

1 **Quasi-Latin designs and their use in glasshouse exper-** 2 **iments**

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12 **Summary.** This paper gives a general method for constructing quasi-Latin square, quasi-Latin
13 rectangle and extended quasi-Latin rectangle designs for symmetric factorial experiments. Two
14 further methods are given for parameter values satisfying certain conditions. Designs are
15 constructed for a range of numbers of rows and columns so that the different construction
16 techniques are illustrated. For some row and column combinations, different designs are com-
17 pared, including designs constructed using computer search algorithms. The construction of
18 designs with rows and columns that are nested or contiguous is also discussed.

19 *Keywords:* Factorial designs; Glasshouse experiments; Greenhouse experiments; Latinized
 designs; Quasi-Latin designs; Row-column designs

20 **1. Introduction**

21 The motivation for the work reported here comes from the need for designs for glasshouse
22 experiments that involved several treatment factors (Tran, 2009), but is also applicable to
23 experiments in other plant houses, such as in greenhouses, shade houses and polytunnels.
24 In one experiment described by Tran (2009), a design was required for an experiment to
25 investigate the effects of five treatment factors on the growth of species of Australian native
26 plants that potentially could be used in the remediation of sites in the rail corridor either

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27 side of railway tracks. Three factors, each at two levels, were whether or not a nurse crop
28 was used, whether or not gypsum was added to the soil and whether or not mycorrhizal fungi
29 were added to the soil. The resulting eight treatments were to be applied to main plots that
30 were arranged in a 4×10 rectangle. It was thought that there would be interactions between
31 the factors and so it was important that the design gave good estimates of all interactions.
32 The other two factors were to be applied to subplots and are not considered in this paper.
33 A design for the main plot treatments is required and, as plots are arranged in a rectangle,
34 designs such as the quasi-Latin square designs cannot be employed. Originally, Design 2 in
35 Section 7.3 was obtained using CycDesign 2.0 (Whitaker et al., 2002) to be used for the
36 experiment, but the same design is produced using CycDesign 4.0 (CycSoftware Ltd, 2009)
37 with default settings. The confounding in this design was derived by inspection. Design 3
38 in Section 7.3 was then constructed by choosing suitable sets of confounding characters,
39 after which a version of the algorithm in Section 4 was used. Before the experiment was
40 run, the researchers decided to reduce the number of replicates of the 8 treatments from 5
41 to 3 and so Design 3 in Section 7.2 for a 4×6 rectangle was produced. It was employed in
42 the experiment. The other designs in Sections 7.2 and 7.3 were constructed subsequently.

43 Usually glasshouses are carefully aligned on North/South and East/West axes as in
44 Edmondson (1989) and Williams and John (1996) and this is the case for the glasshouse
45 experiments described in Tran (2009). It is usually anticipated that, in glasshouse exper-
46 iments, there will be trends along both axes and so designs with rows and columns have
47 long been recommended for these experiments. Youden (1940) recommended the use of
48 Latin and Youden squares and Cochran and Cox (1957, Section 4.3.1) recommended Latin
49 square designs for experiments involving a single treatment factor. Edmondson (1989) used
50 a Graeco-Latin square in a split plot design. Williams and John (1996) used factorial
51 designs with rows and columns in designing glasshouse experiments and Williams et al.
52 (2002, Section 7.5.1) advocated the use of designs with rows and columns for glasshouse
53 experiments.

54 Tran (2009) reports a review of 20 ecological journals over the period from 1980 to
55 2006. The review focussed on articles concerning experiments in glasshouses or greenhouses
56 on native plants that grow in temperate and semi-arid climates like Australia. In total,
57 59 experiments were reported, of which 43 involved factorial treatments. Only one of the
58 59 experiments stated that a design with rows and columns was used and this utilized a
59 strip-plot design. This somewhat surprised us because our experience, and that of other
60 statisticians, is that designs with rows and columns are often used. While this disparity
61 might indicate that the designs being employed are not always correctly reported, it could
62 also mean that a large proportion of ecological experiments are being designed by researchers
63 themselves and that they do not use designs with rows and columns. Hence, while the
64 evidence is not conclusive, it does seem that there is under-utilization of designs with rows

65 and columns. Of course, they will not always be appropriate. However, to ensure they will
 66 be used whenever appropriate, every obstacle to their use needs to be removed. In this
 67 paper we facilitate the design of experiments for situations in which the number of rows
 68 does not equal the number of columns by extending the range of designs that can be readily
 69 constructed for this case.

70 Natural contenders for designs with rows and columns for factorial experiments are quasi-
 71 Latin designs. These designs were introduced by Yates (1937) for factorial experiments in
 72 which, like a Latin square, the treatments are to be applied to units arranged in an equal
 73 number of rows and columns. However, unlike Latin squares, the treatments are arranged
 74 such that no treatment occurs more than once in a row or a column and not all treatments
 75 occur in any row or column. A quasi-Latin square design may consist of one or more quasi-
 76 Latin squares and each quasi-Latin square contains one or more complete sets of treatments
 77 (Rao, 1946; Cochran and Cox, 1957). If there is more than one quasi-Latin square, the design
 78 is usually treated as a nested row-column design that is α -resolved (Shah, 1978), where α
 79 is the number of complete sets of treatments per square.

80 Quasi-Latin square designs extend the factorial designs that have treatment effects con-
 81 founded with blocks to those that allow for two-way elimination of heterogeneity. They
 82 require the (partial) confounding of interactions with rows and columns within squares.
 83 Treatments must be equally replicated and the number of replicates is restricted. For ex-
 84 ample, consider a 2^3 factorial experiment. The eight treatments can be arranged in one or
 85 more 4×4 squares: the number of replicates for treatments must be a multiple of 2. We do
 86 not consider designs like that given by Cochran and Cox (1957, Plan 8.1b) to be quasi-Latin
 87 squares as they consist of Latin squares for subsets of the treatments; they do not fit into
 88 the class considered by Rao (1946) because the squares do not contain complete replicates
 89 and do not have treatment effects confounded within squares.

90 However, as has already been suggested, not all experiments in practice satisfy the
 91 restrictions placed on the number of replicates for a quasi-Latin square design and this was
 92 the case for experiments considered in Tran (2009). To provide more flexible designs we
 93 look to Latin rectangle designs. Adapting Preece (2006), Latin rectangles are defined to
 94 have k rows and ℓ columns for v treatments with $k \neq \ell$, $k \leq v$ and $\ell \leq v$. This differs
 95 from Preece (2006) in not insisting that $k < \ell$, although we will usually present designs
 96 so that this is true. Youden square designs, for which $\ell = v$, are Latin rectangles and are
 97 constructed from balanced incomplete block designs, with columns corresponding to blocks.
 98 Healy (1951) describes Latin rectangle designs for 2^k factorial experiments on a rectangle
 99 of 4×8 units. However, we refer to the subset of Latin rectangles in which a factorial set
 100 of treatments is assigned to a rectangle as *quasi-Latin rectangles*, because of their similarity
 101 to quasi-Latin squares. They retain the property of having no treatment repeated in any
 102 row or column. Then, a *quasi-Latin rectangle design* consists of one or more quasi-Latin

103 rectangles, and may contain more than one complete replicate of the treatments. Thus, the
 104 designs given by Healy (1951) are described as quasi-Latin rectangle designs that assign 2^3 ,
 105 2^4 or 2^5 factorial treatments to a single 4×8 rectangle. In addition, we consider *extended*
 106 *quasi-Latin rectangles* for which $k \neq \ell$ and at least one of k and ℓ exceeds v , so that either
 107 rows or columns or both contain some treatments more than once.

108 We begin in Section 2 by giving notation and some definitions, while Section 3 outlines
 109 some general principles that apply in designing experiments with rows and columns. In Sec-
 110 tion 4, a general method for the construction of row-column designs for symmetric factorial
 111 experiments is described and this is illustrated for a range of combinations of numbers of
 112 rows and columns in Section 5. Section 6 gives two further methods for parameter values
 113 satisfying certain conditions and Section 7 discusses the construction of row-column designs
 114 using the different methods and compares the designs obtained. The examples are only
 115 representative of the designs that can be constructed using the methods. The only type
 116 of design considered up to this point is row-column designs, the construction of designs
 117 with multiple squares or rectangles and choosing between these different types of design
 118 being deferred until Section 8. Some general aspects of quasi-Latin designs are discussed in
 119 Section 9.

120 2. Notation and some definitions

121 We consider designs in which there are s squares or rectangles each with k rows by ℓ columns,
 122 for $s \geq 1$ and $k, \ell \geq 2$. These squares and rectangles will be called *whole frames*. There are
 123 a total of v treatments, each with r replicates in each whole frame, and these treatments
 124 are the combinations of m factors each with p levels, where p divides both k and ℓ . Hence
 125 $v = p^m$ and $vr = k\ell$. Any subrectangle or subsquare of v units which contains one complete
 126 set of treatments will be called a *grid*. A single whole frame often contains grids of different
 127 shapes. The experimental unit, to which a single treatment is to be applied, is referred to
 128 simply as a *unit*. There are $sk\ell$ units in total. We assume that p is prime. This is not
 129 necessarily restrictive because, for a factor whose number of levels is a power of a prime, it
 130 is possible to substitute a combination of pseudofactors all of whose numbers of levels are
 131 equal to that prime.

132 Our construction methods use characters (Bailey, 2008, Section 12.2). Each level of
 133 a p -level factor is coded with the integers $0, 1, \dots, p - 1$. Each treatment combination
 134 of m factors can be written as an m -tuple of these levels. A character specifies a linear
 135 combination of factors that can be evaluated for each treatment combination; the coefficients
 136 are integers modulo p , as is the evaluation. For example, for the 3-level factors A and B ,
 137 the levels are coded 0, 1 and 2 and one of their nine treatment combinations is $(2, 1)$. One
 138 character is $A + 2B$ and, for $(2, 1)$, it evaluates to $1 \times 2 + 2 \times 1 = 2 + 2 = 1$.

139 For sources we use the notation of Brien et al. (2011). In particular, $A\#B$ denotes the
 140 interaction of factors A and B, $R[Q]$ denotes the nested effects of factor R within the levels
 141 of factor Q, and $R[P \wedge Q]$ denotes the nested effects of R within the combinations of the
 142 levels of factors P and Q. For most designs in this paper, the only factors indexing the units
 143 before treatments are allocated are Rows and Columns, which are crossed. Hence, the unit
 144 sources are Rows, Columns and Rows#Columns.

145 3. Some principles in designing experiments with rows and columns

146 Firstly, as will be seen in this paper, often there are competing designs with different
 147 properties for fixed basic design parameters, such as the numbers of treatments, replicates,
 148 rows and columns. They may differ in the unit sources that are taken into account in the
 149 design and the manner in which the different treatment sources are confounded with the unit
 150 sources. Hence, choosing between designs depends on the expected sources of variability
 151 and potential treatment effects. With regard to the potential treatment effects, the issue is
 152 usually about which interactions, if any, need to be allowed for. If the designer decides that
 153 certain interactions are likely, then in view of the likely smaller size of interactions (Yates,
 154 1937), it is especially important to maximize the amount of information about them which
 155 is confounded with the smallest source of variability. Consequently, our objective is not to
 156 obtain the “best design” for a given set of factorial treatments and of units, but to give
 157 several designs each of which is applicable in different situations.

158 Quasi-Latin designs are resolvable so that there is the option of developing a design
 159 that is (i) a *row-column design* with a single whole frame, (ii) a *nested design* being an
 160 α -resolved design consisting of s whole frames within which rows and columns are nested,
 161 or (iii) a *contiguous design*, which is like a nested design except that the contiguity between
 162 frames is acknowledged, so that treatments may be latinized to rows or columns (Williams,
 163 1986). The value of α depends on the choice of s . Latinizing the treatments means that
 164 they are replicated as equally as possible in the direction being latinized. The three types
 165 of design that have been described differ in the sources of unit variability for which they
 166 allow. Hence, they have different unit structures and so different randomizations. A *unit*
 167 *structure* for an experiment is the decomposition of its data vector according to unit sources
 168 only, all treatment sources being disregarded.

169 A row-column design anticipates differences between rows and between columns. In this
 170 case, the factors indexing the units are Rows and Columns and these are crossed, as in, say,
 171 the design in Section 5.1. The randomization that applies is that rows and columns are
 172 permuted independently.

173 A nested design is appropriate when (i) there is a set of frames among which differences
 174 are anticipated and (ii) differences are also anticipated between rows and columns within

175 each frame, these not being consistent across frames. Its factors are Frames, Rows and
176 Columns, with Rows and Columns nested within Frames, as in Design 1 in Section 8.1,
177 where the word Squares is used for Frames. For the randomization of a nested design,
178 frames are permuted, as are rows and columns within each frame.

179 A contiguous design, like a nested design, has frames. It is utilized when, in addi-
180 tion to the unit variability for the nested design, consistent differences between rows, for
181 horizontally-aligned frames, and columns, for vertically-aligned frames, are expected across
182 frames. To account for these consistent differences, the treatments are latinized across
183 frames to rows or columns or rows and columns, depending on the contiguity of the design.
184 For designs in which either rows or columns are contiguous, the factors are Frames, Rows
185 and Columns. If rows are contiguous, because consistent difference between rows across
186 frames are expected, then Rows are crossed with Frames and Columns, and Columns are
187 nested within Frames. An example is Design 3 in Section 8.1, where the word Squares
188 is used for Frames. Randomization involves the permutation of frames, of rows, and of
189 columns within frames.

190 Yates (1937) originally stressed that all rows and all columns should be randomized,
191 because he was concerned to ensure that the Latin square analysis would be unbiased.
192 However, the other randomizations can be applied, provided the appropriate analysis is
193 then employed. The choice between the three options should be based on the expected unit
194 sources of variability, rather than on the availability of a resolvable design. Will there be
195 differences between frames? Will rows or columns differ consistently across frames? If it is
196 likely that there are differences between frames and a row-column design is used, then either
197 the differences between frames may obscure differences between treatments or, depending
198 on the outcome of the randomization, the Residual mean square will give an overestimate
199 of the variance of treatment effects. Similarly, if a nested design is used when a contiguous
200 one is needed, then consistent differences between rows across frames will inflate either the
201 Residual mean square for Rows within Frames or the apparent size of treatment effects
202 confounded with Rows within Frames. On the other hand, if such differences are not
203 appreciable, there is a penalty in employing a nested or contiguous design in that they are
204 often less efficient than a good row-column design. Especially, for smaller designs, the loss
205 in efficiency may outweigh any advantage of a nested or contiguous design.

206 Some authors advocate the use of a nested design, or, if frames are contiguous, a con-
207 tiguous design in which treatments are latinized, whenever feasible and suggest that terms
208 for the extra required sources of variation be omitted if a preliminary data analysis indi-
209 cates that they are minor sources. This amounts to a “sometimes-pool” strategy. Janky
210 (2000), in reviewing this topic, concluded that this strategy should not be used routinely as
211 it generally inflates the probability of Type I errors and offers, at best, insubstantial gains
212 in power. In a similar vein, Gilmour and Goos (2009) warn of the dangers that arise from

213 omitting variance components that happen to have zero estimates and are based on small
 214 degrees of freedom. Our conclusion is that the designer should select a design appropriate
 215 for the anticipated sources of variation and use the analysis appropriate for the chosen de-
 216 sign. This supports our objective of having a range of designs for a specific set of design
 217 parameters.

218 In considering the confounding of factorial treatment sources with unit sources, the
 219 *canonical efficiency factor* will be used because it reflects the amount of information about
 220 a treatment source, adjusted for previously fitted treatment sources, that is (partially) con-
 221 founded with a particular unit source (John and Williams, 1995, Section 2.3). One property
 222 that distinguishes between designs is whether or not they are structure balanced (Brien and
 223 Bailey, 2009). This obtains when, for all treatment sources (partially) confounded with a
 224 particular unit source, (i) each treatment source has a single canonical efficiency factor and
 225 (ii) treatment sources remain orthogonal when estimated from that unit source. Condi-
 226 tion (i) is met for all 2^m factorial experiments. Condition (ii) means that the experiment
 227 has *orthogonal factorial structure* (Bailey, 1985). The advantage of factorial experiments
 228 having structure balance is that the estimates of various factorial effects are independent.
 229 It is desirable that, if at all possible, a structure-balanced design is used, even if it means
 230 sacrificing some efficiency on particular treatment effects. A structure-balanced design has
 231 a much simpler analysis, and conclusions are more straightforward than for designs that
 232 are not structure balanced. Also, the amount of treatment information available from unit
 233 sources is maximized for structure-balanced experiments, the amount of information avail-
 234 able about treatments from a unit source being the weighted sum of the degrees of freedom
 235 of treatment sources confounded with it, the weights being the efficiency factors.

236 We have obtained designs in two distinctly different ways: (i) choosing which treatment
 237 effects to confound with rows and columns and constructing the designs using the methods
 238 introduced in Sections 4 and 6, and (ii) using a package for the computer generation of
 239 designs based on an interchange algorithm, in our case CycDesigN (CycSoftware Ltd, 2009).
 240 The first gives the designer greater control over the design by providing the tools for choosing
 241 how to spread the information about the treatment effects across the different sources of
 242 variability, that is, deciding what to confound a treatment effect with and the amount of
 243 information that will be associated with more variable unit sources.

244 On the other hand, software, such as CycDesigN, has the advantage of giving an auto-
 245 mated procedure, which requires little more than input of the design parameters. CycDesigN
 246 4.0 (CycSoftware Ltd, 2009), optimizes the overall weighted efficiency of a design. A further
 247 significant advantage of such programs is that they can produce designs for a wider range
 248 of design parameters than the construction methods we outline. With CycDesigN, some
 249 control over the properties of the design produced is afforded by varying the weighting of
 250 (i) treatment main effects relative to two-factor treatment interactions and (ii) the row,

251 column and row-column components of a resolved design relative to each other. For (i), the
 252 weighted efficiency is the weighted linear combination of the average efficiency factors of
 253 the treatment main effects and of the treatment two-factor interactions (CycSoftware Ltd,
 254 2009, Section 9.3). Hence, it is not possible to specify the properties of single main effects
 255 or two-factor interactions or anything about the higher order interactions. Also, CycDesigN
 256 generates only 1-resolved designs.

257 It is our contention that, in spite of the availability of software like CycDesigN, con-
 258 struction methods of the type outlined in this paper are useful because they permit more
 259 direct control of the design process and can lead to designs that are better suited to some
 260 particular situations than those produced by such software. Further, they often have nice
 261 properties and provide a point of comparison for computer-generated designs. The methods
 262 of construction will be useful to any designer who does not have access to software like Cyc-
 263 DesigN, provided the experimental conditions correspond to the parameter combinations
 264 available with the methods. In comparing constructed and computer-generated designs, we
 265 generally compare the canonical efficiency factors of the two designs. However, even if the
 266 canonical efficiency factors are the same, the designs themselves may not be isomorphic in
 267 that it is not possible to obtain one by permuting the other in ways that are allowable for
 268 its unit structure.

269 **4. A method for the construction of row-column designs for symmetric factorial** 270 **experiments**

271 In this section we present Method 1 for constructing whole frames of shape $k \times \ell$. We
 272 assume that $v = p^m$, that v divides $k\ell$, that k and ℓ are both divisible by p . The replication
 273 is r , where $r = k\ell/v$. The method produces quasi-Latin squares and rectangles, extended
 274 quasi-Latin rectangles and Latin squares. It involves dividing a whole frame into several
 275 types of frames as illustrated in Figure 1 and consists of the steps below. The crux of the
 276 method is to form what we term box frames, whose dimensions are powers of p such that
 277 each contains one or more complete replicates of the treatments. Then sets of characters
 278 can be confounded with sets of rows and sets of columns in each box frame.

279 **Step 1 Divide up the whole frame:** Having ascertained the values of p , m , k , ℓ and
 280 r , determine values of t and u so that we can write $k = p^t r_1$ and $\ell = p^u r_2$, where
 281 $1 \leq t \leq m$, $1 \leq u \leq m$, $t + u \geq m$. That is, we factorize both k and ℓ as a product
 282 of two integers, of which the first is p raised to a non-zero power. The condition
 283 $t + u \geq m$ means that the product of the powers of p must be divisible by v . Select t
 284 and u as follows:

- 285 (a) If v divides k , then take $t = m$ and $r_1 = k/v$; if k is a power of p smaller than v

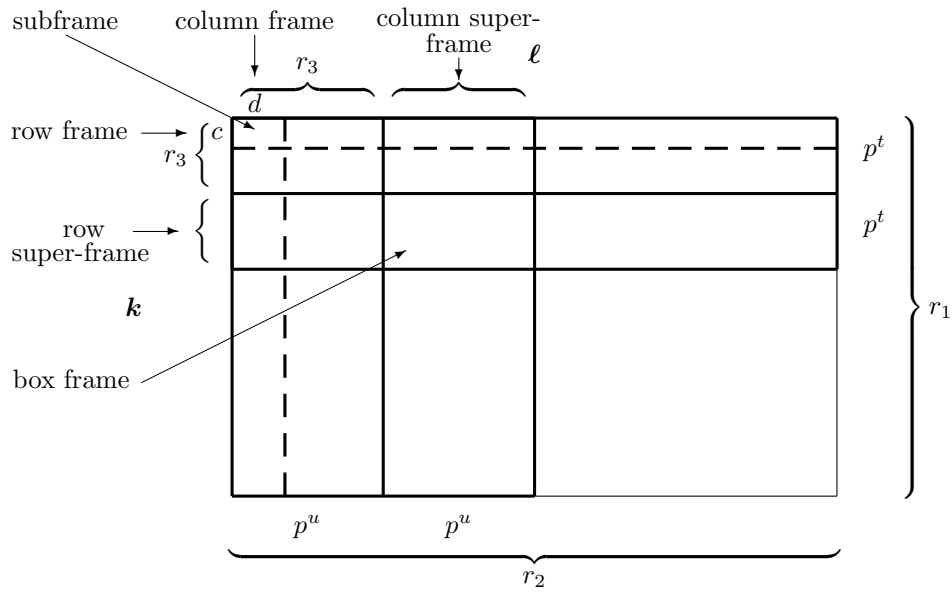


Fig. 1. Division of the whole frame for Method 1: the whole frame is divided into r_1 row super-frames and r_2 column super-frames whose intersections form box frames of shape $p^t \times p^u$; each row super-frame is divided into r_3 row frames and each column super-frame is divided into r_3 column frames; their intersections form subframes of shape $c \times d$.

286 or r is not divisible by p , then p^t is the largest power of p dividing k ; otherwise
 287 there is some choice in the value of t .
 288 (b) If v divides ℓ , then take $u = m$ and $r_2 = \ell/v$; if ℓ is a power of p smaller than v
 289 or r is not divisible by p , then p^u is the largest power of p dividing ℓ ; otherwise
 290 there is some choice in the value of u .

291 Note that if $k < v$, $\ell < v$ and r is a power of p , then $r_1 = r_2 = 1$.

292 Now divide the whole frame into r_1 row super-frames of p^t whole rows and r_2 column
 293 super-frames of p^u whole columns. The intersection of a row super-frame and a column
 294 super-frame forms a box frame of shape $p^t \times p^u$. Of course, if $r_1 = r_2 = 1$, then the
 295 super-frames and box frames are all the same as the whole frame.

296 To set up row and column frames, calculate $c = p^{m-u}$, $d = p^{m-t}$ and $r_3 = p^{t+u-m}$.
 297 Then $r = r_1 r_2 r_3$. Divide each row super-frame into r_3 row frames of shape $c \times \ell$ and
 298 each column super-frame into r_3 column frames of shape $k \times d$. The intersection of a
 299 row frame and a column frame forms a subframe of shape $c \times d$ and each box frame
 300 contains an $r_3 \times r_3$ array of these subframes. Also, each box frame contains r_3 grids
 301 of shape $p^t \times d$, as well as r_3 of shape $c \times p^u$.

302 **Step 2 Specify the row design:** Each row frame consists of r_2 grids of shape $c \times p^u$. If
 303 $u = m$ then $c = 1$. In this case, each $1 \times p^u$ grid contains a complete replicate of the
 304 treatments and there is no need to further consider the row design. If $c > 1$, then
 305 select $r_1 r_3$ sets of characters, one set per row frame, so that each set specifies $(c - 1)$
 306 treatment degrees of freedom to confound with c rows. The characters specifying one
 307 lot of $(c - 1)$ treatment degrees of freedom must be closed under the formation of
 308 sums (modulo p). We shall call these *row characters*. Each set divides the treatments
 309 into c groups of size p^u . If $r_2 = 1$ then the groups in each row frame are completely
 310 confounded with rows. If $r_2 > 1$ then each row frame needs a $c \times r_2$ row-column
 311 design Δ_1 for c treatments as an auxiliary design, where these treatments correspond
 312 to the c groups of treatments defined by the set of row characters for this row frame.
 313 In design Δ_1 , the columns are complete and the row design should be as efficient as
 314 possible.

315 **Step 3 Specify the column design:** Similarly, each column frame consists of r_1 grids of
 316 shape $p^t \times d$. If $t = m$ then $d = 1$. In this case, each $p^t \times 1$ grid contains a complete
 317 replicate of the treatments and there is no need to further consider the column design.
 318 If $d > 1$, then select $r_2 r_3$ sets of characters, each specifying $d - 1$ degrees of freedom
 319 and dividing the treatments into d groups of size p^t . We shall call these *column*
 320 *characters*. If $r_1 > 1$ then each column frame needs a $r_1 \times d$ auxiliary design Δ_2 for
 321 d treatments.

322 **Step 4 Ensure a unique treatment for each unit:** In each box frame, the treatments
 323 in each $1 \times p^u$ subrectangle are specified: if $u = m$, this subrectangle contains a
 324 complete set of treatments; otherwise they are specified by the row characters and,
 325 if $r_2 > 1$, the auxiliary design Δ_1 . Similarly, if $t = m$, then each $p^t \times 1$ subrectan-
 326 gle contains a complete set of treatments; otherwise the treatments it contains are
 327 specified by the column characters and, if $r_1 > 1$, the auxiliary design Δ_2 . If $r_3 = 1$,
 328 this uniquely determines the treatment on each unit. Otherwise, for each box frame,
 329 choose a set of characters which divide the treatments into r_3 groups of size cd . We
 330 shall call these *unit characters*. The groups are assigned to the $r_3 \times r_3$ array of sub-
 331 frames of shape $c \times d$ by using a $r_3 \times r_3$ Latin square Δ_3 as third auxiliary design.
 332 For each box frame, the sets of characters of whichever of the three different types
 333 (row, column and unit) are needed must satisfy the following condition:

any nonempty collection of characters, all of different types, must be linearly (1)
 independent modulo p .

334 If $t = u = m$, then $c = d = 1$ and $r_3 = p^m$ so that no row and column characters
 335 are required and there is no need to specify unit characters. All that is needed is Δ_3 ,

336 which is a $p^m \times p^m$ Latin square. The whole design is an $r_1 \times r_2$ array of such Latin
 337 squares.

338 In general, for each set of c rows, one has to specify either (i) c characters, including 0,
 339 closed under addition, or (ii) $m - u$ linearly independent characters, or (iii) $(c - 1)/(p - 1)$
 340 characters none of which is a multiple of any other. Similarly, for each set of d columns,
 341 one has to specify either (i) d characters, including 0, closed under addition, or (ii) $m - t$
 342 linearly independent characters, or (iii) $(d - 1)/(p - 1)$ characters none of which is a multiple
 343 of any other. A set of characters to be confounded with c rows (d columns) can be repeated
 344 amongst the $r_1 r_3$ ($r_2 r_3$) sets of row (column) characters. If the sets of one type are not all
 345 the same, this results in partial confounding.

346 If r is divisible by p but is not a power of p then there is some choice in the values of
 347 u and t . Different choices may lead to designs with different properties. If $t + u = m$ then
 348 $r_3 = 1$ and there is no need for unit characters, so Condition (1) is easier to satisfy. On
 349 the other hand, there is more freedom of choice for the row characters when c is smaller,
 350 and more freedom of choice for the column characters when d is smaller. The availability of
 351 good $c \times r_2$ and $r_1 \times d$ row-column designs for the possible values of c , d , r_1 and r_2 is also
 352 an issue. When u is larger then c and r_2 are smaller so the former are easier to find, but
 353 there may be more choice when c and r_2 are larger. For designs of practical size, it seems
 354 unlikely that all three of r_1 , r_2 and r_3 will be bigger than one.

355 With $k = \ell$ and $r_1 = r_2 = 1$, the method is equivalent to that of Rao (1946) for
 356 constructing quasi-Latin square designs. That is, our method generalizes that of Rao (1946)
 357 in two ways. The first simply allows $t \neq u$ when $r_1 = r_2 = 1$. The second allows one or
 358 both of r_1 and r_2 to be bigger than one: in either case, another auxiliary design is needed.

359 **5. Examples of quasi-Latin squares and rectangles with dimensions less than the**
 360 **number of treatments**

361 **5.1. A 2^3 factorial in a 4×4 square**

362 For $p = 2$, $m = 3$, $k = \ell = 4$ and $r = 2$, both k and ℓ are powers of p and the only possible
 363 quasi-Latin square is for $t = u = 2$ and $r_1 = r_2 = 1$. Thence, $c = d = 2$ and $r_3 = 2$. Hence,
 364 we can ignore super-frames and take the box frame to be the whole frame. The square is
 365 subdivided into two row frames of shape 2×4 and two column frames of shape 4×2 . The
 366 whole (box) frame is a 2×2 array of subframes, each of which is 2×2 .

367 Construction of the design requires two row characters and two column characters, as
 368 well as a unit character; the unit character splits the 8 treatments into 2 groups of 4 and
 369 the groups are assigned using, as an auxiliary design Δ_3 , a 2×2 Latin square.

370 Let U and V be the row characters, W and X be the column characters and Y be the
 371 unit character. They can be any five characters with the property that none of $U + W$,

Table 1. Quasi-Latin square for a 2^3 factorial experiment in 4 rows \times 4 columns

	$A + B$		$A + C$	
	$= 0$	$= 1$	$= 0$	$= 1$
$B + C = 0$	1, 1, 1	1, 0, 0	0, 0, 0	0, 1, 1
$B + C = 1$	1, 1, 0	1, 0, 1	0, 1, 0	0, 0, 1
$A + B + C = 0$	0, 0, 0	0, 1, 1	1, 0, 1	1, 1, 0
$A + B + C = 1$	0, 0, 1	0, 1, 0	1, 1, 1	1, 0, 0

Table 2. Canonical efficiency factors and Residual degrees of freedom (DF) for a 2^3 factorial experiment in 4 rows \times 4 columns

<i>Unit sources</i>	<i>Treatment sources</i>							<i>Residual DF</i>
	A	B	C	A#B	A#C	B#C	A#B#C	
Rows	0	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1
Columns	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	1
Rows#Columns	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	2

372 $U + X$, $V + W$ and $V + X$ is equal to Y or to 0. The four rows are defined by $U = 0$,
373 $U = 1$, $V = 0$ and $V = 1$, respectively, and the four columns by $W = 0$, $W = 1$, $X = 0$ and
374 $X = 1$, respectively. These restrictions are not enough to define the entries uniquely, so we
375 put $Y = 1$ on the top left-hand and the bottom right-hand subsquares, and put $Y = 0$
376 on the other two subsquares. In the top left-hand corner, the four combinations of levels of
377 U and W , together with the constraint $Y = 1$, define the treatments uniquely, giving all
378 four treatments with $Y = 1$. Similarly, in the top right-hand corner, the four combinations
379 of levels of U and X , together with the constraint $Y = 0$, define the treatments uniquely,
380 giving the remaining four treatments. Hence the first two rows form a complete replicate. In
381 a similar manner, the treatment on each unit is defined uniquely, and the first two columns
382 form a complete replicate, as do the last two rows and also the last two columns.

383 This construction results in U and V each losing half their information to rows, if $U \neq V$,
384 while W and X each lose half their information to columns, if $W \neq X$. The character Y is
385 necessary for the construction, but it remains orthogonal to both rows and columns.

386 For example, if we want full information on all main effects then we can put $U = B + C$,
387 $V = A + B + C$, $W = A + B$, $X = A + C$ and $Y = A$. This gives the design in Table 1.
388 Up to relabelling of the factors, it is the same as Square II in Cochran and Cox (1957,
389 Table 8.1). The canonical efficiency factors and Residual degrees of freedom for the design
390 are in Table 2. Clearly, the design has too few Residual degrees of freedom to be of practical
391 use.

392 To increase the Residual degrees of freedom, two squares ($s = 2$) are usually proposed
393 for a 2^3 factorial. Cochran and Cox (1957) give such a plan, which will be compared with

394 other designs using two squares in Section 8.1. However, there is another possibility that
 395 applies when the rows (or columns) of the two squares are contiguous. Namely, construct
 396 a single 4×8 rectangle, as is done in Section 7.1.

397 5.2. $A 2^5$ factorial in an 8×8 square

398 Here $p = 2$, $m = 5$, $k = \ell = 8$ and $r = 2$ and so both k and ℓ are powers of p . Hence,
 399 the only possible quasi-Latin square is for $t = u = 3$ and $r_1 = r_2 = 1$ so that $c = d = 4$
 400 and $r_3 = 2$. Again, super-frames are superfluous. The row frames are 4×8 and col-
 401 umn frames are 8×4 and there are two of each. To construct the design two sets of
 402 three row characters are needed and two sets of three column characters. Plan 8.3 in
 403 Cochran and Cox (1957) uses the two sets $\{A + B + C, A + D + E, B + C + D + E\}$ and
 404 $\{A + B + D, B + C + E, A + C + D + E\}$ for row characters and the two sets
 405 $\{A + C + E, B + C + D, A + B + D + E\}$ and $\{A + C + D, B + D + E, A + B + C + E\}$
 406 for column characters. Each set of characters is closed under addition (modulo 2). The box
 407 frame for this design is of shape 8×8 or the whole frame. As $r_3 = 2$, the whole frame
 408 consists of a 2×2 array of subframes of shape 4×4 and a unit character is required. The
 409 unit character chosen is $A + B + C + D$ and a 2×2 Latin square is used to assign its levels
 410 to the subframes.

411 In Section 8.2, nested and contiguous designs, based on two 4×8 grids, are explored as
 412 alternatives to the above design.

413 5.3. $A 3^3$ factorial experiment in a 9×12 rectangle

414 In this case $p = 3$, $m = 3$, $k = 9$, $\ell = 12$ and $r = 4$ so that k is a power of p and r is not
 415 divisible by p . Thus, $t = 2$ and $u = 1$. We have $c = 9$, $d = 3$ and $r_3 = 1$. The numbers of
 416 row and column super-frames are $r_1 = 1$ and $r_2 = 4$. Then there is one row frame, the same
 417 as the row super-frame and the whole frame; each column super-frame is also a column
 418 frame and a box frame, and consists of a 9×3 grid.

419 Row characters specifying 8 degrees of freedom and four column characters, each spec-
 420 ifying 2 degrees of freedom, are required. An auxiliary design Δ_1 , for assigning the nine
 421 groups defined by the row characters, is needed and this will be a 9×4 row-column design
 422 for 9 treatments. A suitable design has the following rows: (5, 6, 8, 9), (9, 4, 6, 7), (7, 8, 4, 5),
 423 (8, 9, 2, 3), (3, 7, 9, 1), (1, 2, 7, 8), (2, 3, 5, 6), (6, 1, 3, 4), (4, 5, 1, 2). Use two of the row char-
 424 acters to index treatments 1–9 in lexicographical order. Then, the four degrees of freedom
 425 corresponding to these two row characters have canonical efficiency factor $1/4$ in rows,
 426 while the canonical efficiency factor for the four degrees of freedom for the other two row
 427 characters is $1/16$.

428 No unit characters are required because $r_3 = 1$.

429 For example, one could choose $A + B$ and $B + C$ for row characters, so that $A + 2C$ and
 430 $A + 2B + C$ would be required to make the complete set of row characters. The column
 431 characters could be chosen from $A + B + C$, $A + B + 2C$ and $A + 2B + 2C$. For example,
 432 one could use two of these characters in one frame each and the other in two frames, thus
 433 partially confounding the corresponding effects. Those used in just one would have 75% of
 434 their information orthogonal to rows and columns and, for the other character, it would be
 435 50%.

436 **5.4. $A 2^4$ factorial in an 8×12 rectangle**

437 In this example $p = 2$, $m = 4$, $k = 8$, $\ell = 12$ and $r = 6$. We have $k = 2^3$ and $\ell = 2^2 3^1$ and
 438 so, as k is a power of p , it must be that $t = 3$ and, as r is divisible by p , there is a choice of
 439 values for u ; $u = 2$ is chosen. As a result $c = 4$, $d = 2$ and $r_3 = 2$. The numbers of row and
 440 column super-frames are $r_1 = 1$ and $r_2 = 3$, respectively. Hence, there are two row frames
 441 of shape 4×12 in the one row super-frame, and three column frames, one in each column
 442 super-frame of shape 8×4 .

443 The row and column characters chosen for this design are given in Table 3. Because
 444 $r_2 = 3$, an auxiliary design Δ_1 is needed to assign groups defined by the row characters to
 445 the 4×3 array of subrectangles of shape 1×4 . The transpose of a 3×4 Youden square,
 446 constructed by removing the last row from a Latin square, is suitable. The three rows of
 447 the Youden square are $(1, 2, 3, 4)$, $(2, 3, 4, 1)$ and $(3, 4, 1, 2)$. As the Youden square has $1/9$
 448 of the treatment information confounded with columns, $1/9$ of each of the row characters is
 449 confounded with Rows. Because $r_1 = 1$, an auxiliary design Δ_2 is not needed for assigning
 450 the column characters.

451 The box frames for this design are of shape 8×4 . As $r_3 = 2$, a box frame consists of a
 452 2×2 array of subframes of shape 4×2 and a unit character is required for each box frame.
 453 They are in Table 3. The auxiliary design Δ_3 , used in assigning unit characters, is a 2×2
 454 Latin square with rows $(0,1)$ and $(1,0)$.

455 The canonical efficiency factors and Residual degrees of freedom for this design are
 456 summarized in Table 4. This shows that the design has very good properties. Many other
 457 choices of sets of confounding characters are possible, depending on which interactions are
 458 considered important.

459 **5.5. $A 2^3$ factorial in a 6×12 rectangle**

460 Here $p = 2$, $m = 3$, $k = 6$ and $\ell = 12$, so that $r = 9$ and, as r is not divisible by p , we are
 461 forced to put $t = 1$, $u = 2$ and $r_1 = r_2 = 3$, which give $c = 2$, $d = 4$ and $r_3 = 1$. Thus the
 462 row and column super-frames are the same as the row and column frames.

463 There are three row frames, each of shape 2×12 . We can assign the characters A , B and

Table 3. Quasi-Latin rectangle for a 2^4 factorial experiment in 8 rows \times 12 columns

Unit characters Column characters	Column frame I			Column frame II			Column frame III		
	$A + B + C + D$ $= 0$	A $= 0$	$A + C + D$ $= 1$	$A + B + C$ $= 0$	D $= 1$	$C + D$ $= 1$	$A + B + D$ $= 0$	$A + B + C + D$ $= 1$	$B + C + D$ $= 1$
Upper row frame (Row characters: $A + B, A + C, B + C$)	0,0,0,0 0,0,1,1 0,1,0,1 0,1,1,0	0,0,0,1 0,0,1,0 0,1,0,0 0,1,1,1	1,1,1,0 1,1,0,1 1,0,1,0 1,0,0,1	1,1,0,0 1,0,1,0 0,1,1,0 0,0,0,0	0,0,1,1 0,1,0,0 0,1,1,1 1,1,1,1	1,1,0,1 0,1,0,1 1,0,0,1 0,0,0,1	0,1,0,1 1,0,0,1 1,1,1,1 1,1,0,0	1,0,1,1 0,1,1,0 1,1,1,1 0,0,1,1	0,1,0,0 1,0,0,0 0,0,0,0 1,1,0,1
Lower row frame (Row characters: $A + D, B + D, A + B$)	1,1,1,1 1,1,0,0 1,0,0,1 1,0,1,0	1,1,0,1 1,1,1,0 1,0,1,1 1,0,0,0	0,0,0,0 0,0,1,1 0,1,0,0 0,1,1,1	0,0,0,1 1,0,1,1 0,1,1,1 1,1,0,1	1,1,0,0 0,1,0,0 1,0,0,0 0,0,0,0	1,1,1,0 0,1,1,0 1,0,0,0 0,0,0,0	1,0,1,1 0,1,1,1 0,0,1,0 1,1,1,1	0,1,1,0 1,0,0,0 1,1,0,1 0,0,0,0	0,1,0,0 1,0,1,0 0,0,0,0 0,0,1,1

Table 4. Canonical efficiency factors and Residual degrees of freedom (DF) for the design for a 2^4 factorial experiment in 8 rows \times 12 columns

<i>Treatment sources</i>	<i>Unit sources</i>		
	Rows	Columns	Rows#Columns
A, B, C, D	0	0	1
A#B	$\frac{1}{9}$	0	$\frac{8}{9}$
A#C, A#D, B#C, B#D	$\frac{1}{18}$	0	$\frac{17}{18}$
C#D, A#B#C, A#B#D, A#C#D, B#C#D, A#B#C#D	0	$\frac{1}{6}$	$\frac{5}{6}$
<i>Residual DF</i>	2	5	62

Table 5. Quasi-Latin rectangle for a 2^3 factorial experiment in 6 rows \times 12 columns

	<i>Column frame I</i>	<i>Column frame II</i>	<i>Column frame III</i>
$A = 0, 0, 1$	0, 0, 0 0, 0, 1 0, 1, 0 0, 1, 1	0, 0, 0 0, 0, 1 0, 1, 0 0, 1, 1	1, 1, 1 1, 1, 0 1, 0, 1 1, 0, 0
$A = 1, 1, 0$	1, 1, 1 1, 1, 0 1, 0, 1 1, 0, 0	1, 1, 1 1, 1, 0 1, 0, 1 1, 0, 0	0, 0, 0 0, 0, 1 0, 1, 0 0, 1, 1
$B = 0, 0, 1$	0, 0, 1 1, 0, 1 1, 0, 0 0, 0, 0	0, 0, 1 1, 0, 1 1, 0, 0 0, 0, 0	1, 1, 0 0, 1, 0 0, 1, 1 1, 1, 1
$B = 1, 1, 0$	1, 1, 0 0, 1, 0 0, 1, 1 1, 1, 1	1, 1, 0 0, 1, 0 0, 1, 1 1, 1, 1	0, 0, 1 1, 0, 1 1, 0, 0 0, 0, 0
$C = 0, 0, 1$	0, 1, 0 1, 0, 0 0, 0, 0 1, 1, 0	0, 1, 0 1, 0, 0 0, 0, 0 1, 1, 0	1, 0, 1 0, 1, 1 1, 1, 1 0, 0, 1
$C = 1, 1, 0$	1, 0, 1 0, 1, 1 1, 1, 1 0, 0, 1	1, 0, 1 0, 1, 1 1, 1, 1 0, 0, 1	0, 1, 0 1, 0, 0 0, 0, 0 1, 1, 0

464 C to one row frame each. In each row frame we use, for the two levels of the row character,
 465 the 2×3 auxiliary design Δ_1 whose rows are $(0, 0, 1)$ and $(1, 1, 0)$: this confounds $1/9$ of the
 466 between-level information with rows. There are three column frames, each of shape 6×4 :
 467 in order to satisfy Condition (1), we take $\{A + B, A + C, B + C\}$ to be the set of column
 468 characters in each column frame. This set divides the eight treatments into four groups of
 469 two, so our 3×4 auxiliary design Δ_2 is the Youden square given in Section 5.4. There is
 470 no need for a unit character or a third auxiliary design, because $r_3 = 1$.

471 The complete design is shown in Table 5. All main effects have canonical efficiency
 472 factors $1/27$, 0 and $26/27$ in Rows, Columns and Rows#Columns respectively, while the
 473 corresponding figures for the two-factor interactions are 0 , $1/9$ and $8/9$. The three-factor
 474 interaction is completely confounded with Rows#Columns.

475 **6. Other methods for constructing row-column designs for symmetric factorial ex-**
 476 **periments**

477 We now give two other methods for constructing (extended) quasi-Latin rectangles.
 478 Method 2 applies when one of k and ℓ is a multiple of v . Method 3 divides the design
 479 into unequally-sized segments and a design is constructed for each segment.

480 For Method 2, not only must one of k and ℓ be a multiple of v , but the other must

481 be a proper divisor of v . Take ℓ to be a multiple of v ; for the case of k a multiple of v
 482 interchange the roles of rows and columns. While such designs can be constructed using
 483 Method 1, this requires the specification of both column and unit characters. On the other
 484 hand, Method 2 requires only column characters and so with it there will usually be more
 485 choice for the column characters. Further, Condition (1) is vacuously satisfied, and so there
 486 are no constraints on the choice of column characters. Hence, Method 2 is likely to be
 487 the preferred method for this class of designs, unless the designer is prepared to confound
 488 column characters with multiple column frames. The steps for Method 2 are:

489 **Step 1: Divide up the whole frame:** Divide the design into r_2 column super-frames of
 490 shape $k \times v$, where $r_2 = \ell/v$. Divide each column super-frame into k column frames
 491 of shape $k \times d$, where $d = v/k$.

492 **Step 2: Specify the column design:** In each column super-frame, choose k sets of col-
 493 umn characters each specifying $d - 1$ degrees of freedom. It is not necessary for all
 494 the sets to be different. Each set of characters is confounded with the columns of one
 495 of the column frames.

496 **Step 3: Form the row design:** In each column super-frame, rearrange the treatments
 497 in each column, using the algorithm given in Technique 11.1 of Bailey (2008), so that
 498 each row consists of a complete replicate.

499 The justification for the last step is that the column design can be viewed as a symmetric
 500 incomplete-block design. By Hall's Marriage Theorem, the treatments in each column can
 501 be rearranged so that each row consists of a complete replicate.

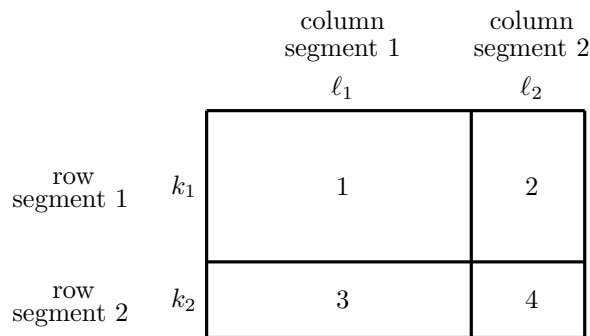


Fig. 2. Segmentation of the whole frame for Method 3 into four segments numbered as shown

502 Method 3 divides the design into segments as illustrated in Figure 2. It is useful when
 503 at least one of k and ℓ is neither a power of p nor a multiple of v ; otherwise, it duplicates

504 Method 1 or Method 2. We assume that the normal conditions for (extended) quasi-Latin
 505 designs apply.

506 **Step 0: Initialize:** Set $k_1 = k$, $k_2 = 0$, $\ell_1 = \ell$ and $\ell_2 = 0$.

507 **Step 1: Choose the row segment sizes:** If k is neither a power of p nor a multiple of
 508 v , then choose a value of t such that $p^t < k$, $t \leq m$, v divides $p^t \ell$, and p^t does not
 509 divide k . If there is no such value of t , then there is nothing to be gained by row
 510 segmentation. Otherwise, it is usually sensible to choose the largest possible value of
 511 t ; in particular, if $k \geq v$ then take $t = m$. Let k_1 be the largest multiple of p^t which
 512 is smaller than k , and put $k_2 = k - k_1$. Then v divides $k_1 \ell$ and $k_2 \ell$, and p divides k_1
 513 and k_2 .

514 **Step 2: Choose the column segment sizes:** If ℓ is neither a power of p nor a multiple
 515 of v , then choose a value of u such that $p^u < \ell$, $u \leq m$, v divides $p^u k$, and p^u does not
 516 divide ℓ . If there is no such value of u , then there is nothing to be gained by column
 517 segmentation. Otherwise, it is usually sensible to choose the largest possible value of
 518 u ; in particular, if $\ell \geq v$ then take $u = m$. Let ℓ_1 be the largest multiple of p^u which
 519 is smaller than ℓ , and put $\ell_2 = \ell - \ell_1$. Since v divides $p^u k$, we must have k divisible
 520 by p^{m-u} . If t is defined, then $m - u \leq t$, and hence p^{m-u} divides k_1 and k_2 ; otherwise
 521 $k_1 = k$ and $k_2 = 0$ and again p^{m-u} divides k_1 and k_2 . Therefore v divides $k_1 \ell_1$, $k_1 \ell_2$,
 522 $k_2 \ell_1$ and $k_2 \ell_2$, and p divides ℓ_1 and ℓ_2 .

523 **Step 3: Divide the whole frame into segments:** If $k_2 = \ell_2 = 0$, then it is not useful
 524 to segment the design and this method does not apply. Otherwise, segment the design
 525 as shown in Figure 2. Only if both $k_2 \neq 0$ and $\ell_2 \neq 0$ will there be four segments. If
 526 only one is nonzero, then there will be two segments.

527 **Step 4: Construct a design for each segment:** Use Method 1, 2 or 3, as appropriate,
 528 on each of the design segments. If there are two segments in the same row segment and
 529 row characters are needed for both, then, to minimize the amount of information on
 530 row characters in Rows in the whole design, the row characters in each row frame of the
 531 first segment should be a subset of those in the corresponding row frame of the second
 532 segment; this will also require the values of these characters in the corresponding
 533 rows of the two designs to be chosen suitably. Similar considerations apply if there
 534 are two segments in the same column segment and column characters are needed
 535 for both. The simplest situation is that $k_1 = \ell_1 = v$ and so a Latin square can be
 536 used for segment 1. In this situation, segment 2 will require only row characters for
 537 its construction, segment 3 will require only column characters and segment 4 will
 538 require both row and column characters, but these can be chosen independently of
 539 the those used for the other segments.

540 **7. Examples comparing (extended) quasi-Latin rectangles constructed using the**
 541 **different methods**

542 In this Section we compare several row-column designs for three sets of basic design param-
 543 eters using all three methods of construction that we have presented, as well as computer
 544 search.

545 **7.1. A 2^3 factorial in a 4×8 rectangle**

546 For this example $p = 2$, $m = 3$, $k = 2^2$, $\ell = 2^3$ and $r = 4$. Here, we compare the properties
 547 of three designs, all of which have orthogonal factorial structure.

548 **Design 1:** As $\ell = v$, Method 2 in Section 6 applies with $r_2 = 1$ and $d = 2$. We use it to
 549 construct a design. There are four column frames of shape 4×2 . A column character
 550 is needed for each column pair. For example, assign each of $A + B$, $A + C$, $B + C$ and
 551 $A + B + C$ to be confounded with one pair of columns, so that a different character is
 552 used for each pair. Then, $1/4$ of the information on each of the corresponding effects is
 553 lost to columns. Table 6 shows one of the possible designs obtained after rearranging
 554 treatments in each column to make each row a complete replicate.

555 **Design 2:** The optimal design produced by CycDesigN 4.0, which can be constructed using
 556 Method 2 by assigning $A + C$ to one pair of columns and $A + B + C$ to three pairs of
 557 columns. It partially confounds $1/4$ of $A \# C$ and $3/4$ of $A \# B \# C$ with its columns.

558 **Design 3:** Healy's (1951) design, which can be constructed using Method 1, with $r_1 =$
 559 $r_2 = 1$, $t = 2$ and $u = 3$ so that $c = 1$ and $d = 2$. Thus, $r_3 = 4$ and the whole
 560 (box) frame consists of a 4×4 array of 1×2 subframes. Four column characters
 561 are needed, as well as one set of three unit characters that are closed under addition
 562 and an auxiliary design Δ_3 , for assigning the four groups of treatments determined
 563 by the four combinations of the values of the unit characters. Healy's design has the
 564 character $A + B + C$ assigned to every pair of columns, uses the set $\{B, C, B + C\}$
 565 for unit characters, and takes a 4×4 Latin square for the auxiliary design. So, the
 566 interaction $A \# B \# C$ is totally confounded with columns and no treatment effects
 567 are confounded with rows.

568 Table 7 compares the canonical efficiency factors and Residual degrees of freedom for
 569 the three designs. In using CycDesigN to produce Design 2, the default weights ratio of
 570 $1 : 0.25$ for main effects relative to two-factor interactions was employed. One might consider
 571 reducing the weights for two-factor interactions in order to produce Design 1, which has
 572 lower efficiency for two-factor interactions. However, for two designs with orthogonal main
 573 effects and so maximal main-effect efficiency, such as Designs 1 and 2, the weighted efficiency

Table 6. Quasi-Latin rectangle for a 2^3 factorial experiment in 4 rows \times 8 columns constructed using Method 2

$A + B$		$A + C$		$B + C$		$A + B + C$	
= 0	= 1	= 0	= 1	= 0	= 1	= 0	= 1
0, 0, 0	1, 0, 0	0, 1, 0	0, 0, 1	0, 1, 1	1, 1, 0	1, 0, 1	1, 1, 1
1, 1, 0	1, 0, 1	0, 0, 0	1, 0, 0	1, 1, 1	0, 0, 1	0, 1, 1	0, 1, 0
0, 0, 1	0, 1, 0	1, 1, 1	0, 1, 1	0, 0, 0	1, 0, 1	1, 1, 0	1, 0, 0
1, 1, 1	0, 1, 1	1, 0, 1	1, 1, 0	1, 0, 0	0, 1, 0	0, 0, 0	0, 0, 1

Table 7. Canonical efficiency factors and Residual degrees of freedom (DF) for the designs for a 2^3 factorial experiment in 4 rows \times 8 columns

<i>Design</i>	<i>Unit sources</i>	<i>Treatment sources</i>							<i>Residual DF</i>
		A	B	C	A#B	A#C	B#C	A#B#C	
1	Rows	0	0	0	0	0	0	0	3
	Columns	0	0	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	3
	Rows#Columns	1	1	1	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	14
2	Rows	0	0	0	0	0	0	0	3
	Columns	0	0	0	0	$\frac{1}{4}$	0	$\frac{3}{4}$	5
	Rows#Columns	1	1	1	1	$\frac{3}{4}$	1	$\frac{1}{4}$	14
3	Rows	0	0	0	0	0	0	0	3
	Columns	0	0	0	0	0	0	1	6
	Rows#Columns	1	1	1	1	1	1	0	15

574 factor for a given set of weights must be greater for the design for which the sum of the
575 efficiency factors for two-factor interactions is greater. Hence, Design 2, or any design
576 whose two-factor efficiencies include both 0.75 and 1 and no other values, will always have
577 higher efficiency than Design 1 and so Design 1 will only be selected as the optimal design
578 if one manages to stop the iterative search procedure prematurely. This is more likely to
579 be possible if a relatively very small weight is used for two-factor interactions, such as in a
580 weights ratio of 1:0.001, because this will make the differences between the efficiency of the
581 designs small (< 0.001).

582 Clearly, Design 3 suits experiments in which it is appropriate to confound the three-
583 factor interaction with the likely more variable Columns, such as when this interaction is
584 anticipated to be negligible. On the other hand, as concluded in Section 3, Design 1 will
585 be preferred if a three-factor interaction is thought to be highly likely and one wants to
586 estimate it with good precision.

587 **7.2. A^{2^3} factorial in a 4×6 rectangle**

588 This example is for $p = 2$, $m = 3$, $k = 4$, $\ell = 6$ and $r = 3$. As always, Method 1 applies.
 589 Neither ℓ nor k are multiples of v and so Method 2 is not applicable. On the other hand, ℓ
 590 is neither a multiple of v nor a power of p , so that Method 3 can be used.

591 Three designs will be constructed, ordered according to the amount of information par-
 592 tially confounded with Rows#Columns: the amount for A#B#C decreases and that for
 593 the two-factor interactions increases. They demonstrate how the designer can influence the
 594 spread of the information about the treatment effects across the unit sources and show the
 595 flexibility of our construction methods.

596 **Design 1:** In this design Method 1 is used. As r is not a multiple of p , it follows that
 597 $t = 2$ and $u = 1$ so that $r_1 = 1$ and $r_2 = 3$. Also, $c = 4$, $d = 2$ and $r_3 = 1$.
 598 Hence, there are 3 column super-frames, each containing a single 4×2 grid that is
 599 both a box frame and a subframe. To construct the design requires, firstly, a set of
 600 row characters specifying 3 treatment degrees of freedom and an auxiliary design Δ_1
 601 for assigning groups of treatments determined by the row characters. Secondly, one
 602 column character for each column super-frame is needed. Unit characters and the
 603 associated auxiliary design Δ_3 are not required.

604 Let $\{U, V, U + V\}$ be the set of row characters and $\{W, X, Y\}$ the set of column
 605 characters. It is not necessary for all the column characters to be different, but
 606 Condition (1) must be satisfied. The three row characters divide the eight treatments
 607 into four groups of two, say S_1, S_2, S_3 and S_4 . The transpose of the 3×4 Youden
 608 square used in Section 5.4 is a suitable auxiliary design for assigning these groups.

609 To maximize the minimum canonical efficiency factor for all treatment effects when
 610 (partially) confounded with Rows# Columns, we can take $U = A$, $V = B$, $W = A + C$,
 611 $X = B + C$ and $Y = A + B + C$. Table 8 shows the final design.

612 **Design 2:** This design uses Method 3. Because k is a power of p , row segmentation is not
 613 useful and $k_2 = 0$. On the other hand, ℓ is not a power of p or a multiple of v , and
 614 $\ell > 4$ so that column segmentation can be employed. Here $u = 2$ so that $\ell_1 = 4$ and
 615 $\ell_2 = 2$. That is, segment 1 is of shape 4×4 and the other segment is 4×2 . The first
 616 can be constructed as a 4×4 quasi-Latin square and the other as a 4×2 quasi-Latin
 617 rectangle, both using Method 1.

618 For the quasi-Latin rectangle, which consists of a single grid, a set of row characters
 619 specifying 3 treatment degrees of freedom and a column character are required. Sup-
 620 pose that, in order to have no main effects involved, the row characters are $A + B$,
 621 $A + C$ and $B + C$ and the column character is $A + B + C$. The row characters divide the
 622 treatments into four groups of two, one for each combination of the values of $A + B$,
 623 $A + C$.

624 For the quasi-Latin square, which has the same basic design parameters as the design
 625 given in Section 5.1, two row and two column characters, as well as a unit character,
 626 are needed. To match the quasi-Latin rectangle, the row characters for the quasi-Latin
 627 square should be a subset of those for the rectangle. Take $A + B$ and $A + C$. For the
 628 column characters, again to have no main effects involved and more information about
 629 $A\#B\#C$ confounded with Columns, suppose the characters $B + C$ and $A + B + C$
 630 are chosen. The unit character is A . The transpose of the 4×4 quasi-Latin square
 631 design in Table 1 is such a design, which is given in the first four columns of Design 2
 632 in Table 8.

633 In combining the 4×2 rectangle and the 4×4 square, assign the values of the row
 634 characters in each row of the combined design so that they differ between the two
 635 segments. The last two columns of Design 2 in Table 8 have the quasi-Latin rectangle.
 636 The last column of the table applies to the last two columns of the design. The
 637 canonical efficiency factors for the combined design are those of Design 2 in Table 9
 638 and it happens that they are the same as those for the design produced by CycDesign
 639 4.0. However, the two designs are not isomorphic.

640 **Design 3:** This design is constructed in the same manner as Design 1, but using different
 641 characters. To completely confound $A\#B\#C$ with Columns, take $U = A + C$, $V =$
 642 $B + C$ and $W = X = Y = A + B + C$.

Table 8. Designs for a 2^3 factorial experiment in 4 rows \times 6 columns

<i>Design 1 — Method 1</i>									
$A + C$		$B + C$		$A + B + C$					
= 0	= 1	= 0	= 1	= 0	= 1				
0, 0, 0	0, 0, 1	1, 0, 0	1, 0, 1	0, 1, 1	0, 1, 0				
1, 0, 1	1, 0, 0	0, 1, 1	0, 1, 0	1, 1, 0	1, 1, 1				
0, 1, 0	0, 1, 1	1, 1, 1	1, 1, 0	0, 0, 0	0, 0, 1				
1, 1, 1	1, 1, 0	0, 0, 0	0, 0, 1	1, 0, 1	1, 0, 0				

<i>Design 2 — Method 3</i>									
	$B + C$		$A + B + C$		$A + B + C$				
	= 0	= 1	= 0	= 1	= 0	= 1	$A + B^\dagger$	$A + C^\dagger$	
$A + B = 0$	1, 1, 1	1, 1, 0	0, 0, 0	0, 0, 1	0, 1, 1	1, 0, 0	= 1,	= 1	
$A + B = 1$	1, 0, 0	1, 0, 1	0, 1, 1	0, 1, 0	0, 0, 0	1, 1, 1	= 0,	= 0	
$A + C = 0$	0, 0, 0	0, 1, 0	1, 0, 1	1, 1, 1	1, 1, 0	0, 0, 1	= 0,	= 1	
$A + C = 1$	0, 1, 1	0, 0, 1	1, 1, 0	1, 0, 0	1, 0, 1	0, 1, 0	= 1,	= 0	

[†]These relations apply only to units in the last two columns of the design

Table 9. Canonical efficiency factors and Residual degrees of freedom (DF) for the designs for a 2^3 factorial experiment in 4 rows \times 6 columns

Design	Unit sources	Treatment sources							Residual DF
		A	B	C	A#B	A#C	B#C	A#B#C	
1	Rows	$\frac{1}{9}$	$\frac{1}{9}$	0	$\frac{1}{9}$	0	0	0	0
	Columns	0	0	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	2
	Rows#Columns	$\frac{8}{9}$	$\frac{8}{9}$	1	$\frac{8}{9}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	8
2	Rows	0	0	0	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	0	0
	Columns	0	0	0	0	0	$\frac{1}{3}$	$\frac{2}{3}$	3
	Rows#Columns	1	1	1	$\frac{8}{9}$	$\frac{8}{9}$	$\frac{5}{9}$	$\frac{1}{3}$	8
3	Rows	0	0	0	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	0	0
	Columns	0	0	0	0	0	0	1	4
	Rows#Columns	1	1	1	$\frac{8}{9}$	$\frac{8}{9}$	$\frac{8}{9}$	0	9

643 The canonical efficiency factors and Residual degrees of freedom for the three designs
 644 are in Table 9. Design 3 has the advantage over the other designs in having more Residual
 645 degrees of freedom for Columns. To achieve this there is no information about A#B#C
 646 confounded with Rows#Columns. This is the design that was used in the experiment
 647 described by Tran (2009), but, in retrospect, Design 1 would have been better. The reason
 648 is that Design 1 has more information about the three-factor interaction confounded with
 649 Rows#Columns, and so is better able to distinguish between models with and without the
 650 three-factor interaction, with little loss of information about the other treatment effects
 651 from Rows#Columns.

652 An alternative to Design 1, when three-factor interactions are likely, is to ignore the
 653 factorial structure in the construction and to generate a design for eight treatments. For
 654 example, treatments 1–8 are assigned to the combinations of *A*, *B* and *C* listed in lexico-
 655 graphical order and CycDesigN 2.0 used to produce a design for the 8 treatments. The
 656 canonical efficiency factors for the factorial effects, when confounded with Rows#Columns,
 657 are more uniform than those for Designs 1–3, but the gain in efficiency for A#B#C is less
 658 than 10%. Also, the design does not have orthogonal factorial structure and cannot be
 659 made to have, no matter how the 2^3 factorial combinations are assigned to treatments 1–8.

660 Section 7.3 investigates constructing a row-column design when the number of replicates
 661 is increased from 3 to 5.

662 **7.3. A 2^3 factorial in a 4×10 rectangle**

663 For this example $p = 2$, $m = 3$, $k = 4$, $\ell = 10$ and $r = 5$. As always, Method 1 applies.
 664 Neither ℓ nor k are multiples of v and so Method 2 is not applicable. On the other hand, ℓ
 665 is neither a multiple of v nor a power of p so that Method 3 can be used. The design is an
 666 extended quasi-Latin rectangle design. We compare the three designs given in Table 10, that
 667 are in order of decreasing information about $A\#B\#C$ confounded with Rows#Columns.
 668 They are constructed as follows:

669 **Design 1:** Method 3 is used and segments the design into a column segment of shape 4×8
 670 and a second of shape 4×2 that contains an additional grid. The first segment uses
 671 Design 1 from Section 7.1 and the second segment is constructed using Method 1. It
 672 uses $A + B$ and $A + C$, and hence $B + C$, for row characters and $A + B + C$ for the
 673 column character.

674 **Design 2:** Either CycDesigN 2.0 or 4.0 (Whitaker et al., 2002; CycSoftware Ltd, 2009) is
 675 used, although the resulting design is not isomorphic to any of those constructed by
 676 our methods. A design with the same (partial) confounding of treatment effects as
 677 the computer-generated design can be constructed using Method 3 in the manner of
 678 Design 1. Here, the 4×8 segment is the same as Design 2 from Section 7.1, except
 679 that $B + C$ is used instead of $A + C$ in one of the column pairs. The 4×2 segment
 680 is the same as for Design 1.

681 **Design 3:** Method 1 is used, dividing the 4×10 rectangle into 1 row super-frame and 5
 682 column super-frames. Each column super-frame contains a single column frame which
 683 is a grid of shape 4×2 . The set of row characters is $\{A + B, A + C, B + C\}$; a 4×5
 684 extended Latin square is used as an auxiliary design to assign the 4 pairs of treatments
 685 defined by the row characters. The column character is $A + B + C$ for all 5 column
 686 super-frames.

687 The canonical efficiency factors and Residual degrees of freedom for the three designs
 688 are given in Table 11. It appears that Designs 2 and 3 are suitable for situations in which it
 689 is appropriate to confound most of the three-factor interaction with Columns. In this cir-
 690 cumstance, Design 3 has the advantage over Design 2 that there is no two-factor interaction
 691 confounded with Columns so that the rather small Residual degrees of freedom are increased
 692 by one. Design 1 would be preferred where the variance of the estimate of the three-factor
 693 interaction is to be minimized and the researcher is prepared to sacrifice some precision in
 694 estimating the two-factor interactions by partially confounding them with Columns; even
 695 so, only 24% of each two-factor interaction is confounded with Rows or Columns.

Table 10. Designs for a 2^3 factorial experiment in 4 rows \times 10 columns

<i>Design 1 — Method 3</i>										
$A + B$		$A + C$		$B + C$		$A + B + C$		$A + B + C$		<i>Relations</i>
= 0	= 1	= 0	= 1	= 0	= 1	= 0	= 1	= 0	= 1	
0,0,0	1,0,0	0,1,0	0,0,1	0,1,1	1,1,0	1,0,1	1,1,1	0,0,0	1,1,1	$A + B = 0, A + C = 0^\dagger$
1,1,0	1,0,1	0,0,0	1,0,0	1,1,1	0,0,1	0,1,1	0,1,0	1,0,1	0,1,0	$A + B = 1, A + C = 0^\dagger$
0,0,1	0,1,0	1,1,1	0,1,1	0,0,0	1,0,1	1,1,0	1,0,0	1,1,0	0,0,1	$A + B = 0, A + C = 1^\dagger$
1,1,1	0,1,1	1,0,1	1,1,0	1,0,0	0,1,0	0,0,0	0,0,1	0,1,1	1,0,0	$A + B = 1, A + C = 1^\dagger$
<i>Design 2 — Generated using CycDesignN: columns re-ordered to show confounding</i>										
$A + B + C$		$A + B + C$		$A + B + C$		$A + B + C$		$B + C$		
= 0	= 1	= 0	= 1	= 0	= 1	= 0	= 1	= 0	= 1	
0,1,1	0,1,0	1,0,1	1,0,0	0,1,1	1,1,1	0,0,0	0,0,1	1,0,0	1,1,0	
1,1,0	1,1,1	1,1,0	0,1,0	0,0,0	0,0,1	1,0,1	1,0,0	0,1,1	0,0,1	
0,0,0	1,0,0	0,0,0	0,0,1	1,1,0	0,1,0	0,1,1	1,1,1	1,1,1	1,0,1	
1,0,1	0,0,1	0,1,1	1,1,1	1,0,1	1,0,0	1,1,0	0,1,0	0,0,0	0,1,0	
<i>Design 3 — Method 1</i>										
$A + B + C$		$A + B + C$		$A + B + C$		$A + B + C$		$A + B + C$		
= 0	= 1	= 0	= 1	= 0	= 1	= 0	= 1	= 0	= 1	
0,0,0	1,1,1	1,1,0	0,0,1	0,1,1	1,0,0	1,0,1	0,1,0	0,0,0	1,1,1	
1,0,1	0,1,0	0,0,0	1,1,1	1,1,0	0,0,1	0,1,1	1,0,0	1,0,1	0,1,0	
1,1,0	0,0,1	0,1,1	1,0,0	1,0,1	0,1,0	0,0,0	1,1,1	1,1,0	0,0,1	
0,1,1	1,0,0	1,0,1	0,1,0	0,0,0	1,1,1	1,1,0	0,0,1	0,1,1	1,0,0	

[†]These relations apply only to units in the last two columns of the design

Table 11. Canonical efficiency factors and Residual degrees of freedom (DF) for the designs for a 2^3 factorial experiment in 4 rows \times 10 columns

<i>Design</i>	<i>Unit sources</i>	<i>Treatment sources</i>							<i>Residual DF</i>
		A	B	C	A#B	A#C	B#C	A#B#C	
1	Rows	0	0	0	$\frac{1}{25}$	$\frac{1}{25}$	$\frac{1}{25}$	0	0
	Columns	0	0	0	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	5
	Rows#Columns	1	1	1	$\frac{19}{25}$	$\frac{19}{25}$	$\frac{19}{25}$	$\frac{3}{5}$	20
2	Rows	0	0	0	$\frac{1}{25}$	$\frac{1}{25}$	$\frac{1}{25}$	0	0
	Columns	0	0	0	0	0	$\frac{1}{5}$	$\frac{4}{5}$	7
	Rows#Columns	1	1	1	$\frac{24}{25}$	$\frac{24}{25}$	$\frac{19}{25}$	$\frac{1}{5}$	20
3	Rows	0	0	0	$\frac{1}{25}$	$\frac{1}{25}$	$\frac{1}{25}$	0	0
	Columns	0	0	0	0	0	0	1	8
	Rows#Columns	1	1	1	$\frac{24}{25}$	$\frac{24}{25}$	$\frac{24}{25}$	0	21

696 8. Choosing a unit structure

697 In the previous Sections we have constructed just row-column designs. However, as outlined
 698 in Section 3, no one type of design applies in all experimental situations and for quasi-Latin
 699 designs there is generally the option of using row-column, nested or contiguous designs
 700 for any specific experiment. That is, it is necessary, for any quasi-Latin design, to decide
 701 whether to employ a row-column design or an r -resolved design with the design divided into
 702 s whole frames and, if so, whether to latinize the divided design across its whole frames. This
 703 amounts to choosing between unit structures, which give different decompositions according
 704 to the unit sources. In addition, there is the question of how to construct r -resolved designs
 705 using our techniques. Choosing between different types of designs and the construction of
 706 r -resolved designs are illustrated for just two of the examples from Sections 5 and 7: the 2^3
 707 factorial in an 4×8 square from Section 7.1 and the 2^5 factorial in an 8×8 square from
 708 Section 5.2.

709 This division of a design into several Latin squares or rectangles is fundamentally dif-
 710 ferent from the division of a whole frame into frames or segments. Firstly, the division of
 711 a whole frame is purely a device for construction and is constrained by the construction
 712 algorithm. On the other hand, the division into several whole frames is done deliberately
 713 by the designer and is based on expected sources of variability in the experiment. Secondly,
 714 the division of whole frames is ignored in the randomization and analysis of the design. On
 715 the other hand, the formation of several whole frames results in randomizations between
 716 and, usually, within whole frames, depending on what is appropriate for the unit structure.
 717 It also results in the inclusion of terms corresponding to the whole frames in the analysis.

718 In constructing both nested and contiguous designs the first step is to specify the number
 719 and size of whole frames, which is akin to Method 3, except that the designer has more
 720 freedom in choosing the size of the whole frames. However, each whole frame needs to meet
 721 the conditions for a quasi-Latin square or (extended) quasi-Latin rectangle. For nested
 722 designs, the second step is to apply the methods we have presented to each whole frame
 723 independently, although the overall pattern of (partial) confounding of the treatment effects
 724 must be considered. For contiguous designs, the whole frames are joined into a single frame
 725 and the methods that we have presented are applied to this combined frame. This ensures
 726 that the same characters are confounded between the contiguous entities (rows or columns).
 727 However, in choosing the characters for the noncontiguous entities, the overall pattern of
 728 their (partial) confounding must be considered. Also, care is needed in choosing the unit
 729 characters and auxiliary design Δ_3 , as this will determine the treatment effects confounded
 730 with the interaction of whole frames and the contiguous entities.

731 A design constructed as one type can often be deployed as a design of a different type.
 732 This requires the randomization and analysis appropriate to the type of design actually
 733 deployed. For example, a Latin square design can be deployed as a randomized complete

734 block design, by randomizing and analysing the Latin square design for one blocking factor
 735 nested within the other. That is, the unit structure differs between the constructed and
 736 deployed designs. Thus, the nested designs constructed in this Section can be deployed as
 737 row-column designs and the contiguous designs as row-column or nested designs.

738 8.1. *The 2^3 factorial in a 4×8 rectangle revisited*

739 In Section 5.1 it was suggested that two 4×4 squares, like the plan given by Cochran
 740 and Cox (1957, Table 8.1), are more useful than a single square. It was also noted that
 741 constructing a single 4×8 rectangle, as is done in Section 7.1, is an alternative. Here, this
 742 alternative and two nested and two contiguous quasi-Latin square designs with two squares
 743 are compared. For the designs constructed here, the first step is to divide them into two
 744 squares (whole frames) of shape 4×4 . Then, $p = 2$, $m = 3$, $s = 2$, $k = \ell = 4$ and $r = 2$ so
 745 $t = u = 2$ and $c = d = 2$.

746 The two nested designs are constructed by applying Method 1 to each square. Design 1
 747 is the quasi-Latin square design given in Cochran and Cox (1957, Table 8.1). The first
 748 square of the design is in Table 1; the second square is obtained from this by swapping row
 749 and column characters. The design involves complete replicates in grids of shape 2×4 and
 750 4×2 in each square, and is a nested, 2-resolved design.

751 Design 2, also a nested, 2-resolved design, is constructed in the same way as Design 1.
 752 For its first square, $U = V = A + B + C$, $W = A + B$, $X = A + C$ and $Y = A$; the second
 753 square uses the same characters, except that $X = B + C$ and $Y = B$.

754 These nested designs do not take advantage of the contiguity of the rows of the two
 755 squares because there is no constraint on the treatments assigned to the same row in different
 756 squares. On the other hand, although the quasi-Latin rectangle design in Table 6 has a
 757 complete replicate in each row and in each of four 4×2 grids, it also not suitable as a
 758 contiguous design, because no attention has been paid to the confounding with rows within
 759 squares. In particular, it does not have orthogonal factorial structure. Our construction
 760 method can be used to choose a better confounding pattern for a contiguous design.

761 Design 3 consists of two row-contiguous 4×4 quasi-Latin squares and is 2-resolved.
 762 To construct it, we apply Method 1 to the whole design, which is of shape 4×8 . The
 763 construction is similar to that of Design 3 in Section 7.1. That is, we require four column
 764 characters, which need not be different, and one set of three unit characters that are closed
 765 under addition. Also, necessary is an auxiliary design Δ_3 for assigning the values of the unit
 766 characters. For example, take as the column characters $B + C$ and $A + C$ in both squares to
 767 leave other interaction characters for unit characters. Take the set $\{A + B + C, A + B, C\}$
 768 for unit characters. In order to have $A + B + C$ and $A + B$, but not C , partially confounded
 769 with the Rows#Squares, number the combinations of the values of the first two characters
 770 as follows: $1 = (0, 0)$, $2 = (0, 1)$, $3 = (1, 0)$ and $4 = (1, 1)$. Then assign assign these groups

771 to the 4×4 array of 1×2 subframes using the particular Latin square whose rows are
 772 $(2, 1, 3, 4)$, $(3, 4, 2, 1)$, $(1, 3, 4, 2)$ and $(4, 2, 1, 3)$. The design is in Table 12.

773 *CycDesignN* cannot produce a design with the same unit structure as Design 3, because
 774 it generates only 1-resolved designs. Instead, Design 4 is a contiguous design for a 2×2
 775 array of 2×4 grids produced using *CycDesignN*; the design is resolved and latinization is
 776 across both rows and columns. The unit factors are *BigRows*, *BigCols*, *Rows* and *Columns*,
 777 all with 2 levels except that *Columns* has 4 levels. Also, they all are crossed, except that
 778 *Rows* is nested within *BigRows* and *Columns* is nested *BigCols*. The design is in Table 12.

Table 12. Contiguous designs for a 2^3 factorial experiment in 4 rows \times 8 columns

<i>Design 3 — Method 1</i>							
<i>B + C</i>		<i>A + C</i>		<i>B + C</i>		<i>A + C</i>	
= 0	= 1	= 0	= 1	= 0	= 1	= 0	= 1
0, 1, 1	1, 0, 1	0, 0, 0	1, 1, 0	1, 1, 1	0, 0, 1	0, 1, 0	1, 0, 0
1, 1, 1	0, 0, 1	0, 1, 0	1, 0, 0	0, 1, 1	1, 0, 1	0, 0, 0	1, 1, 0
0, 0, 0	1, 1, 0	1, 1, 1	0, 0, 1	1, 0, 0	0, 1, 0	1, 0, 1	0, 1, 1
1, 0, 0	0, 1, 0	1, 0, 1	0, 1, 1	0, 0, 0	1, 1, 0	1, 1, 1	0, 0, 1
<i>Design 4 — Generated using CycDesignN</i>							
0, 1, 0	0, 1, 1	0, 0, 1	0, 0, 0	1, 1, 1	1, 0, 0	1, 1, 0	1, 0, 1
1, 1, 1	1, 1, 0	1, 0, 0	1, 0, 1	0, 0, 0	0, 1, 1	0, 0, 1	0, 1, 0
1, 1, 0	0, 1, 0	0, 1, 1	1, 1, 1	1, 0, 1	0, 0, 0	1, 0, 0	0, 0, 1
1, 0, 1	0, 0, 1	0, 0, 0	1, 0, 0	0, 1, 1	1, 1, 0	0, 1, 0	1, 1, 1

779 The canonical efficiency factors and Residual degrees of freedom for the four designs are
 780 given in Table 13. As described by Cochran and Cox (1957), Design 1 has a quarter of the
 781 information of the effects for each of $\{A + B, A + C, B + C, A + B + C\}$ confounded with
 782 both rows and columns. Main effects are orthogonal to rows and columns. Design 2, com-
 783 pared with Design 1, has $A \# B \# C$ completely confounded with *Rows* [Squares] and more
 784 information about the two-factor interactions confounded with *Rows* # *Columns* [Squares].
 785 However, the effect of removing *Squares* variability would appear to be a decrease in the
 786 amount of information about the interactions in the lowest unit source; both designs have
 787 lower efficiencies for the interactions than those for Design 1 in Section 7.1. Of the contigu-
 788 ous designs, Design 3 is better than Design 4, because it has orthogonal factorial structure,
 789 and, for the last unit source, it has (i) main effects completely confounded with it, (ii) more
 790 three-factor information partially confounded with it, and (iii) more Residual degrees of
 791 freedom.

792 Designs 3 and 4, while constructed as contiguous designs, could also be deployed as
 793 row-column or nested designs. The analysis for Design 3 when deployed as a nested de-

Table 13. Canonical efficiency factors and Residual degrees of freedom (DF) for the nested and contiguous designs for a 2^3 factorial experiment in 4 rows \times 8 columns

Design	Unit sources	Treatment sources						Residual DF	
		A	B	C	A#B	A#C	B#C		A#B#C
1	Squares	0	0	0	0	0	0	0	1
	Rows [Squares]	0	0	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	2
	Columns [Squares]	0	0	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	2
	Rows # Columns [Squares]	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	11
2	Squares	0	0	0	0	0	0	0	1
	Rows [Squares]	0	0	0	0	0	0	1	5
	Columns [Squares]	0	0	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	0	3
	Rows # Columns [Squares]	1	1	1	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	0	12
3	Squares	0	0	0	0	0	0	0	1
	Rows	0	0	0	0	0	0	0	3
	Rows # Squares	0	0	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1
	Columns [Squares]	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	4
	Rows # Columns [Squares]	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	11
4	BigRows	0	0	0	0	0	0	0	1
	BigCols	0	0	0	0	0	0	0	1
	BigRows#BigCols	0	0	0	0	0	0	0	1
	Rows [BigRows]	0	0	0	0	0	0	0	2
	Columns [BigCols]	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	0	0 [†]	$\frac{1}{8}$ [†]	0 [†]	2
	Rows # BigCols [BigRows]	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	0	0
	Columns # BigRows [BigCols]	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{2}$	0 [†]	$\frac{1}{8}$ [†]	$\frac{1}{2}$ [†]	0
	Rows # Columns [BigRows \wedge BigCols]	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	5

[†]This treatment source is nonorthogonal to previous treatment sources estimated from the same unit source and its canonical efficiency factor is adjusted for the previous sources.

794 sign combines the Rows and Rows#Squares unit sources from the analysis of the con-
 795 tiguous design. The nested design has the advantage over the contiguous design of hav-
 796 ing more Residual degrees of freedom for Rows, but would subject A#C and A#B#C
 797 to Rows variability. Hence, Design 3 should not be deployed as a nested design when
 798 appreciable differences between long rows are anticipated. The efficiencies for the last
 799 unit source for Design 3, as a nested design, and for Design 1 are equal. However, of
 800 these two nested designs, Design 3 has the advantage that it has more Residual degrees
 801 of freedom for Rows [Squares] and Columns [Squares]. The analysis for Design 3 when de-
 802 ployed as a row-column design can be obtained from its analysis as a contiguous design by

803 combining (i) the Squares and Columns [Squares] sources and (ii) the Rows#Squares and
 804 Rows # Columns [Squares] sources. The result is that A#B and A#B#C are now com-
 805 pletely confounded with Rows#Columns, the latter having 14 Residual degrees of freedom.
 806 Using Design 3 as a row-column design is restricted to situations in which it is anticipated
 807 that squares are similar and the Rows#Squares is not appreciable source of variability.

808 Design 1 in Section 7.1 and Designs 1 and 3 from this Section are row-column, nested
 809 and contiguous designs, respectively, for which the amount of information about A#B#C
 810 partially confounded with the last unit source is maximized for each type of design. Com-
 811 paring them shows that allowing for the removal of the difference between Squares reduces
 812 the last unit source's (i) efficiencies for the interactions and (ii) Residual degrees of freedom
 813 (from 14 to 11). As suggested in Section 3, the choice between these designs depends on
 814 the sources of unit variability that are expected. One might argue that there is little to
 815 be lost in routinely employing a contiguous design because, if unit sources are shown to be
 816 negligible in a preliminary analysis, then one can drop these unit sources and so increase
 817 the efficiencies and Residual degrees of freedom associated for the remaining sources in the
 818 final analysis. However, as noted in Section 3, this is a strategy with undesirable conse-
 819 quences and it is preferable to use the analysis appropriate to the design chosen. Further,
 820 the contiguous design, when analysed as a row-column design, is inferior to Design 1 from
 821 Section 7.1.

822 *8.2. The 2^5 factorial in an 8×8 square revisited*

823 Section 5.2 describes a quasi-Latin square design for arranging a 2^5 factorial in an 8×8
 824 square. The design in Section 5.2, as constructed, is resolvable; grids, each containing a
 825 complete set of treatments, are obtained by dividing either the rows into two 4×8 grids
 826 or the columns into two 8×4 grids. So again, while the design was formulated as a row-
 827 column design, it could be deployed as either a nested or a contiguous design. The resulting
 828 design is a quasi-Latin rectangle design. However, as in Section 8.1, the characters used
 829 in constructing the particular design given in Section 5.2 are not ideal for these other unit
 830 structures and so we again employ our construction method to choose a better confounding
 831 pattern.

832 The first step in constructing either a nested or a contiguous design is to divide the
 833 design into whole frames. We consider a design in which there are two rectangles (whole
 834 frames) of shape 4×8 , each of which is a grid. Thus, $p = 2$, $m = 5$, $s = 2$, $k = 4$, $\ell = 8$ and
 835 $r = 1$.

836 Constructing a nested quasi-Latin rectangle design for two such rectangles is a straight-
 837 forward application of Method 1 to each rectangle. Different column characters can be
 838 chosen in the two rectangles.

839 We now give two column-contiguous quasi-Latin rectangle designs consisting of two rect-

840 angles of shape 4×8 . Design 1 is constructed by applying Method 1 to the whole design.
 841 So, as for the design in Section 5.2, the new design consists of a 2×2 array of 4×4 sub-
 842 squares, and two sets of row characters and two sets of columns characters are required.
 843 Each set contains three characters closed under addition (modulo 2). Also, a unit char-
 844 acter is required and this must be chosen carefully as it, and its sums with the column
 845 characters, are confounded with Columns#Grids. The chosen sets of row characters are
 846 $\{A + B + C, C + D + E, A + B + D + E\}$ and $\{A + B + C + E, B + C + D + E, A + D\}$
 847 and the sets of column characters are $\{A + B + C + D, A + C + E, B + D + E\}$ and
 848 $\{A + C + D + E, B + C + D, A + B + E\}$. The unit character is $B + C + E$. It is se-
 849 lected because none of its sums with column characters results in a main effect.

850 Design 2 was constructed using CycDesign 4.0 and it is also a column-contiguous design.
 851 It was obtained using default weights in a two-stage search in which each stage was allowed
 852 to run for between 210 and 420 seconds on a computer running Windows XP. If one stops
 853 the searches sooner, as discussed in Section 7.1, a design with lower weighted efficiency will
 854 be obtained.

855 The skeleton analysis-of-variance tables for Designs 1 and 2 are in Table 14. They are
 856 divided into subtables according to the unit sources. The treatment sources confounded
 857 with a unit source, along with their efficiencies and degrees of freedom, are incorporated
 858 into the subtable for that unit source. An important difference between the two designs is
 859 that Design 1 has orthogonal factorial structure whereas Design 2 does not. As a result,
 860 the estimation of one treatment effect is independent of another for Design 1, but not for
 861 Design 2. Design 1 also has an advantage over Design 2 in the estimation of two-factor
 862 interactions. Unlike Design 2, Design 1 has no two-factor interactions partially confounded
 863 with Columns; it confounds more with Columns#Grids, which is expected to be less variable
 864 than Columns. The two designs have a similar amount of information about two-factor
 865 interactions confounded with Rows # Columns [Grids]. Design 1 has more information
 866 about three-factor interactions confounded with Rows # Columns [Grids].

867 Design 3 is constructed like Design 1, but using different characters, so that all main ef-
 868 fects are estimated with full efficiency from the last unit source and all two- and three-factor
 869 interactions from the last unit source with as high efficiency as possible. The sets of row char-
 870 acters are $\{A + C + D, B + C + E, A + B + D + E\}$ and
 871 $\{B + C + D, A + C + E, A + B + D + E\}$ and the sets of column characters are
 872 $\{A + B, A + D + E, B + D + E\}$ and $\{D + E, A + B + D, A + B + E\}$. The unit char-
 873 acter is $A + B + C + D + E$.

874 Like Designs 3 and 4 in Section 8.1, these contiguous designs can be deployed as row-
 875 column, nested or contiguous designs, with similar considerations to those outlined in Sec-
 876 tion 8.1.

Table 14. Skeleton analysis-of-variance tables for a 2^5 factorial experiment in 2 column-contiguous rectangles of 4 rows \times 8 columns (G = Grids; R = Rows; L = Columns)

<i>units</i>			<i>treatments</i>				
			<i>Design 1</i>		<i>Design 2</i>		
<i>Source</i>	<i>DF</i>	<i>Eff.</i> [§]	<i>Source</i>	<i>DF</i>	<i>Eff.</i> [§]	<i>Source</i> [¶]	<i>DF</i>
Mean	1	1	Mean	1	1	Mean	1
Grids	1						
Rows [G]	6	$\frac{1}{2}$	A#D	1	$\frac{1}{8}$	B#D	1
		$\frac{1}{2}$	A#B#C	1	$\frac{1}{2}$	A#B#C	1
		$\frac{1}{2}$	C#D#E	1	$\frac{1}{8}$	A#B#D	1
		$\frac{1}{2}$	A#B#C#E	1	$\frac{1}{8}$	A#B#E	1
		$\frac{1}{2}$	A#B#D#E	1	$\frac{1}{4}$	A#D#E [†]	1
		$\frac{1}{2}$	B#C#D#E	1	$\frac{1}{4}$	C#D#E	1
Columns	7	$\frac{1}{2}$	A#B#E	1	$\frac{1}{4}$	A#C	1
		$\frac{1}{2}$	A#C#E	1	$\frac{1}{4}$	B#C	1
		$\frac{1}{2}$	B#C#D	1	$\frac{1}{8}$	A#D	1
		$\frac{1}{2}$	B#D#E	1	$\frac{1}{16}$	A#E [†]	1
		$\frac{1}{2}$	A#B#C#D	1	$\frac{1}{8}$	D#E [†]	1
		$\frac{1}{2}$	A#C#D#E	1	$\frac{1}{8}$	A#B#D	1
			Residual	1	$\frac{1}{8}$	A#C#D	1
L # G	7	$\frac{1}{2}$	A#B	1	$\frac{1}{4}$	A#C	1
		$\frac{1}{2}$	A#C	1	$\frac{1}{4}$	B#C	1
		$\frac{1}{2}$	C#D	1	$\frac{1}{8}$	A#D	1
		$\frac{1}{2}$	D#E	1	$\frac{1}{16}$	A#E [†]	1
		$\frac{1}{2}$	A#B#D	1	$\frac{1}{8}$	D#E [†]	1
		$\frac{1}{2}$	A#D#E	1	$\frac{5}{8}$	A#B#D	1
		1	B#C#E	1	$\frac{1}{40}$	A#C#D [†]	1
R # L [G]	42	1	Main effects	5	1	Main effects	5
		0.67	Two-factor	10	0.68	Two-factor [‡]	10
		0.53	Three-factor	9	0.21	Three-factor [‡]	10
		0.50	Four-factor	5	0.56	Four-factor [‡]	5
		1	A#B#C#D#E	1	$\frac{1}{2}$	A#B#C#D#E [†]	1
			Residual	12		Residual	11

[§]For single-degree-of-freedom sources, the efficiencies are the canonical efficiency factors; when a source is nonorthogonal to previous treatment sources estimated from the same unit source, its canonical efficiency factor is adjusted for the previous sources. Those for the sources with multiple degrees of freedom are the harmonic means of the canonical efficiency factors or the A-optimality criterion (John and Williams, 1995, Section 2.4).

[†]This source is nonorthogonal to previous treatment sources estimated from the same unit source.

[‡]Not all these interactions are orthogonal to each other.

[¶]The following sources are partially confounded with the accompanying unit source, but there is no information about them remaining after the previous treatment sources have been fitted:

Rows [G]: B#E, C#D, C#E, A#C#D, A#C#E, A#B#D#E, B#C#D#E and A#B#C#D#E;
 Columns: A#B#E, A#C#E, B#C#D, B#C#E, B#D#E, C#D#E, A#B#C#D, A#B#C#E and A#C#D#E;

L # G: A#B#E, A#C#E, B#C#D, B#C#E, B#D#E, C#D#E, A#B#C#D, A#B#C#E and A#C#D#E.

877 **9. Discussion**

878 The (extended) quasi-Latin rectangle designs increase the range of situations in which a row-
879 column design with orthogonal factorial structure can be employed for assigning factorial
880 treatments. In particular, they allow for more choice in the number of replicates of each
881 treatment than is available with quasi-Latin square designs.

882 The construction methods that we present are flexible and permit a degree of direct
883 control of the confounding in a quasi-Latin design, as shown in Section 7. For example, in
884 Sections 7.1 and 7.2, all designs can be produced using our construction methods and the
885 designer can choose which two-factor interactions are partially confounded with Columns.
886 It would be helpful to practitioners if software assisted in this by searching through possible
887 confounding characters and listing them, in the way that PLANOR (Kobilinsky, 1994)
888 currently does for fractional factorial designs.

889 The methods can be used to produce row-column, nested or contiguous designs as demon-
890 strated in Section 8. A contiguous design can always be deployed as a row-column or a nested
891 design, by varying the randomization and analysis to suit the deployed design. However,
892 the deployed design may not be the best of its type.

893 The practice of habitually employing a contiguous design on the basis that nonsignificant
894 unit sources can be dropped after a preliminary analysis is discouraged on the grounds that
895 this would mean that (i) often the best design is not used and (ii) an analysis strategy
896 with undesirable consequences is employed. Our recommendation is that, in designing an
897 experiment, the designer identify the expected sources of unit variability for the experiment
898 and these determine the unit structure. Then a purpose-built design is constructed of the
899 type corresponding to the unit structure for the experiment. The chosen design should be
900 randomized, and the results analysed, according to its unit structure.

901 Generating quasi-Latin designs using computer search algorithms, such as the one em-
902 ployed by CycDesigN, may not produce a design with the properties desired by the designer.
903 For example, with CycDesigN as currently implemented, the control over the confounding
904 of two-factor interactions is not as flexible as with our construction method. Also, the de-
905 sign produced can depend upon the seed for the random number generator and the length
906 of the search time. There is also no guarantee that computer-generated designs will have
907 orthogonal factorial structure. On the other hand, software like CycDesigN has the distinct
908 advantages that it can produce designs for a wider range of design parameters than our
909 construction methods and that it is easy to use. Also, a design produced by CycDesigN, if
910 one has access to it, can be used as a benchmark for constructed designs, as was done by
911 Tran (2009).

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