U_{n,\tau,\omega}(\boldsymbol{\delta})$. Combining (3.27)–(3.30) yields (3.26), thus completing the proof of Theorem 3.3. □

4 Smoothed periodograms.

We have seen in Section 3.1 that the Laplace periodogram kernel converges in distribution and that the expectation of the limit is the *scaled* spectral density kernel (at (τ_1,τ_2))

$$2\pi\,\overset{\circ}{\mathfrak{f}}_{\tau_1,\tau_2}(\omega):=2\pi\frac{\mathfrak{f}_{q_{\tau_1}q_{\tau_2}}(\omega)}{f_Y(q_{\tau_1})f_Y(q_{\tau_2})}=\frac{1}{f_Y(q_{\tau_1})f_Y(q_{\tau_2})}\sum_{k=-\infty}^{\infty}\gamma_k(q_{\tau_1},q_{\tau_2})\mathrm{e}^{-\mathrm{i}\omega k}.$$

In practice, however, this is not enough, and consistent estimation is a minimal requirement. For this purpose, we consider, as in traditional spectral estimation, smoothed versions of periodograms, of the form

$$\hat{\mathfrak{f}}_{n,\tau_1,\tau_2}(\omega_{j,n}):=\sum_{|k|\le N_n}W_n(k)\hat{L}_{n,\tau_1,\tau_2}(\omega_{j+k,n}) \quad (4.1)$$

at the Fourier frequencies $\omega_{j,n}=2\pi j/n$, where $N_n\to\infty$ as $n\to\infty$ is a sequence of positive integers, and $W_n=\{W_n(j)\,:\,|j|\le N_n\}$ a sequence of positive weights satisfying

$$W_n(k)=W_n(-k)\ \text{ for all } k \text{ and } \sum_{|k|\le N_n}W_n(k)=1.$$

Extending the definition of $\hat{\mathfrak{f}}_{n,\tau_1,\tau_2}$ to the interval $(0,\pi)$ we introduce

$$\big\{\omega\mapsto\hat{\mathfrak{f}}_{n,\tau_1,\tau_2}(\omega)\ \big|\ (\tau_1,\tau_2)\in[0,1]^2,\ \ \omega\in(0,\pi)\big\}$$

as *smoothed Laplace periodogram kernel*, where

$$\hat{\mathfrak{f}}_{n,\tau_1,\tau_2}(\omega):=\hat{\mathfrak{f}}_{n,\tau_1,\tau_2}(g_n(\omega)), \quad (4.2)$$

and the function $g_n(\omega)$ is defined in (3.4). In order to show that $\hat{\mathfrak{f}}_{n,\tau_1,\tau_2}(\omega)$ is a consistent estimator of the scaled spectral density $\overset{\circ}{\mathfrak{f}}_{\tau_1,\tau_2}(\omega)$ we make the following additional assumptions.

Assumption (A5) $N_n/n \to 0$, and $\sum_{|k|\leq N_n} W_n^2(k) = O(1/n)$ as $n \to \infty$;

Assumption (A6) For any $\tau_1, \tau_2, \tau_3, \tau_4 \in (0,1)$, $\sum_{k=-\infty}^{\infty} |\gamma_k(\tau_1, \tau_2)| < \infty$, and

$$\sum_{k_2,k_3,k_4=-\infty}^{\infty} |\operatorname{cum}(I\{Y_t \leq q_{\tau_1}\}, I\{Y_{t+k_2} \leq q_{\tau_2}\}, I\{Y_{t+k_2} \leq q_{\tau_3}\}, I\{Y_{t+k_4} \leq q_{\tau_4}\})| < \infty;$$

Assumption (A7) The function $\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}$ defined in (2.3) is continuously differentiable for all $(\tau_1, \tau_2) \in (0,1)^2$;

Theorem 4.1 *Let (A1)–(A7) hold, with $\eta_n = O(1/n)$ and $\kappa_n = O(1/n)$. Then the smoothed Laplace periodogram defined in* (4.1) *and* (4.2) *is a consistent estimator for the scaled Laplace spectral density; more precisely,*

$$\hat{\mathfrak{f}}_{n,\tau_1,\tau_2}(\omega) = 2\pi \stackrel{\circ}{\mathfrak{f}}_{\tau_1,\tau_2}(\omega) + O_{\mathrm{P}}\big(R_n + n^{-1/2} + N_n/n\big) = 2\pi \stackrel{\circ}{\mathfrak{f}}_{\tau_1,\tau_2}(\omega) + o_{\mathrm{P}}(1), \qquad (4.3)$$

where $R_n = |f_Y(q_\tau) - f_{n,X}(q_{\tau,n})| \log n + (n^{-1/8} \vee n^{-1/6} m_n^{1/3})(\log n)^2$.

Proof. The proof proceeds in several steps which are sketched here. The technical details can be found in the Appendix B. We first show (Section 7.1) that

$$\hat{L}_{n,\tau_1,\tau_2}(\omega_{j,n}) = L_{n,\tau_1,\tau_2}(\omega_{j,n})/\big(f_Y(q_{\tau_1}) f_Y(q_{\tau_2})\big) + O_{\mathrm{P}}(R_n), \qquad (4.4)$$

uniformly in the Fourier frequencies $\omega_{j,n} := 2\pi j/n$, where

$$L_{n,\tau_1,\tau_2}(\omega_{j,n}) := n^{-1} d_n(\tau_1, \omega_{j,n}) d_n(\tau_2, -\omega_{j,n}),$$

$d_n(\tau, \omega_{j,n}) := \sum_{t=1}^n \mathrm{e}^{\mathrm{i}\omega_{j,n}t}(\tau - I\{Y_t \leq q_\tau\}) = (1, \mathrm{i})' n^{1/2} \mathbf{b}_{n,\tau,\omega_{j,n}} 2^{-1} f_Y(q_\tau)$ and

$$n^{1/2}\mathbf{b}_{n,\tau,\omega_{j,n}} := \frac{2}{f_Y(q_\tau)} n^{-1/2} \sum_{t=1}^n \begin{pmatrix} \cos(\omega_{j,n}t) \\ \sin(\omega_{j,n}t) \end{pmatrix} (\tau - I\{Y_t \leq q_\tau\}).$$

As an immediate consequence we obtain

$$\hat{\mathfrak{f}}_{n,\tau_1,\tau_2}(\omega_{j,n}) = \sum_{|k|\leq N_n} W_n(k) L_{n,\tau_1,\tau_2}(\omega_{j+k,n})/\big(f_Y(q_{\tau_1}) f_Y(q_{\tau_2})\big) + O_{\mathrm{P}}(R_n).$$

In Section 7.2, we show that, for any $\omega_{j,n} = 2\pi j/n$,

$$K_n := \sum_{|k|\leq N_n} W_n(k) \left(\frac{L_{n,\tau_1,\tau_2}(\omega_{j+k,n})}{f_Y(q_{\tau_1}) f_Y(q_{\tau_2})} - \stackrel{\circ}{\mathfrak{f}}_{\tau_1,\tau_2}(\omega_{j+k,n}) \right) = O_{\mathrm{P}}(1/\sqrt{n}). \qquad (4.5)$$

Now, let $\omega_{j_n n}$ be a sequence of Fourier frequencies that converges to a frequency $\omega \in (0,\pi)$ with rate $|\omega_{j_n n} - \omega| = O(N_n/n)$: both for $f \equiv \Re \stackrel{\circ}{\mathfrak{f}}_{\tau_1,\tau_2}$ and $f \equiv \Im \stackrel{\circ}{\mathfrak{f}}_{\tau_1,\tau_2}$, we have

$$\begin{aligned}
&\Big| \sum_{|k|\leq N_n} W_n(k) \left(f(\omega_{j_n+k,n}) - f(\omega) \right) \Big| \leq \sum_{|k|\leq N_n} W_n(k) |f'(\xi_{j_n+k,n})| \, |\omega_{j_n+k,n} - \omega| \\
&\leq C_n \sum_{|k|\leq N_n} W_n(k) \, |2\pi k/n + \omega_{j_n n} - \omega| \leq C_n \sum_{|k|\leq N_n} W_n(k) \, |2\pi k/n| + C_n \sum_{|k|\leq N_n} W_n(k) \, |\omega_{j_n n} - \omega| \\
&\leq C_n \big(2\pi N_n/n + |\omega_{j_n n} - \omega| \big) \sum_{|k|\leq N_n} W_n(k) = O(N_n/n),
\end{aligned}$$

where $|\xi_{j_n+k,n} - \omega| \leq |\omega - \omega_{j_n+k,n}|$ and $C_n := \sup_{\xi \in \Xi_n} |f'(\xi)|$ is the supremum over

$$\Xi_n = \big[\omega - |\omega - \omega_{j_n,n}| - \omega_{N_n,n},\ \omega + |\omega - \omega_{j_n,n}| + \omega_{N_n,n}|\big].$$

Note that, since $|\omega - \omega_{j_n,n}| \to 0$ and $\omega_{N_n,n} = 2\pi N_n/n \to 0$, $C_n \to f'(\omega)$, so that (C_n) is a bounded sequence. This yields

$$\Big| \sum_{|k|\leq N_n} W_n(k)\Big(\overset{\circ}{\mathfrak{f}}_{\tau_1,\tau_2}(\omega_{j_n+k}) - \overset{\circ}{\mathfrak{f}}_{\tau_1,\tau_2}(\omega)\Big)\Big| = O(N_n/n),$$

and completes the proof of Theorem 4.1. □

For a consistent estimation of the (unscaled) Laplace spectral density $\mathfrak{f}_{\tau_1,\tau_2}(\omega)$, we propose a smoothed version

$$\hat{\underset{\sim}{\mathfrak{f}}}_{n,\tau_1,\tau_2}(\omega) := \hat{\underset{\sim}{\mathfrak{f}}}_{n,\tau_1,\tau_2}(g_n(\omega)), \quad \hat{\underset{\sim}{\mathfrak{f}}}_{n,\tau_1,\tau_2}(\omega_{j,n}) := \sum_{|k|\leq N_n} W_n(k)\hat{\underset{\sim}{L}}_{n,\tau_1,\tau_2}(\omega_{j+k,n}),$$

of the rank-based Laplace periodogram $\hat{\underset{\sim}{L}}_{n,\tau_1,\tau_2}(\omega)$. We then have the following result

Theorem 4.2 *Let Assumptions (A1)–(A3) and (A5)–(A7) hold with $\eta_n = O(1/n)$ and $\kappa_n = O(1/n)$. Then the smoothed rank-based Laplace periodogram $\underset{\sim}{L}_n^{\tau_1\tau_2}$ is a consistent estimator of the (unscaled) Laplace spectral density $\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}$. More precisely,*

$$\hat{\underset{\sim}{\mathfrak{f}}}_{n,\tau_1,\tau_2}(\omega) = 2\pi\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega) + O_{\mathrm{P}}\left(o(n^{-1/8}m_n^{1/4}(\log n)^{3/2}) + N_n/n\right) = 2\pi\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega) + o_{\mathrm{P}}(1).$$

Proof. The proof is very similar to the proof of Theorem 4.1. The main difference lies in the asymptotic representation for the quantity $n^{1/2}\mathbf{b}^U_{n,\tau,\omega}$ denoting the second and third coordinate of the quantity $\boldsymbol{\delta}^U_{n,\tau,\omega}$ in (3.25). Here we have to estimate the difference

$$A_n := \sup_{\omega\in\mathcal{F}_n} \Big\| n^{1/2}\mathbf{b}^U_{n,\tau,\omega} - 2n^{-1/2}\sum_{t=1}^n \begin{pmatrix}\cos(\omega t)\\ \sin(\omega t)\end{pmatrix}(\tau - I\{F_Y(Y_t)\leq\tau\})\Big\|.$$

Direct computation shows that

$$\begin{aligned}
A_n &\leq Cn^{-1/2}\sum_{t=1}^n |I\{F_Y(Y_t)\leq\tau\} - I\{F_{n,X}(X_{t,n})\leq\tau\}| \\
&\leq Cn^{-1/2}\sum_{t=1}^n I\{|F_{n,X}(X_{t,n})-\tau| \leq |F_Y(Y_t) - F_{n,X}(X_{t,n})|\} \\
&\leq Cn^{-1/2}\sum_{t=1}^n I\{|F_{n,X}(X_{t,n})-\tau| \leq \|F_Y - F_{n,X}\|_\infty + \tilde{C}|D_{t,n}|\} \\
&\leq Cn^{-1/2}\sum_{t=1}^n I\{|F_{n,X}(X_{t,n})-\tau| \leq \|F_Y - F_{n,X}\|_\infty + \tilde{C}\eta_n\} + I\{|D_{t,n}|\geq\eta_n\}.
\end{aligned}$$

Since

$$|I\{Y_t\leq x\} - I\{X_{t,n}\leq x\}| \leq I\{|X_{t,n}-x|\leq|D_{t,n}|\} \leq I\{|X_{t,n}-x|\leq\eta_n\} + I\{|D_{t,n}|\geq\eta_n\},$$

$\|F_Y - F_{n,X}\|_\infty = O(\kappa_n + \eta_n)$. Thus, $\mathrm{E}|A_n| = O(\sqrt{n}(\kappa_n + \eta_n))$, hence $A_n = O_\mathrm{P}(\sqrt{n}(\kappa_n + \eta_n))$. The rest of the proof follows by the same arguments as in the proof of Theorem 4.1, yielding the estimate

$$\hat{\mathfrak{f}}_{n,\tau_1,\tau_2}(\omega) = \mathfrak{f}_{\tau_1,\tau_2}(\omega) + O_\mathrm{P}\left(\sqrt{n\log n}(\kappa_n + \eta_n) + (n^{-1/8}m_n^{1/4}\log n)\sqrt{\log n} + n^{-1/2} + N_n/n\right),$$

Finally the assumptions imply $\sqrt{n\log n}(\kappa_n + \eta_n) + n^{-1/2} = o(n^{-1/8}m_n^{1/4}(\log n)^{3/2})$, which completes the proof of Theorem 4.2. □

Note that Theorem 4.1 solves an open and important problem raised in Li [2008, 2011], who considered the Laplace periodogram $\hat{L}_{n,\tau_1,\tau_2}$ for $\tau_1 = \tau_2$. This author showed asymptotic unbiasedness, but did not prove consistency of a smoothed version of the Laplace periodogram. Moreover, as pointed out in Theorem 3.1 the smoothed version of $\hat{L}_{n,\tau_1,\tau_2}$ is not consistent for the copula spectral density kernel, which is the object of interest in our paper. Theorem 4.2 shows that the smoothed rank-based Laplace periodogram yields a consistent estimate of this quantity.

5 Finite sample properties.

5.1 Simulation results.

In order to illustrate the finite sample properties of the new estimates we present a small simulation study. We consider AR(1) processes $Y_t = \vartheta Y_{t-1} + \varepsilon_t$ with $\mathcal{N}(0,1)$- and t_1-distributed innovations ε_t, where $\vartheta := -0.3$. All results presented in this section are based on 5000 simulation runs.

We generate pseudo-random time series of lengths $n = 100$, $n = 500$ and $n = 1000$, of the AR(1) processes and calculate the Laplace and rank-based Laplace periodogram, for $\tau_1, \tau_2 \in \{0.05, 0.25, 0.5, 0.75, 0.95\}$. We also determine the smoothed estimates using Daniell kernels (as in Brockwell and Davis [2006], Example 10.4.3), where the parameters are $(2,1)$ for $n = 100$, $(10,4)$ for $n = 500$ and $(10,25)$ for $n = 1000$. From all calculated periodograms we determine the mean to estimate the expectation of the various estimates. Each of the following figures contains 9 subfigures. For any combination of τ_1 and τ_2, the imaginary parts of periodograms and spectra are represented above the diagonal, whereas the real parts are represented below, or, for $\tau_1 = \tau_2$, on the diagonal (note that in the case $\tau_1 = \tau_2$ all quantities under consideration are real). The curves are plotted against $\omega/(2\pi)$. In all figures, the dashed line represents the "true" spectrum (scaled for Figures 1 and 2; unscaled for Figure 3 and 4) and the solid line the (pointwise) mean of the simulated smoothed Laplace periodograms. In order to illustrate the variability the figures contain some additional information. The dotted lines represent the 0.1 and 0.9 (pointwise) sample quantiles of the unsmoothed periodograms from the 5000 simulation runs, while the border of the gray area represents the 0.1 and 0.9 sample quantiles of the smoothed estimates.

For the sake of brevity only results for sample size $n = 500$ are presented here, but further results, which show a similar behavior, are available from the authors. In Figure 1 we present the results of the Laplace and smoothed Laplace periodogram in the case of Gaussian innovations, while the case of t_1-distributed innovations is shown in Figure 2. Inspection of these figures reveals that the imaginary component of the spectrum is vanishing in the case of Gaussian innovations (see Figure 1). This observation reflects the fact that AR processes with

Gaussian innovations are time-reversible. On the other hand for t_1-distributed innovations this phenomenon only takes place for the extreme quantiles ($\tau_1 = 0.05$, $\tau_2 = 0.95$), meaning that $\mathrm{P}(X_t \le q_{0.05}, X_{t+k} \le q_{0.95})$ is approximately equal to $\mathrm{P}(X_t \le q_{0.95}, X_{t+k} \le q_{0.05})$. This, however, does not hold for $\tau_1 = 0.5$ and $\tau_2 = 0.05$ or 0.95 and frequencies above 0.3, which indicates a time-irreversible impact of extreme values on the central ones. Another finding is that in most cases the bias is usually larger for the estimation of the Laplace spectrum with $\tau_1 = \tau_2$, see, for instance the panels in the diagonals of Figures 1 and 2.

The corresponding rank-based Laplace periodograms are shown in Figure 3 and 4 corresponding to standard normal and t_1-distributed innovations, respectively. The results show a similar type of time reversibility as observed for the Laplace periodogram. It is interesting to note that for the rank-based Laplace periodogram the bias appears to be much smaller, and smoothing seems to be more effective (see Figures 3 and 4)

Finally, we investigate the quality of the estimates by their mean squared properties. In Table 1 we present the square roots of the integrated mean squared errors (MSE). We consider two innovation distributions (t_1 and standard normal) and the smoothed rank-based Laplace periodogram for sample sizes $n = 100$, 500, and 1000. Note that because of symmetry we do not display all combinations. For example the spectra corresponding to the quantiles $(.05, .05)$ and $(.95, .95)$ coincide in the scenario under consideration. We observe a reasonable behavior of the rank-based Laplace periodogram with respect to MSE in all cases. It is interesting to note that in both cases the integrated MSE becomes larger if quantiles from a neighbourhood of $\tau = 0.5$ are involved. For example the integrated MSE is increasing for $(0.05, 0.05)$, $(0.05, 0.25)$ and $(0.05, 0.50)$ and then again decreasing for $(0.05, 0.75)$, $(0.05, 0.95)$. This phenomenon is closely related to the fact that the empirical copula has variance zero at the boundaries of the unit cube, see Genest and Segers [2010] for more details on this subject.

		(τ_1, τ_2)							
ε_t	n	(.05, .05)	(.05, .25)	(.05, .5)	(.05, .75)	(.05, .95)	(.25, .25)	(.25,.5)	(.5,.5)
	100	0.0206	0.0411	0.0465	0.0406	0.0221	0.0646	0.0842	0.0872
$\mathcal{N}(0,1)$	500	0.0086	0.0191	0.0223	0.0195	0.0102	0.0355	0.0444	0.0487
	1000	0.0055	0.0122	0.0143	0.0126	0.0066	0.0231	0.0287	0.0320
	100	0.0217	0.0422	0.0472	0.0412	0.0236	0.0666	0.0863	0.0939
t_1	500	0.0093	0.0196	0.0224	0.0196	0.0112	0.0366	0.0464	0.0531
	1000	0.0060	0.0126	0.0144	0.0126	0.0074	0.0236	0.0298	0.0347

Table 1: *Root Integrated Mean Square Errors of smoothed, rank-based Laplace periodograms, for the two AR(1) examples above, and various series lengths.*

5.2 An empirical application: S&P 500 returns.

The smoothed rank-based Laplace periodogram was computed from the series of daily return values of the S&P 500 index (Jan/2/1963–Dec/31/2009, $n = 11844$), for the same quantile orders as in the previous section; the weighting function is given by a Daniell kernel with parameters (200,100). Results for the smoothed traditional periodogram are shown in Figure 5, while the results for the rank-based Laplace periodogram are presented in Figure 6, with the same convention as described in Section 5.1.

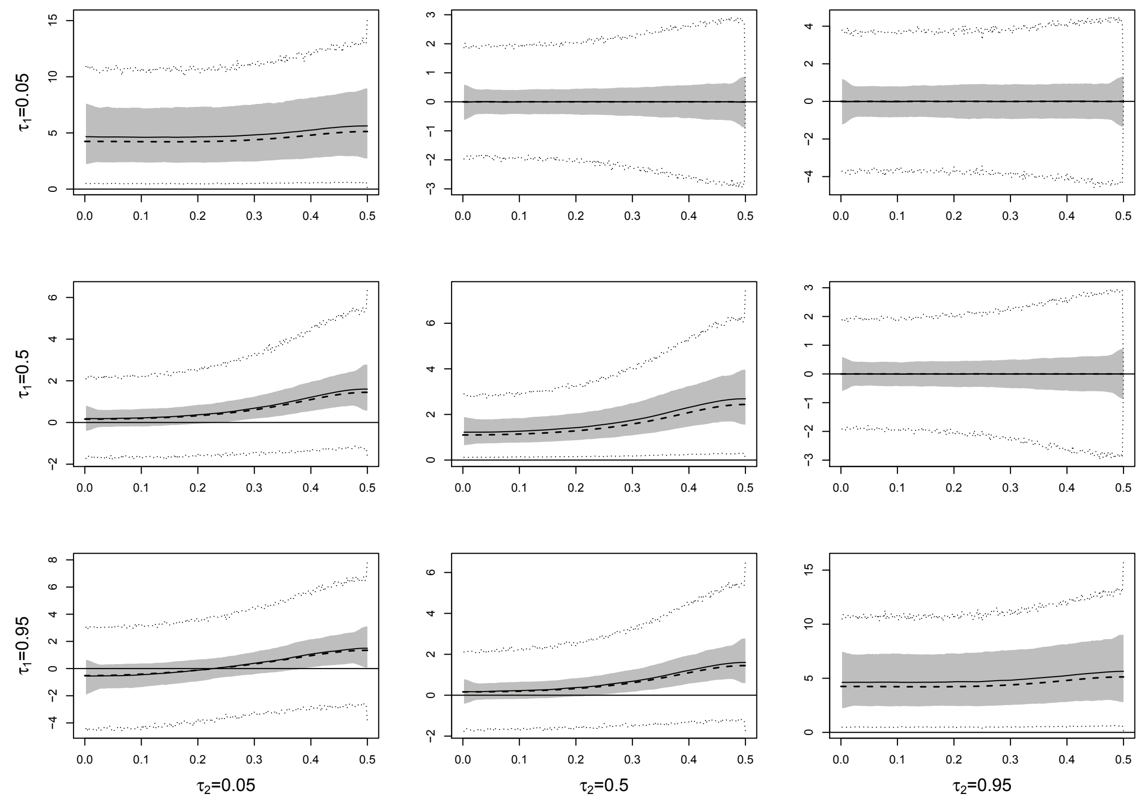

Figure 1: *Smoothed Laplace periodograms and (scaled) spectral densities defined in* (3.3). *The process is an AR(1) process with* $\mathcal{N}(0,1)$*-distributed innovations and the sample size is* 500.

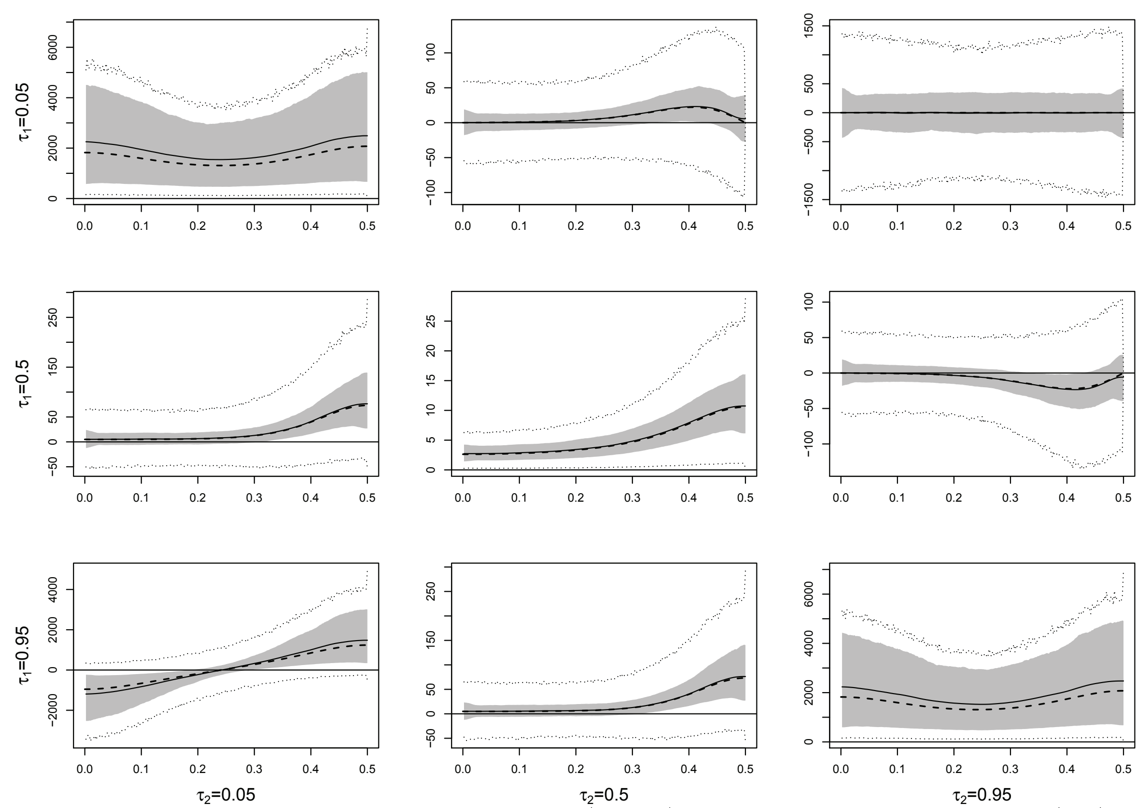

Figure 2: *Smoothed Laplace periodograms and (scaled) spectral densities defined in* (3.3). *The process is an AR(1) process with* t_1*-distributed innovations and the sample size is* 500.

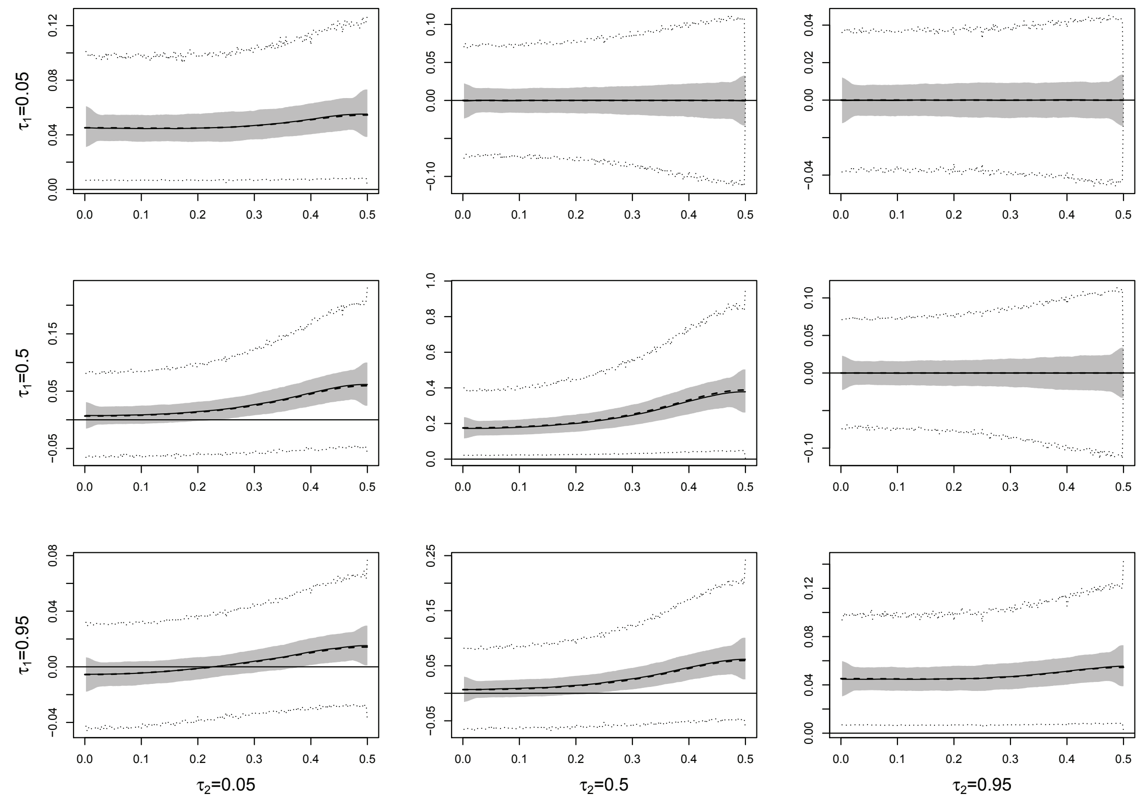

Figure 3: *Smoothed rank-based Laplace periodograms and (unscaled) spectral densities defined in* (2.1). *The process is an AR(1) process with* $\mathcal{N}(0,1)$*-distributed innovations; sample size:* $n = 500$.

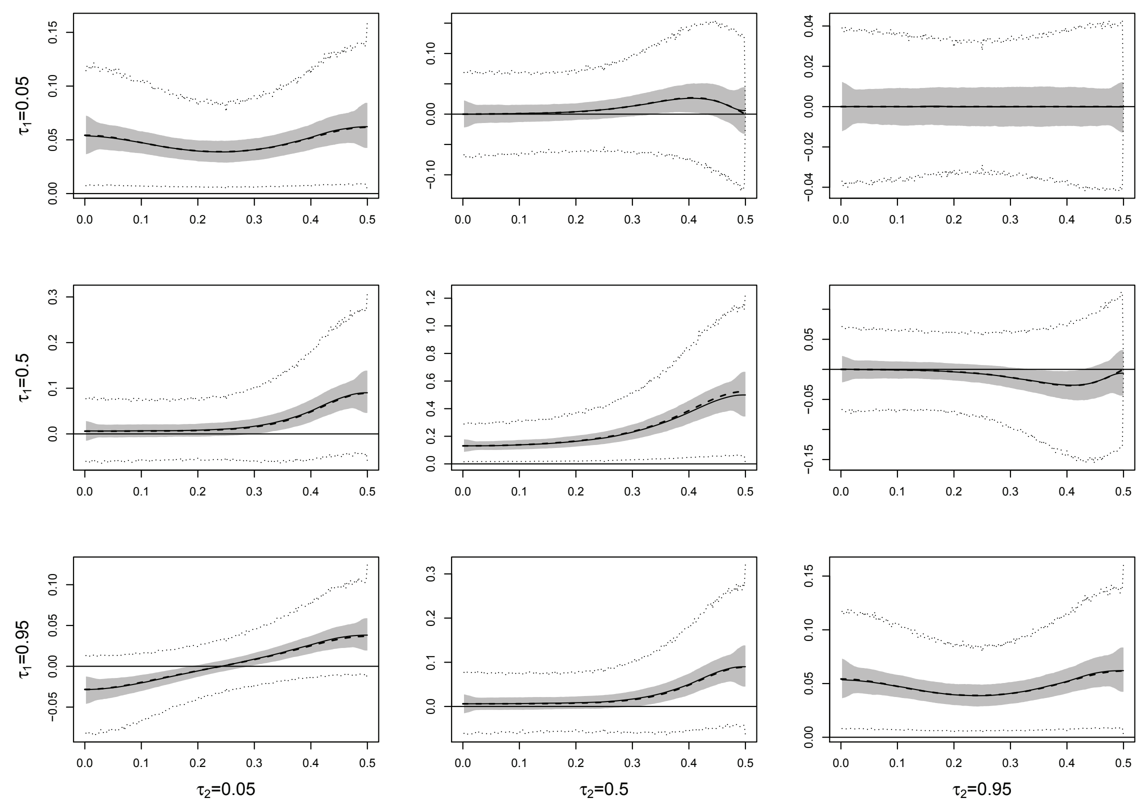

Figure 4: *Smoothed rank-based Laplace periodograms and (unscaled) spectral densities defined in* (2.1). *The process is an AR(1) process with* t_1*-distributed innovations; sample size:* $n = 500$.

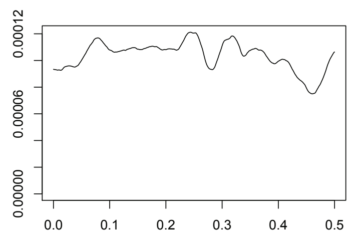

Figure 5: *Smoothed traditional periodogram, S&P 500 returns*

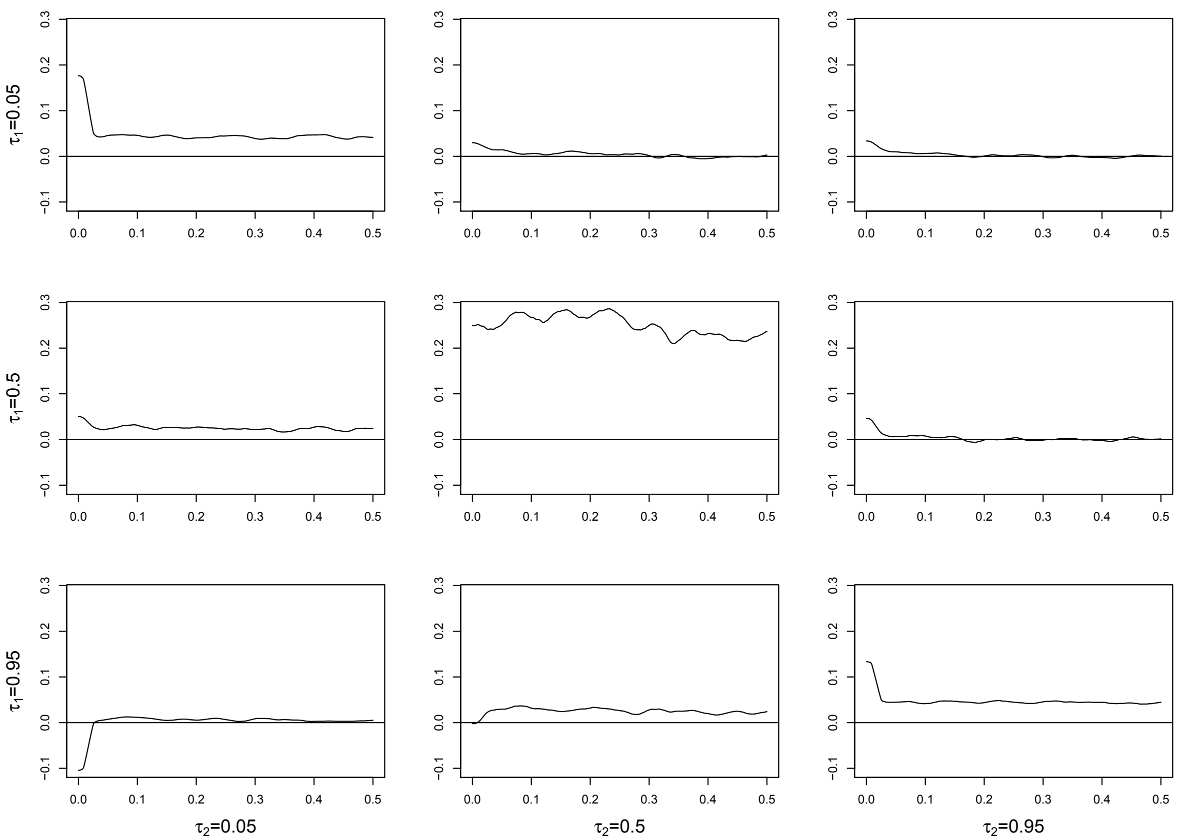

Figure 6: *Smoothed rank-based Laplace periodograms, S&P 500 returns*

The non-linear features of that series have been stressed by many authors (see, e.g. Abhyankar et al. [1997], Berg et al. [2010], Brock et al. [1992], Hinich and Patterson [1990, 1985], Hsieh [1989], Vaidyanathan and Krehbiel [1992]). Those non-linear features cannot be detected by classical correlogram-based spectral methods, and hence do not appear in Figure 5, where the traditional smoothed periodogram is depicted. They do appear, however, in the plots of Figure 6. Whereas the picture for the central quantiles $\tau_1 = \tau_2 = 0.5$ looks quite similar to that in Figure 5, the remaining ones, which involve at least one extreme quantile, are drastically different, indicating a marked discrepancy between tail and central dependence structures. All plots involving at least one extremal quantile yield a peak at the origin, which possibly corresponds to a long-range memory for extremal events. Imaginary parts are not zero, suggesting again time-irreversibility. Such features entirely escape a traditional spectral analysis.

6 Appendix A: Technical details for the proofs in Section 3

In this section we give the technical details for the proofs of Theorem 3.1 and 3.3. The proofs of the results in this section rely on several lemmas. Five of them (Lemma 6.7, 6.8, 6.9, 6.10 and 6.6) are general results, the statement of which we postpone to Section 6.3. Two further ones (Lemmas 6.4 and 6.5) are specific to the proof of (3.25) and presented in Section 6.2.4. Finally, Lemmas 6.1, and 6.2 are auxiliary results used in the proofs of both (3.12) and (3.25); they are regrouped in Section 6.1.1, while Lemma 6.3 (also in Section 6.1.1) is specific to the proof of (3.12).

6.1 Details for the proof of (3.12)

Recall that the estimate (3.12) was obtained by a combination of Lemma 6.1, Lemma 6.2 and (3.14) - (3.15). The Lemmata will be proved in Section 6.1.1 and are also used in a slightly modified form in the proof of (3.25), which is the essential step in the proof of Theorem 3.3. Throughout this section, the notation from the proof of Theorem 3.1 is used.

6.1.1 Three auxiliary Lemmas

The following Lemma generalizes ideas from Pollard [1991].

Lemma 6.1 *Let $B_{a_n}(\mathbf{x})$ denote the closed ball (in $\mathbb{R}^3$) with center $\mathbf{x}$ and radius $a_n > 0$. Assume that, for some sequence of real numbers $a_n = o(1)$,*

$$\Delta_n := \sup_{\omega \in \mathcal{F}_n} \sup_{\boldsymbol{\delta} \in B_{a_n}(\boldsymbol{\delta}^X_{n,\tau,\omega})} |\hat{Z}_{n,\tau,\omega}(\boldsymbol{\delta}) - Z^X_{n,\tau,\omega}(\boldsymbol{\delta})| = o_{\mathrm{P}}(a_n^2).$$

Then, $\sup_{\omega \in \mathcal{F}_n} |\hat{\boldsymbol{\delta}}_{n,\tau,\omega} - \boldsymbol{\delta}^X_{n,\tau,\omega}| = o_{\mathrm{P}}(a_n)$.

Proof. Let $r_{n,\tau,\omega}(\boldsymbol{\delta}) := \hat{Z}_{n,\tau,\omega}(\boldsymbol{\delta}) - Z^X_{n,\tau,\omega}(\boldsymbol{\delta})$. Simple algebra and the explicit form (3.10) of $\boldsymbol{\delta}^X_{n,\tau,\omega}$ yield

$$\hat{Z}_{n,\tau,\omega}(\boldsymbol{\delta}) = \frac{1}{2}(\boldsymbol{\delta} - \boldsymbol{\delta}^X_{n,\tau,\omega})' \mathbf{Q}^X_{n,\tau,\omega}(\boldsymbol{\delta} - \boldsymbol{\delta}^X_{n,\tau,\omega}) - \frac{1}{2}(\boldsymbol{\delta}^X_{n,\tau,\omega})' \mathbf{Q}^X_{n,\tau,\omega} \boldsymbol{\delta}^X_{n,\tau,\omega} + r_{n,\tau,\omega}(\boldsymbol{\delta}). \tag{6.1}$$

Any $\boldsymbol{\delta} \in \mathbb{R}^3 \setminus B_{a_n}(\boldsymbol{\delta}^X_{n,\tau,\omega})$ with distance $l_n := \|\boldsymbol{\delta} - \boldsymbol{\delta}^X_{n,\tau,\omega}\| > a_n$ to $\boldsymbol{\delta}^X_{n,\tau,\omega}$ can be represented as $\boldsymbol{\delta} = \boldsymbol{\delta}^X_{n,\tau,\omega} + l_{n,\tau,\omega}\mathbf{d}_{n,\tau,\omega}$, where $\mathbf{d}_{n,\tau,\omega} := l^{-1}_{n,\tau,\omega}(\boldsymbol{\delta} - \boldsymbol{\delta}^X_{n,\tau,\omega})$. The point $\boldsymbol{\delta}^*_{n,\tau,\omega} = \boldsymbol{\delta}^X_{n,\tau,\omega} + a_n \mathbf{d}_{n,\tau,\omega}$

is on the boundary of the ball $B_{a_n}(\boldsymbol{\delta}^X_{n,\tau,\omega})$. The convexity of $\hat{Z}_{n,\tau,\omega}(\boldsymbol{\delta})$ therefore implies

$$\begin{aligned} a_n l^{-1}_{n,\tau,\omega}\hat{Z}_{n,\tau,\omega}(\boldsymbol{\delta}) \quad + \quad & \big(1-a_n l^{-1}_{n,\tau,\omega}\big)\,\hat{Z}_{n,\tau,\omega}(\boldsymbol{\delta}^X_{n,\tau,\omega}) \geq \hat{Z}_{n,\tau,\omega}(\boldsymbol{\delta}^*_{n,\tau,\omega}) = Z^X_{n,\tau,\omega}(\boldsymbol{\delta}^*_{n,\tau,\omega}) + r_{n,\tau,\omega}(\boldsymbol{\delta}^*_{n,\tau,\omega}) \\ \geq \quad & \frac{1}{2}\mathbf{d}'_{n,\tau,\omega}\mathbf{Q}^X_{n,\tau,\omega}\mathbf{d}_{n,\tau,\omega}a_n^2 - \frac{1}{2}(\boldsymbol{\delta}^X_{n,\tau,\omega})'\mathbf{Q}^X_{n,\tau,\omega}\boldsymbol{\delta}^X_{n,\tau,\omega} - \Delta_n \\ \geq \quad & \frac{1}{2}\mathbf{d}'_{n,\tau,\omega}\mathbf{Q}^X_{n,\tau,\omega}\mathbf{d}_{n,\tau,\omega}a_n^2 + \hat{Z}_{n,\tau,\omega}(\boldsymbol{\delta}^X_{n,\tau,\omega}) - 2\Delta_n. \end{aligned}$$

Rearranging and taking the infimum over ω and $\boldsymbol{\delta}$, we obtain

$$\begin{aligned} &\inf_{\omega\in\mathcal{F}_n}\ \inf_{\boldsymbol{\delta}:|\boldsymbol{\delta}-\boldsymbol{\delta}^X_{n,\tau,\omega}|>a_n}\big(\hat{Z}_{n,\tau,\omega}(\boldsymbol{\delta}) - \hat{Z}_{n,\tau,\omega}(\boldsymbol{\delta}^X_{n,\tau,\omega})\big) \\ &\qquad\qquad \geq \inf_{\omega\in\mathcal{F}_n}\ \inf_{\boldsymbol{\delta}:|\boldsymbol{\delta}-\tilde{\boldsymbol{\delta}}_{n,\tau,\omega}|>a_n} l_{n,\tau,\omega}\, a_n^{-1}\big(\frac{1}{2}\mathbf{d}'_{n,\tau,\omega}\mathbf{Q}^X_{n,\tau,\omega}\mathbf{d}_{n,\tau,\omega}a_n^2 - 2\Delta_n\big). \qquad (6.2) \end{aligned}$$

By assumption, the smallest eigenvalue of $\mathbf{Q}^X_{n,\tau,\omega}$ is bounded away from zero uniformly in $\omega \in \mathcal{F}_n$, for n sufficiently large. Hence, $2\Delta_n < \frac{1}{2}\mathbf{d}'_{n,\tau,\omega}\mathbf{Q}^X_{n,\tau,\omega}\mathbf{d}_{n,\tau,\omega}a_n^2$ with probability tending to one, the right-hand side in (6.2) is strictly positive, and the minimum of $\hat{Z}_{n,\tau,\omega}(\boldsymbol{\delta})$ cannot be attained at any $\boldsymbol{\delta}$ with $|\boldsymbol{\delta} - \boldsymbol{\delta}^X_{n,\tau,\omega}| > a_n$. □

Lemma 6.2 *For any $\tau \in (0,1)$, there exists a finite constant A such that*

$$\lim_{n\to\infty} \mathrm{P}\big(\sup_{\omega\in\mathcal{F}_n}\|\boldsymbol{\delta}^X_{n,\tau,\omega}\|_\infty > A\sqrt{\log n}\big) = 0.$$

Proof. For any $A > 0$,

$$\mathrm{P}\big(\sup_{\omega\in\mathcal{F}_n}\|\boldsymbol{\delta}^X_{n,\tau,\omega}\|_\infty \geq A\sqrt{\log n}\big) \quad\leq\quad n\sup_{\omega\in\mathcal{F}_n}\mathrm{P}\big(\|\mathbf{S}_{n,\omega}\|_\infty \geq A\sqrt{n\log n}\big), \qquad (6.3)$$

where $\mathbf{S}_{n,\omega} := n^{1/2}\boldsymbol{\delta}^X_{n,\tau,\omega} = \sum_{t=1}^n \mathbf{H}_{t,\omega}$ with $\mathbf{H}_{t,\omega} := (\mathbf{Q}^X_{n,\tau,\omega})^{-1}\mathbf{c}_t(\omega)(\tau - I\{X_{t,n} \leq q_{n,\tau}\})$. In order to bound the probabilities on the right-hand side of (6.3), we apply Lemma 6.8, with $x_n = 2\sqrt{\log n}$ and $\psi_n = n^{1/6}m_n^{-2/3}x_n^{-1}$, to each component of $\mathbf{S}_{n,\omega}$. Let $S_{n,\omega,j}$ and $H_{t,\omega,j}$, $j = 1,2,3$ denote the jth components of $\mathbf{S}_{n,\omega}$ and $\mathbf{H}_{t,\omega}$, respectively. The quantities $H_{t,\omega,j}$ form a triangular array of centered, m_n-dependent, real-valued random variables, and each of them is bounded by a finite constant that does not depend on ω. Therefore, Condition 2 of Lemma 6.8 holds. Next, observe that, for $\omega \in \mathcal{F}_n$, we have $\mathrm{E}[\mathbf{S}_{n,\omega}\mathbf{S}'_{n,\omega}] = n\ \mathrm{diag}(1,4,4)\mathbf{W}_n$, where

$$\mathbf{W}_n = \frac{1}{n}\sum_{s=1}^n\sum_{t=1}^n \mathbf{c}_s(\omega)\mathbf{c}'_t(\omega)(\mathrm{E}[I\{X_{s,n} \leq q_{n,\tau}\}I\{X_{t,n} \leq q_{n,\tau}\}] - \tau^2).$$

Each entry of $\mathbf{c}_s(\omega)\mathbf{c}'_t(\omega)$ is uniformly bounded by 1. Together with Assumption (A3), this implies that the diagonal entries of $\mathbf{W}_n$ are bounded uniformly in ω and n. Because all terms are centered, we have $B^2_{n,\omega,j} := \mathrm{E}\,S^2_{n,\omega,j} = O(n)$. Note that $\psi_n(n^{-1/2}m_n^2x_n^3)^{1/3} = 1$. Therefore, the left-hand side in Condition 1 of Lemma 6.8 is of the order $x_nm_n = O(n^{1/4})$ and, since $\psi_n^2(nm_n^2x_n^3)^{2/3} = n$, that condition holds. Condition 3 follows from the fact that $\psi_n \to \infty$ and $n^{-1/2}m_n^2\psi_n^3x_n^3 = 1$, and Condition 4 from the assumption that $m_n = O(n^{1/4-a})$. Lemma 6.8 therefore applies, yielding

$$\mathrm{P}\big(\|\mathbf{S}_{n,\omega}\|_\infty \geq A\sqrt{n\log n}\big) \leq \mathrm{P}\big(\|\mathbf{S}_{n,\omega}\|_\infty \geq 2\sqrt{6}x_n\mu_n\big) \leq 4\exp(-4\log n) = 4n^{-4},$$

where $x_n\mu_n = x_n(B_n^2 + \psi_n^2(nm_n^2x_n^3)^{2/3})^{1/2} = x_n(B_n^2+n)^{1/2} = O(\sqrt{n\log n})$. Consequently, the constant A can always be chosen in such a way that, for n sufficiently large,

$$A\sqrt{n\log n} \geq 2\sqrt{6}x_n(B_n^2 + \psi_n^2(nm_n^2x_n^3)^{2/3})^{1/2} = 2\sqrt{6}x_n\mu_n.$$

The claim follows. $\square$

Lemma 6.3 *For the Fourier frequencies $\omega_{j,n} \in \mathcal{F}_n$, let $\Gamma_n(\boldsymbol{\delta};\tau,\omega_{j,n}) := \sum_{t=1}^n H_t(\boldsymbol{\delta};\tau,\omega_{j,n})$, where*

$$H_t(\boldsymbol{\delta};\tau,\omega_{j,n}) := \int_0^{n^{-1/2}\mathbf{c}_t'(\omega_{j,n})\boldsymbol{\delta}} (I\{X_{t,n} \leq s + q_{n,\tau}\} - I\{X_{t,n} \leq q_{n,\tau}\})\mathrm{d}s.$$

Then, $\mathrm{E}[\Gamma_n(\boldsymbol{\delta};\tau,\omega_{j,n})] = f_{n,X}(q_{n,\tau})(2n)^{-1}\sum_{t=1}^n(\mathbf{c}_t'(\omega_{j,n})\boldsymbol{\delta})^2 + r_1(\tau,\omega_{j,n})$, *where* $|r_1(\omega_{j,n},\tau)| \leq C_4\|\boldsymbol{\delta}\|^3n^{-1/2}$ *for some finite constant C_4. Moreover, if $\|\boldsymbol{\delta}\| \leq A\log n$, there exists a positive constant C_1 such that, for sufficiently large n, any τ and $\omega_{j,n}$,*

$$\mathrm{Var}(\Gamma_n(\boldsymbol{\delta};\tau,\omega_{j,n})) \leq n^{-1/2}C_1\|\boldsymbol{\delta}\|^3.$$

Proof. Assumption (A2) guarantees that all calculations that follow still hold when distributions depend on n. Due to m_n-dependence,

$$\mathrm{Var}(\Gamma_n(\boldsymbol{\delta};\tau,\omega)) = \sum_{t=1}^n \mathrm{Var}(H_t(\boldsymbol{\delta};\tau,\omega)) + \sum_{t_1=1}^n \sum_{\substack{1\leq t_2\neq t_1\leq n\\ |t_2-t_1|\leq m_n}} \mathrm{Cov}(H_{t_1}(\boldsymbol{\delta};\tau,\omega), H_{t_2}(\boldsymbol{\delta};\tau,\omega)).$$

Note that

$$\begin{aligned}
&\mathrm{E}\big[H_t(\boldsymbol{\delta};\tau,\omega_{j,n})^2\big] \\
&= \mathrm{E}\Big[\int_0^{n^{-1/2}\mathbf{c}_t'(\omega_{j,n})\boldsymbol{\delta}}\int_0^{n^{-1/2}\mathbf{c}_t'(\omega_{j,n})\boldsymbol{\delta}} (I\{X_{t,n}\leq u+q_{n,\tau}\} - I\{X_{t,n}\leq q_{n,\tau}\}) \\
&\qquad\qquad \times (I\{X_{t,n}\leq v+q_{n,\tau}\} - I\{X_{t,n}\leq q_{n,\tau}\})\,\mathrm{d}u\mathrm{d}v\Big] \\
&= \mathrm{E}\Big[\int_0^{n^{-1/2}\mathbf{c}_t'(\omega_{j,n})\boldsymbol{\delta}}\int_0^{n^{-1/2}\mathbf{c}_t'(\omega_{j,n})\boldsymbol{\delta}} (I\{X_{t,n}\leq (u\wedge v)+q_{n,\tau}\} - I\{X_{t,n}\leq (u\wedge 0)+q_{n,\tau}\} \\
&\qquad\qquad - I\{X_{t,n}\leq (v\wedge 0)+q_{n,\tau}\} + I\{X_{t,n}\leq q_{n,\tau}\})\mathrm{d}u\mathrm{d}v\Big] \\
&= \int_0^{n^{-1/2}\mathbf{c}_t'(\omega_{j,n})\boldsymbol{\delta}}\int_0^{n^{-1/2}\mathbf{c}_t'(\omega_{j,n})\boldsymbol{\delta}} (u\wedge v - u\wedge 0 - v\wedge 0)\,f_{n,X}(q_{n,\tau}) + r_2(u,v,\tau)\mathrm{d}u\mathrm{d}v \qquad (6.4)\\
&= 3^{-1}n^{-3/2}f_{n,X}(q_{n,\tau})\left|\mathbf{c}_t'(\omega_{j,n})\boldsymbol{\delta}\right|^3 + r_3(\omega_{j,n},\tau), \qquad (6.5)
\end{aligned}$$

where $|r_2(u,v,\tau)| \leq C_1(u^2+v^2)$, hence $|r_3(\omega_{j,n},\tau)| \leq C_2\|\boldsymbol{\delta}\|^4n^{-2}$. Equality (6.4) follows via a Taylor expansion, (6.5) from the fact that $\int_0^x\int_0^x (u\wedge v - u\wedge 0 - v\wedge 0)\,\mathrm{d}u\mathrm{d}v = \frac{1}{3}|x|^3$. Similarly,

$$\mathrm{E}\big[H_t(\boldsymbol{\delta};\tau,\omega_{j,n})\big] = \mathrm{E}\Big[\int_0^{n^{-1/2}\mathbf{c}_t'(\omega_{j,n})\boldsymbol{\delta}} (I\{X_{t,n}\leq u+q_{n,\tau}\} - I\{X_{t,n}\leq q_{n,\tau}\})\Big]\mathrm{d}u \qquad (6.6)$$

$$= \int_0^{n^{-1/2}\mathbf{c}_t'(\omega_{j,n})\boldsymbol{\delta}} (f_{n,X}(q_{n,\tau})u + r_4(u,\tau))\,\mathrm{d}u = \frac{f_{n,X}(q_{n,\tau})}{2n}(\mathbf{c}_t'(\omega_{j,n})\boldsymbol{\delta})^2 + r_1(\tau,\omega_{j,n})$$

where $|r_4(u,\tau)| \leq C_3 u^2$, hence $|r_1(\omega_{j,n},\tau)| \leq C_4 \|\boldsymbol{\delta}\|^3 n^{-3/2}$. Thus,

$$\mathrm{E}[H_t(\boldsymbol{\delta};\tau,\omega_{j,n})]^2 = n^{-2}\Big(\frac{1}{2} f_{n,X}(q_{n,\tau})(\mathbf{c}_t'(\omega_{j,n})\boldsymbol{\delta})^2\Big)^2 + r_5(\omega_{j,n},\tau),$$

where $|r_5(\omega_{j,n},\tau)| \leq C_5(\|\boldsymbol{\delta}\|^5 + \|\boldsymbol{\delta}\|^6) n^{-5/2}$. For the covariances, we have

$$\begin{aligned}
&\mathrm{E}\big[H_{t_1}(\boldsymbol{\delta};\tau,\omega_{j,n})\, H_{t_2}(\boldsymbol{\delta};\tau,\omega_{j,n})\big] \\
= \;& \mathrm{E}\Big[\int_0^{n^{-1/2}\mathbf{c}_{t_1}'(\omega_{j,n})\boldsymbol{\delta}} \int_0^{n^{-1/2}\mathbf{c}_{t_2}'(\omega_{j,n})\boldsymbol{\delta}} (I\{X_{t_1,n} \leq u + q_{n,\tau}\} - I\{X_{t_1,n} \leq q_{n,\tau}\}) \\
& \qquad \times (I\{X_{t_2,n} \leq v + q_{n,\tau}\} - I\{X_{t_2,n} \leq q_{n,\tau}\})\,\mathrm{d}u\mathrm{d}v\Big] \\
= \;& \int_0^{n^{-1/2}\mathbf{c}_{t_1}'(\omega_{j,n})\boldsymbol{\delta}} \int_0^{n^{-1/2}\mathbf{c}_{t_2}'(\omega_{j,n})\boldsymbol{\delta}} F_{t_2-t_1,n,X}(u+q_{n,\tau}, v+q_{n,\tau}) - F_{t_2-t_1,n,X}(q_{n,\tau}, v+q_{n,\tau}) \\
& \qquad - F_{t_2-t_1,n,X}(u+q_{n,\tau}, q_{n,\tau}) + F_{t_2-t_1,n,X}(q_{n,\tau}, q_{n,\tau})\mathrm{d}u\mathrm{d}v \\
= \;& \int_0^{n^{-1/2}\mathbf{c}_{t_1}'(\omega_{j,n})\boldsymbol{\delta}} \int_0^{n^{-1/2}\mathbf{c}_{t_1}'(\omega_{j,n})\boldsymbol{\delta}} r_6(u,v,\tau)\mathrm{d}u\mathrm{d}v \qquad (6.7) \\
= \;& r_7(\omega_{j,n},\tau),
\end{aligned}$$

where $|r_6(u,v,\tau)| \leq C_6(u^2+v^2)$, hence $|r_7(u,v,\tau)| \leq C_7\|\boldsymbol{\delta}\|^4 n^{-2}$; equality (6.7) follows via a Taylor expansion and some straightforward algebra. From (6.4), (6.5) and (6.7), we obtain

$$\mathrm{Var}(\Gamma_n(\boldsymbol{\delta};\tau,\omega_{j,n})) = \frac{1}{n^{3/2}} f_{n,X}(q_{n,\tau}) \sum_{t=1}^n \big|\mathbf{c}_t'(\omega_{j,n})\boldsymbol{\delta}\big|^3 + R_n,$$

with $|R_n| \leq C_3\|\boldsymbol{\delta}\|^4 m_n n^{-1}$ for sufficiently large n, and

$$\sum_{t=1}^n \big|\mathbf{c}_t'(\omega_{j,n})\boldsymbol{\delta}\big|^3 \leq \sum_{t=1}^n \Big|\|\mathbf{c}_t(\omega_{j,n})\|_\infty \sqrt{3}\|\boldsymbol{\delta}\|\Big|^3 \leq 3^{3/2}\sum_{t=1}^n \|\boldsymbol{\delta}\|^3 = 3^{3/2} n \|\boldsymbol{\delta}\|^3.$$

This completes the proof. □

6.1.2 Proof of (3.14)

Applying Knight's identity (Knight [1998]; see p. 121 of Koenker [2005]) to (3.6) and (3.7), we obtain

$$\hat{Z}_{n,\tau,\omega}(\boldsymbol{\delta}) = \hat{Z}_{n,\tau,\omega,1}(\boldsymbol{\delta}) + \hat{Z}_{n,\tau,\omega,2}(\boldsymbol{\delta}) \quad \text{and} \quad \hat{Z}^X_{n,\tau,\omega}(\boldsymbol{\delta}) = \hat{Z}^X_{n,\tau,\omega,1}(\boldsymbol{\delta}) + \hat{Z}^X_{n,\tau,\omega,2}(\boldsymbol{\delta}),$$

where

$$\begin{aligned}
\hat{Z}_{n,\tau,\omega,1}(\boldsymbol{\delta}) &= -\boldsymbol{\delta}' n^{-1/2}\sum_{t=1}^n \mathbf{c}_t(\omega)(\tau - I\{Y_t \leq q_\tau\}), \\
\hat{Z}_{n,\tau,\omega,2}(\boldsymbol{\delta}) &= \sum_{t=1}^n \int_0^{n^{-1/2}\mathbf{c}_t'(\omega)\boldsymbol{\delta}} \big(I\{Y_t \leq q_\tau + s\} - I\{Y_t \leq q_\tau\}\big)\mathrm{d}s, \\
\hat{Z}^X_{n,\tau,\omega,1}(\boldsymbol{\delta}) &= -\boldsymbol{\delta}' n^{-1/2}\sum_{t=1}^n \mathbf{c}_t(\omega)(\tau - I\{X_{t,n} \leq q_{n,\tau}\}), \\
\hat{Z}^X_{n,\tau,\omega,2}(\boldsymbol{\delta}) &= \sum_{t=1}^n \int_0^{n^{-1/2}\mathbf{c}_t'(\omega)\boldsymbol{\delta}} \big(I\{X_{t,n} \leq q_{n,\tau} + s\} - I\{X_{t,n} \leq q_{n,\tau}\}\big)\mathrm{d}s.
\end{aligned}$$

Let us show that $\hat{Z}_{n,\tau,\omega,i}$ and $\hat{Z}^X_{n,\tau,\omega,i}$, $i = 1, 2$, are uniformly close in probability. For $i = 1$, note that, with probability tending to one, for any sequence of positive real numbers η_n, uniformly over $\|\boldsymbol{\delta}\| \le A\sqrt{\log n}$,

$$|\hat{Z}_{n,\tau,\omega,1}(\boldsymbol{\delta}) - \hat{Z}^X_{n,\tau,\omega,1}(\boldsymbol{\delta})| = n^{-1/2}\Big|\boldsymbol{\delta}' \sum_{t=1}^{n} \mathbf{c}_t(\omega)(I\{X_{t,n} \le q_\tau - D_{t,n}\} - I\{X_{t,n} \le q_{n,\tau}\})\Big|$$

$$\le A\sqrt{n \log n}\, \frac{1}{n} \sum_{t=1}^{n} \Big(I\{|D_{t,n}| > \eta_n\} + I\{|X_{t,n} - q_{n,\tau}| \le \eta_n + |q_\tau - q_{n,\tau}|\}\Big),$$

since $I\{X_{t,n} \le q_\tau - D_{t,n}\} - I\{X_{t,n} \le q_{n,\tau}\} \neq 0$ implies $|X_{t,n} - q_{n,\tau}| \le |D_{t,n}| + |q_\tau - q_{n,\tau}|$ which, in turn, implies $|X_{t,n} - q_{n,\tau}| \le \eta_n + |q_\tau - q_{n,\tau}|$ or $|D_{t,n}| > \eta_n$. Taking expectations, part (3.2) of Assumption (A1), the strict stationarity of $(D_{t,n})$ and $(X_{t,n})$, and a Taylor expansion yield

$$\mathrm{E}|\hat{Z}_{n,\tau,\omega,1}(\boldsymbol{\delta}) - \hat{Z}^X_{n,\tau,\omega,1}(\boldsymbol{\delta})| \le A\sqrt{n \log n}\, O(\kappa_n + \eta_n + |q_\tau - q_{n,\tau}|).$$

Using similar arguments and the same reasoning as in Section 6.2.1, we obtain

$$\sup_{\|\boldsymbol{\delta}\| \le A\sqrt{\log n}} |\hat{Z}_{n,\tau,\omega,2}(\boldsymbol{\delta}) - \hat{Z}^X_{n,\tau,\omega,2}(\boldsymbol{\delta})| = O_{\mathrm{P}}\big((\kappa_n + \eta_n + r_n(O(\kappa_n + \eta_n), m_n))\sqrt{n \log n}\big),$$

where the function r_n is defined by

$$r_n(a_n, m_n) := C_2 n^{-1/2}\Big(m_n a_n + \big(\frac{m_n}{n} \vee a_n\big)^{2/3} m_n^{4/3} n^{-1/3} (\log n)^3\Big)^{1/2} (\log n)^{1/2}. \tag{6.8}$$

In order to complete the proof, let us show that $|q_\tau - q_{n,\tau}| = O(\eta_n + \kappa_n)$. Observe that

$$|\mathrm{P}(X_{1,n} + D_{1,n} \le t) - \mathrm{P}(X_{1,n} \le t)| \le \mathrm{E}[I\{|X_{1,n} - t| \le \eta_n\} + I\{|D_{1,n}| > \eta_n\}] = O(\eta_n + \kappa_n).$$

Thus, $\|F_Y - F_{n,X}\|_\infty = O(\eta_n + \kappa_n)$. Let g be strictly increasing, and h increasing, on $[a, b]$, with $|g(x) - g(y)| \ge c|x - y|$ for some $c > 0$ and $\sup_{t \in [a,b]} |g(t) - h(t)| \le L$. Denote by h^{-1} the generalized inverse of h on $[a, b]$ (namely, $h^{-1}(p) := \inf\{t | h(t) \ge p\}$). Then,

$$\sup_{t \in [g(a)+2L/c,\, g(b)-2L/c]} |h^{-1}(t) - g^{-1}(t)| \le 2L/c.$$

Letting $g = F_{n,X}, h = F_Y$, and taking into account the fact that $f_{n,X} > 0$, hence $F_{n,X}$ strictly increasing, on $[F^{-1}_{n,X}(\tau) - d, F^{-1}_{n,X}(\tau) + d]$, we thus obtain

$$|q_{n,\tau} - q_\tau| = O(\eta_n + \kappa_n), \tag{6.9}$$

which completes the proof. □

6.1.3 Proof of (3.15)

Due to Knight's identity, we have

$$K_{2n}(\boldsymbol{\delta}; \tau, \omega) := \hat{Z}^X_{n,\tau,\omega}(\boldsymbol{\delta}) - Z^X_{n,\tau,\omega}(\boldsymbol{\delta}) = \sum_{t=1}^{n} W_{t,n}(\omega, \boldsymbol{\delta}),$$

with

$$W_{t,n}(\omega, \boldsymbol{\delta}) := \int_0^{n^{-1/2}\mathbf{c}_t'(\omega)\boldsymbol{\delta}} \big(I\{X_{t,n} \le s + q_{n,\tau}\} - I\{X_{t,n} \le q_{n,\tau}\}\big) ds - \frac{f_{n,X}(q_{n,\tau})}{2n} \big(\boldsymbol{\delta}'\mathbf{c}_t(\omega)\big)^2.$$

Direct calculations yield, for some finite constants C_1, C_2, C_3 and n large enough,

$$\sup_{\omega\in\mathcal{F}_n}\sup_t |\mathrm{E}[W_{t,n}(\omega,\boldsymbol{\delta})]| \le C_1\|\boldsymbol{\delta}\|^3 n^{-3/2}, \qquad \sup_{\omega\in\mathcal{F}_n}\sup_t |W_{t,n}(\omega,\boldsymbol{\delta})| \le C_2\|\boldsymbol{\delta}\| n^{-1/2} \text{ a.s.,} \tag{6.10}$$

$$\sup_{\omega\in\mathcal{F}_n}\sup_t \mathrm{E}[|W_{t,n}(\omega,\boldsymbol{\delta})|^p] \le C_3\|\boldsymbol{\delta}\|^{1+p} n^{-(1+p)/2} \tag{6.11}$$

and, for $n \ge n_0$ with $n_0 \in \mathbb{N}$ independent of t and ω,

$$\sup_{\omega\in\mathcal{F}_n} |K_{2n}(a;\tau,\omega) - K_{2n}(b;\tau,\omega)| \le 3\sqrt{n}\|a-b\|.$$

It is obviously possible to construct $N = o(n^5)$ points $d_1, ..., d_N$ (the dependence of the points on n is not reflected in the notation) such that for every $\boldsymbol{\delta} \in \{\boldsymbol{\delta} : \|\boldsymbol{\delta}\|_\infty \le A\sqrt{\log n}\}$ there exists an index $j(\delta)$ for which $\|\boldsymbol{\delta} - d_{j(\delta)}\| \le n^{-3/2}$. The computations above then show that

$$\sup_{\omega\in\mathcal{F}_n}\sup_{\|\boldsymbol{\delta}\|_\infty\le A\sqrt{\log n}} |K_{2n}(\boldsymbol{\delta};\tau,\omega)| \le \sup_{\omega\in\mathcal{F}_n}\sup_{j=1,\dots,N} |K_{2n}(d_j;\tau,\omega)| + O_{\mathrm{P}}(n^{-1}).$$

Since $\#\mathcal{F}_n \le n$,

$$\mathrm{P}\Big(\sup_{\omega\in\mathcal{F}_n}\sup_{j=1,\dots,N} |K_{2n}(d_j;\tau,\omega)| \ge a_n\Big) \le Nn \sup_{\omega\in\mathcal{F}_n}\sup_{j=1,\dots,N} \mathrm{P}\big(|K_{2n}(d_j;\tau,\omega)| \ge a_n\big). \tag{6.12}$$

Let us show that, for a sequence a_n with $a_n = o((n^{-1/4} \vee n^{-1/3}m_n^{2/3})(\log n)^3)$, we can make the probabilities in (6.12) tend to zero as fast as n^{-D}, for any $D \in \mathbb{N}$, which is fast enough to compensate for the factor $nN = o(n^6)$ and hence will complete the proof.
For this purpose, we apply Lemma 6.8 again. Fix ω and d_j, and observe that the collection of random variables

$$V_{t,n}(\omega, d_j) := n^{1/2}(\log n)^{-1/2}(W_{t,n}(\omega,d_j) - \mathrm{E}W_{t,n}(\omega,d_j)), \quad t = 1, ..., n$$

are centered and m_n-dependent. Note that, for $\|\boldsymbol{\delta}\| \le A\sqrt{\log n}$, we have

$$\sup_{\omega\in\mathcal{F}_n}\sup_t |V_{t,n}(\omega,\boldsymbol{\delta})| \le AC_2(1+o(1)), \quad \sup_{\omega\in\mathcal{F}_n}\sup_t \mathrm{E}[|V_{t,n}(\omega,\boldsymbol{\delta})|^3] \le A^4 C_3 n^{-1/2}(\log n)^{1/2}, \tag{6.13}$$

and, as a consequence of (6.10), (6.11) and Lemma 6.3 in Section 6.1.1,

$$\sup_{\omega\in\mathcal{F}_n}\sup_t \mathrm{E}\Big[\Big(\sum_t V_{t,n}(\omega,\boldsymbol{\delta})\Big)^2\Big] \le A^3 C_4 n^{1/2}(\log n)^{1/2}(1+o(1)).$$

Let $x_n = D\sqrt{\log(n)}$ and $\psi_n = (n^{-1/3}m_n^{1/3} \vee \sup_t(\mathrm{E}|V_{t,n}(\omega,\boldsymbol{\delta})|^3)^{1/3})\log n$. Since $n^{-1/3}m_n^{1/3}$ is $O(n^{-1/4-a/3})$ and $\sup_t(\mathrm{E}|V_{t,n}(\omega,\boldsymbol{\delta})|^3)^{1/3}$ is $O(n^{-1/6}(\log n)^{1/6})$, ψ_n is $O(n^{-1/6}(\log n)^{7/6})$, hence $o(1)$. In view of (6.13), the sufficient condition in part (ii) of Lemma 6.8 is satisfied, so that Conditions 1 and 3 of Lemma 6.8 hold. Obviously, Condition 2 also follows from (6.13). Because of the definition of x_n and the assumption that $m_n = O(n^{1/4-a})$, Condition 4 holds as well. It thus follows from Lemma 6.8 that there exists a finite constant C_D such that, for sufficiently large n,

$$\mathrm{P}\Big(\Big|\sum_t V_{t,n}(\omega,d_j)\Big| \ge C_D\big((n\log n)^{1/2} \vee (n^{1/3}m_n^{4/3}(\log n)^{10/3})\big)^{1/2}\Big) \le 4n^{-D}(1+o(1)).$$

Setting

$$a_n := n^{-1/2}(\log n)^{1/2} C_D\big((n\log n)^{1/2} \vee (n^{1/3} m_n^{4/3}(\log n)^{10/3})\big)^{1/2},$$

we have $a_n = o((n^{-1/4} \vee n^{-1/3} m_n^{2/3})(\log n)^3)$; it thus follows that

$$\begin{aligned}
\mathrm{P}\left(|K_{2n}(d_j;\tau,\omega)| \geq a_n\right) &\leq \mathrm{P}\Big(\Big|\sum_t (W_{t,n}(\omega,d_j) - \mathrm{E}W_{t,n}(\omega,d_j))\Big| \geq a_n - n\sup_t |\mathrm{E}W_{t,n}(\omega,d_j)|\Big) \\
&\leq \mathrm{P}\Big(\Big|\sum_t V_{t,n}(\omega,d_j)\Big| \geq n^{1/2}(\log n)^{-1/2} a_n - O(\log n)\Big) \leq 4n^{-D}(1+o(1)).
\end{aligned}$$

This completes the proof. □

6.2 Details for the proof of (3.25)

Subsections 6.2.1–6.2.3 contain the proofs of (3.27) - (3.29) that were basic in establishing (3.25). Some auxiliary results used in the proofs are collected in Section 6.2.4. Denote by $\hat{F}_{n,Y}$ and $\hat{F}_{n,X}$ the empirical distribution functions of $Y_1,\ldots,Y_n$ and $X_{1,n},\ldots,X_{n,n}$, respectively. Throughout this section, the notation from the proof of Theorem 3.3 is used.

6.2.1 Proof of (3.27)

From Knight's identity, we have

$$\hat{\underset{\sim}{Z}}_{n,\tau,\omega}(\boldsymbol{\delta}) = \hat{\underset{\sim}{Z}}_{n,\tau,\omega,1}(\boldsymbol{\delta}) + \hat{\underset{\sim}{Z}}_{n,\tau,\omega,2}(\boldsymbol{\delta}) \quad \text{and} \quad \hat{\underset{\sim}{Z}}^X_{n,\tau,\omega}(\boldsymbol{\delta}) = \hat{\underset{\sim}{Z}}^X_{n,\tau,\omega,1}(\boldsymbol{\delta}) + \hat{\underset{\sim}{Z}}^X_{n,\tau,\omega,2}(\boldsymbol{\delta}),$$

where

$$\begin{aligned}
\hat{\underset{\sim}{Z}}_{n,\tau,\omega,1}(\boldsymbol{\delta}) &:= -\boldsymbol{\delta}' n^{-1/2}\sum_{t=1}^n \mathbf{c}_t(\omega)(\tau - I\{Y_t \leq \hat{F}^{-1}_{n,Y}(\tau)\}), \\
\hat{\underset{\sim}{Z}}_{n,\tau,\omega,2}(\boldsymbol{\delta}) &:= \sum_{t=1}^n \int_0^{n^{-1/2}\mathbf{c}_t'(\omega)\boldsymbol{\delta}} \big(I\{Y_t \leq \hat{F}^{-1}_{n,Y}(s+\tau)\} - I\{Y_t \leq \hat{F}^{-1}_{n,Y}(\tau)\}\big)\mathrm{d}s, \\
\hat{\underset{\sim}{Z}}^X_{n,\tau,\omega,1}(\boldsymbol{\delta}) &:= -\boldsymbol{\delta}' n^{-1/2}\sum_{t=1}^n \mathbf{c}_t(\omega)(\tau - I\{X_{t,n} \leq \hat{F}^{-1}_{n,Y}(\tau)\}) \\
\hat{\underset{\sim}{Z}}^X_{n,\tau,\omega,2}(\boldsymbol{\delta}) &:= \sum_{t=1}^n \int_0^{n^{-1/2}\mathbf{c}_t'(\omega)\boldsymbol{\delta}} \big(I\{X_{t,n} \leq \hat{F}^{-1}_{n,Y}(s+\tau)\} - I\{X_{t,n} \leq \hat{F}^{-1}_{n,Y}(\tau)\}\big)\mathrm{d}s.
\end{aligned}$$

The assertion in (3.27) follows if we can show that $\hat{\underset{\sim}{Z}}_{n,\tau,\omega,i}$ and $\hat{\underset{\sim}{Z}}^X_{n,\tau,\omega,i}$ are uniformly close for $i = 1,2$. For $i = 1$, note that, with probability tending to one, we have, for any $\epsilon > 0$,

$$\begin{aligned}
&|\hat{\underset{\sim}{Z}}_{n,\tau,\omega,1}(\boldsymbol{\delta}) - \hat{\underset{\sim}{Z}}^X_{n,\tau,\omega,1}(\boldsymbol{\delta})| \\
&\quad = \Big|\frac{\boldsymbol{\delta}'}{\sqrt{n}}\sum_{t=1}^n \mathbf{c}_t(\omega)(I\{X_{t,n} \leq \hat{F}^{-1}_{n,Y}(\tau) - D_{t,n}\} - I\{X_{t,n} \leq \hat{F}^{-1}_{n,Y}(\tau)\})\Big| \\
&\quad \leq A\sqrt{n^{-1}\log n}\Big(\sum_{t=1}^n I\{|D_{t,n}| > \eta_n\} + \sup_{|s - F^{-1}_{n,X}(\tau)| \leq \epsilon}\sum_{t=1}^n I\{|X_{t,n} - s| \leq \eta_n\}\Big) \qquad (6.14)
\end{aligned}$$

for any sequence of real numbers (η_n), where the inequality is due to the fact that $I\{Y_t \leq \hat{F}^{-1}_{n,Y}(\tau)\} - I\{X_{t,n} \leq \hat{F}^{-1}_{n,Y}(\tau)\} \neq 0$ implies $|D_{t,n}| > \eta_n$ or $|X_{t,n} - \hat{F}^{-1}_{n,Y}(\tau)| \leq \eta_n$, along with the fact that, by Lemma 6.4, $F_{n,X}(F_{n,Y}^{-1}(\tau)) - \tau = o_{\mathrm{P}}(1)$. By (3.2) in Assumption (A1) and the strict stationarity of the process $(D_{t,n})$, the first sum in (6.14) is of order $O_{\mathrm{P}}(\kappa_n\sqrt{n\log n})$. The absolute value of the second sum in (6.14) is bounded by

$$A\sqrt{n\log n} \sup_{|s-F_{n,X}^{-1}(\tau)|\le\epsilon} \frac{1}{n}\sum_{t=1}^{n}\big(I\{X_{t,n}\le s+\eta_n\} - I\{X_{t,n} < s-\eta_n\}\big)$$

$$\le A\sqrt{n\log n} \sup_{|s-F_{n,X}^{-1}(\tau)|\le\epsilon} \sup_{x:|x|\le 2\eta_n} \big|\hat{F}_{n,X}(s+x) - \hat{F}_{n,X}(y) - F_{n,X}(s+x) + F_{n,X}(y)\big| \qquad (6.15)$$

$$+ \sqrt{n\log n} \sup_{|s-F_{n,X}^{-1}(\tau)|\le\epsilon} \Big|F_{n,X}(s+2\eta_n) - F_{n,X}(s-2\eta_n)\Big|. \qquad (6.16)$$

Therefore, it follows from Lemma 6.9 that (6.15) is of the order $\sqrt{n\log n}\; r_n(2\eta_n, m_n)$. For the second term, a Taylor expansion yields the order $O(\eta_n\sqrt{n\log n})$. For $i=2$, note that

$$|\underset{\sim}{\hat{Z}}_{n,\tau,\omega,2}(\boldsymbol{\delta}) - \underset{\sim}{\hat{Z}}^{X}_{n,\tau,\omega,2}(\boldsymbol{\delta})| = \big|A^{(1)}_{n,\tau,\omega}(\boldsymbol{\delta}) - A^{(2)}_{n,\tau,\omega}(\boldsymbol{\delta})\big| \le \big|A^{(1)}_{n,\tau,\omega}(\boldsymbol{\delta})\big| + \big|A^{(2)}_{n,\tau,\omega}(\boldsymbol{\delta})\big|$$

where

$$\begin{aligned}
A^{(1)}_{n,\tau,\omega}(\boldsymbol{\delta}) &:= \sum_{t=1}^{n}\int_0^{n^{-1/2}\mathbf{c}_t'(\omega)\boldsymbol{\delta}} \big(I\{Y_t \le \hat{F}_{n,Y}^{-1}(s+\tau)\} - I\{X_{t,n}\le \hat{F}_{n,Y}^{-1}(s+\tau)\}\big)\mathrm{d}s \\
&= \int_{-2\|\boldsymbol{\delta}\|}^{2\|\boldsymbol{\delta}\|} n^{-1/2}\sum_{t=1}^{n}\Big(I\{Y_t\le \hat{F}_{n,Y}^{-1}(n^{-1/2}s+\tau)\} - I\{X_{t,n}\le \hat{F}_{n,Y}^{-1}(n^{-1/2}s+\tau)\}\Big) \\
&\qquad\qquad \times \big(I\{0\le s\le \mathbf{c}_t'(\omega)\boldsymbol{\delta}\} - I\{0\ge s\ge \mathbf{c}_t'(\omega)\boldsymbol{\delta}\}\big)\mathrm{d}s, \\
A^{(2)}_{n,\tau,\omega}(\boldsymbol{\delta}) &:= \frac{\mathbf{c}_t'(\omega)\boldsymbol{\delta}}{\sqrt{n}}\sum_{t=1}^{n}\big(I\{Y_t\le \hat{F}_{n,Y}^{-1}(\tau)\} - I\{X_{t,n}\le \hat{F}_{n,Y}^{-1}(\tau)\}\big).
\end{aligned}$$

For any $\epsilon>0$ we have, with probability tending to one,

$$\begin{aligned}
|A^{(1)}_{n,\tau,\omega}(\boldsymbol{\delta})| &\le \int_{-2\|\boldsymbol{\delta}\|}^{2\|\boldsymbol{\delta}\|} n^{-1/2}\sum_{t=1}^{n}\big(I\{|D_{t,n}|>\eta_n\} + I\{|X_{t,n} - \hat{F}_{n,Y}^{-1}(\tau+sn^{-1/2})|\le\eta_n\}\big)\mathrm{d}s \\
&\le 4\|\boldsymbol{\delta}\|\Big(n^{-1/2}\sum_{t=1}^{n} I\{|D_{t,n}|>\eta_n\} + n^{-1/2}\sup_{|w-F_{n,X}^{-1}(\tau)|\le\epsilon}\sum_{t=1}^{n} I\{|X_{t,n}-w|\le\eta_n\}\Big)
\end{aligned}$$

and

$$\begin{aligned}
|A^{(2)}_{n,\tau,\omega}(\boldsymbol{\delta})| &\le \frac{\|\boldsymbol{\delta}\|}{\sqrt{n}}\sum_{t=1}^{n}\big|I\{X_{t,n}\le \hat{F}_{n,Y}^{-1}(\tau) - D_{t,n}\} - I\{X_{t,n}\le \hat{F}_{n,Y}^{-1}(\tau)\}\big| \\
&\le \|\boldsymbol{\delta}\|\Big(n^{-1/2}\sum_{t=1}^{n} I\{|D_{t,n}|>\eta_n\} + n^{-1/2}\sup_{|w-F_{n,X}^{-1}(\tau)|\le\epsilon}\sum_{t=1}^{n} I\{|X_{t,n}-w|\le\eta_n\}\Big).
\end{aligned}$$

Since the supremum in (3.27) is over $\|\boldsymbol{\delta}\|\le A\sqrt{\log n}$, the upper bound for $|A^{(1)}_{n,\tau,\omega}(\boldsymbol{\delta})|$ and $|A^{(2)}_{n,\tau,\omega}(\boldsymbol{\delta})|$ is of the same order as for $|\underset{\sim}{\hat{Z}}_{n,\tau,\omega,1}(\boldsymbol{\delta}) - \underset{\sim}{\hat{Z}}^{X}_{n,\tau,\omega,1}(\boldsymbol{\delta})|$ above. This completes the proof. □

6.2.2 Proof of (3.28)

Plugging into (3.28) the definition of $\underset{\sim}{\hat{Z}}^X_{n,\tau,\omega,1}(\boldsymbol{\delta})$, it remains to show that [recall that $c_{t,1}(\omega) = 1$]

$$\max_{k=2,3} \sup_{\omega\in\mathcal{F}_n} \Big| n^{-1/2} \sum_{t=1}^{n} c_{t,k}(\omega)\big(I\{U_{t,n} \leq F_{n,X}(\hat{F}^{-1}_{n,Y}(\tau))\} - I\{U_{t,n} \leq \tau\}\big)\Big| \\ = O_{\mathrm{P}}(n^{-1/4} m_n^{1/2} (\log n)^{3/2}) \qquad (6.17)$$

and

$$\Big| n^{-1/2} \sum_{t=1}^{n} (I\{U_{t,n} \leq F_{n,X}(\hat{F}^{-1}_{n,Y}(\tau))\} - I\{U_{t,n} \leq \tau\}) - \sqrt{n}(F_{n,X}(\hat{F}^{-1}_{n,Y}(\tau)) - \tau)\Big| \\ = O_{\mathrm{P}}(n^{-1/4} m_n^{1/2} (\log n)^{3/2}). \qquad (6.18)$$

First consider (6.17). Since, by Lemma 6.4, $|F_{n,X}(\hat{F}^{-1}_{n,Y}(\tau)) - \tau| = O_{\mathrm{P}}(n^{-1/2}\sqrt{\log n})$, we obtain

$$\sup_{\omega\in\mathcal{F}_n} \Big| n^{-1/2} \sum_{t=1}^{n} c_{t,k}(\omega)(I\{U_{t,n} \leq F_{n,X}(\hat{F}^{-1}_{n,Y}(\tau))\} - I\{U_{t,n} \leq \tau\})\Big| \\ \leq \sup_{\omega\in\mathcal{F}_n} n^{-1/2} \sup_{|x-\tau|\leq n^{-1/2}\log n} \Big| \sum_{t=1}^{n} c_{t,k}(\omega)(I\{U_{t,n} \leq x\} - I\{U_{t,n} \leq \tau\} - (x-\tau))\Big| \\ + \sup_{\omega\in\mathcal{F}_n} n^{-1}\log n \Big| \sum_{t=1}^{n} c_{t,k}(\omega)\Big| \qquad (6.19)$$

for $k = 2, 3$, with probability tending to one. The second term in (6.19) vanishes, because, for all $\omega \in \mathcal{F}_n$, $\sum_{t=1}^n \cos(\omega t) = \sum_{t=1}^n \sin(\omega t) = 0$. In order to bound the first term, cover the set $\mathcal{Z} := \{u : |u - \tau| \leq n^{-1/2}\log n\}$ with $N < n$ balls of radius $1/n$ and centers $u_1, ..., u_N \in \mathcal{Z}$, and define $\mathbb{G}_{n,\omega,k}(u) := n^{-1/2}\sum_{t=1}^n c_{t,k}(\omega)(I\{U_{t,n} \leq u\} - u)$. Then,

$$\sup_j \sup_{\omega\in\mathcal{F}_n} \sup_{|u-u_j|\leq n^{-1}} \Big|\mathbb{G}_{n,\omega,k}(u) - \mathbb{G}_{n,\omega,k}(u_j)\Big| \\ \leq \sup_{u\in Z} n^{-1/2} \sum_{t=1}^{n} \Big(I\{U_{t,n} \leq u + 2n^{-1}\} - I\{U_{t,n} \leq u - 2n^{-1}\} + 4n^{-1}\Big) + O(n^{-1/2}) \\ \leq \sqrt{n} \sup_{j=1,\dots,N} \Big|\hat{F}_{n,U}(u_j + 2n^{-1}) - \hat{F}_{n,U}(u_j - 2n^{-1}) - 4n^{-1}\Big| + O(n^{-1/2}),$$

where the latter bound, in view of Lemma 6.9, is $o_{\mathrm{P}}(n^{-1/4})$ since $\sqrt{n}r_n(4n^{-1}, m_n)$ is of order $o(m_n n^{-1/2}(\log n)^3)$. Thus,

$$\sup_j \sup_{\omega\in\mathcal{F}_n} \sup_{|u-u_j|\leq n^{-1}} \Big|\mathbb{G}_{n,\omega,k}(u) - \mathbb{G}_{n,\omega,k}(u_j)\Big| = o_{\mathrm{P}}(n^{-1/4}), \quad k = 2, 3,$$

and therefore

$$\max_{k=2,3} \sup_{\omega\in\mathcal{F}_n} \Big| n^{-1/2} \sum_{t=1}^{n} c_{t,k}(\omega)(I\{U_{t,n} \leq F_{n,X}(F^{-1}_{n,Y}(\tau))\} - I\{U_{t,n} \leq \tau\})\Big| \\ \leq \max_{k=2,3} \sup_{j=1,\dots,N} \sup_{\omega\in\mathcal{F}_n} \Big|\mathbb{G}_{n,\omega,k}(u_j) - \mathbb{G}_{n,\omega,k}(\tau)\Big| + o_{\mathrm{P}}(n^{-1/4}). \qquad (6.20)$$

Now, by construction, $\max_j |u_j - \tau| \le n^{-1/2}\log n$. Moreover,

$$\mathbb{G}_{n,\omega,k}(u_j) - \mathbb{G}_{n,\omega,k}(\tau) = n^{-1/2}\sum_{t=1}^{n} V_{n,t,\omega,k}(u_j,\tau),$$

where $V_{n,t,\omega,k}(u_j,\tau) := c_{t,k}(\omega)\big(I\{U_{t,m_n} \le u_j\} - I\{U_{t,n} \le \tau\} - (u_j - \tau)\big)$ are centered, m_n-dependent random variables. Direct calculations show that, for some finite constant C independent of n, ω, τ and j,

$$|V_{n,t,\omega,k}(u_j,\tau)| \le 2 \text{ a.s.}, \quad \mathrm{E}|V_{n,t,\omega,k}(u_j,\tau)|^3 \le C|u_j - \tau| \le Cn^{-1/2}\log n,$$

and

$$\mathrm{E}\Big[\big(\sum_{t=1}^{n} V_{n,t,\omega,k}(u_j,\tau)\big)^2\Big] \le Cn^{1/2}m_n \log n.$$

An application of Lemma 6.8 with $x_n = A\sqrt{\log n}$ and $\psi_n = C^{1/3}n^{-1/6}\log n$ thus yields, for $k = 1,2$, some finite constant $\tilde{C}$ and n sufficiently large,

$$\sup_{\omega\in\mathcal{F}_n}\sup_{j=1,\dots,N} \mathrm{P}\big(n^{-1/2}\big|\sum_{t=1}^{n} V_{n,t,\omega,k}(u_j,\tau)\big| > \tilde{C}n^{-1/2}\sqrt{\log n}(n^{1/2}m_n\log n)^{1/2}\big) \le 4e^{-3\log n}.$$

Hence, in view of the fact that $N \le n$ and $\#\mathcal{F}_n \le n$,

$$\max_{k=2,3}\sup_{j=1,\dots,N}\sup_{\omega\in\mathcal{F}_n}\Big|\mathbb{G}_{n,\omega,k}(u_j) - \mathbb{G}_{n,\omega,k}(\tau)\Big| = O_{\mathrm{P}}(n^{-1/4}m_n^{1/2}(\log n)^{3/2}).$$

Together with (6.20) this yields (6.17). Turning to (6.18), Lemma 6.9 yields

$$\begin{aligned}
&\Big|n^{-1/2}\sum_{t=1}^{n}\big(I\{U_{t,n} \le F_{n,X}(\hat{F}_{n,Y}^{-1}(\tau))\} - I\{U_{t,n} \le \tau\} - (F_{n,X}(\hat{F}_{n,Y}^{-1}(\tau)) - \tau)\big)\Big| \\
&\quad \le \sup_{|u-\tau|\le n^{-1/2}\log n}\Big|n^{-1/2}\sum_{t=1}^{n}\big(I\{U_{t,n} \le u\} - I\{U_{t,n} \le \tau\} - (u-\tau)\big)\Big| \\
&\quad = n^{1/2}\sup_{|u-\tau|\le n^{-1/2}\log n}\big|\hat{F}_{n,U}(u) - \hat{F}_{n,U}(\tau) - (u-\tau)\big| = O_{\mathrm{P}}(n^{-1/4}m_n^{1/2}(\log n)^{3/2}). \quad \square
\end{aligned}$$

6.2.3 Proof of (3.29)

Observe the decomposition

$$\begin{aligned}
&\hat{\underset{\sim}{Z}}^{X}_{n,\tau,\omega,2}(\boldsymbol{\delta}) - \sum_{t=1}^{n}\int_0^{n^{-1/2}\mathbf{c}_t'(\omega)\boldsymbol{\delta}}\big(I\{U_{t,n} \le s+\tau\} - I\{U_{t,n} \le \tau\}\big)\mathrm{d}s \\
&= \sum_{t=1}^{n}\int_0^{n^{-1/2}\mathbf{c}_t'(\omega)\boldsymbol{\delta}}\big(I\{U_{t,n} \le F_{n,X}(\hat{F}_{n,Y}^{-1}(s+\tau))\} - I\{U_{t,n} \le F_{n,X}(\hat{F}_{n,Y}^{-1}(\tau))\} - I\{U_{t,n} \le s+\tau\} \\
&\qquad + I\{U_{t,n} \le \tau\}\big)\mathrm{d}s \\
&= \int_{-2\|\boldsymbol{\delta}\|}^{2\|\boldsymbol{\delta}\|} n^{-1/2}\sum_{t=1}^{n}\big(I\{U_{t,n} \le F_{n,X}(\hat{F}_{n,Y}^{-1}(n^{-1/2}s+\tau))\} - I\{U_{t,n} \le F_{n,X}(\hat{F}_{n,Y}^{-1}(\tau))\} \\
&\qquad - I\{U_{t,n} \le n^{-1/2}s+\tau\} + I\{U_{t,n} \le \tau\}\big)\big(I\{0 \le s \le \mathbf{c}_t'(\omega)\boldsymbol{\delta}\} - I\{0 \ge s \ge \mathbf{c}_t'(\omega)\boldsymbol{\delta}\}\big)\mathrm{d}s \\
&= A_n^{(1)} - A_n^{(2)} - A_n^{(3)} + A_n^{(4)}, \text{ say,}
\end{aligned}$$

where

$$A_n^{(1)} =: \int_{-2\|\boldsymbol{\delta}\|}^{2\|\boldsymbol{\delta}\|} \big(\mathbb{S}^{(+)}_{n,\omega,\boldsymbol{\delta}}(F_{n,X}(\hat{F}^{-1}_{n,Y}(n^{-1/2}s+\tau)), n^{-1/2}s+\tau; s) - \mathbb{S}^{(+)}_{n,\omega,\boldsymbol{\delta}}(F_{n,X}(\hat{F}^{-1}_{n,Y}(\tau)), \tau; s)\big)\mathrm{d}s$$

$$A_n^{(2)} =: \int_{-2\|\boldsymbol{\delta}\|}^{2\|\boldsymbol{\delta}\|} n^{-1/2}\sum_{t=1}^{n}\Big[(F_{n,X}(\hat{F}^{-1}_{n,Y}(n^{-1/2}s+\tau)) - (n^{-1/2}s+\tau)) - (F_{n,X}(\hat{F}^{-1}_{n,Y}(\tau)) - \tau)\Big] \times I\{0 \le s \le \mathbf{c}_t'(\omega)\boldsymbol{\delta}\}\mathrm{d}s$$

$$A_n^{(3)} =: \int_{-2\|\boldsymbol{\delta}\|}^{2\|\boldsymbol{\delta}\|} \big(\mathbb{S}^{(-)}_{n,\omega,\boldsymbol{\delta}}(F_{n,X}(\hat{F}^{-1}_{n,Y}(n^{-1/2}s+\tau)), n^{-1/2}s+\tau; s) - \mathbb{S}^{(-)}_{n,\omega,\boldsymbol{\delta}}(F_{n,X}(\hat{F}^{-1}_{n,Y}(\tau)), \tau; s)\big)\mathrm{d}s$$

$$A_n^{(4)} =: \int_{-2\|\boldsymbol{\delta}\|}^{2\|\boldsymbol{\delta}\|} n^{-1/2}\sum_{t=1}^{n}\Big[(F_{n,X}(\hat{F}^{-1}_{n,Y}(n^{-1/2}s+\tau)) - (n^{-1/2}s+\tau)) - (F_{n,X}(\hat{F}^{-1}_{n,Y}(\tau)) - \tau)\Big] \times I\{0 \ge s \ge \mathbf{c}_t'(\omega)\boldsymbol{\delta}\}\mathrm{d}s$$

and

$$\mathbb{S}^{(+)}_{n,\omega,\boldsymbol{\delta}}(u,v;s) := n^{-1/2}\sum_{t=1}^{n}\big(I\{U_{t,n} \le u\} - I\{U_{t,n} \le v\} - (u-v)\big)I\{0 \le s \le \mathbf{c}_t'(\omega)\boldsymbol{\delta}\},$$

$$\mathbb{S}^{(-)}_{n,\omega,\boldsymbol{\delta}}(u,v;s) := n^{-1/2}\sum_{t=1}^{n}\big(I\{U_{t,n} \le u\} - I\{U_{t,n} \le v\} - (u-v)\big)I\{0 \ge s \ge \mathbf{c}_t'(\omega)\boldsymbol{\delta}\}.$$

First note that, in view of Lemma 6.4,

$$|A_n^{(2)}| \le 4\|\boldsymbol{\delta}\|\sqrt{n} \sup_{|u-\tau|\le 2\|\boldsymbol{\delta}\|/\sqrt{n}} |F_{n,X}(\hat{F}^{-1}_{n,Y}(u)) - u - (F_{n,X}(\hat{F}^{-1}_{n,Y}(\tau)) - \tau)| = o_{\mathrm{P}}(\beta_n\sqrt{n\log n}),$$

with $\beta_n := \kappa_n + \eta_n + r_n(\eta_n, m_n) + O(n^{-3/4}m_n^{1/2}\log n)$. A similar bound can be obtained for $A_n^{(4)}$. Next, for sufficiently large n, still in view of Lemma 6.4,

$$\begin{aligned}
&\int_{-2\|\boldsymbol{\delta}\|}^{2\|\boldsymbol{\delta}\|} \big|\mathbb{S}^{(+)}_{n,\omega,\boldsymbol{\delta}}(F_{n,X}(\hat{F}^{-1}_{n,Y}(n^{-1/2}s+\tau)), n^{-1/2}s+\tau; s)\big|\mathrm{d}s \\
&\quad\le \int_{-2\|\boldsymbol{\delta}\|}^{2\|\boldsymbol{\delta}\|} \sup_{v:|v-\tau|\le 2\|\boldsymbol{\delta}\|/\sqrt{n}} \big|\mathbb{S}^{(+)}_{n,\omega,\boldsymbol{\delta}}(F_{n,X}(\hat{F}^{-1}_{n,Y}(v)), v; s)\big|\mathrm{d}s \\
&\quad\le \int_{-2\|\boldsymbol{\delta}\|}^{2\|\boldsymbol{\delta}\|} \sup_{v:|v-\tau|\le 2\|\boldsymbol{\delta}\|/\sqrt{n}} \sup_{u:|u-v|\le n^{-1/2}\log n} \big|\mathbb{S}^{(+)}_{n,\omega,\boldsymbol{\delta}}(u,v;s)\big|\mathrm{d}s \\
&\quad\le 4\|\boldsymbol{\delta}\| \sup_{s:|s|\le 2\|\boldsymbol{\delta}\|} \sup_{v:|v-\tau|\le 2\|\boldsymbol{\delta}\|/\sqrt{n}} \sup_{u:|u-v|\le n^{-1/2}\log n} \big|\mathbb{S}^{(+)}_{n,\omega,\boldsymbol{\delta}}(u,v;s)\big|.
\end{aligned}$$

Similar inequalities hold for $\int_{-2\|\boldsymbol{\delta}\|}^{2\|\boldsymbol{\delta}\|} \big|\mathbb{S}^{(+)}_{n,\omega,\boldsymbol{\delta}}(F_{n,X}(\hat{F}^{-1}_{n,Y}(\tau)), \tau; s)\big|\mathrm{d}s$. Let us show that

$$\sup_{\omega\in\mathcal{F}_n} \sup_{\boldsymbol{\delta}:\|\boldsymbol{\delta}\|\le A\sqrt{\log n}} \sup_{s:|s|\le 2\|\boldsymbol{\delta}\|} \sup_{\substack{(u,v):|v-\tau|\le 2\|\boldsymbol{\delta}\|/\sqrt{n} \\ |u-v|\le n^{-1/2}\log n}} \big|\mathbb{S}^{(+)}_{n,\omega,\boldsymbol{\delta}}(u,v;s)\big| = O_{\mathrm{P}}(n^{-1/4}m_n^{1/2}\log n). \quad (6.21)$$

For any $C > 0$ we have $I\{0 \le s \le \mathbf{c}_t'\boldsymbol{\delta}\} = I\{0 \le Cs \le C\mathbf{c}_t'\boldsymbol{\delta}\}$. Thus, it is sufficient to consider vectors $\boldsymbol{\delta}$ satisfying $\|\boldsymbol{\delta}\|_2 = 1$. Since by definition $\|\mathbf{c}_t(\omega)\|_2 = \sqrt{2}$, it also is sufficient to consider values of s in the interval $[0,\sqrt{2}]$. Finally, note that if $I\{0 \le s_1 \le \mathbf{c}_t'\boldsymbol{\delta}_1\} = I\{0 \le s_2 \le \mathbf{c}_t'\boldsymbol{\delta}_2\}$ for all $t = 1, ..., n$, then also $\mathbb{S}^{(+)}_{n,\omega,\boldsymbol{\delta}_1}(u,v;s_1) = \mathbb{S}^{(+)}_{n,\omega,\boldsymbol{\delta}_2}(u,v;s_2)$. We thus can rewrite (6.21) as

$$G_n := \sup_{T\in\mathcal{M}_n} \sup_{\substack{(u,v):|v-\tau|\le 2\|\boldsymbol{\delta}\|/\sqrt{n} \\ |u-v|\le n^{-1/2}\log n}} |\tilde{\mathbb{S}}^{(+)}_n(u,v;T)| = O_{\mathrm{P}}(n^{-1/4}m_n^{1/2}\log n) \tag{6.22}$$

where

$$\mathcal{M}_n := \big\{T = \{t \in \{1,...,n\} : 0 \le s \le \mathbf{c}_t'\boldsymbol{\delta}\}\big|\omega \in \mathcal{F}_n, s \in [0,\sqrt{2}], \|\boldsymbol{\delta}\| = 1\big\} \tag{6.23}$$

and

$$\bar{\mathbb{S}}^{(+)}_n(u,v;T) \;\; := \;\; n^{-1/2}\sum_{t\in T}\big(I\{U_{t,n}\le u\} - u - (I\{U_{t,n}\le v\} - v)\big) =: n^{-1/2}\sum_{t\in T} V_{t,n}(u,v).$$

In order to prove (6.21) (or (6.22)), define the set

$$\mathcal{Z} := \big\{(u,v)\in\mathbb{R}^2 : |u-v| \le n^{-1/2}\log n, |v-\tau| \le 2An^{-1/2}\sqrt{\log n}\big\}$$

and cover it with $N < n^2$ balls of radius $1/n$ with centers $z_1, ..., z_N \in \mathcal{Z}$. For any $(u,v)\in\mathcal{Z}$ there exists an index j such that $\|(u,v) - (z_{1j}, z_{2j})\|_\infty \le 1/n$ and, letting $z_j := (z_{1j}, z_{2j})$,

$$\begin{aligned}
\rho(u,v,z_j) :=& |\bar{\mathbb{S}}^{(+)}_n(u,v;T) - \bar{\mathbb{S}}^{(+)}_n(z_{1j},z_{2j};T)| \\
&\le n^{-1/2}\sum_{t=1}^n\big(I\{|U_{t,n} - z_{1j}| \le n^{-1}\} + I\{|U_{t,n} - z_{2j}| \le n^{-1}\} + |u - z_{1j}| + |v - z_{2j}|\big) \\
&\le 2n^{-1/2} + n^{-1/2}\sum_{t=1}^n\big(I\{|U_{t,n} - z_{1j}| \le n^{-1}\} + I\{|U_{t,n} - z_{2j}| \le n^{-1}\}\big) \\
&\le 2n^{-1/2} + n^{-1/2}\sum_{t=1}^n\big(I\{U_{t,n} \le z_{1j} + n^{-1}\} - I\{U_{t,n} < z_{1j} - n^{-1}\} \\
&\qquad + I\{U_{t,n} \le z_{2j} + n^{-1}\} - I\{U_{t,n} < z_{2j} - n^{-1}\}\big) \\
&\le n^{1/2}\big(\hat{F}_{n,U}(z_{1j} + 2n^{-1}) - (z_{1j} + 2n^{-1}) - \big(\hat{F}_{n,U}(z_{1j} - 2n^{-1}) - (z_{1j} - 2n^{-1})\big) \\
&\quad + \hat{F}_{n,U}(z_{2j} + 2n^{-1}) - (z_{2j} + 2n^{-1}) - \big(\hat{F}_{n,U}(z_{2j} - 2n^{-1}) - (z_{2j} - 2n^{-1})\big)\big) + O(n^{-1/2})
\end{aligned}$$

where $\hat{F}_{n,U}$ denotes the empirical distribution function of $U_{1,n}, \ldots, U_{n,n}$. From Lemma 6.9,

$$\begin{aligned}
\sup_{z_1,\ldots,z_N} \sup_{\substack{(u,v)\in[0,1]^2 \\ \|z_j-(u,v)\|_\infty < n^{-1}}} |\rho(u,v,z_j)| &\le n^{1/2}\sup_{z_j\in\mathcal{Z}}\big|\hat{F}_{n,U}(z_{1j} + 2n^{-1}) - \hat{F}_{n,U}(z_{1j} - 2n^{-1}) - 4n^{-1}\big| \\
&\quad + n^{1/2}\sup_{z_j\in\mathcal{Z}}\big|\hat{F}_{n,U}(z_{2j} + 2n^{-1}) - \hat{F}_{n,U}(z_{2j} - 2n^{-1}) - 4n^{-1}\big| + O(n^{-1/2}) \\
&= O\big(m_n n^{-1/2}\log n\big).
\end{aligned}$$

With this, we have, for G_n defined in (6.22),

$$G_n \le \sup_{T\in\mathcal{M}_n}\sup_{z_1,\ldots,z_N}|\bar{\mathbb{S}}^{(+)}_n(z_{1j},z_{2j};T)| + O_{\mathrm{P}}\big(m_n n^{-1/2}\log n\big).$$

Because $|\mathcal{M}_n| \le (n+1)^4$ by Lemma 6.5 and $N < n^2$ by construction, it follows that

$$\mathrm{P}\big(\sup_{T\in\mathcal{M}_n}\sup_{z_1,\dots,z_N}|\bar{\mathbb{S}}_n^{(+)}(z_{1j},z_{2j};T)|>n^{-1/4}m_n^{1/2}\log n\big)$$
$$\le(n+1)^4n^2\sup_{T\in\mathcal{M}_n}\sup_{z_1,\dots,z_N}\mathrm{P}\big(|\bar{\mathbb{S}}_n^{(+)}(z_{1j},z_{2j};T)|>n^{-1/4}m_n^{1/2}\log n\big).$$

It therefore suffices to show that for some $D>6$ and finite constant C independent of T and z_j,

$$\mathrm{P}\big(|\bar{\mathbb{S}}_n^{(+)}(z_{1j},z_{2j};T)|\ge Cn^{-1/4}m_n^{1/2}\log n\big)\le e^{-D\log n} \tag{6.24}$$

for every $T\in\mathcal{M}_n$ and $z_j\in\mathcal{Z}$. To this end, note that $n^{1/2}\bar{\mathbb{S}}_n^{(+)}(z_{1j},z_{2j};T)$ is a sum of centered m_n-dependent random variables $V_{t,n}(z_{1j},z_{2j})$. Simple computations yield (recall that $z_j\in\mathcal{Z}$)

$$\sup_{\omega\in\mathcal{F}_n}\sup_t|V_{t,n}(z_{1j},z_{2j})| \le 2, \tag{6.25}$$

$$\sup_{\omega\in\mathcal{F}_n}\sup_t\mathrm{E}[|V_{t,n}(z_{1j},z_{2j})|^3] \le 8|z_{1j}-z_{2j}|=O(n^{-1/2}\log n), \tag{6.26}$$

$$\sup_{\omega\in\mathcal{F}_n}\mathrm{E}\Big[\Big(\sum_t V_{t,n}(z_{1j},z_{2j})\Big)^2\Big] = O((\#T)m_n|z_{1j}-z_{2j}|)=O(m_nn^{1/2}\log n). \tag{6.27}$$

We now apply Lemma 6.8 with $x_n=D\sqrt{\log n}$. Because of (6.25) and (6.26), part (ii) of Lemma 6.8 implies that Conditions 1–3 hold if we choose

$$\psi_n:=\big(n^{-1/3}m_n^{1/3}\vee\big(\sup_t\mathrm{E}|V_{t,n}(u,v)|^3\big)^{1/3}\big)(\log n)^{2/3}=O\big(n^{-1/6}\log n\big).$$

Condition 4 follows because $m_n=O(n^{1/4-a})$. Therefore (6.24) follows from Lemma 6.8, since

$$2\sqrt{6}ADn^{-1/2}\sqrt{\log n}(B_n^2+\psi_n^2n^{2/3}m_n^{4/3}x_n^2)^{1/2}=O\big(n^{-1/4}m_n^{1/2}\log n\big).$$

A similar result can be derived for $\mathbb{S}_{n,\omega,\boldsymbol{\delta}}^{(-)}$. This completes the proof. □

6.2.4 Two auxiliary Lemmas

Lemma 6.4 *(i) If $\kappa_n+\eta_n+r_n(2\eta_n,m_n)=o(n^{-1/2})$ and, for any $\delta>0$ such that $[\alpha-\delta,\beta-\delta]\subset(0,1)$, $\inf_{u\in[\alpha-\delta,\beta+\delta]}f_{n,X}(F_{n,X}^{-1}(u))>0$, then*

$$\sup_{u\in[\alpha,\beta]}|F_{n,X}(\hat{F}_{n,Y}^{-1}(u))-u|=O_{\mathrm{P}}(n^{-1/2}\sqrt{\log n}).$$

(ii) If, moreover, $\kappa_n+\eta_n+r_n(2\eta_n,m_n)+r_n(a_n,m_n)=o(a_n)$, then

$$\sup_{u,v\in[\alpha,\beta],|u-v|\le a_n}|F_{n,X}(\hat{F}_{n,Y}^{-1}(u))-F_{n,X}(\hat{F}_{n,Y}^{-1}(v))-(u-v)|=O_{\mathrm{P}}\big(\kappa_n+\eta_n+r_n(2\eta_n,m_n)+r_n(2a_n,m_n)\big).$$

Proof. (i) Elementary analytic considerations show that, for any non-decreasing function g, $\sup_{w\in[u,v]}|g(w)-w|\le a_n$ implies $\sup_{w\in[u+2a_n,v-2a_n]}|g^{-1}(w)-w|\le a_n$. This, for $g(w)=\hat{F}_{n,Y}(F_{n,X}^{-1}(w))$, $u=\alpha-\delta$, $v=\beta+\delta$, in combination with Lemmas 6.9 and 6.10, yields the desired result.

(ii) By Lemmas 6.9 and 6.10, for any bounded $\mathcal{Y}\subset\mathbb{R}$,

$$\sup_{y\in\mathcal{Y}}\sup_{|x|\le a_n}|\hat{F}_{n,Y}(y+x)-\hat{F}_{n,Y}(y)-F_{n,X}(x+y)+F_{n,X}(y)|$$
$$=O_{\mathrm{P}}(\kappa_n+\eta_n+r_n(2\eta_n,m_n)+r_n(a_n,m_n)).$$

Since, for any $A \subset [0,1]$, $\sup_{u,v\in A} |F_{n,X}^{-1}(u) - F_{n,X}^{-1}(v)| \leq C_A|u-v|$ for some positive constant C_A,

$$\sup_{u,v\in[\alpha-\delta,\beta+\delta],|u-v|\leq a_n} |\hat{F}_{n,Y}(F_{n,X}^{-1}(u)) - \hat{F}_{n,Y}(F_{n,X}^{-1}(v)) - (u-v)|$$
$$= O_{\mathrm{P}}(\kappa_n + \eta_n + r_n(2\eta_n, m_n) + r_n(a_n, m_n)).$$

We now apply Lemma 3.5 from Wendler [2011], with $F(w) = \hat{F}_{n,Y}(F_{n,X}^{-1}(w))$, $l = a_n$, $c = D(\kappa_n + \eta_n + r_n(2\eta_n, m_n) + r_n(a_n, m_n))$, $C_1 = \hat{F}_{n,Y}(F_{n,X}^{-1}(\alpha-\delta)), C_2 = \hat{F}_{n,Y}(F_{n,X}^{-1}(\beta+\delta))$. By assumption,

$$l + 2c = a_n + 2D(\kappa_n + \eta_n + r_n(2\eta_n, m_n) + r_n(a_n, m_n)) \leq 2a_n$$

for sufficiently large n. By Lemma 6.9 and 6.10, we have $C_1 = \alpha + \delta + o_{\mathrm{P}}(1), C_2 = \beta - \delta + o_{\mathrm{P}}(1)$ and, for any strictly increasing continuous function G, $(F \circ G^{-1})^{-1} = G \circ F^{-1}$ (see Exercise 3 in Chapter 1 of Shorack and Wellner [1986]); moreover, by part (i) of the present lemma, $F_{n,X}(\hat{F}_{n,Y}^{-1}(u))$ is uniformly close to u for large n. Hence,

$$\sup_{u,v\in[\alpha,\beta],|u-v|\leq 2a_n} |F_{n,X}(\hat{F}_{n,Y}^{-1}(u)) - F_{n,X}(\hat{F}_{n,Y}^{-1}(v)) - (u-v)| > D(\kappa_n + \eta_n + r_n(2\eta_n, m_n) + r_n(2a_n, m_n))$$

implies

$$\sup_{u,v\in[\alpha-\delta,\beta+\delta],|u-v|\leq a_n} |\hat{F}_{n,Y}(F_{n,X}^{-1}(u)) - \hat{F}_{n,Y}(F_{n,X}^{-1}(v)) - (u-v)| > D(\kappa_n + \eta_n + r_n(2\eta_n, m_n) + r_n(a_n, m_n));$$

part (ii) of the Lemma follows on letting D tend to infinity. □

Lemma 6.5 *The cardinality of the set $\mathcal{M}_n$ defined in (6.23) is at most $(n+1)^4$.*

Proof. Fix a Fourier frequency $\omega_{j,n} = 2\pi j/n \in \mathcal{F}_n$ and note that

$$\mathbf{c}_t(\omega_{j,n})'\boldsymbol{\delta} = \delta_1 + \delta_2 \cos(\omega_{j,n}t) + \delta_3 \sin(\omega_{j,n}t) = \delta_1 + \sqrt{\delta_2^2 + \delta_3^2}\cos(\omega_{j,n}t + \phi(\delta_2, \delta_3))$$

where $\phi(\delta_2, \delta_3) \in [0, 2\pi]$ denotes a phase shift. Moreover, for any $v \in [0,1]$, noting that $x \mapsto \cos(\omega_{j,n}x + \phi)$ is n/j-periodic,

$$\begin{aligned} &\big\{t \in \{1,...,n\}\big|0 \leq v \leq \delta_1 + \sqrt{\delta_2^2 + \delta_3^2}\cos(\omega_{j,n}t + \phi)\big\} \\ = \;&\big\{\frac{nk}{j} + w\big|w \in [C_{1,\phi,v,\boldsymbol{\delta}} - C_{0,\phi,v,\boldsymbol{\delta}}, C_{1,\phi,v,\boldsymbol{\delta}} + C_{0,\phi,v,\boldsymbol{\delta}}], k = 0,...,n\big\} \cap \{1,...,n\} \end{aligned}$$

where $C_{0,\phi,v,\boldsymbol{\delta}}$ and $C_{1,\phi,v,\boldsymbol{\delta}}$ denote two real-valued constants (depending on $\phi, v, \boldsymbol{\delta}$) with $C_{0,\phi,v,\boldsymbol{\delta}} \in [0, n/2j]$ and $C_{1,\phi,v,\boldsymbol{\delta}} \in [0, n/j]$. Now, we have

$$\big\{\frac{nk}{j} + v\big|v \in [a_1, b_1], k = 0,1,...,n\big\} \cap \{1,...,n\} = \big\{\frac{nk}{j} + v\big|v \in [a_2, b_2], k = 0,1,...,n\big\} \cap \{1,...,n\}$$

provided that $\lceil ja_1 \rceil = \lceil ja_2 \rceil, \lceil jb_1 \rceil = \lceil jb_2 \rceil$ where $\lceil a \rceil$ denotes the smallest integer larger or equal than a. The argument above holds for any Fourier frequency. In particular, it implies that

$$\begin{aligned} \mathcal{M}_n \;\subset\; &\Big\{T = \Big\{t \in \{1,...,n\} \cap \Big\{\frac{kn}{j} + v\Big|v \in [\frac{a-b}{j}, \frac{a+b}{j}]\Big\}\Big\}\Big| \\ &\qquad b = 0,..., \lceil n/2 \rceil,\; a,k = 0,...,n,\; j = 1,...,n\Big\}. \end{aligned}$$

Since the set (of sets) on the right-hand side of the above equality contains less than $(n+1)^4$ elements, the proof is complete. □

6.3 Five basic lemmas

Lemma 6.6 *Let $(X_{j,n})_{j=1,...,n}$ be a triangular array of uniformly bounded p-variate random variables, $p \in \mathbb{N}$, such that the sequences $(X_{j,n})_{j=1,...,n}$ are m_n-dependent for every $n \in \mathbb{N}$. Let Σ_n be the covariance matrix of $S_n := \sum_{t=1}^{n} X_{tn}$ and assume that the limit $\Sigma_0 := \lim_{n\to\infty} \frac{1}{n}\Sigma_n$, possibly singular, exists. If $m_n = O(n^\delta)$, for some $\delta \in [0, 0.25)$, then $n^{-1/2}(S_n - \mathrm{E}(S_n))$ is asymptotically $\mathcal{N}_p(0, \Sigma_0)$.*

Proof. For the proof of the assertion consider $\lambda' S_n$ and apply the Cramér-Wold device. The degenerate case is proved by applying Chebyshev's inequality to establish $\lambda' S_n = o_{\mathrm{P}}(1)$. For the nondegenerate case we apply a modification of the Central Limit Theorem 7.3.1 for uniformly bounded m-dependent random variables in Chung [1968] that holds also if m varies with n, such that $m_n = O(n^\delta)$, for some $\delta \in [0, 0.25)$, and $n^{-2(1+2\delta)/3}\,\mathrm{Var}(S_n) \to \infty$. For a proof use the blocking factor $k_n := \lfloor n^{2(1-\delta)/3} \rfloor$ as suggested by Li [2007]. □

Lemma 6.7 *Let $X_{1,n}, X_{2,n}, ..., X_{n,n}$ be a triangular array of centered, m_n-dependent, real-valued random variables. Define $S_n := \sum_{t=1}^{n} X_{t,n}$ and $B_n^2 := \mathrm{E}[S_n^2]$, denote by $x_n \geq 1$ a sequence of real numbers and assume that*

$$m_n \sup_t |X_{t,n}| < B_n/x_n \quad a.s., \qquad \sup_t \mathrm{E}|X_{t,n}|^3 < \infty \quad \text{for all } n, \quad \text{and} \tag{6.28}$$

$$n m_n^2 x_n^3 \max_t \mathrm{E}|X_{t,n}|^3 B_n^{-3} = o(1) \quad \text{as } n \to \infty. \tag{6.29}$$

Then, denoting by Φ the standard normal distribution function,

$$1 - \mathrm{P}(S_n \leq B_n x_n) = (1 - \Phi(x_n))\big(1 + O\big(n m_n^2 x_n^3 \max_t \mathrm{E}|X_{t,n}|^3 B_n^{-3}\big)\big).$$

Proof. The assertion directly follows from Lemma 2 of Heinrich [1985]. Define the new variables $Y_{k,n} := X_{(k-1)m_n,n} + ... + X_{km_n-1,n}$ for $k = 1, ..., \lfloor n/m_n \rfloor$, and $Y_{\lfloor n/m_n \rfloor,n}$ as the sum of the remaining $X_{j,n}$'s. Direct calculations show that the conditions stated above imply the conditions of Lemma 2 of Heinrich [1985]. □

Lemma 6.7 cannot be applied when condition (6.29) does not hold, i.e. when $\mathrm{E}|X_{j,n}|^3$ is too large compared to B_n^2. A slight modification of the result above allows to handle those situations by replacing B_n^2 with a larger quantity.

Lemma 6.8 *(i) Let $X_{1,n}, X_{2,n}, ..., X_{n,n}$ be a triangular array of centered, m_n-dependent, real-valued random variables. Define $S_n := \sum_{j=1}^{n} X_{j,n}$ and $B_n^2 := \mathrm{E}[S_n^2]$, denote by $x_n \geq 1$ and $\psi_n \geq 0$ sequences of real numbers, and assume that*

(1) $x_n m_n\big(\sup_j |X_{j,n}| + \sqrt{3}\psi_n (n^{-1/2} m_n^2 x_n^3)^{1/3}\big) < \mu_n := \big(B_n^2 + \psi_n^2 (n m_n^2 x_n^3)^{2/3}\big)^{1/2}$ *a.s.,*

(2) $\sup_j \mathrm{E}|X_{j,n}|^3 < \infty$ *for every fixed n,*

(3) $\sup_j \mathrm{E}|X_{j,n}|^3/\psi_n^3 \to 0$ *and* $n^{-1/2} m_n^2 \psi_n^3 x_n^3 = O(1)$,

(4) $m_n^2 x_n^3 n^{-1/2} = o(1)$ *as n tends to infinity.*

Then,

$$\mathrm{P}(|S_n| \geq 2\sqrt{6}x_n\mu_n) \leq 4\exp(-x_n^2)\big(1 + O\big(nm_n^2x_n^3 \sup_j \mathrm{E}|X_{j,n}|^3/\mu_n^3\big)\big).$$

In particular, the $O(...)$ *quantity does not depend on any property of the distribution of the random variables* $X_{1,n}, X_{2,n}, ..., X_{n,n}$ *other than* $nm_n^2x_n^3 \max_j \mathrm{E}|X_{j,n}|^3/\mu_n^3$*. Moreover, the assumptions of the Lemma imply that* $O(nm_n^2x_n^3 \max_j \mathrm{E}|X_{j,n}|^3/\mu_n^3) = o(1)$*.*

(ii) It is sufficient, for Conditions (1) and (3) to hold for n *large enough, with* $\psi_n = \big((n^{-1/3}m_n^{1/3}) \vee (\sup_j \mathrm{E}|X_{j,n}|^3)^{1/3}\big)\alpha_n$ *for any* $\alpha_n \to \infty$ *such that* $\psi_n = o(1)$*, that* $\sup_n \sup_j |X_{j,n}| < \infty$ *a.s. and* $\sup_j \mathrm{E}|X_{j,n}|^3 = o(1)$ *as* $n \to \infty$*.*

Proof. Define $W_{1,n}, W_{2,n}, ..., W_{n,n}$ as a collection of i.i.d. random variables (for every n) that are independent of $X_{1,n}, X_{2,n}, ..., X_{n,n}$ and have distribution $\psi_n(n^{-1/2}m_n^2x_n^3)^{1/3}\mathcal{U}[-\sqrt{3}, \sqrt{3}]$ where the constant $\sqrt{3}$ is chosen in such a way that $\mathrm{Var}(\mathcal{U}[-\sqrt{3}, \sqrt{3}]) = 1$. Define $X_{j,n}^{(A)} := X_{j,n} + W_{j,n}, j = 1, ..., n$ and $S_n^{(A)} := \sum_{t=1}^n X_{t,n}^{(A)}$. The quantities $X_{1,n}^{(A)}, X_{2,n}^{(A)}, ..., X_{n,n}^{(A)}$ form a triangular array of m_n-dependent random variables with the following properties:

(a) $(B_n^{(A)})^2 := \mathrm{E}[(S_n^{(A)})^2] = B_n^2 + \psi_n^2(nm_n^2x_n^3)^{2/3}$;

(b) $\sup_j |X_{j,n}^{(A)}| \leq \sup_j |X_{j,n}| + \psi_n\sqrt{3}(n^{-1/2}m_n^2x_n^3)^{1/3}$;

(c) $\sup_j \mathrm{E}|X_{j,n}^{(A)}|^3 \leq 7\big(\sup_j \mathrm{E}|X_{j,n}|^3 + 3^{3/2}\psi_n^3n^{-1/2}m_n^2x_n^3\big)$.

Let us show that the results of Lemma 6.7 apply to $X_{1,n}^{(A)}, X_{2,n}^{(A)}, ..., X_{n,n}^{(A)}$. From (a), (b) and (1) which we obtain the first condition of (6.28) of Lemma 6.7. Condition (2) together with (c) yields the second condition of (6.28) of Lemma 6.7. Finally, note that

$$\frac{nm_n^2x_n^3 \sup_j \mathrm{E}|X_{j,n}^{(A)}|^3}{(B_n^{(A)})^3} \leq \frac{7nm_n^2x_n^3\Big(\max_j \mathrm{E}|X_{j,n}|^3 + 3^{3/2}\psi_n^3n^{-1/2}m_n^2x_n^3\Big)}{\mu_n^3} = o(1)$$

where the first inequality follows from (c) and the last identity from Conditions 3 and 4. Thus Lemma 6.7 yields

$$\mathrm{P}\big(S_n^{(A)} \geq B_n^{(A)}x_n\big) = (1 - \Phi(x_n))\big(1 + O\big(nm_n^2x_n^3 \max_j \mathrm{E}|X_{j,n}|^3(B_n^{(A)})^{-3}\big)\big).$$

Next, observe that $S_n = S_n^{(A)} - \sum_{k=1}^n W_{k,n}$. Thus

$$\begin{aligned}\mathrm{P}\big(S_n \geq B_n^{(A)}x_n\big) &\leq \mathrm{P}\big(\{S_n^{(A)} \geq B_n^{(A)}x_n/2\} \cup \{-\sum_{k=1}^n W_{k,n} \geq B_n^{(A)}x_n/2\}\big)\\ &\leq \mathrm{P}\big(S_n^{(A)} \geq B_n^{(A)}x_n/2\big) + \mathrm{P}\big(-\sum_{k=1}^n W_{k,n} \geq B_n^{(A)}x_n/2\big).\end{aligned}$$

For the second probability, Hoeffding's inequality [see e.g. Pollard (1984), Appendix B] yields

$$\mathrm{P}\big(-\sum_{k=1}^n W_{k,n} \geq A_n\big) \leq \exp\big(-\frac{A_n^2}{6n\psi_n^2(n^{-1/2}m_n^2x_n^3)^{2/3}}\big) = \exp(-6^{-1}A_n^2\psi_n^{-2}n^{-2/3}m_n^{-4/3}x_n^{-2}).$$

Setting $A_n = \sqrt{6}x_n^2\psi_nn^{1/3}m_n^{2/3}$, the right-hand side becomes $\exp(-x_n^2)$. Finally, repeat all the arguments so far with $X_{j,n}$ and $W_{j,n}$ replaced by $-X_{j,n}$ and $-W_{j,n}$, respectively. Straightforward calculations yield

$$\mathrm{P}\big(|S_n| \geq 2\sqrt{6}B_n^{(A)}x_n\big) \leq 4\exp(-x_n^2)\big(1 + O\big(nm_n^2x_n^3 \max_j \mathrm{E}|X_{j,n}|^3(B_n^{(A)})^{-3}\big)\big)$$

and thus the proof of the first part of Lemma 6.8 is complete.

For a proof of the second part, in order to see that Condition 1 holds, it suffices to verify that $m_n x_n < \psi_n n^{1/3} m_n^{2/3} x_n$ since by assumption $\psi_n (n^{-1/2} m_n^2 x_n^3)^{1/3} = o(1)$. Now $m_n x_n < \psi_n n^{1/3} m_n^{2/3} x_n$ holds iff $\psi_n > n^{-1/3} m_n^{1/3}$ does. Condition 3 holds by construction. $\square$

In the next two Lemmas, we denote by $\hat{F}_{n,Y}$ and $\hat{F}_{n,X}$ the empirical distribution functions of the random variables $Y_1, \ldots, Y_n$ and $X_{1,n}, \ldots, X_{n,n}$, respectively.

Lemma 6.9 *Let Assumptions (A1) and (A2) hold.*

(i) For any bounded set $\mathcal{Y} \subset \mathbb{R}$, there exist finite constants C_1, C_2 such that, for any sequence $a_n = o((\log n)^{-1})$ and sufficiently large n,

$$\mathrm{P}\big(\sup_{y\in\mathcal{Y}} \sup_{|x|\le a_n} |\hat{F}_{n,X}(y+x)-\hat{F}_{n,X}(y)-F_{n,X}(x+y)+F_{n,X}(y)| \ge D^2 r_n(a_n, m_n)\big) \le C_1 n^2 e^{-D^2 \log n}$$

where the function r_n is defined in (6.8). *It follows that*

$$\sup_{y\in\mathcal{Y}} \sup_{|x|\le a_n} |\hat{F}_{n,X}(y+x) - \hat{F}_{n,X}(y) - F_{n,X}(x+y) + F_{n,X}(y)| = O_{\mathrm{P}}(r_n(a_n, m_n)).$$

(ii) For any bounded $\mathcal{Y} \subset \mathbb{R}$, $\sup_{x\in\mathcal{Y}} |\hat{F}_{n,X}(x) - F_{n,X}(x)| = O_{\mathrm{P}}(n^{-1/2}\sqrt{\log n})$.

Proof. Observe that

$$\hat{F}_{n,X}(y+x) - \hat{F}_{n,X}(y) - F_{n,X}(x+y) + F_{n,X}(y) = \frac{1}{n}\sum_{t=1}^{n} W_{n,t}(x,y)$$

with centered, m_n-dependent random variables $W_{n,t}(x,y)$ that satisfy (with probability one) $\sup_{n,t,x,y} |W_{n,t}(x,y)| \le 2$, $\sup_{n,t,y} \mathrm{E}|W_{n,t}(x,y)|^3 \le C|x|$, and

$$\sup_{y} \mathrm{E}\big[\big(\sum_{t=1}^{n} W_{n,t}(x,y)\big)^2\big] \le n m_n C|x|$$

for some finite constant C independent of m_n. Set $\psi_n := ((n^{-1}m_n) \vee a_n)^{1/3} \log n$ and $x_n = D\sqrt{\log n}$. Part Lemma 6.8 implies that, for sufficiently large n,

$$\sup_{y\in\mathbb{R}} \sup_{|v|\le a_n} \mathrm{P}\Big(\Big|\sum_{t=1}^{n} W_{n,t}(v,y)\Big| \ge nD^2 r_n(a_n, m_n)\Big) \le 6\exp(-D^2 \log n). \tag{6.30}$$

Next, cover the bounded set $Z := \{(x,y) \in \mathbb{R}^2 | y \in \mathcal{Y}, |x| \le a_n\}$ with $N = O(n^2)$ spheres of radius $\frac{1}{2}n^{-1}$ with centers $(z_{1j}, z_{2j}) \in Z, j = 1, ..., N$. A Taylor expansion yields

$$\begin{aligned}\sup_{\|(x,y)-(z_{1j},z_{2j})\|_\infty \le 1/2n} & |W_{n,t}(x,y) - W_{n,t}(z_{1j}, z_{2j})| \\ & \le I\{|X_{t,n} - z_{2j}| \le n^{-1}\} + I\{|X_{t,n} - (z_{1j} + z_{2j})| \le n^{-1}\} + C_3 n^{-1} =: V_{t,j}.\end{aligned}$$

The random variables $V_{t,j}$ are m_n-dependent, uniformly bounded, and have an expectation of order $O(n^{-1})$. Define $\tilde{V}_{t,j} := V_{t,j} - \mathrm{E}[V_{t,j}]$. It is easy to see that $\mathrm{E}|\tilde{V}_{t,j}|^p \le \tilde{C}_p n^{-1}$ for finite constants $\tilde{C}_p$, $p = 2, 3$, and $B_n^2 := \mathrm{E}\big[\big(\sum_{t=1}^{n} \tilde{V}_{t,j}\big)^2\big] = O(m_n)$. Setting $\psi_n = n^{-1/3} m_n^{1/3} \log n$

and $x_n = D\sqrt{\log n}$, we thus see that $(B_n^2 + \psi_n^2(nm_n^2x_n^3)^{2/3})^{1/2} \leq C_4 m_n(\log n)^{3/2}$ for sufficiently large n. Lemma 6.8 thus yields, for sufficiently large n,

$$\sup_{j=1,\dots,N} \mathrm{P}\Big(\sup_{\|(x,y)-(z_{1j},z_{2j})\|_\infty \leq n^{-1}} \Big| \sum_{t=1}^n W_{n,t}(x,y) - W_{n,t}(z_{1j},z_{2j}) \Big| \geq D^2 C_5 (\log n)^2 m_n \Big)$$
$$\leq 6\exp(-D^2 \log n).$$

Now (6.30) yields $\mathrm{P}\Big(\sup_{j=1,\dots,N} \Big|\sum_{t=1}^n W_{n,t}(z_{1j},z_{2j})\Big| \geq nD^2 r_n(a_n,m_n)\Big) \leq 6N\exp(-D^2\log n)$. In particular, choosing D large enough, the right-hand side of the expression above is $o(1)$. Combining these results with the fact that, by after enlarging C_2 if necessary, $nD^2 r_n(a_n,m_n) \geq D^2(\log n)^2 m_n$ for sufficiently large n, we obtain the first part of the Lemma. The second part follows along the same lines. □

Lemma 6.10 *Let Assumptions (A1) and (A2) hold. Then, for any bounded $\mathcal{Y} \subset \mathbb{R}$,*

$$\sup_{s\in\mathcal{Y}} |\hat{F}_{n,Y}(s) - \hat{F}_{n,X}(s)| = O_{\mathrm{P}}(\kappa_n + \eta_n + r_n(2\eta_n, m_n))$$

with $r_n(\eta_n, m_n)$ defined in (6.8).

Proof. Observe that $|I\{Y_t \leq s\} - I\{X_{t,n} \leq s\}| \leq I\{|D_{t,n}| \geq \eta_n\} + I\{|Y_t - s| \leq \eta_n\}$. Thus,

$$\sup_{s\in\mathcal{Y}} |\hat{F}_{n,Y}(s) - \hat{F}_{n,X}(s)| \leq \frac{1}{n}\sum_{t=1}^n I\{|D_{t,n}| \geq \eta_n\} + \sup_{s\in\mathcal{Y}} \Big|\hat{F}_{n,X}(s+2\eta_n) - \hat{F}_{n,X}(s-2\eta_n)\Big|. \quad (6.31)$$

The first term on the right-hand side of (6.31) is a.s. non negative; its expected value is bounded by κ_n, and thus the term is of order $O_{\mathrm{P}}(\kappa_n)$. As for the second term, we have (recall that, by assumption, $F_{n,X}$ has uniformly bounded derivative)

$$\begin{aligned}
&\sup_{s\in\mathcal{Y}} \big|\hat{F}_{n,X}(s+2\eta_n) - \hat{F}_{n,X}(s-2\eta_n)\big| \\
&\quad\leq \sup_{s\in\mathcal{Y}} \big|\hat{F}_{n,X}(s+2\eta_n) - \hat{F}_{n,X}(s-2\eta_n) - (F_{n,X}(s+2\eta_n) - F_{n,X}(s-2\eta_n))\big| + O(\eta_n) \\
&\quad= O_{\mathrm{P}}(r_n(2\eta_n, m_n)) + O(\eta_n),
\end{aligned}$$

where the last identity follows from Lemma 6.9. This completes the proof. □

7 Appendix B: Technical details for the proof of Theorem 4.1

7.1 Proof of (4.4)

As a first step, let us show that the vectors $n^{1/2}\mathbf{b}^X_{n,\tau,\omega_{j,n}}$ appearing in the linearity result in (3.12) can be approximated by a similar vector with $X_{t,n}$ replaced by Y_t and $q_{n,\tau}$ replaced by q_τ. More precisely we show that

$$n^{1/2} \sup_{\omega_{j,n}\in\Omega_n} \|\mathbf{b}^X_{n,\tau,\omega_{j,n}} - \mathbf{b}_{n,\tau,\omega_{j,n}}\| = O_{\mathrm{P}}(\sqrt{\log n}|f(q_\tau) - f_{n,X}(q_{n,\tau})| + \sqrt{n}(\kappa_n + \eta_n)). \quad (7.1)$$

To see this, note that

$$n^{1/2}(\mathbf{b}^X_{n,\tau,\omega_{j,n}} - \mathbf{b}_{n,\tau,\omega_{j,n}}) = 2\frac{f_Y(q_\tau) - f_{n,X}(q_{n,\tau})}{f_Y(q_\tau)} n^{1/2}\mathbf{b}^X_{n,\tau,\omega_{j,n}}$$
$$- \frac{1}{f_Y(q_\tau)} 2n^{-1/2}\sum_{t=1}^{n}\begin{pmatrix}\cos(\omega_{j,n}t)\\ \sin(\omega_{j,n}t)\end{pmatrix}(I\{Y_t \le q_\tau\} - I\{X_{t,n} \le q_{n,\tau}\}) \tag{7.2}$$

Since $n^{1/2}\mathbf{b}^X_{n,\tau,\omega_{j,n}}$, by Lemma 6.2, is $O_{\mathrm{P}}(\sqrt{\log n})$, uniformly in $\omega_{j,n}$, the norm of the first term in (7.2) is $O_{\mathrm{P}}(\sqrt{\log n}|f_Y(q_\tau) - f_{n,X}(q_{n,\tau})|)$. For the norm of the second term, the argument used in the proof of (3.14) together with (6.9) yields an order $O_{\mathrm{P}}(\sqrt{n}(\kappa_n + \eta_n))$; (7.1) follows.

Next, note that by (3.12) we have

$$n^{1/2}\sup_{\omega_{j,n}\in\Omega_n}\|\hat{\mathbf{b}}_{n,\tau,\omega_{j,n}} - \mathbf{b}^X_{n,\tau,\omega_{j,n}}\| = o_{\mathrm{P}}\big((n^{-1/8} \vee n^{-1/6}m_n^{1/3})(\log n)^{3/2}\big).$$

Putting

$$4n^{-1}\tilde{\Delta}_n := (\hat{\mathbf{b}}_{n,\tau,\omega_{j,n}} - \mathbf{b}_{n,\tau,\omega_{j,n}})'\begin{pmatrix}1 & -\mathrm{i}\\ \mathrm{i} & 1\end{pmatrix}\mathbf{b}_{n,\tau,\omega_{j,n}} + (\mathbf{b}_{n,\tau,\omega_{j,n}})'\begin{pmatrix}1 & -\mathrm{i}\\ \mathrm{i} & 1\end{pmatrix}(\hat{\mathbf{b}}_{n,\tau,\omega_{j,n}} - \mathbf{b}_{n,\tau,\omega_{j,n}})$$
$$+(\hat{\mathbf{b}}_{n,\tau,\omega_{j,n}} - \mathbf{b}_{n,\tau,\omega_{j,n}})'\begin{pmatrix}1 & -\mathrm{i}\\ \mathrm{i} & 1\end{pmatrix}(\hat{\mathbf{b}}_{n,\tau,\omega_{j,n}} - \mathbf{b}_{n,\tau,\omega_{j,n}}),$$

we obtain, from the definition of the Laplace periodogram,

$$L_{n,\tau_1,\tau_2}(\omega_{j,n}) := \frac{n}{4}(\hat{\mathbf{b}}_{n,\tau,\omega_{j,n}})'\begin{pmatrix}1 & -\mathrm{i}\\ \mathrm{i} & 1\end{pmatrix}\hat{\mathbf{b}}_{n,\tau,\omega_{j,n}} = \frac{n}{4}(\mathbf{b}_{n,\tau,\omega_{j,n}})'\begin{pmatrix}1 & -\mathrm{i}\\ \mathrm{i} & 1\end{pmatrix}\mathbf{b}_{n,\tau,\omega_{j,n}} + \tilde{\Delta}_n$$
$$= \frac{1}{f_Y(q_{\tau_1})f_Y(q_{\tau_2})}\Big(n^{-1}d_n(\tau_1,\omega_{j,n})d_n(\tau_2,-\omega_{j,n})\Big) + \tilde{\Delta}_n.$$

Note that, for $\tau \in \{\tau_1, \tau_2\}$,

$$n^{1/2}\sup_{\omega_{j,n}\in\mathcal{F}_n}\|\hat{\mathbf{b}}_{n,\tau,\omega_{j,n}} - \mathbf{b}_{n,\tau,\omega_{j,n}}\|$$
$$\le n^{1/2}\sup_{\omega_{j,n}\in\mathcal{F}_n}\|\hat{\mathbf{b}}_{n,\tau,\omega_{j,n}} - \mathbf{b}^X_{n,\tau,\omega_{j,n}}\| + n^{1/2}\sup_{\omega_{j,n}\in\mathcal{F}_n}\|\mathbf{b}^X_{n,\tau,\omega_{j,n}} - \mathbf{b}_{n,\tau,\omega_{j,n}}\|$$
$$= o_{\mathrm{P}}\big((n^{-1/8} \vee n^{-1/6}m_n^{1/3})(\log n)^{3/2}\big) + O_{\mathrm{P}}(\sqrt{\log n}|f(q_\tau) - f_{n,X}(q_{n,\tau})| + \sqrt{n}(\kappa_n + \eta_n)).$$

Lemma 6.2 and (7.1) imply $n^{1/2}\sup_{\omega_{j,n}\in\mathcal{F}_n}\|\mathbf{b}_{n,\tau,\omega_{j,n}}\| = n^{1/2}\sup_{\omega_{j,n}\in\mathcal{F}_n}\|\mathbf{b}^X_{n,\tau,\omega_{j,n}}\| + o_{\mathrm{P}}(1) = O_{\mathrm{P}}(\sqrt{\log n})$, so that $\|\tilde{\Delta}_n\| = O_{\mathrm{P}}(n\|\hat{\mathbf{b}}_{n,\tau,\omega_{j,n}} - \mathbf{b}_{n,\tau,\omega_{j,n}}\| \cdot \|\mathbf{b}_{n,\tau,\omega_{j,n}}\|) = O_{\mathrm{P}}(R_n)$. □

7.2 Proof of (4.5)

Note that $L_{n,\tau_1,\tau_2}(\omega_{j,n})$ is the cross-periodogram of the bivariate time series

$$\big(\tau_1 - I\{Y_t \le q_{\tau_1}\}, \tau_2 - I\{Y_t \le q_{\tau_2}\}\big). \tag{7.3}$$

Let $\omega_{j,n}, \omega_{kn} \in (0,\pi)$ be two Fourier frequencies. From Corollary 7.2.2 in Brillinger [1975], we know that

$$\mathrm{Var}(L_{n,\tau_1,\tau_2}(\omega_{j,n})) = \mathfrak{f}_{1,1}(\omega_{j,n})\mathfrak{f}_{2,2}(\omega_{j,n}) + \frac{2\pi}{n}\mathfrak{f}_{1,2,1,2}(\omega_{j,n}, -\omega_{j,n}, -\omega_{k,n}) + O(1/n) \tag{7.4}$$

and, for $\omega_{j,n} \neq \omega_{kn}$,

$$\text{Cov}\left(L_{n,\tau_1,\tau_2}(\omega_{j,n}), L_{n,\tau_1,\tau_2}(\omega_{k,n})\right) = \frac{2\pi}{n}\mathfrak{f}_{1,2,1,2}(\omega_{j,n}, -\omega_{j,n}, -\omega_{k,n}) + O(1/n^2), \tag{7.5}$$

where $\mathfrak{f}_{1,1}$, $\mathfrak{f}_{2,2}$ and $\mathfrak{f}_{1,2,1,2}$ are the spectra and cumulant spectra of the bivariate time series (7.3), which exist by Assumption (A6). Note that the orders $O(1/n)$ and $O(1/n^2)$ of the remainders in (7.4) and (7.5) hold uniformly with respect to the frequencies. The aforementioned spectra coincide with the Laplace spectra $\mathfrak{f}_{\tau_1,\tau_1}$, and $\mathfrak{f}_{\tau_2,\tau_2}$ and the cumulant spectra are also bounded (see Brillinger [1975], p. 26). Therefore,

$$\text{Cov}\left(L_{n,\tau_1,\tau_2}(\omega_{j,n}), L_{n,\tau_1,\tau_2}(\omega_{k,n})\right) = \begin{cases} \mathfrak{f}_{\tau_1,\tau_1}(\omega_{j,n})\mathfrak{f}_{\tau_2,\tau_2}(\omega_{j,n}) + \bar{R}_n & \omega_{j,n} = \omega_{k,n} \\ \bar{R}_n & \omega_{j,n} \neq \omega_{k,n}, \end{cases}$$

where $\bar{R}_n = O(1/n)$ does not depend on the frequencies. The assertion follows by the fact that the variance and the bias of the random variable K_n in (4.5) both are of the order $O(1/n)$. For the variance, note that

$$\begin{aligned} \text{Var}(K_n) = \frac{1}{f_Y^2(q_{\tau_1})f_Y^2(q_{\tau_2})}\Big[\sum_{|k|\leq N_n} W_n^2(k)\,\text{Var}\left(L_{n,\tau_1,\tau_2}(\omega_{j+k,n})\right) \\ + \sum_{|k_1|\leq N_n} W_n(k_1) \sum_{\substack{|k_2|\leq N_n \\ k_2\neq k_1}} W_n(k_2)\,\text{Cov}\left(L_{n,\tau_1,\tau_2}(\omega_{j+k_1,n}), L_{n,\tau_1,\tau_2}(\omega_{j+k_2,n})\right)\Big] = O(1/n), \end{aligned}$$

due to the second part of Assumption (A5) and (7.5). As for the bias, $\text{E}[K_n] = O(1/n)$ due to the fact that $\text{E}L_{n,\tau_1,\tau_2}(\omega_{j+k,n}) = \sum_{k=-\infty}^{\infty} \gamma_k(q_{\tau_1}, q_{\tau_2})\text{e}^{-\text{i}\omega_{j+k,n}k} + O(1/n)$ uniformly with respect to the frequencies (see Theorem 4.3.2 in Brillinger [1975]). $\square$

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