

# UNTWISTING A TWISTED CALABI-YAU ALGEBRA

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ABSTRACT. Twisted Calabi-Yau algebras are a generalisation of Ginzburg’s notion of Calabi-Yau algebras. Such algebras  $A$  come equipped with a *modular automorphism*  $\sigma \in \text{Aut}(A)$ , the case  $\sigma = \text{id}$  being precisely the original class of Calabi-Yau algebras. Here we prove that every twisted Calabi-Yau algebra may be extended to a Calabi-Yau algebra. More precisely, we show that if  $A$  is a twisted Calabi-Yau algebra with modular automorphism  $\sigma$ , then the smash product algebras  $A \rtimes_{\sigma} \mathbb{N}$  and  $A \rtimes_{\sigma} \mathbb{Z}$  are Calabi-Yau.

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## 1. INTRODUCTION

Twisted Calabi-Yau algebras, that is, algebras  $A$  that satisfy a twisted Poincaré duality in Hochschild (co)homology, have been the object of intense recent study due to their natural prevalence in various flavours of noncommutative geometry, see e.g. [1–3, 7, 15, 18–20, 24]. More precisely, such algebras have isomorphic Hochschild homology and cohomology up to a twist in the coefficients given by some algebra automorphism  $\sigma \in \text{Aut}(A)$ ,

$$H^{\bullet}(A, M) \cong H_{d-\bullet}(A, \sigma M)$$

This generalises Ginzburg’s notion of a Calabi-Yau algebra [10] which cover the case where  $\sigma = \text{id}$ . See the main text for notation and precise definitions.

By analogy with the situation for Poisson manifolds [7] or in Tomita-Takesaki theory [5] it is natural to expect that the smash product  $A \rtimes_{\sigma} \mathbb{Z}$  is an (untwisted) Calabi-Yau algebra. Some partial results in that direction have been obtained in, or can be derived from [8, 20, 24]. However, to the best of our knowledge no

general proof of this fact has been obtained yet. Specifically, we make no assumptions about the existence of a grading on  $A$ , or about properties of the automorphism  $\sigma$ , e.g. finiteness of order or more generally, semisimplicity. Hence the aim of the present paper is to prove:

**Theorem 1.1.** *Let  $A$  be a  $\sigma$ -twisted Calabi-Yau algebra of dimension  $d$ . Then  $A \rtimes_{\sigma} \mathbb{N}$  and  $A \rtimes_{\sigma} \mathbb{Z}$  are Calabi-Yau algebras of dimension  $d + 1$ .*

The structure of the paper is as follows: In the first section, we provide some background on the Hochschild (co)homology of algebras and the cup and cap products. In Section 2, we recall a theorem of Van den Bergh on Poincaré type duality in Hochschild (co)homology and define twisted Calabi-Yau algebras. The third section describes twisted cyclic homology and the underlying notion of a paracyclic module of which the Hochschild complex  $C_{\bullet}(A, {}_{\sigma}A)$  of an algebra  $A$  with coefficients twisted by an algebra automorphism  $\sigma$  is an example. We also show, as a consequence of the basic paracyclic theory, that the Hochschild homology  $H_{\bullet}(A, {}_{\sigma}A)$  is invariant under the natural action of  $\sigma$  for any algebra  $A$  and  $\sigma \in \text{Aut}(A)$  which implies the invariance of the cohomology  $H^{\bullet}(A, M)$  of a twisted Calabi-Yau algebra under the action of the modular automorphism. In Section 4, we introduce the smash products  $A \rtimes_{\sigma} \mathbb{N}$  and  $A \rtimes_{\sigma} \mathbb{Z}$  and recall a result of Farinati [8] which shows that the Calabi-Yau property of  $A \rtimes_{\sigma} \mathbb{Z}$  is implied by that of  $A \rtimes_{\sigma} \mathbb{N}$ . The section is then concluded by the proof of the main theorem.

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## 2. PRELIMINARIES

**2.1. General Notations.** Fix a field  $\mathbb{k}$  of characteristic zero. All algebras under consideration will be unital and associative  $\mathbb{k}$ -algebras. Unadorned  $\otimes$  and  $\text{Hom}$  shall denote  $\otimes_{\mathbb{k}}$  and  $\text{Hom}_{\mathbb{k}}$  respectively. If  $A$  is an algebra, we shall denote by  $A^{\text{op}}$  and  $A^e := A \otimes A^{\text{op}}$  its opposite and enveloping algebras. The antipodal map

$$S: A^e \longrightarrow A^e$$

satisfying  $S(a \otimes b) = (b \otimes a)$  is an involution and induces inverse functors

$$S: A^e\text{-Mod} \rightleftarrows \text{Mod-}A^e: S$$

identifying the categories of left and right  $A^e$ -modules. There are further equivalences between the categories of left and right  $A^e$ -modules and the category of  $A$ -bimodules with a symmetric action of  $\mathbb{k}$ . If  $M$  and  $N$  are left (resp. right)  $A^e$ -modules, we shall denote by  $\triangleright$  and  $\triangleleft$  (resp.  $\blacktriangleright$  and  $\blacktriangleleft$ ) the induced left and right actions of  $A$ .

**2.2. Twisted Bimodules.** Let  $M$  be an  $A$ -bimodule. For each pair of automorphisms  $\rho, \sigma \in \text{Aut}(A)$ , we define the *twisted bimodule*  ${}_{\rho}M_{\sigma}$  to be the  $\mathbb{k}$ -space  $M$  together with the  $A$ -bimodule structure:

$$a \cdot m \cdot b := \rho(a)m\sigma(b), \quad a, b \in A, m \in M$$

If either  $\rho$  or  $\sigma$  is the identity, we shall suppress it from the notation, writing for example  $M_{\sigma}$  instead of  ${}_{\text{id}}M_{\sigma}$ . The following lemma is standard.

**Lemma 2.1.** *Let  $\rho, \sigma, \tau \in \text{Aut}(A)$  be automorphisms. Then*

(i) *The map*

$${}_{\rho}A_{\sigma} \longrightarrow {}_{\tau \circ \rho}A_{\tau \circ \sigma}, \quad a \mapsto \tau(a)$$

*is an isomorphism of bimodules. In particular,  $A_{\sigma} \cong {}_{\sigma^{-1}}A$ .*

(ii) *The bimodules  $A$  and  $A_{\sigma}$  are isomorphic if and only if  $\sigma$  is an inner automorphism.*

*Proof.* See, for example, [3]. □

**2.3. Hochschild Homology.** The *Hochschild homology* of an algebra  $A$  with coefficients in a right  $A^e$ -module  $M$  is given by

$$H_\bullet(A, M) := \text{Tor}_\bullet^{A^e}(M, A)$$

and is realised as the homology of the *Hochschild (chain) complex*  $(C_\bullet(A, M), \mathbf{b}_\bullet)$  where

$$C_n(A, M) := M \otimes A^{\otimes n}$$

and the boundary map  $\mathbf{b}_\bullet$  satisfies

$$\begin{aligned} \mathbf{b}_n(m \otimes a_1 \otimes \cdots \otimes a_n) &= m \blacktriangleleft a_1 \otimes a_2 \otimes \cdots \otimes a_n \\ &+ \sum_{i=1}^{n-1} (-1)^i m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ &+ (-1)^n a_n \blacktriangleright m \otimes a_1 \otimes \cdots \otimes a_{n-1}. \end{aligned}$$

**2.4. Hochschild Cohomology.** The *Hochschild cohomology* of an algebra  $A$  with coefficients in a left  $A^e$ -module  $N$  is

$$H^\bullet(A, N) := \text{Ext}_{A^e}^\bullet(A, N)$$

and is realised as the cohomology of the *Hochschild (cochain) complex*  $(C^\bullet(A, N), \mathbf{b}^\bullet)$  where

$$C^n(A, N) := \text{Hom}_{\mathbb{k}}(A^{\otimes n}, N)$$

and the coboundary map  $\mathbf{b}^\bullet$  satisfies

$$\begin{aligned} \mathbf{b}^n(f)(a_1 \otimes \cdots \otimes a_{n+1}) &= a_1 \triangleright f(a_2 \otimes \cdots \otimes a_{n+1}) \\ &+ \sum_{i=1}^n (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) \\ &+ (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) \triangleleft a_{n+1}. \end{aligned}$$

**2.5. Cup Product.** The Hochschild cohomology  $H^\bullet(A, A)$ , *a priori* just a  $\mathbb{k}$ -space, may also be considered as a graded commutative algebra whose multiplication

$$\cup: H^\bullet(A, A) \otimes H^\bullet(A, A) \longrightarrow H^\bullet(A, A),$$

called the *cup product*, is induced by the associative product

$$C^n(A, A) \otimes C^m(A, A) \longrightarrow C^{n+m}(A, A), \quad f \otimes g \mapsto f \cup g$$

on the Hochschild complex  $C^\bullet(A, A)$  where  $f \cup g: A^{\otimes(n+m)} \longrightarrow A$  is the map satisfying:

$$(f \cup g)(a_1 \otimes \cdots \otimes a_n \otimes a_{n+1} \otimes \cdots \otimes a_{n+m}) := (-1)^{mn} f(a_1 \otimes \cdots \otimes a_n) g(a_{n+1} \otimes \cdots \otimes a_{n+m}).$$

**2.6. Cap Product.** The *cap product* is the map

$$\cap: H_p(A, M) \otimes H^n(A, N) \longrightarrow H_{p-n}(A, M \otimes_A N), \quad p \geq n$$

defined on the level of Hochschild (co)chains by

$$m \otimes a_1 \otimes \cdots \otimes a_p \cap f = (-1)^{pn} m \otimes_A f(a_1 \otimes \cdots \otimes a_n) \otimes a_{n+1} \otimes \cdots \otimes a_p.$$

The tensor product  $M \otimes_A N$  is formed using the right and left actions  $M \blacktriangleleft$  and  $\triangleright N$  respectively and is considered as a right  $A^e$ -module using the remaining actions  $\blacktriangleright M$  and  $N \triangleleft$ .

**Remark 2.2.** Taking the cohomology coefficients in the above to be  $N = A$ , the cap product endows the Hochschild homology  $H_\bullet(A, M)$  with the structure of a (graded) right  $H^\bullet(A, A)$ -module via

$$H_p(A, M) \otimes H^n(A, A) \xrightarrow{\cap} H_{p-n}(A, M \otimes_A A) \xrightarrow{\cong} H_{p-n}(A, M).$$

### 3. DUALITY AND THE FUNDAMENTAL CLASS

**3.1. Van den Bergh's Theorem.** The most general framework for Poincaré type duality in Hochschild (co)homology is provided by a well known theorem of Van den Bergh which we state below. First, we need the following definition.

**Definition 3.1.** An algebra  $A$  is said to be *homologically smooth* if there exists a finite length resolution of the  $A^e$ -module  $A$  by finitely generated projective  $A^e$ -modules.

**Theorem 3.2.** ([27]) *Let  $A$  be a homologically smooth algebra. Assume that there exists an integer  $d \geq 0$  such that  $\text{Ext}_{A^e}^i(A, A^e) \cong 0$  for all  $i \neq d$  and further assume that  $U_A := H^d(A, A^e)$  is an invertible  $A^e$ -module. Then, for all left  $A^e$ -modules  $M$ , there are natural isomorphisms*

$$H^\bullet(A, M) \cong H_{d-\bullet}(A, U_A \otimes_A M).$$

**Remark 3.3.** When the conditions of the theorem are satisfied, the algebra  $A$  is said to have *Van den Bergh duality of dimension  $d$*  and the (right)  $A^e$ -module  $U_A := H^d(A, A^e)$  is called the *dualising bimodule* of  $A$ . In this case, we necessarily have that  $d$  is equal to the dimension  $\dim(A)$  of  $A$  which is, by definition, the projective dimension of  $A$  as an  $A^e$ -module (see [4]).

**3.2. The Fundamental Class.** For an  $n$ -dimensional closed orientable manifold  $M$ , the Poincaré duality isomorphism

$$H^\bullet(M) \cong H_{n-\bullet}(M)$$

in singular cohomology is given by taking the cap product with a fundamental class  $[M] \in H_n(M)$ . In [17] and [13] it is shown that the dualising isomorphism in the (co)homology of an algebra with Van den Bergh duality may be realised analogously.

First, we must define the fundamental class. Given any algebra  $A$  and any integer  $d \geq 0$ , abbreviate  $U_A := H^d(A, A^e)$  as in the statement of Theorem 2.2 (however, for the moment we do not assume any of the further conditions to be satisfied). Then, the cap product

$$H_d(A, U_A) \otimes U_A \longrightarrow H_0(A, U_A \otimes_A A^e) \cong U_A$$

provides a  $\mathbb{k}$ -linear map

$$F: H_d(A, U_A) \longrightarrow \text{Hom}_{A^e}(U_A, U_A), \quad F(z) = (z \cap -).$$

In [17], it is proven that if  $A$  has Van den Bergh duality of dimension  $d$ , then  $F$  is an isomorphism. One then defines the *fundamental class* of  $A$  to be the unique element  $\omega_A \in H_d(A, U_A)$  such that  $F(\omega_A) = \text{id}$ . Using this terminology, we then have:

**Proposition 3.4.** *If  $A$  is an algebra with Van den Bergh duality of dimension  $d$  and  $M$  is any left  $A^e$ -module, then*

$$\omega_A \cap - : H^\bullet(A, M) \longrightarrow H_{d-\bullet}(A, U_A \otimes_A M)$$

*is an isomorphism.*

*Proof.* See [17] or [13]. □

**Remark 3.5.** Taking  $M = A$ , the proposition says that  $H_\bullet(A, U_A)$  is freely generated as a  $H^\bullet(A, A)$ -module by the fundamental class  $\omega_A$ .

**3.3. Twisted Calabi-Yau Algebras.** In [10], Ginzburg introduced the study of Calabi-Yau algebras, a noncommutative generalisation of the coordinate rings of Calabi-Yau varieties. Specifically, a Calabi-Yau algebra is an algebra  $A$  with Van den Bergh duality whose dualising bimodule  $U_A$  is isomorphic to  $A$  as right  $A^e$ -modules. More generally, we have:

**Definition 3.6.** An algebra  $A$  is said to be  *$\sigma$ -twisted Calabi-Yau* if it has Van den Bergh duality and the dualising bimodule  $U_A$  is isomorphic to  $A_\sigma$  for some  $\sigma \in \text{Aut}(A)$ . The automorphism  $\sigma$  is variously called the *modular* or *Nakayama automorphism*<sup>1</sup> of  $A$ .

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<sup>1</sup>The term *modular automorphism* was used in [7] since in the case of a deformation quantization of a Poisson variety, the modular automorphism quantizes the flow of the modular vector field of the Poisson structure whereas the authors in [3] use the term *Nakayama automorphism* since for a Frobenius algebra, it coincides with the classical Nakayama automorphism.

**Remark 3.7.** Observe that the modular automorphism is only unique up to inner automorphisms. If  $\sigma$  is the identity (or more generally an inner automorphism), then the algebra  $A$  is a Calabi-Yau algebra in the sense of Ginzburg.

**Remark 3.8.** If  $A$  is a  $\sigma$ -twisted Calabi-Yau algebra, any choice of isomorphism  $U_A \rightarrow \sigma^{-1}A$  identifies the fundamental class  $\omega_A \in H_d(A, U_A)$  with an element of  $H_d(A, \sigma^{-1}A)$  which we shall refer to as  $a$  fundamental class for  $A$  and by abuse of notation, shall denote by  $\omega_A$ .

The twisted Calabi-Yau condition appears to have first been explicitly defined in [3] where the authors used the term *rigid Gorenstein*. They showed that a wide class of noetherian Hopf algebras, including for example, the quantised function algebras  $\mathcal{O}_q(G)$  of connected complex semisimple algebraic groups  $G$ , are what we now refer to as twisted Calabi-Yau. Other examples of twisted Calabi-Yau algebras include quantum homogeneous spaces such as Podleś quantum 2-sphere [15], the deformation quantization algebra of a Calabi-Yau Poisson variety [7], Koszul algebras whose Koszul dual is Frobenius [27], group algebras of Poincaré duality groups (e.g. the group algebras of fundamental groups of closed aspherical manifolds) [17] and AS-regular algebras [28].

#### 4. TWISTED CYCLIC HOMOLOGY AND THE PARACYCLIC STRUCTURE OF $C_\bullet(A, \sigma A)$

**4.1. (Twisted) Cyclic Homology.** The cyclic cohomology of a noncommutative algebra was introduced in the early 1980s independently by Connes [6] and Tsygan [26]. Shortly afterwards, the corresponding theory of cyclic homology was formulated by Loday and Quillen [22]. The cyclic homology of an algebra  $A$  is based on the *cyclic operator*

$$\mathbf{t}_\bullet: C_\bullet(A, A) \rightarrow C_\bullet(A, A)$$

on the Hochschild complex whose  $n$ th degree component is given by the signed permutation

$$a_0 \otimes a_1 \otimes \cdots \otimes a_n \mapsto (-1)^n a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}.$$

This endows the simplicial module  $C_\bullet(A, A)$  with the additional structure of a *cyclic module*. Associated to this structure is *Connes-Tsygan boundary map*  $\mathbf{B}: C_\bullet(A, A) \rightarrow C_{\bullet+1}(A, A)$  which anticommutes with the Hochschild boundary  $\mathbf{b}_\bullet$  and leads to Connes' mixed  $(\mathbf{b}, \mathbf{B})$ -bicomplex whose total homology is the cyclic homology  $HC_\bullet(A)$  of  $A$ . We refer the reader to [21] for more detail and precise definitions.

In [16], in order to develop an appropriate analogue of the trace of a  $C^*$ -algebra for the compact quantum groups of Woronowicz, the authors defined the twisted Hochschild and cyclic cohomologies of an algebra  $A$  relative to an automorphism  $\sigma \in \text{Aut}(A)$ . These extend the usual Hochschild and cyclic cohomology of  $A$ , which may then be viewed as the corresponding twisted theories relative to the automorphism  $\sigma = \text{id}$ .

The dual theories of twisted Hochschild and cyclic homology were then observed to be often less degenerate than the untwisted theories and in particular to avoid the so called 'dimension drop' in Hochschild homology that occurs in examples of quantum deformations. More precisely, for  $q = 1$  the only automorphism for which the twisted Hochschild homology of  $\mathcal{O}_q(SL_2)$  does not vanish in degree 3 is by the Hochschild-Kostant-Rosenberg theorem the identity. However, for  $q$  not a root of unity this happens precisely for the positive powers of the modular automorphism of  $\mathcal{O}_q(SL_2)$ , see [12]. This phenomenon has also been observed in similar examples, see e.g. [?Sit].

**4.2. Paracyclic  $\mathbb{k}$ -modules.** We now define the notion of a paracyclic module [9] (introduced independently under the name 'duplicial module' in [?DwyKa]) which generalises cyclic modules and underlies twisted cyclic homology. First, we recall the following:

**Definition 4.1.** A *simplicial  $\mathbb{k}$ -module* is a graded  $\mathbb{k}$ -module  $C_\bullet = \bigoplus_{n \in \mathbb{N}} C_n$  together with maps

$$\mathbf{d}_{n,i}: C_n \rightarrow C_{n-1}, \quad \mathbf{s}_{n,j}: C_n \rightarrow C_{n+1}$$

for all  $n \in \mathbb{N}$  and  $0 \leq i, j \leq n$  such that

$$\mathbf{d}_{n-1,i} \mathbf{d}_{n,j} = \mathbf{d}_{n-1,j-1} \mathbf{d}_{n,i}, \quad i < j, \quad \mathbf{s}_{n-1,i} \mathbf{s}_{n,j} = \mathbf{s}_{n-1,j+1} \mathbf{s}_{n,i}, \quad i \leq j,$$

$$\mathbf{d}_{n+1,i} \mathbf{s}_{n,j} = \begin{cases} \mathbf{s}_{n-1,j-1} \mathbf{d}_{n,i}, & i < j \\ \text{id}, & i = j, j+1 \\ \mathbf{s}_{n-1,j} \mathbf{d}_{n,i-1}, & i > j+1. \end{cases}$$

**Definition 4.2.** A *paracyclic*  $\mathbb{k}$ -module is a simplicial  $\mathbb{k}$ -module  $C_\bullet$  together with maps  $\mathfrak{t}_n : C_n \rightarrow C_n$  for all  $n \in \mathbb{N}$  satisfying

$$\mathfrak{d}_{n,i}\mathfrak{t}_n = \begin{cases} -\mathfrak{t}_{n-1}\mathfrak{d}_{n,i-1}, & 1 \leq i \leq n \\ (-1)^n \mathfrak{d}_{n,n}, & i = 0 \end{cases}, \quad \mathfrak{s}_{n,i}\mathfrak{t}_n = \begin{cases} -\mathfrak{t}_{n+1}\mathfrak{s}_{n,i-1}, & 1 \leq i \leq n \\ (-1)^n \mathfrak{t}_{n+1}^2 \mathfrak{s}_{n,n}, & i = 0. \end{cases}$$

A *cyclic*  $\mathbb{k}$ -module is a paracyclic  $\mathbb{k}$ -module  $C_\bullet$  such that  $\mathfrak{T}_n := \mathfrak{t}_n^{n+1} = \text{id}$ . In particular, for every paracyclic  $\mathbb{k}$ -module  $C_\bullet$ , we define the *associated cyclic module*  $C_\bullet^{\text{cyc}} := C_\bullet / \text{im}(\text{id} - \mathfrak{T}_\bullet)$ .

The motivating example of a paracyclic module is the Hochschild complex  $C_\bullet(A, \sigma A)$  associated to an algebra  $A$  together with an automorphism  $\sigma \in \text{Aut}(A)$ . Indeed, one easily checks the paracyclic relations to prove:

**Proposition 4.3.** *The Hochschild complex  $C_\bullet(A, \sigma A)$  together with the maps  $(\mathfrak{d}_{n,i}, \mathfrak{s}_{n,j}, \mathfrak{t}_n)$  satisfying*

$$\mathfrak{d}_{n,i}(a_0 \otimes \cdots \otimes a_n) = \begin{cases} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n, & 0 \leq i < n \\ \sigma(a_n) a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}, & i = n \end{cases}$$

$$\mathfrak{s}_{n,i}(a_0 \otimes \cdots \otimes a_n) = a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n$$

$$\mathfrak{t}_n(a_0 \otimes \cdots \otimes a_n) = (-1)^n \sigma(a_n) \otimes a_0 \otimes \cdots \otimes a_{n-1}$$

is a paracyclic module. □

**4.3. The Homotopy Formula and Quasicyclic Modules.** For any paracyclic module  $C_\bullet$  we define the following maps:

(i) The *simplicial and acyclic boundaries*

$$\mathfrak{b}_n := \sum_{i=0}^n (-1)^i \mathfrak{d}_{n,i} \quad \text{and} \quad \mathfrak{b}'_n := \sum_{i=0}^{n-1} (-1)^i \mathfrak{d}_{n,i};$$

(ii) The *norm operator*

$$\mathfrak{N}_n := \sum_{i=0}^n \mathfrak{t}_n^i;$$

(iii) The *extra degeneracy*

$$\mathfrak{s}_n := (-1)^{n+1} \mathfrak{t}_{n+1} \mathfrak{s}_{n,n};$$

(iv) The *Connes-Tsygan boundary map*

$$\mathfrak{B}_n := (\text{id} - \mathfrak{t}_{n+1}) \mathfrak{s}_n \mathfrak{N}_n.$$

Unlike as is the case for cyclic modules, for a paracyclic module the operator  $\mathfrak{B}_\bullet$  does in general neither anticommute with the simplicial boundary  $\mathfrak{b}_\bullet$ , nor does it square to zero. However, we do have the following:

**Proposition 4.4** ([9, Theorem 2.3 (2)]). *For any paracyclic module  $C_\bullet$ , the simplicial boundary  $\mathfrak{b}_\bullet$  and the Connes-Tsygan boundary map  $\mathfrak{B}_\bullet$  satisfy the relations*

$$\mathfrak{b}_{n+1} \mathfrak{B}_n + \mathfrak{B}_{n-1} \mathfrak{b}_n = \text{id} - \mathfrak{T}_n, \quad \mathfrak{B}_{n+1} \mathfrak{B}_n = (\text{id} - \mathfrak{T}_{n+2})(\text{id} - \mathfrak{t}_{n+2}) \mathfrak{s}_{n+1} \mathfrak{s}_n \mathfrak{N}_n.$$

*Proof.* One establishes the formula directly using the subsidiary relations

$$\mathfrak{b}_n(\text{id} - \mathfrak{t}_n) = (\text{id} - \mathfrak{t}_{n-1}) \mathfrak{b}'_n, \quad \mathfrak{b}'_{n+1} \mathfrak{s}_n + \mathfrak{s}_{n-1} \mathfrak{b}'_n = \text{id}, \quad \mathfrak{b}'_n \mathfrak{N}_n = \mathfrak{N}_{n-1} \mathfrak{b}_n$$

which are easily verified by computation. □

The proposition shows that for any paracyclic module  $C_\bullet$ , the images of  $\mathfrak{b}_\bullet$  and  $\mathfrak{B}_\bullet$  on the associated cyclic complex  $C_\bullet^{\text{cyc}}$  anticommute. The cyclic homology of  $C_\bullet$  is then defined to be the total homology of



**Remark 5.1.** For all smash products  $A \rtimes_{\sigma} \mathbb{N}$ , there is an isomorphism

$$A \rtimes_{\sigma} \mathbb{N} \xrightarrow{\cong} A[x; \sigma], \quad a \otimes x^i \mapsto ax^i$$

where the *skew polynomial algebra*  $A[x; \sigma]$  is formed by adjoining the variable  $x$  to  $A$  subject to the commutation relation  $xa = \sigma(a)x$ . Similarly, the smash products  $A \rtimes_{\sigma} \mathbb{Z}$  are isomorphic to the *skew Laurent algebras*  $A[x^{\pm 1}; \sigma]$ . Whenever it causes no confusion, we shall tacitly make use of these identifications writing, for example  $x$  to denote  $1 \otimes x \in A \rtimes_{\sigma} \mathbb{N}$ .

**Remark 5.2.** The smash products  $A \rtimes_{\sigma} \mathbb{Z}$  are examples of the general notion of the smash product  $A \rtimes H$  of a Hopf algebra  $H$  with an  $H$ -module algebra  $A$ . We refer to [23] for a full definition. This construction still makes sense if  $H$  is only a bialgebra which justifies our use of terminology in calling  $A \rtimes_{\sigma} \mathbb{N}$  a smash product.

**5.2. Noncommutative Localisation.** The following is a short excursus, recalling a general notion of non-commutative localisation proposed by Farinati in [8], of which the algebra extension  $A \rtimes_{\sigma} \mathbb{N} \hookrightarrow A \rtimes_{\sigma} \mathbb{Z}$  is an example. It is also proven in [8] that Van den Bergh duality is stable under such localisation and furthermore the dualising module of a localisation is explicitly described. In particular, it follows that the Calabi-Yau property of  $A \rtimes_{\sigma} \mathbb{Z}$  is implied by that of  $A \rtimes_{\sigma} \mathbb{N}$ .

**Definition 5.3.** A map of  $\mathbb{k}$ -algebras  $A \rightarrow B$  is a *localisation* (in the sense of Farinati) if

- (i) The map  $A \rightarrow B$  is flat; that is  $B \otimes_A -$  and  $- \otimes_A B$  are exact functors
- (ii) Multiplication in  $B$  induces an isomorphism  $B \otimes_A B \rightarrow B$  of  $B^e$ -modules.

**Theorem 5.4.** *Let  $A$  be an algebra with Van den Bergh duality of dimension  $d$  and dualising bimodule  $U_A$  and let  $A \rightarrow B$  be a localisation such that  $B \otimes_A U_A \cong U_A \otimes_A B$  as  $A^e$ -modules. Then,  $B$  has Van den Bergh duality of dimension  $d$  with dualising bimodule  $B \otimes_A U_A \otimes_A B$ .  $\square$*

**Corollary 5.5.** *If  $A \rtimes_{\sigma} \mathbb{N}$  is Calabi-Yau, then  $A \rtimes_{\sigma} \mathbb{Z}$  is Calabi-Yau.*

*Proof.* That  $A \rtimes_{\sigma} \mathbb{N} \rightarrow A \rtimes_{\sigma} \mathbb{Z}$  is a localisation is [8, Example 8.]. The conclusion then follows from the description of dualising bimodule provided by the theorem.  $\square$

**5.3. The Calabi-Yau property of  $A \rtimes_{\sigma} \mathbb{N}$ .** In this section we shall prove that the smash product  $A \rtimes_{\sigma} \mathbb{N}$  of a  $\sigma$ -twisted Calabi-Yau algebra  $A$  is Calabi-Yau using the following result:

**Theorem 5.6** ([20, Propositions 3.1. and 3.2.]). *Let  $A$  be a  $\sigma$ -twisted Calabi-Yau algebra of dimension  $d$ . Then the smash product  $A \rtimes_{\sigma} \mathbb{N}$  is homologically smooth and*

$$H^{\bullet}(A \rtimes_{\sigma} \mathbb{N}, (A \rtimes_{\sigma} \mathbb{N})^e) \cong \begin{cases} H^d(A, A \otimes A_{\sigma^{-1}}) \otimes \mathbb{k}[x], & \bullet = d + 1 \\ 0, & \bullet \neq d + 1 \end{cases}$$

as  $(A \rtimes_{\sigma} \mathbb{N})^e$ -modules where the actions of  $A \rtimes_{\sigma} \mathbb{N}$  on  $H^d(A, A \otimes A_{\sigma^{-1}}) \otimes \mathbb{k}[x]$  are given by

$$\begin{aligned} a \blacktriangleright ([f] \otimes x^k) &= \sigma^{-k}(a) \cdot [f] \otimes x^k \\ x \blacktriangleright ([f] \otimes x^k) &= [f] \otimes x^{k+1} \\ ([f] \otimes x^k) \blacktriangleleft a &= [f] \cdot a \otimes x^k \\ ([f] \otimes x^k) \blacktriangleleft x &= [(\sigma^{-1})^{\otimes 2} \circ f \circ \sigma^{\otimes d}] \otimes x^{k+1}. \end{aligned}$$

Here the actions of  $A$  on  $H^d(A, A \otimes A_{\sigma^{-1}})$  are induced by those of the right  $A^e$ -module structure on the coefficients  $A \otimes A_{\sigma^{-1}}$ .  $\square$

**Remark 5.7.** It follows from this theorem that  $A \rtimes_{\sigma} \mathbb{N}$  is twisted Calabi-Yau and that the modular automorphism extends the identity automorphism of  $A$ . Such automorphisms of  $A \rtimes_{\sigma} \mathbb{N}$  are parameterised by central units  $u$  of  $A$  where the automorphism associated to  $u$  satisfies  $x \mapsto ux$  (cf. [8, Proposition. 22.]). It therefore remains to show that the modular automorphism is the one corresponding to  $u = 1$ .

*Proof of Theorem 1.1.* Fix an isomorphism  $U_A \rightarrow \sigma^{-1}A$ . Recall that the choice of such an isomorphism fixes a fundamental class  $\omega_A \in H_d(A, \sigma^{-1}A)$ . The cap product with  $\omega_A$  thus yields a  $\mathbb{k}$ -linear isomorphism

$$H^d(A, A \otimes A_{\sigma^{-1}}) \otimes \mathbb{k}[x] \xrightarrow{(\omega_A \cap -) \otimes \text{id}} H_0(A, \sigma^{-1}A \otimes A_{\sigma^{-1}}) \otimes \mathbb{k}[x] \xrightarrow{([a \otimes b] \mapsto ba) \otimes \text{id}} A \otimes \mathbb{k}[x].$$

The  $(A \rtimes_{\sigma} \mathbb{N})^e$ -module structure on  $H^d(A, A \otimes A_{\sigma^{-1}}) \otimes \mathbb{k}[x]$  from Theorem 5.6 then induces a right action of  $(A \rtimes_{\sigma} \mathbb{N})^e$  on  $A \otimes \mathbb{k}[x]$  by transport of structure: it is clear that for all  $(a \otimes x^k) \in A \otimes \mathbb{k}[x]$  and  $b \in A$ , this action is given by

$$\begin{aligned} b \blacktriangleright (a \otimes x^k) &= \sigma^{-k}(b)a \otimes x^k \\ x \blacktriangleright (a \otimes x^k) &= a \otimes x^{k+1} \\ (a \otimes x^k) \blacktriangleleft b &= ab \otimes x^k \end{aligned}$$

together with the right action of  $x$  on  $A \otimes \mathbb{k}[x]$  that we shall now determine.

Pick a cycle  $z \in C_d(A, \sigma^{-1}A)$  such that  $\omega_A = [z]$ . Then, we have the commutative diagram of (co)chain complexes:

$$\begin{array}{ccc} C^{\bullet}(A, A \otimes A_{\sigma^{-1}}) \otimes \mathbb{k}[x] & \xrightarrow{(z \cap -) \otimes \text{id}} & C_{d-\bullet}(A, \sigma^{-1}A \otimes A_{\sigma^{-1}}) \otimes \mathbb{k}[x] \\ \downarrow (\sigma^{-1})^{\otimes 2} \circ \sigma^{\otimes d} \otimes \text{id} & & \downarrow (\sigma^{-1})^{\otimes d-\bullet+2} \otimes \text{id} \\ C^{\bullet}(A, A \otimes A_{\sigma^{-1}}) \otimes \mathbb{k}[x] & \xrightarrow{(\mathbb{T}_d(z) \cap -) \otimes \text{id}} & C_{d-\bullet}(A, \sigma^{-1}A \otimes A_{\sigma^{-1}}) \otimes \mathbb{k}[x] \end{array}$$

where  $\mathbb{k}[x]$  is viewed as a (co)chain complex concentrated in degree 0 and  $\mathbb{T}_d(z) = (\sigma^{-1})^{\otimes d}(z)$  comes from the paracyclic structure on  $C_{\bullet}(A, \sigma^{-1}A)$  described in Proposition 4.3. Now Corollary 4.5 gives

$$[\mathbb{T}_d(z)] = \omega_A,$$

so the induced diagram on (co)homology takes in degree  $d$  the form

$$\begin{array}{ccc} H^d(A, A \otimes A_{\sigma^{-1}}) \otimes \mathbb{k}[x] & \xrightarrow{(\omega_A \cap -) \otimes \text{id}} & H_0(A, \sigma^{-1}A \otimes A_{\sigma^{-1}}) \otimes \mathbb{k}[x] \xrightarrow{([a \otimes b] \mapsto ba) \otimes \text{id}} & A \otimes \mathbb{k}[x] \\ \blacktriangleleft x \downarrow & & & \downarrow \sigma^{-1} \otimes \text{id} \\ H^d(A, A \otimes A_{\sigma^{-1}}) \otimes \mathbb{k}[x] & \xrightarrow{(\omega_A \cap -) \otimes \text{id}} & H_0(A, \sigma^{-1}A \otimes A_{\sigma^{-1}}) \otimes \mathbb{k}[x] \xrightarrow{([a \otimes b] \mapsto ba) \otimes \text{id}} & A \otimes \mathbb{k}[x] \end{array}$$

Consequently, the right action of  $x$  on  $A \otimes \mathbb{k}[x]$  is given by

$$(a \otimes x^k) \blacktriangleleft x = \sigma^{-1}(a) \otimes x^{k+1}.$$

Finally, the bijection

$$A \otimes \mathbb{k}[x] \rightarrow A \rtimes_{\sigma} \mathbb{N}, \quad a \otimes x^k \mapsto \sigma^k(a) \otimes x^k$$

yields an isomorphism  $H^{d+1}(A \rtimes_{\sigma} \mathbb{N}, (A \rtimes_{\sigma} \mathbb{N})^e) \cong A \rtimes_{\sigma} \mathbb{N}$  as right  $(A \rtimes_{\sigma} \mathbb{N})^e$ -modules thus completing the proof.  $\square$

**Remark 5.8.** In [8], using a spectral sequence due to Stefan [25], it is shown that if  $H$  is a Calabi-Yau Hopf algebra and  $A$  is an  $H$ -module algebra with Van den Bergh duality, then the smash product  $A \rtimes H$  also has Van den Bergh duality although the argument used there does not completely determine the dualising bimodule  $U_{A \rtimes H}$ . Considering the special case where  $H = \mathbb{k}[\mathbb{Z}]$  and  $A$  is a  $\sigma$ -twisted Calabi-Yau algebra whose  $H$ -module structure is given by the modular automorphism  $\sigma$ , a straightforward adaptation of the main argument in the proof above shows that the terms  $H^{\bullet}(A, (A \rtimes_{\sigma} \mathbb{Z})^e)$  of the spectral sequence are invariant under  $\sigma$ , which can be used to fully determine the dualising module and show that it is indeed isomorphic to  $A \rtimes_{\sigma} \mathbb{Z}$ . The same result does not directly apply to smash products with the Calabi-Yau bialgebra  $\mathbb{k}[\mathbb{N}]$  since in particular it does not satisfy the hypotheses of Stefan's spectral sequence. However, one can still use the arguments of [25] to show that a similar spectral sequence for smash products  $A \rtimes_{\sigma} \mathbb{N}$  exists and then the proof in [8] proceeds in the same way.

However, referring to Theorem 5.6 allowed us to formulate the proof in a shorter and more direct way. The results in [20] show more generally that any Ore extension  $A[x; \sigma, \delta]$  of a twisted Calabi-Yau algebra  $A$  is also twisted Calabi-Yau. The authors prove this result by means of a bicomplex computing  $A[x; \sigma, \delta]$ , based

on one from [11]. Similarly as with [8], the proof does not explicitly determine the modular automorphism. In the case where  $\delta = 0$  so that the Ore extension  $A[x; \sigma, \delta]$  is the skew polynomial ring  $A[x; \sigma] \cong A \rtimes_{\sigma} \mathbb{N}$ , this bicomplex is an explicit construction of the spectral sequence for  $A \rtimes_{\sigma} \mathbb{N}$  along the lines of Stefan's one as mentioned above. Specialising further to the case where  $\sigma$  is the modular automorphism of  $A$ , one obtains Theorem 5.6 as stated here.

**Remark 5.9.** One interesting feature of the actions of  $A \rtimes_{\sigma} \mathbb{N}$  on  $A \otimes \mathbb{k}[x]$  is that they would appear to be more naturally stated as a left rather than right  $(A \rtimes_{\sigma} \mathbb{N})^e$ -module structure. Although this distinction may seem unimportant, it becomes more pronounced when one views Hochschild (co)homology as an instance of the (co)homology of Hopf algebroids as in [13]. The duality isomorphism takes right cohomology modules and produces left modules in homology. We refer the interested reader to [13, 14] and the references contained therein for further details.

## REFERENCES

- [1] Roland Berger and Andrea Solotar, *A criteria for homogeneous potentials to be 3-Calabi-Yau*, Preprint, arXiv:math/1203.3029 (2013).
- [2] Raf Bocklandt, Travis Schedler, and Michael Wemyss, *Superpotentials and higher order derivations*, J. Pure Appl. Algebra **214** (2010), no. 9, 1501–1522.
- [3] K. A. Brown and J. J. Zhang, *Dualising complexes and twisted Hochschild (co)homology for Noetherian Hopf algebras*, J. Algebra **320** (2008), 1814–1850.
- [4] Henri Cartan and Samuel Eilenberg, *Homological algebra*, Princeton University Press, Princeton, N. J., 1956.
- [5] Alain Connes, *Une classification des facteurs de type III*, Ann. Sci. ?Ecole Norm. Sup. **6** (1973), 133–252.
- [6] ———, *Cohomologie cyclique et foncteurs  $\text{Ext}^n$* , C. R. Acad. Sci. Paris Sér. I Math. **296** (1983), no. 23, 953–958 (French, with English summary).
- [7] Vasilij Dolgushev, *The Van den Bergh duality and the modular symmetry of a Poisson variety*, Selecta Math. (N.S.) **14** (2009), no. 2, 199–228.
- [8] Marco Farinati, *Hochschild duality, localization, and smash products*, J. Algebra **284** (2005), no. 1, 415–434.
- [9] Ezra Getzler and John D. S. Jones, *The cyclic homology of crossed product algebras*, J. Reine Angew. Math. **445** (1993), 161–174.
- [10] V. Ginzburg, *Calabi-Yau Algebras*, Preprint, arXiv:math/0612139 (2006).
- [11] Jorge A. Guccione and Juan J. Guccione, *Hochschild and cyclic homology of Ore extensions and some examples of quantum algebras*, K-Theory **12** (1997), no. 3, 259–276.
- [12] Tom Hadfield and Ulrich Krähmer, *Twisted homology of quantum  $\text{SL}(2)$* , K-Theory **34** (2005), no. 4, 327–360.
- [13] Niels Kowalzig and Ulrich Krähmer, *Duality and products in algebraic (co)homology theories*, J. Algebra **323** (2010), no. 7, 2063–2081.
- [14] ———, *Batalin-Vilkovisky Structures on Ext and Tor*, to appear in J. Reine Angew. Math.
- [15] Ulrich Krähmer, *On the Hochschild (co)homology of quantum homogeneous spaces*, Israel J. Math. **189** (2012), 237–266.
- [16] J. Kustermans, G. J. Murphy, and L. Tuset, *Differential calculi over quantum groups and twisted cyclic cocycles*, J. Geom. Phys. **44** (2003), no. 4, 570–594.
- [17] Thierry Lambre, *Dualité de Van den Bergh et structure de Batalin-Vilkoviskii sur les algèbres de Calabi-Yau*, J. Noncommut. Geom. **4** (2010), no. 3, 441–457 (French, with English and French summaries).
- [18] Stéphane Launois and Lionel Richard, *Twisted Poincaré duality for some quadratic Poisson algebras*, Lett. Math. Phys. **79** (2007), no. 2, 161–174.
- [19] L.-Y. Liu and Q.-S. Wu, *Rigid dualizing complexes over quantum homogeneous spaces*, J. Algebra **353** (2012), 121–141, DOI 10.1016/j.jalgebra.2011.12.007. MR2872439
- [20] L.-Y. Liu, S.-Q. Wang, and Q.-S. Wu, *Twisted Calabi-Yau Property of Ore Extensions*, Preprint, arXiv:math/1205.0893 (2012).
- [21] Jean-Louis Loday, *Cyclic homology*, 2nd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 301, Springer-Verlag, Berlin, 1998. Appendix E by María O. Ronco; Chapter 13 by the author in collaboration with Teimuraz Pirashvili.
- [22] Jean-Louis Loday and Daniel Quillen, *Homologie cyclique et homologie de l'algèbre de Lie des matrices*, C. R. Acad. Sci. Paris Sér. I Math. **296** (1983), no. 6, 295–297 (French, with English summary).
- [23] Susan Montgomery, *Hopf algebras and their actions on rings*, CBMS Regional Conference Series in Mathematics, vol. 82, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1993.
- [24] Manuel Reyes, Daniel Rogalski, and James Zhang, *Skew Calabi-Yau algebras and homological identities*, Preprint, arXiv:math/1302.0437 (2013).
- [25] Dragoş Ştefan, *Hochschild cohomology on Hopf Galois extensions*, J. Pure Appl. Algebra **103** (1995), no. 2, 221–233.
- [26] B. L. Tsygan, *Homology of matrix Lie algebras over rings and the Hochschild homology*, Uspekhi Mat. Nauk **38** (1983), no. 2(230), 217–218 (Russian).
- [27] Michel van den Bergh, *A relation between Hochschild homology and cohomology for Gorenstein rings*, Proc. Amer. Math. Soc. **126** (1998), no. 5, 1345–1348, Erratum: Proc. Amer. Math. Soc. **130**, no. 9, 2809?2810 (electronic) (2002).

- [28] Amnon Yekutieli and James J. Zhang, *Homological transcendence degree*, Proc. London Math. Soc. (3) **93** (2006), no. 1, 105–137.

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