ASYMPTOTIC DENSITY AND COMPUTABLY ENUMERABLE SETS

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Abstract. We study the connections between classical asymptotic density, computable enumerability, and computability. In an earlier paper, the second two authors proved that there is a computably enumerable set A of density 1 with no computable subset of density 1. In the current paper, we extend this result in three different ways: (i) The degrees of such sets A are precisely the nonlow c.e. degrees. (ii) There is a c.e. set A of density 1 with no computable subset of nonzero density. (iii) There is a c.e. set A of density 1 such that every subset of A of density 1 is of high degree. We also study the extent to which c.e. sets A can be approximated by their computable subsets B so that $A \setminus B$ has small density, as well as the arithmetical complexity of the densities of computable and computably enumerable sets. Finally, we study connections between density and classical smallness notions such as immunity, hyperimmunity, and cohesiveness.

1. Introduction

Perhaps the first *explicit* questions concerning whether procedures in mathematics were always computational were those of Max Dehn in 1911 [5] who studied word, conjugacy and isomorphisms in finitely presented groups. Of course, implicitly, mathematicians had always been concerned with algorithmic procedures, but it was the introduction of non-computable methods such as the Hilbert Basis Theorem which brought effective procedures into focus, such as the early work of Grete Herman [11]. It was only through the efforts of Turing, Kleene and Church that methods enabling us to demonstrate *non*-computability of procedures became available. With such tools, we have seen famous examples of non-computable aspects of mathematics such as Hilbert's 10th problem, the Novikov-Boone proof of the undecidability of Dehn's word, conjugacy and isomorphism problems, and other similar questions in topology, Julia sets, ergodic theory etc. The late 20th century also involved the apparent clarification (as polynomial time) of the notion of *feasible* computation with the rise of computational complexity theory.

However, there has been a growing realization that the picture above often has little to do with modelling the behavior of *actual* computational procedures and questions in both complexity theory and mathematics. For example, we know that

2000 *Mathematics Subject Classification.* Primary 20F69, Secondary 20F65, 20E07.

Key words and phrases. asymptotic density, computably enumerable sets, Turing degrees, low degrees.

Downey was supported by the Marsden Fund of New Zealand, and by the University of Chicago, and and the Isaac Newton Institute at Cambridge for portions of this paper.

Sat-solvers work extremely well in practice in spite of the fact that SAT is NP-complete, and we don't know why, though Gaspers and Szeider [9] suggest that parameterized complexity (of Downey and Fellows [6]) might provide the explanation. We know that the simplex algorithm in linear programming runs well on actual data, although it is well-known that we can construct examples where it is provably exponential time (Klee and Minty [20]). There have been several attempts towards explaining this phenomenon, the most recent being the use of *smooth analysis* by Spielman and Tang [30]. This can be seen as a more sophisticated version of the notion of average case complexity introduced by Gurevich [10] and Levin [21]. These methodologies are not widely applied as the notion of average case is highly dependent on the choice of distribution.

We know that most of the undecidable problems in group theory *never* occur in practice. A possible explanation of this phenomenon in group theory was suggested by Kapovich, Myasnikov, Schupp, Shpilrain [17] using a complexity measure which seems widely applicable and much easier to deal with. It is called *generic case complexity* and considers partial algorithms which give no incorrect answers and where the collection of inputs where the algorithm fails to converge is "negligible" in the sense that it has asymptotic density 0. (Formal definitions will be given below.) This idea has been very effectively applied in on a number of problems in combinatorial group theory such as [18, 19], and more applications are being found all the time. *Most* natural problems are generically decidable on finitely presented groups.

On the other hand, the general theory of generic decision problems is completely undeveloped. This is in complete contrast to the classical theory of computation which has seen continuous development since the classic paper of Turing [31].

The first paper towards such a general theory was by Jockusch and Schupp [16]. The present paper represents a significant extension to their work. As we will see, we discover quite unexpected connections between the notions from generic computation to notions from classical computability.

Here are the fundamental definitions.

Definition 1.1. Let $S \subseteq \omega$, where $\omega = \{0, 1, \dots\}$ is the set of all natural numbers. For every $n \geq 0$ let $S \restriction n$ denote the set of all $s \in S$ with $s < n$. For $n > 0$, let

$$\rho_n(S) := \frac{|S \restriction n|}{n}$$

The *upper density* $\overline{\rho}(S)$ of S is

$$\overline{\rho}(S) := \limsup_{n\to\infty} \rho_n(S)$$

and the *lower density* $\underline{\rho}(S)$ of S is

$$\underline{\rho}(S) := \liminf_{n\to\infty} \rho_n(S)$$

If the actual limit $\rho(S) = lim_{n\to\infty}\rho_n(S)$ exists, then $\rho(S)$ is the *(asymptotic) density* of S.

Of course, density is finitely additive but not countably additive. A problem (set) A is called *generically decidable* if there is a partial computable function φ such that for all n, if $\varphi(n)\downarrow$ then $\varphi(n) = A(n)$, and the domain of φ has density 1.

Jockusch and Schupp [16], Observations 1.5 and 1.6, proved that every nonzero Turing degree contains a set that is generically decidable and one that is not. In [16], Jockusch and Schupp introduce the related notion of being *coarsely computable*. Here A is *coarsely computable* if there is a (total) computable function f such that $\{n : f(n) = A(n)\}$ has density 1. Every finitely generated group has a coarsely computable word problem ([16], Observation 2.14), but it is an apparently difficult open question whether there is a finitely presented group which does not have a generically decidable word problem. There is a finitely presented semigroup which has a generically undecidable word problem by a theorem of Myasnikov and Rybalov [25], and also Myasnikov and Osin [24], Corollary 1.4, have shown that there is a finitely generated, recursively presented group with a generically undecidable word problem. (Here the notion of density is defined for sets of words on a finite alphabet in the natural way (see [16], Definition 1.1).)

In [16], Proposition 2.15 and Theorem 2.26, Jockusch and Schupp demonstrate that there are c.e. sets that are coarsely computable but not generically decidable, and c.e. sets that are generically decidable which are not coarsely computable. Jockusch and Schupp essay a number of basic properties about coarse and generic computation in the paper [16].

Our starting point here is the observation that the domain of a generic decision algorithm is a c.e. set of density 1. This observation leads us to concentrate upon the relationship between *density*, *computability*, and *computable enumerability*. One basic question which we address is to what what extent a c.e. set A can be approximated by a *computable* subset B so that the difference $A \setminus B$ has "small" density in various senses.

For example, it is natural to ask whether every c.e. set of density 1 has a computable subset of density 1. This is easily seen to hold if and only if every generically computable set has a generic algorithm φ with a computable domain ([16], Corollary 2.24).

Jockusch and Schupp [16], Theorem 2.22, established that the answer is no: there is a c.e. set of density 1 with no computable subset of density 1. We extend this result in several ways, revealing a deep connection between notions from classical computability and generic computation.

The natural question to ask is "what kinds of c.e. sets *do* have computable subset of density 1?" The answer lies in the the complexity of the sets as measured by the information content. The reader should recall that the natural operation in the Turing degrees is the *jump* operator, the relativization of the halting problem, where the jump A' of a set A is given by $A' = \{n : \Phi_n^A(n)\downarrow\}$, and the jump operation on sets naturally induces the jump operation on degrees. The operator is not injective, and we call sets A with $A' \equiv_T \emptyset'$ *low*, and the degrees of low sets are also called *low*. Low sets resemble computable sets modulo the jump operator and share some of the properties of computable sets. They occupy a central role in classical computability.

On the other hand, there are almost no known natural properties of the c.e. sets which occur in exactly the low c.e. degrees.

We introduce a new nonuniform technique to prove the following.

Theorem 1.2. *A c.e. degree* $\mathbf{a}$ *is not low if and only if it contains a c.e. set* A *of density* 1 *with no computable subset of density* 1*.*

The technique we introduce for handling non-lowness is quite flexible, and we illustrate this fact with some easy applications. For example, recall that if A be a c.e. set, its complement $\overline{A}$ is called *semilow* if $\{e : W_e \cap \overline{A} \neq \emptyset\} \leq_T \emptyset'$, and is called *semilow*$_{1.5}$ if $\{e : |W_e \cap \overline{A}| = \infty\} \leq_m \{e : |W_e| = \infty\}$. The implications low implies semilow implies semilow$_{1.5}$ hold, and it can be shown that they cannot be reversed. These notions were introduced by Soare [27], and Maass [22] in connection with both computational complexity and the lattice of computably enumerable sets.

We also prove the following characterization of non-lowness.

Theorem 1.1. *If* $\mathbf{a}$ *is a c.e. degree then* $\mathbf{a}$ *is not low if and only if there is a c.e. set* A *of degree* $\mathbf{a}$ *such that* $\overline{A}$ *is not semilow*$_{1.5}$*.*

We remark in passing that our technique has also found applications in effective algebra. Downey and Melnikov [7] use the methodology to characterize the Δ^0_2-categorical homogeneous completely decomposable torsion-free abelian groups in terms of the semilowness of the type sequence.

Another direction we take is to ask what kinds of densities are guaranteed for computable or c.e. subsets. We prove the following.

Theorem 1.2. *There is a c.e. set of density 1 with no computable subset of* nonzero *density. Such sets exist in each non-low c.e. degree.*

This result stands in contrast to the low case, where we show that all possible densities for computable subsets are achieved.

Theorem 1.3. *If* A *is c.e. and low and has density* r*, then for any* Δ^0_2 *real* $\hat{r}$ *with* $0 \leq \hat{r} \leq r$*,* A *has a computable subset of density* $\hat{r}$*.*

Finally with Eric Ashton we prove the following.

Theorem 1.4 (with Ashton)**.** *There is a c.e. set* A *of density 1, such that the degrees of subsets of* A *of density are exactly the high degrees.*

On the other hand, we obtain a number of positive results on approximating c.e. sets by computable subsets. The following is a sample.

Theorem 1.5. *If* A *is a c.e. set, then for every real number* $\epsilon > 0$ *there is a computable set* $B \subseteq A$ *such that* $\underline{\rho}(B) > \underline{\rho}(A) - \epsilon$*.*

It turns out that there is a very close correlation between the complexity of a set and the complexity, as real numbers, of its densities. We measure the complexity of a real number by classifying its upper or lower cut in the rationals in the arithmetical hierarchy.

In [16], Theorem 2.21, it was shown that the densities of computable sets are exactly the Δ^0_2 reals in the interval $[0, 1]$. In this article we characterize the upper and lower densities of both computable and c.e. sets. We assume that we have fixed a computable bijection between the natural numbers and the rational numbers. We thus say that a set of rational numbers is Σ_n if the corresponding set of natural numbers is Σ_n, and similarly for other classes in the arithmetic hierarchy. The following definition is fundamental and standard:

Definition 1.3. A real number r is *left*-Σ^0_n if the corresponding lower cut in the rationals, $\{q \in \mathbb{Q} : q < r\}$, is Σ^0_n. We define "left-Π^0_n" analogously.

We first characterize the lower densities of the computable sets.

Theorem 1.4. *Let r be a real number in the interval $[0, 1]$. Then the following hold:*
(i) *r is the lower density of some computable set if and only if r is left-Σ^0_2.*
(ii) *r is the upper density of some computable set if and only if r is left-Π^0_2.*
(iii) *r is the lower density of some c.e. set if and only if r is left-Σ^0_3.*
(iv) *r is the upper density of some c.e. set A if and only if r is left-Π^0_2.*
(v) *r is the density of some c.e. set if and only if r is left-Π^0_2.*

We also explore the relationship between coarse computability and generic computability. The proof that there is a generically computable c.e. set that is not coarsely computable strongly resembles the proof that there is a density 1 c.e. set without a computable subset of density 1. Thus we would expect a similar characterization of the low degrees using the interactions of those notions. Here we get a surprise.

Theorem 1.6. *Every nonzero c.e. degree contains a c.e. set that is generically computable but not coarsely computable.*

We also discuss the relationship of our concepts with classical smallness concepts such as immunity, hyperimmunity, and cohesiveness. In particular we study the extent to which various immunity properties imply that a set has small upper or lower density in various senses. We show that the results for many standard immunity properties are different, thus again bringing out the connection between density and computability theory.

Earlier phases of our work included open questions which were subsequently resolved by Igusa [12] and by Bienvenu, Day, and Hölzl [2]. In the final two sections we state their surprising and beautiful results and mention some related work.

2. Terminology and notation

As usual, we let φ_e be the eth partial computable function in a fixed standard enumeration, and we let W_e be the domain of φ_e. We write Φ_e for the eth Turing functional.

As in [16], Definition 2.5, define:

$$R_k = \{m : 2^k \mid m \ \& \ 2^{k+1} \nmid m\}$$

Note that the sets R_k are pairwise disjoint, uniformly computable sets of positive density, and the union of these sets is $\omega \setminus \{0\}$. These sets were used frequently in [16] and we will also use them several times in this paper.

3. Approximating c.e. sets by computable subsets

We consider the extent to which it is true that every c.e. set A has a computable subset B which is almost as large as A. More precisely, we require that the difference $A \setminus B$ should have small density. Here, "density" may refer to either upper or lower density, and "small" may mean 0 or less than a given positive real number. For convenience in stating results in this area, we introduce the following notation, which is not restricted to the case $B \subseteq A$.

Definition 3.1. Let $A, B \subseteq \omega$.

(i) Let $d(A, B)$ be the lower density of the symmetric difference of A and B (so $d(A, B) = \underline{\rho}(A \triangle B)$).

(ii) Let $D(A, B)$ be the upper density of the symmetric difference of A and B (so $D(A, B) = \overline{\rho}(A \triangle B)$).

Intuitively, $d(A, B)$ is small if there are infinitely many initial segments of the natural numbers on which A and B disagree on only a small proportion of numbers, and $D(A, B)$ is small if there are cofinitely many such initial segments.

The following easy proposition lists some basic properties of d and D.

Proposition 3.2. *Let A, B, and C be subsets of ω.*

(i) $0 \leq d(A, B) \leq D(A, B) \leq 1$

(ii) *(Triangle Inequality)* $D(A, C) \leq D(A, B) + D(B, C)$

Since $D(A, B) = D(B, A)$ and $D(A, A) = 0$ for all A, B, it follows from the above proposition that D is a pseudometric on Cantor space. On the other hand the triangle inequality fails for d. For example, if A is any set with lower density 0 and upper density 1, we have $d(\emptyset, A) = 0, d(A, \omega) = 0$, and $d(\emptyset, \omega) = 1$.

The following elementary lemma gives upper bounds for $d(A, B)$ and $D(A, B)$ in terms of the upper and lower densities of A and B in the case where $B \subseteq A$.

Lemma 3.3. *Let A and B be sets such that $B \subseteq A$.*

(i) $d(A, B) \leq \overline{\rho}(A) - \overline{\rho}(B)$

(ii) $d(A, B) \leq \underline{\rho}(A) - \underline{\rho}(B)$

(iii) $D(A, B) \leq \overline{\rho}(A) - \underline{\rho}(B)$

Proof. Since $B \subseteq A$, we have that $A \triangle B = A \setminus B$, and hence

$$\rho_n(A \triangle B) = \rho_n(A) - \rho_n(B)$$

for all n. The lemma follows in a straightforward way from the above equation and the definitions of upper and lower density. For example, to prove the first part, let a real number $\epsilon > 0$ be given. Let

$$I = \{n : \rho_n(A) \leq \overline{\rho}(A) + \epsilon/2 \quad \& \quad \rho_n(B) \geq \overline{\rho}(B) - \epsilon/2\}$$

Then I is infinite because the first inequality in its definition holds for all sufficiently large n, and the second inequality in its definition holds for infinitely many n. By subtracting these inequalities, we see that

$$\rho_n(A \setminus B) = \rho_n(A) - \rho_n(B) \leq \overline{\rho}(A) - \overline{\rho}(B) + \epsilon$$

holds for infinitely many n. Since $\epsilon > 0$ was arbitrary, we conclude that

$$d(A, B) = \liminf_n \rho_n(A \setminus B) \leq \overline{\rho}(A) - \overline{\rho}(B)$$

The other parts are proved similarly and are left to the reader. $\square$

We begin with a result of Barzdin' from 1970 showing that every c.e. set can be well approximated by a computable subset on infinitely many intervals. We thank Evgeny Gordon for bringing Barzdin's work to our attention.

Theorem 3.4. *(Barzdin' [1]) For every c.e. set A and real number $\epsilon > 0$, there is a computable set $B \subseteq A$ such that $\overline{\rho}(B) > \overline{\rho}(A) - \epsilon$, and hence (by Lemma 3.3) $d(A, B) < \epsilon$.*

Proof. Given such A and ϵ, let q be a rational number such that $\overline{\rho}(A) - \epsilon < q < \overline{\rho}(A)$, and let $\{A_s\}$ be a computable enumeration of A. We now define two computable sequences $\{s_n\}_{n\in\omega}$, $\{t_n\}_{n\in\omega}$ simultaneously by recursion. Let $s_0 = t_0 = 0$. Given s_n and t_n let (s_{n+1}, t_{n+1}) be the first pair (s, t) such that $s > s_n$ and $\rho_s(A_t \setminus [0, s_n)) \geq q$. Such a pair exists because $q < \overline{\rho}(A) = \overline{\rho}(A \setminus [0, s_n))$, so there are infinitely many s with $\rho_s(A \setminus [0, s_n) \geq q$. Now, for each x, put x into B if and only if $x \in A_{t_{n+1}}$, where n is the unique number such that x belongs to the interval $[s_n, s_{n+1})$. Note that

$$\rho_{s_{n+1}}(B) \geq \rho_{s_{n+1}}(A_{t_{n+1}} \setminus [0, s_n)) \geq q$$

for all n. It follows that $\overline{\rho}(B) \geq q > \overline{\rho}(A) - \epsilon$, as needed to complete the proof. $\square$

On the other hand, as pointed out by Barzdin', the above result fails for D. We prove this in a strong form.

Theorem 3.5. *There is a c.e. set A such that $D(A, B) = 1$ for every co-c.e. set B.*

Proof. Let I_n denote the interval $[n!, (n + 1)!)$. Define $A = \cup_n(W_n \cap I_n)$. Fix n, and let $S = \{e : W_e = W_n\}$. Then $A \triangle \overline{W_n} \supseteq \cup_{e\in S} I_e$. The latter set has upper density 1 since S is infinite, so $D(A, \overline{W_n}) = 1$. $\square$

Also it is easy to see that Barzdin's result does not hold for $\epsilon = 0$.

Theorem 3.6. *([16]) There is a c.e. set A such that $d(A, B) > 0$ for every co-c.e. set B.*

This follows at once from the proof of Theorem 2.16 of [16]. We will extend it below in Theorem 3.8 by adding the requirement that the density of A exists.

In [16], it was pointed out just after the proof of Theorem 2.21 that every c.e. set of upper density 1 has a computable subset of upper density 1. We now extend this result using the same method of proof as in Theorem 3.4.

Theorem 3.7. *Let A be a c.e. set such that $\overline{\rho}(A)$ is a Δ^0_2 real. Then A has a computable subset B such that $\overline{\rho}(B) = \overline{\rho}(A)$, and hence, by Lemma 3.3, $d(A, B) = 0$.*

Proof. Let $\{q_s\}_{s\in\omega}$ be a computable sequence of rational numbers converging to $\overline{\rho}(A)$. Define a sequence of pairs of natural numbers $(s_n, t_n)_{n\in\omega}$ recursively as follows. Let $(s_0, t_0) = (0, 0)$. Given (s_n, t_n), let (s_{n+1}, t_{n+1}) be the first pair (s, t) such that:

$$s > s_n \quad \& \quad t > n \quad \& \quad \rho_s(A_t \setminus [0, s_n)) \geq q_t - 2^{-n}$$

We claim that such a pair (s,t) exists. First, choose $s > s_n$ such that $\rho_s(A \setminus [0, s_n)) \geq \overline{\rho}(A) - 2^{-(n+1)}$. There are infinitely many s which satisfy this inequality since $\overline{\rho}(A) = \overline{\rho}(A \setminus [0, s_t))$. Now choose $t > n$ such that $\rho_s(A \setminus [0, s_n)) = \rho_s(A_t \setminus [0, s_n))$ and $q_t \leq \overline{\rho}(A) + 2^{-(n+1)}$. Any sufficiently large t meets these conditions since $\lim_t q_t = \overline{\rho}(A)$. Then

$$\rho_s(A_t \setminus [0, s_n)) = \rho_s(A \setminus [0, s_n)) \geq \overline{\rho}(A) - 2^{-(n+1)} \geq q_t - 2^{-(n+1)} - 2^{-(n+1)} = q_t - 2^{-n}$$

Hence the chosen pair (s,t) meets the condition above to be chosen as (s_{n+1}, t_{n+1}). It is easy to see that the sequence (s_n, t_n) is computable. Let S_n be the interval $[s_n, s_{n+1})$, so that every natural number belongs to S_n for exactly one n.

We now define the desired computable $B \subseteq A$. For $k \in S_n$, put k into B if and only if $k \in A_{t_{n+1}}$. Clearly, B is a computable subset of A. Hence $\overline{\rho}(B) \leq \overline{\rho}(A)$. To get the opposite inequality, note that B and $A_{t_{n+1}}$ agree on the interval S_n. Further, by definition of (s_{n+1}, t_{n+1}), we have $\rho_{s_{n+1}}(A_{t_{n+1}} \setminus [0, s_n)) \geq q_{t_{n+1}}$. It follows from the definition of B that

$$\rho_{s_{n+1}}(B) \geq q_{t_{n+1}}$$

for all n. Therefore:

$$\overline{\rho}(B) \geq \limsup_n \rho_{s_{n+1}}(B) \geq \limsup_n q_{t_{n+1}} = \overline{\rho}(A)$$

as needed to complete the proof. $\square$

We now show that we cannot omit the hypothesis that $\overline{\rho}(A)$ is a Δ^0_2 real from the above theorem, even if we assume in addition that $\rho(A)$ exists.

Theorem 3.8. *There is a c.e. set A such that density of A exists, yet for every Π^0_1 subset B of A, we have $\underline{\rho}(A \setminus B) > 0$ and hence, by Lemma 3.3, $d(A,B) > 0$.*

Proof. Recall that $R_e = \{x : 2^e \mid x \ \& \ 2^{e+1} \nmid x\}$. For $x \in R_e$, put x into A if and only if every $y \leq x$ with $y \in R_e$ is in W_e. We first show that, for each e, if $\overline{W_e} \subseteq A$, then $\overline{\rho}(\overline{W_e}) < \overline{\rho}(A)$, and then show A has a density. Let e be given.

Case 1. $R_e \subseteq W_e$. Then $R_e \subseteq A$ by definition of A. So $R_e \subseteq (A \setminus \overline{W_e})$, and hence $A \setminus \overline{W_e}$ has positive lower density.

Case 2. Otherwise. Take $x \in R_e \setminus W_e$. Then $x \notin A$ by definition of A , so $x \notin A \cup W_e$. Hence $\overline{W_e}$ is not a subset of A. Note further in this case that $R_e \cap A$ is finite.

To see that A has a density, note by the above that, for all e, either $R_e \subseteq A$ or $R_e \cap A$ is finite, so that $R_e \cap A$ has a density for all e. It follows by restricted countable additivity (Lemma 2.6 of [16]) that A has a density, namely

$$\rho(A) = \sum_e \rho(A \cap R_e) = \sum \{2^{-(e+1)} : R_e \subseteq W_e\}$$

$\square$

We now look at analogues of some of the above results where we study $D(A,B)$ instead of $d(A,B)$, for B a computable subset of a given c.e. set A. It was shown in Theorem 3.4 that for every c.e. set A and real number $\epsilon > 0$ there is a computable set $B \subseteq A$ with $d(A,B) < \epsilon$. We pointed out in Theorem 3.6 that the corresponding result fails for D in place of d, but we now show that this corresponding result does hold if we assume A has a density.

Theorem 3.9. *Let A be a c.e. set and ϵ a positive real number. Then A has a computable subset B such that $\underline{\rho}(B) > \underline{\rho}(A) - \epsilon$.*

We give a corollary before proving this result. Roughly speaking, this corollary asserts that the computable sets are topologically dense among the c.e. sets which have an asymptotic density (pretending that the pseudometric D is a metric).

Corollary 3.10. *Let A be a c.e. set which has a density and ϵ a positive real number. Then A has a computable subset B such that $D(A, B) < \epsilon$.*

Proof. (of corollary). By the theorem, let B be a computable subset of A such that $\underline{\rho}(B) > \underline{\rho}(A) - \epsilon$. By Lemma 3.3,

$$D(A, B) \leq \overline{\rho}(A) - \underline{\rho}(B) = \underline{\rho}(A) - \underline{\rho}(B) < \epsilon$$

□

Proof. (of theorem) Let A be a c.e. set and let ϵ be a positive real number. We must construct a computable set $B \subseteq A$ such that $\underline{\rho}(B) > \underline{\rho}(A) - \epsilon$. Let q be a rational number such that $\underline{\rho}(A) - \epsilon < q < \underline{\rho}(A)$. Since $q < \underline{\rho}(A)$ there is a number n_0 such that $\rho_n(A) \geq q$ for all $n \geq n_0$. Given $n \geq n_0$, let $s(n)$ be the least number s such that $\rho_n(A_s) \geq q$, where such an s exists because $\rho_n(A) \geq q$. Then, for each $k \geq \sqrt{n_0}$, define

$$t(k) = \max\{s(n) : n_0 \leq n \leq k^2\}$$

Finally, define

$$B = \{k : k \in A_{t(k)}\}$$

The set B is computable because the functions s and t are computable. Note that in deciding whether to put k into B, we are waiting for sufficient elements to be enumerated in A on the interval $[0, k^2)$, which for large k is much bigger than the interval $[0, k)$. Such a "look-ahead" is crucial to our argument.

Suppose now that $k \geq \sqrt{n}$, and $n \geq n_0$. Then $n \leq k^2$, so $s(n) \leq t(k)$, and hence $A_{s(n)} \subseteq A_{t(k)}$. Thus, for $n \geq n_0$, every number $k \in A_{s(n)}$ with $k \geq \sqrt{n}$ is in B, by the definition of B. It follows that

$$|B \cap [0, n)| \ \geq \ |A_{s(n)}| - \sqrt{n}$$

Since $\rho_n(A_{s(n)}) \geq q$, division by n yields that

$$\rho_n(B) \geq q - 1/\sqrt{n}$$

for $n \geq n_0$. As n approaches infinity, $1/\sqrt{n}$ tends to 0, and hence $\underline{\rho}(B) \geq q$. □

If A is a c.e. set of density 1, it would be tempting to try to show that A has a computable subset B of density 1 by using the method of the previous theorem applied to values of q closer and closer to 1. However, this breaks down because n_0 need not depend effectively on q, so we do not have an effective way to handle the finitely many "bad" $n < n_0$ as q varies. Indeed, this breakdown is essential, as it is shown in [16], Theorem 2.22, there is a c.e. set of density 1 with no computable subset of density 1. On the other hand, if we *assume* that A is such that n_0 depends effectively on q, this plan goes through. We make this explicit in the following definition and theorem.

Definition 3.11. Let A be a set of density 1.

(i) A function w *witnesses* that A has density 1 if $(\forall k)(\forall n \geq w(k))[\rho_n(A) \geq 1 - 2^{-k}]$.

(ii) The set A has density 1 *effectively* if there is a computable function w which witnesses that A has density 1.

Theorem 3.12. *If A is c.e. and has density 1 effectively, then A has a computable subset B which has density 1 effectively.*

Proof. Let w be a computable function which witnesses that A has density 1 and let $\{A_s\}$ be a computable enumeration of A. For $n \geq 0$ let $s(n)$ be the least s such that $\rho_n(A_s) \geq 1 - 2^{-z}$ for all $z \leq n$ such that $w(z) \leq n$. The function s is total because w witnesses that A has density 1. We now define the function t and the set B exactly as in the previous theorem, namely

$$t(k) = \max\{s(n) : n \leq k^2\}$$
$$B = \{k : k \in A_{t(k)}\}$$

As before, B is computable because the functions s and t are computable, and clearly $B \subseteq A$. Further, we can argue exactly as in the previous theorem that if $w(z) \leq n$ and $z \leq n$, then

$$\rho_n(B) \geq 1 - 2^{-z} - 1/\sqrt{n}$$

Let $h(n)$ be the greatest number $z \leq n$ with $w(z) \leq n$. (We may assume without loss of generality that $w(0) = 0$, so such a z always exists.) By the inequality above, we have, for $n > 0$,

$$\rho_n(B) \geq 1 - 2^{-h(n)} - 1/\sqrt{n}$$

Since $h(n)$ tends to infinity as n tends to infinity, it follows that B has density 1. Let $b(n) = 1 - 2^{-h(n)} - 1/\sqrt{n}$ be the lower bound for $\rho_n(B)$ obtained above. Since the function h is nondecreasing and computable, the function b is also nondecreasing, and $b(n)$ is a computable real, uniformly in n. Also $\lim_n b(n) = 1$. It follows that B has density 1 effectively. $\square$

Note that if A has density 1, then there is a function $w_A \leq_T A'$ which witnesses that A has density 1, namely

$$w_A(k) = (\mu y)(\forall n \geq y)[\rho_n(A) \geq 1 - 2^{-k}]$$

We call w_A the *minimal witness function* for A. In particular, if A is a c.e. set of density 1, then there is a function $w \leq_T 0''$ which witnesses that A has density 1. The next result shows that if there is such a $w \leq_T 0'$, then A has a computable subset of B of density 1.

Theorem 3.13. *Let A be a c.e. set of density 1. Then the following are equivalent:*

(i) *A has a computable subset B of density 1.*

(ii) *There is a function $w \leq_T 0'$ which witnesses that A has density 1.*

Proof. First, assume that (i) holds. Then by the remark just above the statement of the theorem, there is a function $w_B \leq_T 0'$ which witnesses that B has density 1. Since $A \supseteq B$, we have that $\rho_n(A) \geq \rho_n(B)$ for all n, and so w_B also witnesses that A has density 1.

Assume now that (ii) holds. We will now prove (i) using the method of Theorem 3.12, but using a computable approximation to w in place of w. The basic trick in proving Theorem 3.12 was to enumerate elements in A until sufficient elements appeared to show that the density of A on a given interval is at least as big as the lower bound given by w. This would seem to carry the danger now that if our approximation to w is incorrect, A may not have sufficient elements in the interval to make its density at least as big as predicted by the approximation, and we would wait forever, causing the construction to bog down. The solution to this is both simple and familiar. As we wait for the elements to appear in A we recompute the approximation. Since the approximation converges to w, eventually sufficient elements must appear in A for some sufficiently late approximation.

We now implement the above strategy. Let $g(.,.)$ be a computable function such that $(\forall k)[w(k) = \lim_s g(k,s)]$. Define

$$s(n) = (\mu s \geq n)(\forall k \leq n)[g(k,s) \leq n \to \rho_n(A_s) \geq 1 - 2^{-k}]$$

Note that the variable s occurs both as an argument of g and as a stage of enumeration of A, in accordance with our informal description of the strategy. The function s is total because all sufficiently large numbers s satisfy the defining property for $s(n)$, since w witnesses that A has density 1. We now define the computable function t and the computable set $B \subseteq A$ exactly as in Theorems 3.9 and 3.12. This yields that B is computable, $B \subseteq A$, and for each $n \geq 0$, $\{k \geq \sqrt{n} : k \in A_{s(n)}\} \subseteq B$. These are proved just as in the proof of Theorem 3.12.

We now show that B has density 1. Let b be given. Suppose n is sufficiently large that $n > b$, $n \geq w(b)$, and $(\forall s \geq n)[g(b,s) = w(b)]$. Then by definition of $s(n)$ (with $k = b$), $\rho_n(A_{s(n)}) \geq 1 - 2^{-b}$. We then have, as in the proof of Theorem 3.13, for all sufficiently large n,

$$\rho_n(B) \geq \rho_n(A_{s(n)}) - 1/\sqrt{n} \geq 1 - 2^{-b} - 1/\sqrt{n} \geq 1 - 2^{-b} - 1/\sqrt{b}$$

Since $\lim_b(1 - 2^{-b} - 1/\sqrt{b}) = 1$, it follows that $\rho(B) = \lim_n \rho_n(B) = 1$. □

Corollary 3.14. *Suppose that A is a low c.e. set of density 1. Then A has a computable subset of density 1.*

Proof. As remarked just before the statement of Theorem 3.13, there is a function $w \leq_T A'$ which witnesses that A has density 1. Since A is low, we have $w \leq_T 0'$, and hence A has a computable subset of density 1 by Theorem 3.13. □

In the next section we will see that, conversely, every nonlow c.e. degree contains a c.e. set of density 1 with no computable subset of density 1.

We now use similar ideas to extend Corollary 3.14 from sets of density 1 to sets whose lower density is a Δ^0_2 real.

Theorem 3.15. *Let A be a low c.e. set such that $\underline{\rho}(A)$ is a Δ^0_2 real. Then A has a computable subset B such that $\underline{\rho}(B) = \underline{\rho}(A)$ and hence, by Lemma 3.3, $d(A, B) = 0$.*

Proof. The proof is similar to that of Theorem 3.13. Let $\{q_n\}$ be a computable sequence of rational numbers converging to $\underline{\rho}(A)$. Define

$$w(k) = (\mu y)(\forall n \geq y)[\rho_n(A) \geq q_n - 2^{-k}]$$

Observe that w is a total function since for each k, whenever n is sufficiently large we have $\rho_n(A) \geq q_n - 2^{-k}$, because $\{q_n\}$ converges to $\liminf_n \rho_n(A)$. Note also that $\rho_n(A)$ is a rational number which can be computed from n and an oracle for A. Hence $w \leq_T A' \leq_T 0'$, so there is a computable function g such that, for all k, $w(k) = \lim_s g(k, s)$. Now define:

$$s(n) = (\mu s \geq n)(\forall k \leq n)[g(k, s) \leq n \implies \rho_n(A_s) \geq q_n - 2^{-k}]$$
$$t(k) = \max\{s(n) : n \leq k^2\}$$
$$B = \{k : k \in A_{t(k)}\}$$

The function s is total because for each n and $k \leq n$ all sufficiently large numbers s satisfy the matrix of the definition of $s(n)$. It follows that the functions s and t and the set B are computable, and obviously $B \subseteq A$. It follows from the latter that $\underline{\rho}(B) \leq \underline{\rho}(A)$, so it remains only to verify that $\underline{\rho}(B) \geq \underline{\rho}(A)$. For this, note that, just as in the proof of Theorem 3.13, for all $n > 0$

$$\{k \geq \sqrt{n} : k \in A_{s(n)}\} \subseteq B \text{ and hence } \rho_n(B) \geq \rho_n(A_{s(n)}) - 1/\sqrt{n}$$

Now let $b > 0$ be given. Let n be sufficiently large that $n > b, n \geq w(b), (\forall s \geq n)[g(b, s) = w(b)]$, and $|\underline{\rho}(A) - q_n| < 2^{-b}$. It then follows that

$$\rho_n(A_{s(n)}) \geq q_n - 2^{-b}$$

by using the above conditions on n and the definition of $w(n)$ with $k = b$. We now have:

$$\rho_n(B) \geq \rho_n(A_{s(n)}) - 1/\sqrt{n} \geq q_n - 2^{-b} - 1/\sqrt{n} \geq \underline{\rho}(A) - 2^{-b} - 1/\sqrt{b}$$

Hence $\underline{\rho}(B) \geq \underline{\rho}(A) - 2^{-b} - 1/\sqrt{b}$. Since $b > 0$ was arbitrary and $\lim_b(2^{-b} + 1/\sqrt{b}) = 0$, we have $\underline{\rho}(B) \geq \underline{\rho}(A)$. □

It was shown in [16], Theorem 2.21, that if a computable set A has a density d, then d is a Δ^0_2 real. (Actually, this part of the theorem is an immediate consequence of the Limit Lemma.) It follows by relativizing the proof that if a low set A has density d, then d is a Δ^0_2 real. This gives the following corollary.

Corollary 3.16. *If A is a low c.e. set and $\rho(A)$ exists, then A has a computable subset B with $\rho(B) = \rho(A)$ and hence, by Lemma 3.3, $D(A, B) = 0$. (Recall that $D(A, B)$ is the upper density of the symmetric difference of A and B.)*

Proof. As noted just above, $\rho(A)$ is a Δ^0_2 real, so by Theorem 3.15, A has a computable subset B with $\underline{\rho}(B) = \underline{\rho}(A) = \rho(A)$. Further, $\overline{\rho}(B) \leq \rho(A)$ since $B \subseteq A$. Finally, $\overline{\rho}(B) \geq \underline{\rho}(B) = \rho(A)$, so $\overline{\rho}(B) = \rho(A)$. As $\underline{\rho}(B) = \overline{\rho}(B) = \rho(A)$, we have $\rho(B) = \rho(A)$. □

The next result uses our previous work to characterize the densities of computable subsets of those low c.e. sets A which have a density d. For d_0 to be the density of a computable subset of A it is clearly necessary that $0 \leq d_0 \leq d$ and (by Theorem 2.21 of [16]) that d_0 be a Δ^0_2 real. We now show that these conditions are also sufficient.

Corollary 3.17. *Let A be a low c.e. set of density d and let d_0 be a Δ^0_2 real such that $0 \leq d_0 \leq d$. Then A has a computable subset B of density d_0.*

Proof. By Corollary 3.16, A has a computable subset A_0 of density d. Thus, we may assume without loss of generality that A is computable, since we can simply replace A by A_0.

By [16], Theorem 2.21, every Δ^0_2 real in $[0, 1]$ is the density of a computable set. Using the same proof but working within A we get that every Δ^0_2 real s is the relative density within A of a computable subset B of A, i.e. $\rho(B|A) = s$. The result to be proved is immediate if $d = 0$, so assume $d > 0$ and hence $s = \rho(B \mid A) = \rho(B)/\rho(A)$. We now choose $s = d_0/d$. (s is a Δ^0_2 real since the Δ^0_2 reals form a field by relativizing to $0'$ the result that the computable reals form a field. Also $0 \leq s_0 \leq 1$ since $0 \leq d_0 \leq d$.) Let B be a computable subset of A such that $\rho(B \mid A) = \rho(B)/\rho(A) = s = d_0/d$. Multiply both sides by $d = \rho(A)$, to obtain $\rho(B) = d_0$ as needed. $\square$

The following theorem greatly strengthens Theorem 2.22 of [16], which asserts that there is a c.e. set of density 1 which has no computable subset of density 1. It contrasts strongly with Corollary 3.17.

Theorem 3.18. *There is a c.e. set A of density 1 such that no computable subset of A has nonzero density.*

Proof. For each e, let $S_e = \{n : \varphi_e(n) = 1\}$, so that the computable sets are exactly the sets S_e with φ_e total. Let N_e be the requirement:

$$N_e : (\varphi_e \text{ total } \ \& \ S_e \subseteq A \ \& \ \rho(S_e)\downarrow) \implies \rho(S_e) = 0$$

To prove the theorem, it suffices to construct a c.e. set A of density 1 which meets all the requirements N_e. The strategy for meeting N_e is as follows. We define a sequence of finite intervals $I_{e,0}, I_{e,1}, \dots$, and this sequence may or may not terminate, and the strategy affects A only on these intervals. These intervals are pairwise disjoint and also disjoint from all intervals used for other requirements, so distinct requirements N_e don't interact. The intervals are defined in the order listed above. When $I_{e,j}$ is chosen, its least element $a_{e,j}$ should be the least number not in any interval already chosen for any requirement. (The purpose of this is to ensure that every number belongs to some interval for some requirement.) Further, we will carefully choose a certain large initial segment $J_{e,j}$ of $I_{e,j}$, but we defer the definition of $J_{e,j}$ for the moment. As soon as $I_{e,j}$ (and hence $J_{e,j}$) are chosen, put all elements of $J_{e,j}$ into A. (This is done to help ensure that A has density 1.) Then wait for a stage $s_{e,j}$ at which φ_e is defined on all elements of $I_{e,j}$. (If this never occurs, it follows that φ_e is not total and hence N_e is met vacuously.) If $\varphi_e(x) = 1$ for some $x \in I_{e,j} \setminus J_{e,j}$, we let $I_{e,j}$ be the final interval for N_e and take no further action for N_e. In this case, N_e is met because S_e is not a subset of A, as $x \in S_e \setminus A$, for the x just mentioned. If there is no such x, put all elements of $I_{e,j} \setminus J_{e,j}$ into A at stage $s_{e,j}+1$, thus ensuring $I_{e,j} \subseteq A$. Then define $I_{e,j+1}$ as above at the next stage devoted to N_e.

The idea of the above strategy is that if we define $I_{e,j+1}$ and S_e has density d, then the density of S_e up to $\max J_{e,j}$ should be approximately d, while the density of S_e on the interval $(\max J_{e,j}, \max I_{e,j}]$ is surely 0, as S_e does not intersect this interval. If the latter interval is large, this suggests that d is close to 0, and in fact we get $d = 0$ by taking a limit. Of course, we also must make $|J_{e,j}|$ a large fraction of $|I_{e,j}|$ to ensure that A has density 1. Although these two largeness requirements go in opposite directions, it is easy to meet both of them, as the following calculations show.

Holding e, j fixed for now, let $a = \min I_{e,j}$, $b = \max J_{e,j}$, and $c = \max I_{e,j}$. Note that $a \leq b \leq c$ because $J_{e,j}$ is an initial segment of $I_{e,j}$. We have already determined a as the least number not in any previously defined interval. In order to meet N_e, we make the ratio b/c strictly less than 1 and independent of k. Specifically, we require that $b/c = 1 - 2^{-(e+1)}$. In order to ensure that A has density 1 we also wish $|J_{e,j}|/|I_{e,j}| = \frac{b-a+1}{c-a+1}$ to have a lower bound which depends only on e and approaches 1 as e approaches infinity. But, for fixed a, if b approaches infinity and b and c are large and related as above, then $\frac{b-a+1}{c-a+1}$ approaches b/c, which equals $1 - 2^{-(e+1)}$. Thus, we may choose b sufficiently large that $\frac{b-a+1}{c-a+1} \geq 1 - 2^{-e}$, and of course this determines c, so the intervals $I_{e,j}, J_{e,j}$ are determined.

We claim that the above strategy suffices to satisfy N_e. This is obvious if there are only finitely many intervals $I_{e,j}$, since in this case either φ_e is not total or $S_e \nsubseteq A$, and N_e is satisfied vacuously. Suppose now there are infinitely many such intervals, so that $I_{e,j}$ is defined for every j. Note that $S_e \cap I_{e,j} \subseteq J_{e,j}$ for all j. For the moment, let e, j be fixed and drop the subscript (e, j) from a, b, and c. We now calculate the decrease in density of S_e as we go from b to c without seeing any elements of S_e. Let $r = |S_e \cap [0, b]$, so $\rho_b(S_e) = r/b$. Then:

$$\rho_b(S_e) - \rho_c(S_e) = \frac{r}{b} - \frac{r}{c} = \frac{r}{b}(1 - \frac{b}{c}) = \rho_b(S_e)2^{-(e+1)}$$

Assume now that $\rho(S_e)$ exists, since otherwise N_e is vacuously met. Letting the (unwritten) j in the above equation tend to infinity yields:

$$\rho(S_e) - \rho(S_e) = \rho(S_e)2^{-(e+1)}$$

It follows that $\rho(S_e) = 0$, and so N_e is met.

It remains to show that A has density 1. Let E be the set of all points of the form $\max I + 1$, where I is any interval used in the construction. We first show that $\lim_{c \in E} \rho_c(A) = 1$. Since every element of ω belongs to one and only one interval used in the construction, we see that, for $c \in E$, $\rho_c(A)$ is the weighted average of the density of A for each interval I used in the construction with $\max I < c$, where I has weight $|I|$. (Here the density of A on I is $|A \cap I|/|I|$.) If I is used for the sake of N_e (i.e. $I = I_{e,j}$ for some j), by construction the density of A on I is either equal to 1 or is at least $1 - 2^{-e}$, where for each e, there is a most one j with this density not equal to 1 (i.e. the greatest j such that $I_{e,j}$ exists). Thus, for each real $q < 1$, A has density at least q on all but finitely many intervals used in the construction. Given

$q < 1$, let $b \in E$ be sufficiently large that A has density at least q on every interval I used in the construction with $\min I \geq b$. If $c \in E$ and $c > b$, then $\rho_c(A)$ is the weighted average of the density of A on $[0, b)$ and the density of A on $[b, c)$, where the weight of each interval is its size. The latter density is at least q, and its weight approaches infinity as c goes to infinity, while the weight of the former density stays fixed. It follows that $\liminf_{c \in E} \rho_c(A) \geq q$. As $q < 1$ was arbitrary, it follows that $\lim_{c \in E} \rho_c(A) = 1$.

We now complete the proof that A has density 1. Let I be any interval used in the construction, and let $J = I \cap A$. Let $I = [a, c]$. By construction, J is an initial segment of I, so as we examine $\rho_b(A)$ for $b-1 \in I$, we note that this density increases until we reach $\max J + 1$ and then decreases until we reach $c + 1$. It follows that for every b with $b - 1 \in I$, either $\rho_b(A) \geq \rho_{a+1}(A)$ or $\rho_b(A) \geq \rho_{c+1}(A)$. Furthermore, $a \in A$, so $\rho_{a+1}(A) \geq \rho_a(A)$ and $a, c+1 \in E$. As b goes to infinity, the points $a, c+1$ also go to infinity, and so $\rho_a(A), \rho_{c+1}(A)$ each approach 1, since $\lim_{c \in E} \rho_c(A) = 1$. Since $\rho_b(A) \geq \min\{\rho_a(A), \rho_{c+1}(A)\}$, it follows that $\rho(A) = \lim_b \rho_b(A) = 1$. □

4. Turing degrees, density, and the outer splitting property

It was shown in [16], Theorem 2.22, that there is a c.e. set of density 1 which has no computable subset of density 1. In this section we study the degrees of such sets and of their subsets of density 1. We also apply the techniques developed for this problem to study the degrees of sets with properties arising in the study of the lattice of c.e. sets.

Theorem 4.1. *There is a c.e. set A such that A has density 1 and every set $B \subseteq A$ of density 1 is high, i.e. $B' \geq_T 0''$.*

Proof. Recall that $R_e = \{x : 2^e \mid x \ \&\ 2^{e+1} \nmid x\}$. As shown in the proof of Theorem 2.22 of [16], to ensure the A has density 1, it suffices to meet the following positive requirements:

$$P_n : R_n \subseteq^* A$$

To ensure that every subset of A of density 1 is high, we make the minimal witness function w_A for A grow very fast. Specifically, define

$$w_A(n) = (\mu b)(\forall k \geq b)[\rho_k(A) \geq 1 - 2^{-n}]$$

In order to ensure that every subset B of A of density 1 is high, it suffices to meet the following negative requirements:

$$N_n : |W_n| < \infty \quad \Longrightarrow \quad w_A(n+2) \geq \max(W_n \cup \{0\})$$

To see that it suffices to meet the given requirements, assume that A satisfies all the positive and negative requirements. Let B be a subset of A of density 1, and let w_B be the corresponding minimal witness function for B, defined as above with A replaced by B. Clearly, $w_B(n) \geq w_A(n)$ for all n, since $B \subseteq A$, and so each requirement N_n holds with A replaced by B. Also, w_B is total because B has density 1, and $w_B \leq_T B'$. Let $\mathrm{Inf} = \{n : |W_n| = \infty\}$. Then for all n,

$$n \in \mathrm{Inf} \iff (\exists x \in W_n)[x > w_B(n+2)]$$

It follows that

$$0'' \leq_T \mathrm{Inf} \leq_T w_B \oplus 0' \leq_T B'$$

since B' can calculate $w_B(n+2)$ and then $0'$ can determine whether W_n has an element exceeding $w_B(n+2)$. It follows that B is high, as needed.

The strategy for meeting P_n is, at each stage s, to enumerate each $x \in R_n$ with $x \leq s$ into A unless x is restrained by N_n at the end of stage s, as described below. This will succeed in meeting P_n because there will be only finitely many numbers permanently restrained by N_n.

We now give the strategy for meeting the requirement N_n, where this strategy is similar to that used in Theorem 2.22 of [16]. This strategy restrains A only on R_n and so interacts only with the requirement P_n. Say that a finite nonempty set $I \subseteq R_n$ is *n-large* if $\rho_m(I) > 2^{-(n+2)}$, where $m = \max I$. Since $\rho(R_n) > 2^{-(n+2)}$, for each a, the set $[a,b] \cap R_n$ is n-large for all sufficiently large b. Also, if $I \subseteq R_n$ is n-large and disjoint from A, we have $\rho_m(A) \leq 1 - \rho_m(I) < 1 - 2^{-(n+2)}$, where $m = \max I$. It follows in this case that $w_A(n+2) \geq m$. Thus to meet N_n, it suffices to ensure that, if W_n is finite, there is an n-large set I which is disjoint from A with $\max I > \max(W_n \cup \{0\})$. To achieve this, start with any n-large set $I_0 \subseteq R_n$ currently disjoint from A and with $\max I_0$ exceeding all elements currently in $W_n \cup \{0\}$. Restrain all elements of I_0 from entering A until, if ever, a stage s_0 is reached at which a number exceeding $\max I_0$ is enumerated in W_n. At stage s_0, enumerate all elements of I_0 into A (for the sake of P_n), and start over with a new interval I_1 which is n-large and currently disjoint from A and satisfies $\max(I_1) > \max(I_0)$. Proceed in the same way, restraining all elements of I_1 from A until, if ever, W_e enumerates an element greater than $\max(I_1)$, in which case you proceed to I_2, etc. Now if I_k exists for every k, then W_n is infinite, since $\max(I_0) < \max(I_1) < \dots$ and, for each k, W_n contains an element exceeding $\max(I_k)$. Thus N_n is met vacuously in this case. Also, P_n is met because $R_n \subseteq A$, as infinitely often all restraints are dropped. Otherwise, there is a largest k such that I_k exists. Then, for this k, I_k is the desired n-large set disjoint from A with $\max(I_k) > \max(W_n \cup \{0\}$, so N_n is met. The requirement P_n is met because $R_n \setminus I_k \subseteq A$. □

Eric Aston (private communication) has observed that every c.e. set of density 1 has subsets of density 1 in every high degree. This allows us to strengthen the theorem as follows:

Corollary 4.2. *(with Aston) There is a c.e. set A of density 1 such that the degrees of the subsets of A which have density 1 are precisely the high degrees.*

Proof. Let A be any c.e. set of density 1 such that every subset of A of density 1 is high. To complete the proof, it suffices to show that, for every set B of high degree, A has a subset C of density 1 which is Turing equivalent to B. Let w_A be the minimal witness function for A as defined just before the statement of Theorem 3.13, and suppose that B has high degree. Note that $w_A \leq_T A' \leq_T 0'' \leq_T B'$. By relativizing the proof of Theorem 3.13 to B, we see that A has a subset $C_0 \leq_T B$ such that $\rho(C_0) = 1$. We now use a simple coding argument so obtain a set $C \subseteq A$ which has density 1 and is Turing equivalent to B. Let R be an infinite computable subset of A which has density 0. (To obtain R, first choose an infinite computable subset R_0 of A, and then show that R_0 has an infinite computable subset of density 0.) Then let C_1 be a subset of R which is Turing equivalent to B. Finally, let $C = (C_0 \setminus R) \cup C_1$. Then $C_1 \subseteq R \subseteq A$, so $C \subseteq C_0 \cup C_1 \subseteq A$. Also, C has density 1 because $C_0 \setminus R$ has density 1. Further, $C \leq_T B$, because $C_0 \leq_T B$ and $C_1 \leq_T B$. Finally, $B \leq_T C_1 \leq_T C$, where $C_1 \leq_T C = (C_0 \setminus R) \cup C_1$ because $C_0 \setminus R$ and C_1 are separated by the computable set R. Thus, C is the desired subset of A which has density 1 and is Turing equivalent to B. □

Recall that it was shown in Corollary 3.14 that every low c.e. set of density 1 has a computable subset of density 1. We now show that, conversely, every nonlow c.e. degree computes a c.e. set of density 1 with no computable subset of density 1. This result extends Theorem 2.22 of [16], which asserts the existence of a c.e. set of density 1 with no computable subset of density 1, and gives an example (the first?) of a simple, natural property P of c.e. sets such that the degrees containing c.e. sets with the property P are exactly the nonlow c.e. degrees. We use a similar technique to show that every nonlow c.e. degree contains a c.e. set which is not semilow$_{1.5}$, and use this to show that every such degree contains a set without the outer splitting property, answering a question raised by Peter Cholak.

Theorem 4.3. *If* **a** *is any non-low c.e. degree then it contains a c.e. set A of density 1 with no computable subset of density 1.*

Proof. The existence of a c.e. set A of density 1 with no computable subset of density 1 was proved in [16], Theorem 2.22, and our proof here uses a similar strategy, but with permitting added in. Familiarity with the proof of [16], Theorem 2.22, would be helpful to the reader.

Given a c.e. set C of nonlow degree **a**, we construct a c.e. set $A \leq_T C$ which has density 1 but has no computable (or even co-c.e.) subset of density 1. This suffices to prove the theorem, since we can then define

$$\hat{A} = (A \setminus \{2^n : n \in \omega\}) \cup \{2^n : n \in C\}$$

and show that $\hat{A}$ is a c.e. set of degree **a** which has density 1 but has no computable subset of density 1.

Recall that

$$R_k = \{m : 2^k \mid m \;\&\; 2^{k+1} \nmid m\}$$

As shown in the proof of Theorem 2.22 of [16], to ensure the A has density 1, it suffices to meet the following positive requirements:

$$P_n : R_n \subseteq^* A$$

To help us meet these positive requirements, as stage s we put s into A unless it is restrained for the sake of some negative requirement as described below. Thus, it is clear that P_n will be met if the restraint associated with R_n comes to a limit. As in the proof of Theorem 2.22 of [16], we will show that $R_n \subseteq A$ if the restraint associated with R_n does not come to a limit, so that P_n is met in either case.

We make $A \leq_T C$ by a slight modification of simple permitting. Namely, if x enters A at stage s, we require that either some number $y \leq x$ enters C at s, or $x = s$. This obviously implies that $A \leq_T C$.

As before, let N_e be the statement:

$$N_e : W_e \cup A = \omega \Rightarrow \overline{\rho}(W_e) > 0$$

The conjunction of the N_e's asserts that A has no co-c.e. subset of density 1. Rather than meet the N_e's directly, we split up each N_e into weaker statements $N_{e,i}$ which will be our actual requirements.

To do this we will define a computable function $g(e, i, s)$ which "threatens" to be a computable approximation to C'. Let $L_{e,i}$ be the statement:

$$\lim_s g(e, i, s) = C'(i)$$

Then define the **requirement** $N_{e,i}$ as follows:

$$N_{e,i} : N_e \text{ or } L_{e,i}$$

Suppose all requirements $N_{e,i}$ are met. If N_e is not met, then all $L_{e,i}$ hold and C is low, a contradiction. Hence, to meet N_e it suffices to meet $N_{e,i}$ for all i.

We will meet $N_{e,i}$ by restraining certain elements of $R_{e,i}$ from entering A. We do this in such a way that either the restraint comes to a limit, or infinitely often all restraint is dropped.

The strategy to meet $N_{e,i}$ is as follows. We fix e, i and refer to sets I of the form $[a, b] \cap R_{e,i}$ as *intervals*. An interval I is called *large* if at least half of the elements of $R_{e,i}$ less than $\max I$ are in I. Since $R_{e,i}$ has positive density, any set which contains infinitely many large intervals has positive lower density. At the beginning of each stage s, we have at most one interval, denoted $I[s]$, which is active for the strategy. The idea of the strategy is that we set $g(e, i, s) = 0$ while $i \notin C'[s]$, thus threatening to satisfy $L_{e,i}$ via $C'(i) = 0 = \lim_s g(e, i, s)$ unless i enters C'. If i enters C', we choose our first interval I. We require that $\min I$ exceed the use of the computation showing $i \in C'$, so that if i leaves C', the elements of I are permitted to enter A. We choose I so that it does not contain elements already in A, and we restrain elements of I from entering A. Thus, if $W_e \cup A = \omega$, W_e must eventually cover I. While we are waiting for W_e to cover I, we keep $g(e, i, s) = 0$, but when W_e covers I, we change $g(e, i, s)$ to 1, thus threatening to meet $L_{e,i}$ via $C'(i) = 1 = \lim_s g(e, i, s)$ unless i leaves C'. If i leaves C', we dump all elements of I into A (which is permitted because C changed below $\min I$) and start over, again setting $g(e, i, s) = 0$ and waiting for i to re-enter C', so we can choose a new interval, etc. Note that we start over in this fashion whenever i leaves C', whether or not W_e has covered our interval. If W_e has not covered the interval when we cancel it, we have made progress on satisfying $L(e, i)$ via $C'(i) = 0 = \lim_s g(e, i, s)$ because $C'(i)$ has changed and we have kept $g(e, i, s) = 0$. If W_e has covered our interval when we cancel it, then we have made progress on showing that W_e has positive lower density because W_e contains a new large interval. The formal construction and verification are given below.

Stage s. If $I[s]$ is not defined and $i \in C'[s]$, choose a large interval $I \subseteq R_{e,i}$ with $I \cap A_s = \emptyset$ and $\min(I)$ larger than the use of the computation showing $i \in C'[s]$. Let

$I[s+1] = I$. Let u_I be the use of the computation showing $i \in C'[s]$ and associate it with I until, if ever it is cancelled. Restrain all elements of I from entering A until, if ever, the interval I is cancelled.

If $I[s]$ is defined and $C_{s+1} - C_s$ contains an element $y \leq u_I$, then cancel $I[s]$, and enumerate all elements of $I[s]$ into A. Note that we do this whether or not $W_{e,s} \supseteq I[s]$, but if $W_{e,s} \supseteq I[s]$, we designate $I[s]$ as a *successful* interval. Of course, this enumeration is consistent with our permitting condition since $u_I \leq \min(I)$.

If neither of the above cases apply, we maintain the current interval and restraints, if any.

Finally, in any case define $g(e,i,s)$ to be 1 if $I[s]$ is defined and $I[s] \subseteq W_{e,s}$, and otherwise let $g(e,i,s) = 0$. Furthermore, if $s \in R_{e,i}$ and s is not restrained at the end of stage s, (i.e. $I[s+1]$ is undefined or $s \notin I[s+1]$), enumerate s into A, in addition to any enumeration required above. This is done to help meet the positive requirement $P_{e,i}$ and is allowed by our modified permitting condition. This completes the description of the construction.

To verify that the construction succeeds in meeting $N_{e,i}$, we consider four cases.

Case 1. For all sufficiently large s, $I[s]$ is undefined. Then, for all sufficiently large s, $i \notin C'[s]$ and $g(e,i,s) = 0$. It follows that $\lim_s g(e,i,s) = 0 = C'(i)$ so that $L(e,i)$ holds and hence $N_{e,i}$ is met.

Case 2. There is an interval I with $I[s] = I$ for all sufficiently large s. Then $i \in C'$ via the same computation as when I was first chosen, since otherwise I would have been cancelled. If $W_e \supseteq I$, we have $g(e,i,s) = 1$ for all sufficiently large s. In this case, $\lim_s g(e,i,s) = 1 = C'(i)$ and hence $L(e,i)$ holds. If $W_e \not\supseteq I$, then $W_e \cup A \not\supseteq I$, since I is disjoint from A, by the way it was chosen and the restraint imposed. It follows that $W_e \cup A \neq \omega$, and thus N_e is met.

Case 3. There are infinitely many successful intervals I. Then W_e contains all of them and so has positive lower density. It follows that N_e is met.

Case 4. None of Cases 1-3 apply. In other words, there are infinitely many intervals, but only finitely many of them are successful. Then $i \notin C'$, since infinitely often the computation showing $i \in C'$ is destroyed. Note that $g(e,i,s) = 1$ only if the interval $I[s]$ is successful or is the final interval. By the failure of Cases 1-3, there are only finitely many such s. Hence $\lim_s g(e,i,s) = 0 = C'(i)$, and $L(e,i)$ holds.

We now show that the construction also meets $P_{e,i}$. Let U be the union of all intervals ever chosen for $R_{e,i}$. If $s \in R_{e,i} - U$, then $s \in A$ by construction, so $R_{e,i} - U \subseteq A$. Thus, if U is finite, then $P_{e,i}$ is met. If U is infinite, then every interval every chosen is cancelled, at which time all of its elements enter A, so $U \subseteq A$. In this case, $R_{e,i} \subseteq (R_{e,i} - U) \cup U \subseteq A$, so again $P_{e,i}$ is met.

Note that this construction affects A only on $R_{e,i}$, so the constructions for the various requirements operate independently, and there is no injury. Thus the full construction is simply a combination of the above, over all pairs (e,i). To ensure that only finitely many actions are taken at each stage over all (e,i), one could require that the (e,i)-construction act at s only for $\langle e,i \rangle \leq s$, and this would clearly not affect the success of the individual constructions.

$\square$

Corollary 4.4. *Let* **a** *be a c.e. degree. Then the following are equivalent:*

(i) **a** *is not low*
(ii) *There is a c.e. set A of degree* **a** *such that A has density 1 but no computable subset of A has density 1.*
(iii) *There is a c.e. set A of degree* **a** *such that A has density 1 but no computable subset of A has nonzero density.*

Proof. The implication (i) $\Rightarrow$ (ii) is Theorem 4.3, and the implication (ii) $\Rightarrow$ (i) is Corollary 3.14. The implication (i) $\Rightarrow$ (iii) is proved by combining the methods of Theorem 4.3 and Theorem 3.18. We omit the details. The implication (iii) $\Rightarrow$ (ii) is immediate. $\square$

In [16], a set A was defined to be *coarsely computable* if there is a computable set B such that $A \triangle B$ (the symmetric difference of A and B) has density 0. It was shown in Proposition 2.15 of [16] that there is a c.e. set which is coarsely computable but not generically computable and in Theorem 2.26 of [16] that there is a c.e. set which is generically computable but not coarsely computable. The proof of the latter result is similar to the proof that there is a c.e. set of density 1 with no computable subset of density 1 (Theorem 2.22 of [16]). Further the existence of a c.e. set which is generically computable but not coarsely computable immediately implies the existence of a c.e. set of density 1 with no computable subset of density 1. Since sets of the latter sort exist only in nonlow degrees, one might conjecture that c.e. sets which are generically computable but not coarsely computable exist only in nonlow c.e. degrees. The next result refutes this conjecture.

Theorem 4.5. *Every nonzero c.e. degree contains a c.e. set which is generically computable but not coarsely computable.*

Proof. The proof is similar to that of Theorem 2.26 of [16], but with permitting added in. Let B be a noncomputable c.e. set. We must construct a c.e. set $A_1 \equiv_T B$ such that A_1 is generically computable but not coarsely computable. To make $A_1 \leq_T B$, we require that if if x is enumerated in A_1 at stage s, then some $y \leq x$ is enumerated in B at stage s. We can then make $A_1 \equiv_T B$ by coding B into A_1 on a computable set of density 0, as in the proof of Corollary 4.2, and clearly this operation affects neither the generic nor the coarse computability of A_1. To ensure that A_1 is generically computable it suffices to construct a c.e. set A_0 such that $A_0 \cap A_1 = \emptyset$ and $A_0 \cup A_1$ has density 1, since the partial computable function which takes the value 0 on A_0 and 1 on A_1 would then witness that A_1 is generically computable. As in Corollary 4.2, to ensure that $A_0 \cup A_1$ has density 1, it suffices to meet the following positive requirements:

$$P_e : R_e \subseteq^* A_0 \cup A_1$$

Let $S_e = \{n : \varphi_e(n) = 1\}$, so that the sets S_e for φ_e total are precisely the computable sets. To ensure that A_1 is not coarsely computable, it suffices to meet the following negative requirements:

$$N_e : \text{ If } \varphi_e \text{ is total, then } S_e \triangle A_1 \text{ is not of density } 0$$

As in the proof of Theorem 4.1, the requirements P_e and N_e act only on the set N_e. We describe the strategy for meeting those two requirements. As in the proof of that theorem, call a set $I \subseteq R_e$ *large* if at least half of the elements of R_e less than $\max I$ are in I . Choose a large interval I_0 not containing any element already in $A_0 \cup A_1$. Restrain elements of I_0 from entering $A_0 \cup A_1$. Wait until the set I_0 becomes *realized*, meaning that φ_e becomes defined on all of its elements. If this never occurs, we meet N_e because φ_e is not total. We now appoint a new large set I_1 with $\min I_1 > \min I_0$ and continue in this fashion. Whenever a realized set I_j which has not yet intersected $A_0 \cup A_1$ is *permitted* in the sense that some number $\leq \min I_j$ enters B, we force $I_j \subseteq S_e \triangle A_1$ by enumerating all elements of $I_j \cap S_e$ into A_0 and all other elements of I_j into A_1. Also, for all $k < j$, if I_k has not yet intersected $A_0 \cup A_1$, we enumerate all elements of I_k into A_1. (Note that no permission is needed for the latter enumerations.) Further, for the sake of P_e, at each stage $s \in R_e$, if s is not restrained by N_e, we enumerate s into A_1. If infinitely many intervals are permitted as above, then we ensure that $S_e \triangle A$ contains infinitely many large sets and so does not have density 0. Suppose now that only finitely many intervals are permitted and φ_e is total. Let s_0 be a stage such that no interval appointed after s_0 is permitted after it is realized. Note that infinitely many intervals are appointed, and all are realized. Hence, we can show that B is computable, since if I_k is any interval appointed after s_0, B never changes below $\min I_k$ after I_k is realized.

Finally, P_e is met because only finitely many elements of R_e are permanently restrained. This is clear if only finitely many intervals are ever appointed. If infinitely many intervals are appointed then all intervals must be realized. Furthermore, infinitely many intervals must be permitted after they are realized, by the above argument. Whenever an interval I_k is permitted, we ensure that all elements of R_e less than or equal to $\max I_k$ belong to $A_0 \cup A_1$. Thus, if infinitely many intervals are appointed, we have $R_e \subseteq A_0 \cup A_1$ and again P_e is met.

□

The technique we have introduced for meeting infinitary requirements via permitting with a nonlow c.e. oracle has applications beyond the study of asymptotic density. We illustrate this point by proving two theorems.

Let A be a c.e. set. Recall that its complement $\overline{A}$ is called *semilow* if $\{e : W_e \cap \overline{A} \neq \emptyset\} \leq_T \emptyset'$, and is called *semilow*$_{1.5}$ if $\{e : |W_e \cap \overline{A}| = \infty\} \leq_m \{e : |W_e| = \infty\}$. The notions of semilow and semilow$_{1.5}$ first arose in the study of the automorphisms of the lattice $\mathcal{E}^*$ of computably enumerable sets modulo finite sets. Let $\mathcal{L}^*(A)$ be the lattice of c.e. supersets of A, modulo finite sets. Maass [23] showed that, if A is coinfinite, $\mathcal{L}^*(A)$ is effectively isomorphic to $\mathcal{E}^*$ if and only if $\overline{A}$ is semilow$_{1.5}$. Clearly the implications low implies semilow implies semilow$_{1.5}$ hold, and it can be shown that they cannot be reversed in general.

We prove the following. An elegant proof not using permitting is given in Soare's forthcoming book [29].

Theorem 4.1. *Let* **a** *be a c.e. degree. Then the following are equivalent:*

(i) *There is a c.e. set A of degree* **a** *such that $\overline{A}$ is not semilow*$_{1.5}$.
(ii) **a** *is not low.*

Proof. For the nontrivial direction, it is enough to show that a nonlow c.e. degree $\mathbf{a}$ bounds a non-semilow$_{1.5}$ c.e. set A since then we can consider $A \oplus C$ for any c.e. set $C \in \mathbf{a}$. Fix a c.e. set C of degree $\mathbf{a}$.

The construction is analogous to the proof of Theorem 4.3. We make $A \leq_T C$ by ordinary permitting.

As before, we use the sets R_n, although any infinite uniformly computable family of pairwise disjoint sets would do just as well. We must satisfy the following conditions:

$$Q_e : \varphi_e \text{ does not witness that } A \text{ is semilow}_{1.5}.$$

As before, we define a computable function $g(e, i, s)$ which threatens to witness that C is low. Let $L_{e,i}$ be the assertion that $C'(i) = \lim_s g(e, i, s)$. Finally, define the *requirement* $Q_{e,i}$ as follows:

$$Q_{e,i} : Q_e \text{ or } L_{e,i}$$

The requirement $Q_{e,i}$ will affect the construction only on the set $R_{\langle e,i\rangle}$, which we denote $R_{e,i}$ for short.

For the sake of $Q_{e,i}$, we will build sets $V_{e,i} = W_{h(e,i)}$, where h is computable and the index $h(e, i)$ is available during the construction by the Recursion Theorem. Let $Y_{e,i} = W_{\varphi_e(h(e,i))}$ if $\varphi_e(h(e, i)) \downarrow$, and otherwise, let $Y_{e,i} = \emptyset$. Of course, $Y_{e,i}$ is c.e., uniformly in e and i. The construction will ensure that, if $L_{e,i}$ fails, the following both hold:

(i) If $Y_{e,i}$ is finite, then $V_{e,i} \cap \overline{A}$ is infinite.
(ii) If $Y_{e,i}$ is infinite, then $V_{e,i} \subseteq A$.

This suffices, since each of the above conditions implies that N_e is met.

We now describe the strategy for $Q_{e,i}$. Initially, we define $g(e, i, 0) = 0$ and in the construction change this from 0 to 1 or 1 to 0 only when explicitly told to, else $g(e, i, s + 1) = g(e, i, s)$. Next, we await the first stage $s_0 > 0$ with $i \in C'[s_0]$, say with use $u_0 < s_0$. If there is no such stage s_0, we satisfy $L_{e,i}$ and hence $Q_{e,i}$ via $\lim_s g(e, i, s) = 0 = C'(i)$. For stages $s > s_0$ we enumerate s into $V_{e,i}$ if $s \in R_{e,i}$ until, if ever, we reach a stage s_1 such that either

(i) some $y < u_0$ enters C at s_1, or
(ii) $|Y_{e,s_1}| > s_0$.

If no such stage s_1 exists, we satisfy N_e because $Y_{e,i}$ is finite and yet $V_{e,i} \cap \overline{A}$ is infinite. Suppose now that s_1 exists.

If (i) occurs, we enumerate an element of $R_{e,i}$ greater than $\max(A[s_1] \cup \{s_1\})$ into $V_{e,i}$ and restart the strategy.

If (ii) occurs, we set $g(e, i, s_1) = 1$. We then await a stage $s_2 > s_1$ such that some $y < u_0$ enters C at s_2. If no such stage occurs, we meet $L(e, i)$ via $\lim_s g(e, i, s) = 1 = C'(i)$. If such a stage occurs, we set $g(e, i, s_2) = 0$, enumerate all of $V_{e,i}[s_2]$ into A, and restart the strategy.

We now show that this strategy succeeds in meeting $Q_{e,i}$. If we wait forever for some stage s_i as above (in some cycle) to occur, then $Q_{e,i}$ is met by remarks in the description of the strategy. Suppose that we never wait in vain and so go through infinitely many cycles. If $Y_{e,i}$ is infinite, then (ii) occurs in infinitely many cycles, and we ensure that $V_e \subset A$ by the action at s_2. If $Y_{e,i}$ is finite, then (ii) occurs in only finitely many cycles. It follows that $\lim_s g(e,i,s) = 0$ because $g(e,i,s)$ changes from 0 to 1 only finitely often, and after each such change it is reset to 0. Also, $i \notin C'$ because in each cycle there is a stage at which a number below the use of the computation showing $i \in C'$ enters C. Hence and we meet $L_{e,i}$ via $\lim_s g(e,i,s) = 0 = C'(i)$. It follows that $Q_{e,i}$ is met in all cases. As before, the requirements don't interact, and we omit further details.

□

The proof above can be modified for another similar property. Cholak [3] proved a result related to Maass's by showing that if A is semilow$_2$ (a generalization of being semilow$_{1.5}$) and has the *outer splitting property* then $\mathcal{L}^*(A)$ is isomorphic to $\mathcal{E}^*$. A has the outer splitting property if there are two total computable functions f and g such that for all e,

(i) $W_e = W_{f(e)} \sqcup W_{g(e)}$ (that is, they split W_e.)
(ii) $|W_{f(e)} \cap \overline{A}| < \infty$.
(iii) $|W_e \cap \overline{A}| = \infty$ implies $W_{f(e)} \cap \overline{A} \neq \emptyset$.

Cholak and Shore showed that there is a low$_2$ c.e. set without the outer splitting property [3]. The following classifies the degrees, and extends the result above since if A is c.e. and $\overline{A}$ is semilow$_{1.5}$, then A has the outer splitting property [3].

Theorem 4.6. *A c.e. degree* **a** *contains a c.e. set A without the outer splitting property if and only if* **a** *is non-low. Hence having the outer splitting property is not definable in the lattice of c.e. sets.*

Proof. The second part of the statement follows from the first part and Rachel Epstein's recent result [8] that there is a nonlow c.e. degree **c** such that every c.e. set in that degree can be sent to a low degree by an automorphism.

The proof of the first part is completely analogous to the proof of the previous theorem. We modify Q_e so that it now requires us to kill the e-th pair of candidates for f and g (say ψ_e and ξ_e) in the definition of the outer splitting property. We define a computable function g as before and use it to determine the statements $L_{e,i}$ and $Q_{e,i}$ as before. We also define sets $V_{e,i} = W_{h(e,i)}$ as witnesses for $Q_{e,i}$. Let $Y_{e,i} = W_{\psi_e(h(e,i))}$ and $Z_{e,i} = W_{\xi_e(h(e,i))}$. We ensure that, if $L_{e,i}$ fails to hold, then either

(i) $Y_{e,i}, Z_{e,i}$ fail to split $W_{h(e,i)}$, or
(ii) $Y_{e,i} \cap \overline{A}$ is infinite, or
(iii) $W_{h(e,i)}$ is infinite and $Y_{e,i} \subseteq A$.

This clearly suffices to meet $Q_{e,i}$. Our strategy to achieve the above (acting only on $R_{e,i}$) is as follows: Set $g(e,i,0) = 0$ and wait for s_0 with $i \in C'[s_0]$. At stage s_0, we start putting elements of $R_{e,i}$ into $W_{h(e,i)}$. We continue until we reach a stage s_1 such that either the computation $i \in C'$ is destroyed or $|Z_{e,i}| > s_0$. In the former case, we dump $Y_{e,i}$ into A and restart the strategy. In the latter case, we set $g(e,i,s_1) = 1$ and wait for a stage s_2 at which the computation $i \in C'$ is injured. At stage s_2, we dump $Y_{e,i}$ into A and restart the strategy, in particular setting $g(e,i,s_2) = 0$.

As before, it is easy to see that $Q_{e,i}$ is met if there are only finitely many cycles. In particular, if s_1 fails to exist, then $W_{h(e,i)}$ is infinite and $Z_{e,i}$ is finite, so either (i) or (ii) above holds. If we set $g(e, i, s_1) = 1$ in infinitely many cycles, then (iii) holds. In the remaining case, we have $\lim_s g(e, i, s) = 0 = C'(i)$, so $L(e, i)$ holds.

□

5. Arithmetical complexity of densities

In [16], Theorem 2.21, it was shown that the densities of the computable sets are exactly the Δ^0_2 reals in the interval $[0, 1]$. In this section we obtain analogous results for c.e. sets in place of computable sets, and we also study upper and lower densities as well as densities. Throughout, we assume we have fixed a computable bijection between the natural numbers and the rational numbers. We say that a set of rational numbers is Σ^0_n if the corresponding set of natural numbers is Σ^0_n, and similarly for other notions. The following definition is fundamental and standard:

Definition 5.1. A real number r is *left-*Σ^0_n if the corresponding lower cut in the rationals, $\{q \in \mathbb{Q} : q < r\}$, is Σ^0_n. We define "left-Π^0_n" analogously.

We first characterize the lower densities of the computable sets.

Theorem 5.2. *Let r be a real number in the interval $[0, 1]$. Then the following are equivalent:*

(i) *$r = \underline{\rho}(A)$ for some computable set A.*
(ii) *r is the lim inf of a computable sequence of rational numbers.*
(iii) *r is left-Σ^0_2.*

Proof. It is obvious that (i) implies (ii), since $\underline{\rho}(A) = \liminf_n \rho_n(A)$ by definition.

To see that (ii) implies (iii), suppose that $r = \liminf_n q_n$, where $\{q_n\}$ is a computable sequence of rational numbers. If r is a rational number, then it is clear that (iii) holds. Thus, we assume that r is irrational. Then, for all rational numbers q,

$$q < r \iff (\forall^\infty n)[q < q_n]$$

where $(\forall^\infty n)$ means "for all but finitely many n." This follows easily from the definition of the lim inf, using the assumption that r is irrational to prove the implication from right to left. Expanding the right-side of the above equivalence shows that r is left-Σ^0_2.

We now show that (iii) implies (ii). We begin with a lemma which characterizes Σ^0_2 sets of natural numbers in terms of computable approximations. The following lemma improves the standard result that for every Σ^0_2 set A there are uniformly computable sets A_s such that, for all k, $k \in A$ if and only if $(\forall^\infty s)[k \in A_s]$.

Lemma 5.3. *Let A be a Σ^0_2 set. Then there is a uniformly computable sequence of sets $\{A_s\}$ such that*

(i) *For all $k \in A$, we have $k \in A_s$ for all sufficiently large s*
(ii) *There exist infinitely many s such that $A_s \subseteq A$*

Proof. (of lemma) To prove this lemma, we use the Lachlan-Soare "hat trick" ([28], page 131), with which we assume the reader is familiar. Since A is c.e. in $\mathbf{0}'$ there exists an e such that A is the domain of Φ_e^K. Now let $A_s = \{k : \hat{\Phi}_{e,s}(K_s, k)\downarrow\}$. Then if $k \in A$, then $\Phi_e^K(k)\downarrow$, and so $\hat{\Phi}_{e,s}(K_s, k)\downarrow$ for all sufficiently large s. Now let T be the set of true stages. T is infinite. If $s \in T$ and $\hat{\Phi}_{e,s}(K_s, k)\downarrow$, then $\Phi_e^K(k)\downarrow$. It follows that $A_s \subseteq A$ for all s in T. $\square$

We now show that (iii) implies (ii) in the theorem. Let r be a real number which is left-Σ_2^0. We must produce a computable sequence $\{q_s\}$ of rational numbers such that $\liminf_s q_s = r$. Let $A = \{q \in \mathbb{Q} : q < r\}$, so that A is Σ_2^0 by hypothesis. Let the uniformly computable sets A_s be related to A as in the lemma. By truncating the sets if necessary, we may assume that every rational in A_s corresponds to a natural number $< s$ under our coding of rationals. Thus A_s is finite, and we may effectively compute a canonical index for the finite set of natural numbers corresponding to it. Let q_s be the greatest rational number in A_s if A_s is nonempty, and otherwise let $q_s = 0$. Then $\{q_s\}$ is a computable sequence of rational numbers. By hypothesis, there are infinitely many s such that $A_s \subseteq A$ and thus every element of A_s is less than r. Using the definition of q_s and the hypothesis that $r \geq 0$, it follows that there are infinitely many s such that $q_s \leq r$, and thus $\liminf_s q_s \leq r$. To show that $r \leq \liminf_s q_s$, note that if $q < r$, then $q \in A$, so $q \in A_s$ for all sufficiently large s, so $q \leq q_s$ for all sufficiently large s. It follows that every rational number $q < r$ is $\leq \liminf_s q_s$, so $r \leq \liminf_s q_s$. This completes the proof that (iii) implies (ii).

We complete the proof of the Theorem by showing that (ii) implies (i). Thus we must show that if $r \in [0, 1]$ and $r = \liminf_s q_s$ where $\{q_s\}$ is a computable sequence of rationals, then there is a computable set A such that $\underline{\rho}(A) = r$. Essentially, this follows from the proof of Theorem 2.21 of [16], where it was shown that every Δ_2^0 real in $[0, 1]$ is the density of a computable set. For the convenience of the reader and for use in Corollary 5.6, we put in the details in the following lemma.

Lemma 5.4. *Let $\{q_s\}$ be a computable sequence of rational numbers such that $0 \leq \liminf_s q_s$ and $\limsup_s q_s \leq 1$. Then there is a computable set A such that $\underline{\rho}(A) = \liminf_s q_s$ and $\overline{\rho}(A) = \limsup_s q_s$.*

Proof. First, we may assume that each q_s lies in the interval $(0, 1)$, by replacing q_s by $1/(s+1)$ if $q_s \leq 0$ and by $1 - 1/(s+1)$ if $q_s \geq 1$ and otherwise leaving q_s unaltered. Since $r \in [0, 1]$, the resulting sequence of rationals still has r as its lim inf. We define a computable set A and an increasing sequence $\{s_n\}$ of natural numbers such that, for all n:

(i) $|\rho_{s_n}(A) - q_n| \leq 1/(n+1)$
(ii) For all natural numbers k in the interval (s_n, s_{n+1}), $\rho_k(A)$ is between $\rho_{s_n}(A)$ and $\rho_{s_{n+1}}(A)$.

Let $s_0 = 1$ and put 0 into A. Now assume inductively that s_n and $A \upharpoonright s_n$ are defined, so that $\rho_{s(n)}(A)$ is defined. There are now two cases.

If $\rho_{s_n}(A) > q_{n+1}$, let s_{n+1} be the least number $t > s_n$ such that $\rho_t(A \upharpoonright s_n) \leq q_{n+1}$. (Such a t exists because $q_{n+1} > 0$ and $\rho_t(A \upharpoonright s_n)$ approaches 0 as t approaches infinity.) Let $A \upharpoonright s_{n+1} = A \upharpoonright s_n$.

If $\rho_{s_n}(A) \leq q_{n+1}$, let s_{n+1} be the least number $t > s_n$ such that $\rho_t((A \upharpoonright s_n) \cup [s_n t)) \geq q_{n+1}$, and let $A \upharpoonright s_{n+1} = A \upharpoonright s_n \cup [s_n, s_{n+1})$.

To verify (i), use that $s_n \geq n$ and the minimality of t in each case. To verify (ii), use that the interval $[s_n, s_{n+1})$ is either contained in or disjoint from A, so that $\rho_t(A)$ is either increasing or decreasing in t on this interval. We omit the details.

It remains to show that $\underline{\rho}(A) = \liminf_s \rho_s(A) = \liminf_n q_n$. Since $\lim_n(q_n - \rho_{s(n)}) = 0$ and $\{\rho_{s(n)}(A)\}$ is a subsequence of $\{\rho_s(A)\}$, we have $\liminf_s \rho_s(A) \leq \liminf_n \rho_{s(n)} = \liminf_n q_n$. To show that $\liminf_n \rho_{s(n)}(A) \leq \liminf_s \rho_s(A)$, note that for every $k > 0$ there is a number $t(k)$ such that $\rho_{s(t(k))}(A) \leq \rho_k(A)$, namely if $s(n) \leq k < s(n+1)$, let $t(k)$ be n if $\rho_{s(n)} \leq \rho_k(A)$, and otherwise let $t(k)$ be $n+1$. Further, by this definition, the function t is finite-one, and hence $t(k)$ approaches infinity as k approaches infinity. We thus have $\liminf_k \rho_k(A) \leq \liminf_n \rho_{s(n)}(A)$, which completes the proof of the Lemma. □

The theorem follows. □

Corollary 5.5. *Let r be a real number in the interval $[0, 1]$. Then the following are equivalent:*

(i) *r is the upper density of some computable set.*
(ii) *r is left-Π^0_2*

Proof. Note that $\overline{\rho}(A) = 1 - \underline{\rho}(\overline{A})$ for every set A, and that for every real number r, $1 - r$ is left-Σ^0_2 if and only if r is left-Π^0_2. Since the computable sets are closed under complementation, the corollary follows. □

The following corollary sums up our results on upper and lower densities of computable sets.

Corollary 5.6. *Let a and b be real numbers such that $0 \leq a \leq b \leq 1$. Then the following are equivalent:*

(i) *There is a computable set R with lower density a and upper density b*
(ii) *a is left-Σ^0_2 and b is left-Π^0_2.*

Proof. It follows at once from Theorem 5.2 and Corollary 5.5 that (i) implies (ii). For the converse, assume that (ii) holds of a and b. Since a is left-Σ^0_2, by Theorem 5.2, there is a computable sequence of rationals $\{q_n\}$ with $\liminf_n q_n = a$. Since b is left-Π^0_2, by the proof of Corollary 5.5, there is a computable sequence of rationals $\{r_n\}$ with $\limsup_n r_n = b$.

If $a = b$, then a is a Δ^0_2 real and hence is the density of a computable set by Theorem 2.21 of [16]. Thus, we may assume that $a < b$. Fix a rational number q^* such that $a \leq q* \leq b$. By replacing q_n by $\max\{q_n, q^*\}$, we may assume that $q_n \leq q^*$ for all n, and hence $\limsup_n q_n \leq q^* \leq b$. Similarly, we may assume that $\liminf_n r_n \geq a$. Now define a computable sequence of rationals $\{s_n\}$ by $s_{2n} = q_n$ and $s_{2n+1} = r_n$. Then $\liminf_n s_n = \min\{\liminf_n q_n, \liminf_n r_n\} = a$ and $\limsup_n s_n = \max\{\limsup_n q_n, \limsup_n r_n\} = b$. The corollary now follows by applying Lemma 5.4 to the sequence $\{s_n\}$. □

We now consider the complexity of the various kinds of density associated with c.e. sets. The first result follows easily from what we have done for computable sets.

Theorem 5.7. *Let r be a real number in the interval $[0,1]$. Then the following are equivalent:*

(i) *r is the upper density of a c.e. set.*
(ii) *r is left-Π^0_2.*

Proof. It follows immediately from Corollary 5.5 that (ii) implies (i). To show that (i) implies (ii), let r be the upper density of a c.e. set A. We may assume without loss of generality that r is irrational. Then for q rational, we have

$$q < r \iff (\exists^\infty n)[q < \rho_n(A)]$$

Since the predicate $q < \rho_n(A)$ is Σ^0_1, (ii) follows. □

Theorem 5.8. *Let r be a real number in the interval $[0,1]$. Then the following are equivalent:*

(i) *r is the lower density of a c.e. set.*
(ii) *r is left-Σ^0_3.*

Proof. By relativizing Theorem 5.2 to $0'$ and applying Post's Theorem, we see that, for $r \in [0,1]$, r is the lower density of a Δ^0_2 set if and only if r is left-Σ^0_3. It follows immediately that (i) implies (ii) above. To prove the converse, it suffices to show that for every Δ^0_2 set B there is a c.e. set A such that A and B have the same lower density. Let the Δ^0_2 set B be given. We will give a strictly increasing Δ^0_2 function t and a c.e. set A such that, for all n, $\rho_{t(n)}(A) = \rho_n(B)$. This implies that $\underline{\rho}(B) \geq \underline{\rho}(A)$. To obtain the opposite inequality (and hence $\underline{\rho}(A) = \underline{\rho}(B)$), we require further that, for each n, $A \cap [t(n), t(n+1))$ be an initial segment of the interval $[t(n), t(n+1))$, so that the least value of $\rho_k(A)$ for $k \in [t(n), t(n+1))$ occurs when $k = t(n)$ or $k = t(n+1)$. It then follows that $\liminf_k \rho_k(A) \geq \liminf_n \rho_{t(n)}(A)$. Hence, $\underline{\rho}(B) = \liminf_n \rho_n(B) = \liminf_n \rho_{t(n)}(A) \leq \liminf_k \rho_k(A) = \underline{\rho}(A)$.

The following straightforward lemma will be useful in defining t as described above.

Lemma 5.9. *Let $F \subseteq \omega$ be a finite set, $a \in \omega$, and r be a rational number in the interval $(0,1)$. Then there is a finite set $G \supseteq F$ and $c \in \omega$ such that:*

(i) $G \upharpoonright a = F \upharpoonright a$
(ii) $c > a$
(iii) $\rho_c(G) = r$
(iv) $G \cap [a, \infty)$ *is an initial segment of* $[a, \infty)$.

Proof. For any b, let $G_b = F \cup [a,b)$. Then for every sufficiently large b, we have $\rho_b(G_b) > r$, since $\lim_b \rho_b(G_b) = 1 > r$. We will set $G = G_b$ for a suitable choice of b. Then (i) above holds with $G = G_b$ for all b, and (iv) above holds with $G = G_b$ for all $b > \max(F)$. To make (ii) and (iii) hold, we set $c = |G_b|/r$, where b is chosen so that $|G_b|/r$ is an integer greater than b, $b > a$ and $b > \max(F)$. To see that such a b exists (and in fact infinitely many such b exist), note that there is a constant k such that $|G_b| = b - k$ for all sufficiently large b. For any such b, we have

$c = |G_b|/r > b > \max F$, so $c > \max(G_b)$. Hence $\rho_c(G_b) = |G_b|/c = r$, and therefore (i)-(iv) all hold with $G = G_b$ and $c = |G_b|/r$. $\square$

We now define a c.e. set A and a strictly increasing Δ^0_2 function t as described above. We enumerate A effectively and obtain t as $\lim_s t(n,s)$, where $t(.,.)$ is computable. Let $\{B_s\}$ be a computable approximation to B. At stage 0, let $A_0 = \emptyset$, and $t(n,0) = n+1$ for all n. At stage $s+1$, suppose inductively that A_s and all values of $t(n,s)$ have been defined. Let n_s be the least $n \leq s$ such that $\rho_{t(n,s)}(B_s) \neq \rho_n(A_s)$. Apply Lemma 5.9 with $F = A_s$ and $a = t(n_s,s)$ to obtain a finite set $G \supseteq A_s$ and a number $c > t(n_s,s)$ such that $\rho_c(G) = \rho_n(A_s)$ and $G \cap [t(n,s),\infty)$ is an initial segment of $[t(n_s,s),\infty)$. Let $A_{s+1} = G$ and $t(n_s,s+1) = c$. (To apply Lemma 5.9 we need $0 < \rho_n(A_s) < 1$, so if $\rho_n(A_s) = 0$, replace it by $1/(n+1)$, and if $\rho_n(A_s) = 1$, replace it by $1 - 1/(n+1)$. In the limit, these replacements have no effect.) For $m < n_s$, define $t(m,s+1) = t(m,s)$, and for $m > n_s$ define $t(m,s+1) = c+m-n_s$.

We now show that, for each $k > 0$ that there are only finitely many s with $n_s = k$, $\lim_s t(k,s)$ exists, and, if $t(k)$ is this limit, $\rho_{t(k)}(A) = \rho_k(B)$. This result is proved by induction on k, so assume it holds for all $m < k$. Let b be a stage $\geq k$ such that, for all $m < k$ and all $s \geq b$, $t(m,s) = t(m)$, $n_s \neq m$, $\rho_{t(m,s)}(A_s) = \rho_m(B_s)$, and $B_s \upharpoonright k = B \upharpoonright k$. If $\rho_{t(k,b)}(A_b) \neq \rho_k(B_b)$, we set $n_b = k$, and the construction ensures that $\rho_{t(k,b+1)}(A_b) = \rho_k(B_b)$. Then, by construction, there is no $s > b$ with $n_s = k$. It follows that there are only finitely many s with $n_s = k$, and that $\lim_s t(k,s) = t(k)$ exists. It also follows that $\rho_{t(k)}(A) = \rho_k(B)$, since this holds at stage $b+1$ (whether or not $n_b = k$) and is preserved forever thereafter.

$\square$

It remains to consider the densities of c.e. sets. Note that if A is c.e., then $\rho_n(A) = \lim_s g(n,s)$ where g is a computable function taking rational values, $g(n,s) \leq g(n,s+1)$ for all n and s, and for each n there are only finitely many s such that $g(n,s) \neq g(n,s+1)$, namely $g(n,s) = \rho_n(A_s)$, where $\{A_s\}$ is a computable enumeration of A. Hence, $\underline{\rho}(A) = \liminf_n \lim_s g(n,s)$ and $\overline{\rho}(A) = \limsup_n \lim_s g(n,s)$. The next result turns this around to show how computable functions g with these stability and monotonicity properties can be used to produce c.e. sets with corresponding upper and lower densities.

Theorem 5.10. *Let $g : \omega^2 \to \mathbb{Q} \cap (0,1)$ be a computable function such that:*

(i) *$g(n,s) \leq g(n,s+1)$ for all n and s, and*
(ii) *$\{s : g(n,s) \neq g(n,s+1)\}$ is finite for all n.*

Let $h(n) = \lim_s g(n,s)$, so $h : \omega \to \mathbb{Q}$ is total by (ii). Then there is a c.e. set A such that:

(iii) *$\underline{\rho}(A) = \liminf_n h(n)$, and*
(iv) *$\overline{\rho}(A) = \limsup_n h(n)$.*

Proof. By changing $g(n,s)$ by at most $1/n$ we may assume without loss of generality that $g(n,s)$ is an integer multiple of $1/n$ for all s and all $n > 0$. Of course, this changes $h(n)$ by at most $1/n$ and has no effect on $\liminf_n h(n)$ or $\limsup_n h(n)$. Partition each interval $[n!,(n+1)!)$ into disjoint subintervals $I_{n,1}, I_{n,2}, \ldots, I_{n,r_n}$ of size n, where $r_n = ((n+1)! - n!)/n = n!$. From each interval $I_{n,i}$ enumerate exactly $nh(n) = n \max_s g(n,s)$ numbers into the c.e. set A. Note that this can be done effectively since g is computable, and $nh(n)$ is an integer not exceeding n. Define the density of a set A on a nonempty finite interval I to be $|A \cap I|/|I|$. Thus we have ensured that the density of A on each interval $I_{n,i}$ for $1 \leq i \leq r(n)$ is $h(n)$. From this, it is easily seen that, modulo error terms which approach 0 as n approaches infinity, $\rho_{(n+1)!}(A) = h(n)$ and if $k \in [n!, (n+1)!)$, then $\rho_k(A)$ is between $h(n-1)$ and $h(n)$. We then get that $\underline{\rho}(A) = \liminf_k \rho_k(A) = \liminf_n \rho_{(n+1)!}(A) = \liminf_n h(n)$, and similarly for $\overline{\rho}(A)$.

We now spell out the details of the above approximations. It is easy to see that if $I_1, I_2, \ldots, I_t$ are disjoint intervals, then the density of A on $I_1 \cup I_2 \cup \cdots \cup I_t$ is at least the minimum of the density of A on the intervals $I_1, \ldots, I_t$, and at most the maximum of the density of A on these intervals. Hence, the density of A on the interval $[n!, (n+1)!)$ is $h(n)$, since the density of A on each subinterval $I_{n,i}$ is $h(n)$. Using this to calculate the cardinality of $A \cap [n!, (n+1)!)$ and noting that $0 \leq |A \cap [0, n!)| \leq n!$, it follows that

$$h(n)((n+1)! - n!) \leq |A \cap [0, (n+1)!)| \leq n! + h(n)((n+1)! - n!)$$

Dividing through by $(n+1)!$ yields that

$$h(n) - \frac{h(n)}{n+1} \leq \rho_{(n+1)!}(A) \leq \frac{1}{n+1} + h(n) - \frac{h(n)}{n+1}$$

and hence

$$-\frac{h(n)}{n+1} \leq \rho_{(n+1)!}(A) - h(n) \leq \frac{1}{n+1} - \frac{h(n)}{n+1}$$

It follows that $\lim_n(\rho_{(n+1)!}(A) - h(n)) = 0$.

We now show that if $k \in (n!, (n+1)!]$, then $\rho_k(A)$ is between $h(n-1)$ and $h(n)$ with an error term which approaches 0 as n approaches infinity. Consider first the case where k has the form $\max(I_{n,i})+1$ for some i. Then $[0,k)$ is the disjoint union of $[0, n!)$ and the intervals $I_{n,j}$ for $j \leq i$. Hence $\rho_k(A)$ is between $\rho_{n!}(A)$ and $h(n)$, and, as we have noted, $\lim_n(\rho_{n!}(A) - h(n-1)) = 0$. Since the intervals of $I_{n,k}$ have size n, every $c \in (n!, (n+1)!]$ differs from at most n by a number of the form $\max(I_{n,i}) + 1$. Finally if $a, b \geq n!$ and $|a - b| \leq n$, we have that $|\rho_a(A) - \rho_b(A)| \leq (n+1)/(n-1)!$. To see this, let $u = |A \upharpoonright a|$ and $v = |A \upharpoonright b|$. Since $|a-b| \leq n$, we also have $|u-v| \leq n$. Note that $\rho_a(A) - \rho_b(A) = u/a - v/b$. Thus, it suffices to show that both $v/b - u/a$ and $u/a - v/b$ are less than or equal to $(n+1)/(n-1)!$. We may assume without loss of generality that $a \leq b$ and hence $u \leq v$. We have

$$\frac{v}{b} - \frac{u}{a} \leq \frac{v}{a} - \frac{u}{a} = \frac{v-u}{a} \leq \frac{n}{n!} \leq \frac{n+1}{(n-1)!}$$

since $0 < a \leq b$ and $v - u \leq n$. Also,

$$\frac{u}{a} - \frac{v}{b} \leq \frac{u}{a} - \frac{u}{b} = \frac{u}{ab}(b-a) \leq \frac{(n+1)!}{(n!)^2} n = \frac{n+1}{(n-1)!}$$

since $b > 0$, $u \leq v$, $a, b \geq n!$, and $b - a \leq n$.

Hence $\underline{\rho}(A) = \liminf_a \rho_a(A) = \liminf_n h(n)$ and similarly for $\overline{\rho}(A)$. $\square$

Theorem 5.11. *Let r be a real number in the interval $[0,1]$. Then the following are equivalent:*

(i) *r is the density of some c.e. set.*
(ii) *r is left-Π^0_2.*

Proof. The implication (i) implies (ii) is immediate from Theorem 5.7, which implies that the upper density of every c.e. set is a left-Π^0_2 real.

For the implication (ii) implies (i), assume that r is left-Π^0_2. Then by the dual of Theorem 5.2, there is a computable sequence of rationals $\{q_n\}$ such that $r = \limsup_n q_n$ and $0 \leq q_n \leq 1$ for all n. We now define a computable function $g : \omega^2 \to \mathbb{Q} \cap [0,1]$ to which to apply Theorem 5.10. We define $g(n,s)$ by recursion on s. Let $g(n,0) = 0$. For the inductive step, define

$$g(n,s+1) = \begin{cases} q_s & \text{if } q_s \geq g(n,s) + \frac{1}{n+1} \text{ and } s \geq n \\ g(n,s) & \text{otherwise} \end{cases}$$

Clearly, g satisfies the hypotheses of Theorem 5.10, so by that result there is a c.e. set A such that $\underline{\rho}(A) = \liminf_n h(n)$ and $\overline{\rho}(A) = \limsup_n h(n)$, where $h(n) = \lim_s g(n,s)$. Thus, it suffices to show that $\lim_n h(n) = \limsup_n q_n$. To this end, define $b(n) = \sup_{s \geq n} q_s$ and note that $\limsup_n q_n = \lim_n b(n)$. By the definition of g and the fact that $h(n) = \lim_s h(n,s)$, we have

$$b(n) - \frac{1}{n+1} \leq h(n) \leq b(n)$$

for all n. It follows that $\lim_n h(n) = \lim_n b(n) = \limsup_n q_n$.

□

6. Density and immunity properties

In computability theory, a whole spectrum of immunity type properties has been studied, with the weakest being immunity itself and the strongest one commonly studied being cohesiveness. In this section, we study results relating immunity properties and asymptotic density. It was already shown in the proof of Proposition 2.15 of [16] that there is a simple set of density 0, and hence an immune set of density 1. We observe in this section, for example, that every hyperimmune set has lower density 0, every strongly hyperhyperimmune (shhi) set has upper density less than 1, and that every cohesive set has density 0. We also prove contrasting results – for example shhi sets can have upper density arbitrarily close to 1.

Theorem 6.1. (i) *Every hyperimmune set has lower density* 0.
(ii) *There is a co-hypersimple set with upper density* 1.

Proof. (Sketch) Let I_n be the interval $[n!, (n+1)!)$. If A is hyperimmune, then $A \cap I_n = \emptyset$ for infinitely many n, from which it follows that $\underline{\rho}(A) = 0$. For the second part, it is a straightforward finite injury argument to construct a hypersimple set B such that $B \cap I_n = \emptyset$ for infinitely many n, so that $\overline{B}$ is the desired co-hypersimple set with upper density 1. □

Theorem 6.2. (i) *Every co-c.e. hyperhyperimmune (hhi) set has density* 0.

(ii) *Every Δ^0_2 hhi set has upper density less than* 1.
(iii) *There is a hhi set with upper density* 1.

Proof. For the first part, recall that by [23] every co-c.e. hhi set A is dense immune, i.e. the principal function of A dominates every computable function. From this it easily follows that A has density 0. For the second part, use the known fact (see the lemma below) that every Δ^0_2 hhi set is shhi, and apply the first part of the next theorem. For the third part, note that every 2-generic set is hhi and has upper density 1.

The lemma below is due to S. B. Cooper [4], and we include a proof for the convenience of the reader.

Lemma 6.3. *([4]) If A is both Δ^0_2 and hhi, then A is shhi.*

Proof. Let $\{A_s\}$ be a computable approximation to A, and suppose that A is infinite and not shhi. We prove that A is not hhi. Let $\{U_n\}$ be a weak array witnessing that A is not shhi, so the sets U_n are uniformly c.e., pairwise disjoint, and all intersect A. To show that A is not hhi, it suffices to produce uniformly c.e. sets $\{V_n\}$ with each V_n a finite subset of U_n so that each V_n intersects A. Let $V_{n,s}$ be the set of numbers enumerated in U_n before stage s, and define $U_{n,s}$ analogously. At each stage s, if $V_{n,s} \cap A_s = \emptyset$, let $V_{n,s+1} = V_{n,s} \cup U_{n,s}$, and otherwise let $V_{n,s+1} = V_{n,s}$.

Clearly, $V_n \subseteq U_n$. Assume for a contradiction that V_n is infinite. Then $V_n = U_n$ so $V_n \cap A \neq \emptyset$. It follows that $V_{n,s} \cap A_s \neq \emptyset$ for all sufficiently large s, so V_n is finite, which is the desired contradiction. Hence V_n is finite. Now assume for a contradiction that $V_n \cap A = \emptyset$. Then $V_n \cap A_s = \emptyset$ for all sufficiently large s, and hence $V_n = U_n$. It follows that $V_n \cap A \neq \emptyset$, which is the desired contradiction. □

□

Theorem 6.4. (i) *No shhi set has upper density* 1.
(ii) *For every $\epsilon > 0$ there is a shhi set with upper density at least $1 - \epsilon$.*

Proof. For (i), let A be shhi, and consider the sets $\{R_n\}$ where, as usual, $R_n = \{k : 2^n \mid k \ \& \ 2^{n+1} \nmid k\}$. Since these sets are pairwise disjoint and uniformly computable, there exists n such that $R_n \cap A = \emptyset$. Since $\rho(R_n) > 0$, it follows that $\overline{\rho}(A) < 1$.

For (ii) we use a special kind of Mathias forcing. Let q_0 be a rational number such that $1 - \epsilon < q_0 < 1$. Let P be the set of pairs (F, I) where F, I are subsets of ω, F is finite, I is infinite, $F \cap I = \emptyset$, and $\overline{\rho}(I) > q_0$. Thus, we are using Mathias forcing with conditions of upper density strictly greater than q_0. If $(F, I) \in P$, say that A *satisfies* (F, I) if $F \subseteq A \subseteq F \cup I$. If $p, q \in P$ say that q *extends* p if every set which satisfies q also satisfies p. We must construct an shhi set A with upper density at least $1 - \epsilon$, and for this it suffices to meet the following requirements:

$$N_{2e} : (\exists s \geq e)[\rho_s(A) \geq q_0]$$

N_{2e+1} : If φ_e is total $\ \& \ (\forall a)(\forall b)[a \neq b \to W_{\varphi_e(a)} \cap W_{\varphi_e(b)} = \emptyset]$ then $(\exists a)[W_{f(a)} \cap A = \emptyset]$

The result to be proved is an easy consequence of the following lemma.

Lemma 6.5. *For any $p \in P$ and $n \in \omega$ there exists $q \in P$ such that q extends P and every set which satisfies q also satisfies the requirement N_n.*

Proof. To prove the lemma, let $p = (F, I)$. Consider first the case where $n = 2e$. Since $\overline{\rho}(I) > q_0$, there exists $s > e$ with $\rho_s(I) \geq q_0$. Let $q = (\hat{F}, \hat{I})$, where $\hat{F} = F \cup \{i \in I : i < s\}$, and $\hat{I} = \{i \in I : i \geq s\}$. Then $q \in P$, q extends p, and $\rho_s(\hat{F}) \geq q_0$. Furthermore, if A satisfies q, then $\rho_s(A) \geq q_0$ because $A \supseteq \hat{F}$, so A meets N_n.

For the case where $n = 2e + 1$, we prove the following combinatorial lemma.

Lemma 6.6. *Suppose the sets $S_0, S_1, \ldots$ are pairwise disjoint and I is a set such that $\overline{\rho}(I) > q_0$. Then, for all sufficiently large j, $\overline{\rho}(I \setminus S_j) > q_0$.*

Proof. Assume the result fails, so for infinitely many j, we have $\overline{\rho}(I \setminus S_j) \leq q_0$. In fact, we may assume without loss of generality that this inequality holds for all j, since we may replace the sequence of all S_j's by the sequence of those S_j's for which it holds. Choose rational numbers q_1, q_2 such that $q_0 < q_1 < q_2 < \overline{\rho}(I)$. Since $q_2 < \overline{\rho}(I)$, we may choose numbers $n_0 < n_1 < \ldots$ such that $\rho_{n_i}(I) \geq q_2$ for all i. Then we have

$$\rho_{n_i}(I) = \rho_{n_i}(I \cap S_j) + \rho_{n_i}(I \setminus S_j)$$

for all i, j. Since, for all j, $\overline{\rho}(I \setminus S_j) \leq q_0 < q_1$, we have that

$$\rho_{n_i}(I \setminus S_j) \leq q_1$$

for all j and all sufficiently large i (dependent on j). It follows that for all j, if i is sufficiently large,

$$\rho_{n_i}(I \cap S_j) = \rho_{n_i}(I) - \rho_{n_i}(I \setminus S_j) \geq q_2 - q_1 > 0$$

Choose $n > (q_2 - q_1)^{-1}$, and then choose i sufficiently large that the above inequalities hold for all $j < n$. Then

$$\rho_{n_i}(I \cap \cup_{j<n} S_j) = \sum_{j<n} \rho_{n_i}(I \cap S_j) \geq n(q_2 - q_1) > 1$$

which is absurd because densities can never exceed 1. This contradiction proves the lemma. □

Now return to the case where $n = 2e + 1$ in the proof of Lemma 6.5, and assume that the hypotheses of N_{2e+1} are satisfied. Let $S_k = W_{\varphi_e(k)}$. Let $p \in P$ be given, and suppose that $p = (F, I)$. Since $S_0, S_1, \ldots$ are pairwise disjoint, there are only finitely many k such that $S_k \cap F \neq \emptyset$. Hence, by Lemma 6.6, there exists k such that $S_k \cap F = \emptyset$ and $\overline{\rho}(I \setminus S_k) > q_0$. Define $q = (F, I \setminus S_k)$. Then $q \in P$, and q extends p. If A satisfies q, then $A \cap S_k = \emptyset$, so A satisfies R_n □

The proof of part (2) of Theorem 6.4 is now standard. Namely, we recursively choose $p_0, p_1, \ldots$ such that each p_n is in P, p_{n+1} extends p_n for all n, and every set which satisfies p_{n+1} meets the requirement N_n. This is possible by letting $p_0 = (\emptyset, \omega)$ and applying Lemma 6.5. If $p_n = (F_n, I_n)$, let $A = \cup_n F_n$. Then A satisfies every p_n and hence meets every requirement.

□

Theorem 6.7. (i) *If A is r-cohesive, then $\rho(A) = 0$.*

(ii) *There is a cohesive set A such that no c.e. set of density 1 is disjoint from A, and hence A is not generically computable.*

Proof. For (i), note that if A is r-cohesive, and $n > 0$ is given, there exists $i < n$ such that all but finitely many members of A are congruent to i modulo n. Since the set of numbers congruent to i mod n has density $1/n$ by the proof of Lemma 7.2, it follows that the upper density of A is at most $1/n$. Since n was arbitrary, it follows that A has density 0.

For (ii), construct A using Mathias forcing with conditions of positive lower density, i.e. pairs (F, I) such that F is a finite set, I is a set of positive lower density, and $F \cap I = \emptyset$. One proceeds in the style of Theorem 6.4, but the necessary combinatorial lemma asserts that if $\underline{\rho}(I) > 0$ and S is a set, then either $\underline{\rho}(I \cap S) > 0$ or $\underline{\rho}(I \cap \overline{S}) > 0$. This is clear since, for any disjoint sets B and C, $\underline{\rho}(B \cup C) \geq \underline{\rho}(B) + \underline{\rho}(C)$. We leave the further details to the reader.

□

7. The minimal pair problem and relative generic computability

The notion of generic computability can be relativized in the obvious way. Specifically, if $A, C \subseteq \omega$, we define C to be generically A-computable if there is a partial A-computable function ψ such that $\psi(n) = C(n)$ for all n in the domain D of ψ and, further, the domain D has density 1. In this section, we show that if A, B are noncomputable Δ^0_2 sets, there is a set C such that C is generically A-computable and generically B-computable and yet C is not generically computable. After we obtained this result, Gregory Igusa [12] greatly strengthened it by showing that it holds even without the assumption that A and B are Δ^0_2 sets. Thus, there are no minimal pairs for relative generic computability, even though minimal pairs exist in abundance for relative Turing computability, i.e. Turing reducibility. Even though our result has been superseded by Igusa's subsequent work, we include it here because the case where A and B are Δ^0_2 is one case in his proof and in fact is a major stepping stone towards his remarkable result.

Note that relative generic computability is not transitive, as shown in [16], Section 3. A stronger transitive notion called "generic reducibility" is defined in Section 4 of [16], and studied further in [12]. The existence of minimal pairs for generic reducibility remains open.

The following result is fundamental to our approach. Recall that D_n is the finite set with canonical index n. Here in fact it is important that we use the standard canonical indexing, i.e. $D_0 = \emptyset$ and, and if $n_1, n_2, \ldots, n_k$ are distinct nonnegative integers and $n = \sum_{i=1}^{k} 2^{n_i}$, then $D_n = \{n_1, n_2, \ldots, n_k\}$.

Theorem 7.1. *Suppose that A, B are infinite sets such that $A \cup B$ is hyperimmune, and let*

$$C = \{n : D_n \cap (A \cup B) \neq \emptyset\}.$$

Then C is generically A-computable and generically B-computable but not generically computable.

Proof. To prove this result, we need the following lemma.

Lemma 7.2. *Let I be an infinite set. Then $\{n : D_n \cap I \neq \emptyset\}$ has density 1.*

Proof. Note first that, for any $m > 0$ and any b, the set S of numbers congruent to b modulo m has density $1/m$, as one would expect. To see this, let k be any number, and write k as $mb + r$, where $0 \leq r < m$. We then have

$$\rho_k(S) \leq \frac{m+1}{k} \leq \frac{m+1}{mb} = \frac{1+1/m}{b}$$

Further,

$$\rho_k(S) \geq \frac{m}{k} \geq \frac{m}{(m+1)b} = \frac{1}{(1+1/m)b}$$

Since both the upper and lower bounds above on $\rho_k(S)$ approach $1/b$ as k approaches infinity, it follows that $\rho(S) = 1/b$, as claimed.

Now let D be a finite set, and let $T = \{n : D_n \cap D = \emptyset\}$. We now claim that $\rho(T) = 2^{-|D|}$. We may assume without loss of generality that $D \neq \emptyset$. Let $m = \max D$. By our choice of indexing of finite sets, the elements of T are exactly the numbers which have a 0 in places of their binary expansion corresponding to elements of D, so for all a, $a \in T$ iff $(a + 2^{m+1}) \in T$. Hence T is a finite union of residue classes modulo 2^{m+1}. Since each of these residue classes has a density, T has a density. To calculate this density, note that if k is a multiple of 2^{m+1}, then each of the 2^k ways of filling in the places of the binary expansion corresponding to elements of D occurs equally often in numbers below k, so $\rho_k(S) = 2^{-k}$. Since S has a density and $\rho_k(S) = 2^{-k}$ for infinitely many k, we have that $\rho(S) = 2^{-k}$.

It now follows that, for every k, $\{n : D_n \cap I = \emptyset\}$ has upper density at most 2^{-k}, so this set has density 0 and its complement has density 1. $\square$

We now complete the proof of the theorem. Recall that A and B are infinite, $A \cup B$ is hyperimmune, and $C = \{n : D_n \cap (A \cup B) \neq \emptyset\}$. To show that C is generically A-computable, define $\psi(n) = 1$ if $D_n \cap A \neq \emptyset$. Obviously, $\psi(n)\downarrow$, then $\psi(n) = 1 = C(n)$, since $D_n \cap A \neq \emptyset$. Also, the domain of ψ has density 1 by the lemma and the assumption that A is infinite. The proof that C is generically B-computable is the same.

It remains to show that C is not generically computable. Suppose for a contradiction that C were generically computable. Note that $\rho(C) = 1$ by the lemma. If ψ is a computable partial function which witnesses that C is generically computable, then $T = \{n : \psi(n) = 1\}$ is a c.e. set of density 1 which is a subset of C. We now obtain a contradiction by showing that $A \cup B$ is not hyperimmune. Construct a strong array $F_0, F_1, \ldots$ of pairwise disjoint finite sets intersecting $A \cup B$. Suppose that F_i has been defined for all $i < j$. Let m exceed all elements of $\cup_{i<j} F_i$. Let $U = \{n : D_n \cap [0, m) = \emptyset\}$. As shown in the proof of the lemma, $\rho(U) = 2^{-m}$. Since $\rho(T) = 1$, it follows that $T \cap U \neq \emptyset$. By effective search, one can find $n \in T \cap U$. Let $F_j = D_n$. $\square$

Theorem 7.3. *Let A_0 and B_0 be noncomputable Δ^0_2 sets. Then there is a set C which is both generically A_0-computable and generically B_0-computable but is not generically computable.*

Proof. Note that the family of generically A_0-computable sets depends only on the degree of A_0. Hence, By the previous theorem, it suffices to show that any two nonzero degrees $\mathbf{a}, \mathbf{b} \leq \mathbf{0}'$ are *jointly hyperimmune*, meaning that there are sets A, B of degree $\mathbf{a}, \mathbf{b}$ respectively with $A \cup B$ hyperimmune. The following lemma is helpful for this.

Lemma 7.4. *If A is hyperimmune and B is A-hyperimmune, then $A \cup B$ is hyperimmune.*

Proof. Suppose that $A \cup B$ is infinite and not hyperimmune. Then there is a strong array $\{F_n\}$ which witnesses this. If $F_n \cap A \neq \emptyset$ for all but finitely many n, then we can conclude that A is not hyperimmune. Otherwise, there are infinitely many n such that $F_n \cap A = \emptyset$. Then the family of such sets F_n can be made into an A-computable array, and this array witnesses that B is not A-hyperimmune. □

If $\mathbf{a} = \mathbf{b}$, then $\mathbf{a}, \mathbf{b}$ are jointly hyperimmune by the theorem of Miller and Martin [26] that every nonzero degree below $\mathbf{0}'$ is hyperimmune. Otherwise, we may assume without loss of generality that $\mathbf{b} \not\leq \mathbf{a}$. In this case we can argue that $\mathbf{b}$ is $\mathbf{a}$-hyperimmune. By the proof of the Miller-Martin result [26] there is a function f of degree $\mathbf{b}$ such that every function which g which majorizes f can compute f. Since $\mathbf{b} \not\leq \mathbf{a}$, no $\mathbf{a}$-computable function can compute f. By the standard majorization characterization of hyperimmunity, relativized to $\mathbf{a}$, it follows that $\mathbf{b}$ is $\mathbf{a}$-hyperimmune. Hence by the above lemma $\mathbf{a}, \mathbf{b}$ are jointly hyperimmune. As remarked above, this suffices to prove the theorem.

□

8. Absolute undecidability

In this section we mention some results on a very strong form of generic noncomputability introduced by Myasnikov and Rybikov [25]. For comparison, recall that a set A is *generically computable* if there is a partial computable function which agrees with the characteristic function of A on its domain and has a domain of density 1.

Definition 8.1. [25] A set $A \subseteq \omega$ is *absolutely undecidable* if every partial computable function agreeing with the characteristic function of A on its domain has a domain of density 0.

It is clear that no absolutely undecidable set is generically computable and that there are sets which are generically noncomputable but not absolutely undecidable.

As pointed out in [16], Observation 2.11, every nonzero Turing degree contains a set which is not generically computable. Bienvenu, Day, and Hözl [2] have obtained a remarkable generalization of this result.

Theorem 8.1. [2] *There exists a Turing functional Φ such that for every noncomputable set A, Φ^A is absolutely undecidable and truth-table equivalent to A. Hence, every nonzero Turing degree contains an absolutely undecidable set.*

The idea of the proof of [2] is to code A into Φ^A using an error correcting code (the Hadamard code) with sufficient redundancy that, given any partial description of Φ^A defined on a set of positive upper density, it is possible to effectively recover A.

In the other direction, we have the following result, which shows that it is impossible to strengthen Theorem 8.1 by requiring Φ^A or its complement to be immune.

Theorem 8.2. *There is a noncomputable set A such that for every absolutely undecidable set $B \leq_T A$, neither B nor $\overline{B}$ is immune.*

This result is proved by analyzing a version of the construction of a non-computable set of bi-immune free degree [14]. We omit the details. Note that Theorem 8.2 immediately implies the existence of a non-zero bi-immune free degree since every bi-immune set is absolutely undecidable.

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