COLOR ALGEBRA IN QUANTUM CHROMODYNAMICS

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Abstract. Quantum chromodynamics (QCD) is a theory for the strong interaction, one of the four fundamental interactions of Nature, based on the color quantum index. In this article we introduce a color algebra and its mathematical theory. This theory offers a mathematical foundation for QCD, is used to resolve a number of difficulties encountered in QCD, provides a good explanation of gluon radiations, and gives detailed structures of mediator clouds around sub-atomic particles.

Contents

1. Introduction and Motivations

There are four fundamental interactions of Nature: the strong interaction, the electromagnetic interaction, the weak interaction and the gravity. The current theory for strong interaction is the quantum chromodynamics (QCD), which is an $SU(3)$ YangMills gauge theory of colored fermions (the quarks); see among others [2, 5] and the references therein. QCD theory is based on the color quantum number which was first introduced by O. Greenberg [1] to solve the inconsistency between

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the Pauli exclusion principle and the constituents of some spin-3/2 baryons such as

$$\triangle^{++} = uuu(\uparrow\uparrow\uparrow, \downarrow\downarrow\downarrow), \quad \triangle^{-} = ddd(\uparrow\uparrow\uparrow, \downarrow\downarrow\downarrow), \quad \Omega^{-} = sss(\uparrow\uparrow\uparrow, \downarrow\downarrow\downarrow).$$

With the color quantum number, one important component of QCD is an $SU(3)$ gauge theory, demonstrating the existence of eight spin-1, massless and electric-neutral gluons g^k $(k = 1, \cdots, 8)$.

Basically, each flavor of quarks is endowed with three different colors: red r, green g, and blue b, and the corresponding antiparticle possesses three anti-colors: $\bar{r}$, $\bar{g}$ and $\bar{b}$:

$$q_r, \quad q_g, \quad q_b, \quad \bar{q}_r, \quad \bar{q}_g, \quad \bar{q}_b. \tag{1.1}$$

These colors obey the following color neutral laws of color group operation:

$$r\bar{r} = g\bar{g} = b\bar{b} = w, \quad rgb = w, \tag{1.2}$$

where w is white. These color neutral laws are based on the fact that the hadrons are colorless (white), and the constituents of a baryon are given by

$$\text{baryon} = q_{1r}q_{2q}q_{3b}, \qquad \text{anti-baryon} = \bar{q}_{1r}\bar{q}_{2q}\bar{q}_{3b} \tag{1.3}$$

and the constituents of a meson are given by

$$\text{meson} = q_{1r}\bar{q}_{2r}, \quad q_{1g}\bar{q}_{2g}, \quad \text{or} \quad q_{1b}\bar{q}_{2b}. \tag{1.4}$$

As we shall see in the beginning of Section 2, the above color neutral laws lead quickly that one cannot define the group operation of colors and anti-colors, i.e.

$$c_i\bar{c}_j \neq \text{ any of } r, g, b, \bar{r}, \bar{g}, \bar{b} \quad \text{for } i \neq j, \tag{1.5}$$

where $c_1 = r$, $c_2 = g$, and $c_3 = b$.

The lack of proper color group operation in (1.5) clearly demonstrates that a complete and consistent algebraic system for the color quantum number is still not available. The main objectives of this article are

(1) to introduce a color group CG with two additional colors, which we call yellow y and anti-yellow $\bar{y}$ added to the three basic colors r, g, b and their anti-colors $\bar{r}$, $\bar{g}$, $\bar{b}$,
(2) to establish a consistent and complete mathematical theory: color algebra, associated with the new color group CG, and
(3) to study the gluon radiation and the structure of mediator clouds around sub-atomic particles.

One important remark here is that as we shall later, all the quarks are still having only the three basic colors and their anti-colors.

Hereafter we examine some difficulties encountered in QCD associated with the lack of proper group operation of the existing color theory, and point out important ingredients and physical conclusions of the new color algebra introduced in this article.

First, the lack of proper definition of group operation in (1.5) leads to many difficulties in classical QCD theory. These difficulties suggest that the introduction of a new color algebraic system is crucial for the further development of QCD theory for strong interactions.

In particular, the inconsistency (1.5) demonstrates that in addition to the six basic colors $r, g, b, \bar{r}, \bar{g}, \bar{b}$, we need to incorporate the following color extensions:

$$r\bar{g}, \quad \bar{r}g, \quad r\bar{b}, \quad \bar{r}b, \quad g\bar{b}, \quad \bar{g}b.$$

It is easy to check that only two of these added colors are independent and are called yellow y and anti-yellow $\bar{y}$:

$$y \overset{\text{def}}{=} r\bar{g} = b\bar{r} = g\bar{b}, \qquad (1.6)$$
$$\bar{y} \overset{\text{def}}{=} g\bar{r} = r\bar{b} = b\bar{g}.$$

Namely, the new color group CG with the two added color is a commutative group:

$$CG = \{r, g, b, y, \bar{r}, \bar{g}, \bar{b}, \bar{y}, w\}. \qquad (1.7)$$

Second, with the color group CG, it is then natural to introduce a new color algebra with quarks and gluons as generators with integer coefficients:

$$P = \left\{ \sum_{k=1}^{19} n_k e_k \;\middle|\; e_k \text{ are the 18 colored quarks and 1 yellow gluon, } n_k \in \mathbb{Z} \right\}.$$

The color index of each element $\omega = \sum_{k=1}^{19} n_k e_k \in P$ can then be naturally defined, leading to color index formulas for hadrons and explanations of quark radiation. Elements in the color algebra P represent colored constituents of particle systems, and the corresponding color index formula provides an easy and comprehensive tool for QCD.

Third, decay and reaction behavior of subatomic particles offers us a doorway to the secret of sub-atomic world. Unfortunately, there are many puzzling issues regarding sub-atomic decays. Nevertheless, sub-atomic decays and reactions show clearly that there must be interior structure of charged leptons, quarks and mediators. For example, the electron radiations and the electron-positron annihilation into photons or quark-antiquark pair clearly shows that there must be interior structure of electrons, and the constituents of an electron contribute to the making of photon or the quark in the hadrons formed in the process.

Recently in [4, 3], based on the layered structure of strong and weak interaction potentials, a careful examination of subatomic decays and reactions leads us to propose six elementary particles, $w^*, w_1, w_2, \nu_e, \nu_\mu, \nu_\tau$, and their anti-particles. Both w^* and $\bar{w}^*$ are the only weaktons carrying strong charge g_s and color indices.

The w^* particle carries the three basic colors r, g, b, and $\bar{w}^*$ carries their anti-colors $\bar{r}, \bar{g}, \bar{b}$. All other elementary particle does not carry any color. Consequently, we are able to introduce a new algebra, which we call. It is a triplet (CG, P^3, Ind_c), consisting 1) the color group CG, 2) an algebra P^3 on the integer group $\mathbb{Z}$:

$$P^3 = \{ \sum_{k=r,g,b} n_k w_k^* \;\Big|\; n_k \in \mathbb{Z} \}, \qquad (1.8)$$

and a homomorphism $\text{Ind}_c : P^3 \to CG$. The color contribution/color index of any particle system is unique determined by an element in P^3, and its image under the homomorphism Ind_c. A complete theory on calculating color index of any particle system is proved in Theorem 3.1.

Fourth, the layered strong and interaction potentials demonstrate that in the sub-atomic scale, there is always an attracting shell-region near a naked particle, such as the electron structure as shown in Figure 4.1; see also [3, 4]. Mediators are then attracted to this attracting shell-region, forming a cloud of mediators.

With the color algebra theory, we are able to study more detailed structures of the mediator clouds of sub-atomic particles. This study offers good explanations of such phenomena as bremsstrahlung and gluon radiations.

FIFTH, QCD is based on the assumption that each quark carries a color quantum number with three color indices r, g, b and their anti-color indices $\bar{r}$, $\bar{g}$, $\bar{b}$. Gluons are also assumed to carry color indices. Different from the color index of quarks, each gluon carries two color charges: one color charge and one anti-color charge such as $r\bar{g}$. Namely, the 8 gluons can be considered as a color octet of the following irreducible representation of $SU_c(3) \otimes SU_c(3)$:

$$SU_c(3) \otimes SU_c(3) = 8 \oplus 1. \tag{1.9}$$

The color constituents of these color octet states can be given by

$$\begin{aligned} &g^1 = r\bar{g}, \quad g^2 = r\bar{b}, \quad g^3 = g\bar{b}, \quad g^4 = (r\bar{r} - b\bar{b})/\sqrt{2}, \\ &g^5 = g\bar{r}, \quad g^6 = b\bar{r}, \quad g^7 = b\bar{g}, \quad g^8 = (r\bar{r} + b\bar{b} - 2g\bar{g})/\sqrt{6}. \end{aligned} \tag{1.10}$$

The additional color singlet is

$$(r\bar{r} + b\bar{b} + g\bar{g})/\sqrt{3}.$$

Here g^4, g^8 and the color singlet are white, representing different quantum states. However, the color of any of the other gluons does not equal to any of known color r, g, b and their anti-colors $\bar{r}, \bar{g}, \bar{b}$. Of course it is clear now that they are either yellow y or anti-yellow $\bar{y}$.

SIXTH, in the classical QCD theory, gluons are mediators for the strong interaction in which the color index plays a crucial role. As both quarks and gluons carry color indices, strong interaction can only take place between particles carrying color indices, including interactions between quark and quark, between quark and gluon, or between gluons. In other words, classically, one views color index as a source of strong interaction. However, this point of view leads to several difficulties. One such difficulty is that no proper color correspondence of gluons prevents us to establish any interaction potential of the following form

$$\phi_s = \sum_{i=1}^{3} \varphi_{c_i}(r), \tag{1.11}$$

where c_i are the three colors. In fact, the situation is even worse when considering strong force between hadrons as hadrons are colorless. It is worth mentioning that with the unified field theory [3], we derive that the strong charge g_s is the main source of strong interaction, and the layered strong interaction potentials can then be derived; see also (4.1) and (4.2).

SEVENTH, in the classical QCD theory, we know that a quark can radiate or absorb gluons, as an electron can radiate or absorb photons in the quantum electrodynamics (QED). However, the lack of color operation (1.5) also leads to difficulties in the QCD theory on gluon radiation and absorption. For example, a red quark q_r can be transformed to a green quark q_g, radiating an $r\bar{g}$ gluon $g_{r\bar{g}}$:

$$q_r \to g_{r\bar{g}} + q_g,$$

which leads to a correct color operation:

$$r = (r\bar{g})g = r\bar{g}g = r.$$

However, if q_r radiates a $g_{b\bar{g}}$ gluon, due to the lack of color algebra, then it is difficult to determine the type of quark in which q_r transform:

$$q_r \to g_{b\bar{g}} + q_X, \qquad X = ?.$$

With the new color algebra theory and in particular with the introduction of the colors y and $\bar{y}$, together with the color index formula developed in Section 2, we know that

$$X = rg\bar{b} = r\bar{y} = g.$$

Namely, a red quark becomes a blue quark after radiating $g_{b\bar{g}}$ gluon.

The article is organized as follows. Section 2 introduces the color group CG and the color algebra based on the classical QCD theory, and Section 3 develops w^* color algebra theory based on the weakton model. The application to the study of the structure of mediator clouds of sub-atomic particles and the gluon radiations.

2. Color Algebra

The main objective of this section is to introduce a consistent color algebraic structure, and to establish a color index formula for hadrons and color transformation exchange for gluon radiation.

The color algebra for quantum chromodynamics (QCD) established in this section is based on color neutral principle of hadrons, and is uniquely determined. Hence it serves as the mathematical foundation of QCD.

2.1. **Color algebra of color quantum numbers.** First we examine the crucial problems encountered in the existing theory for color algebra. The color neutral principle of hadrons requires that the three colors must obey the following laws:

$$rgb = w, \qquad \bar{r}\bar{g}\bar{b} = w, \tag{2.1}$$

$$r\bar{r} = g\bar{g} = b\bar{b} = w. \tag{2.2}$$

Basic physical considerations imply that the product operation of color is commutative and associative, i.e.

$$rg = gr, \qquad rb = br, \qquad bg = gb, \qquad rgb = (rg)b = r(bg).$$

Hence we infer from (2.1) and (2.2) that

$$\begin{aligned} &rg = \bar{b}, \quad rb = \bar{g}, \quad bg = \bar{r}, \\ &\bar{r}\bar{g} = b, \quad \bar{r}\bar{b} = g, \quad \bar{b}\bar{g} = r. \end{aligned} \tag{2.3}$$

Notice that the white color w is the unit element, i.e.

$$wr = r, \quad wg = g, \quad wb = b, \quad w\bar{r} = \bar{r}, \quad w\bar{g} = \bar{g}, \quad w\bar{b} = \bar{b}.$$

Then again, we infer from (2.1) and (2.2) that

$$\begin{aligned} &rr = \bar{r}, \quad gg = \bar{g}, \quad bb = \bar{b}, \\ &\bar{r}\bar{r} = r, \quad \bar{g}\bar{g} = g, \quad \bar{b}\bar{b} = b. \end{aligned} \tag{2.4}$$

Multiplying (2.1) by b and using (2.4), we deduce that

$$r(g\bar{b}) = b, \tag{2.5}$$

which leads to inconsistency, no matter what color we assign for $g\bar{b}$. For example, if we assign $g\bar{b} = r$, then we derive from (2.5) that $rr = b$, which is inconsistent with $rr = \bar{r}$. If we assume $g\bar{b} = \bar{b}$, then $r\bar{b} = b$, which, by multiplying by b, leads to $r = bb = \bar{b}$, a contradiction again.

The above inconsistency demonstrates that in addition to the six basic colors

$$r, g, b, \bar{r}, \bar{g}, \bar{b},$$

we need to incorporate the following color extensions:

$$r\bar{g}, \quad \bar{r}g, \quad r\bar{b}, \quad \bar{r}b, \quad g\bar{b}, \quad \bar{g}b.$$

Only two of these added colors are independent, and in fact, we derive from (2.3) that

$$r\bar{g} = b\bar{r} = g\bar{b}, \qquad g\bar{r} = r\bar{b} = b\bar{g}.$$

Hence we define then as yellow y and anti-yellow $\bar{y}$ as follows:

$$\begin{aligned} y &= r\bar{g} = b\bar{r} = g\bar{b}, \\ \bar{y} &= g\bar{r} = r\bar{b} = b\bar{g}. \end{aligned} \tag{2.6}$$

In a nutshell, in order to establish a consistent color algebra, it is necessary to add two quantum numbers yellow y and anti-yellow $\bar{y}$ to the six color quantum numbers, giving rise to a consistent and complete mathematical theory: color algebra.

Definition 2.1. 1. *The generators of color algebra consists of quarks and gluons, which possess 9 color indices as*

$$r, \quad g, \quad b, \quad y, \quad \bar{r}, \quad \bar{g}, \quad \bar{b}, \quad \bar{y}, \quad w, \tag{2.7}$$

which form a finite commutative group. Here y and $\bar{y}$ are given by (2.6), and the group product operation is defined by

1) *w is the unit element, i.e.*

$$cw = c \qquad \text{for any color in (2.7);}$$

2) *$\bar{c}$ is the inverse of c:*

$$c\bar{c} = w \qquad \text{for } c = r, g, b, y;$$

3) *In addition to the basic operations given in (2.1)-(2.4), we have*

$$\begin{aligned} &yr = b, \quad yg = r, \quad yb = g, \quad yy = \bar{y}, \\ &\bar{y}r = g, \quad \bar{y}g = b, \quad \bar{y}b = r, \quad \bar{y}\bar{y} = y. \end{aligned} \tag{2.8}$$

2. *Color algebra is an algebra with quarks and gluons as generators with integer coefficients, and its element space is given by*

$$P = \left\{ \sum_{k=1}^{19} n_k e_k \;\middle|\; e_k \text{ are the 18 colored quarks and 1 yellow gluon, } n_k \in \mathbb{Z} \right\},$$

and $-e_k$ represent anti-quarks and gluons with anti-yellow color index $\bar{y}$.

3. *The color index of $\omega = \sum_{k=1}^{19} n_k e_k \in P$ is defined by*

$$\text{Ind}_c(\omega) = \prod_{k=1}^{19} c_k^{n_k}, \tag{2.9}$$

where c_k is the color of e_k and $c_k^{n_k} = w$ if $n_k = 0$.

Two remarks are now in order.

Remark 2.1. Each element $\omega = \sum n_k e_k \in P$ represents a particle system, and n_k is the difference between the number of particles with color index c_k and the number of antiparticles with color index $\bar{c}_k$. In particular, particles with colors r, g, b must be quarks, particles with colors $\bar{r}, \bar{g}, \bar{b}$ must be anti-quarks, and particles with colors $y, \bar{y}$ must be gluons.

Remark 2.2. In (2.6), $r\bar{g}$, $b\bar{r}$ and $g\bar{b}$ are all yellow. Consequently, the gluons $g_{r\bar{g}}$, $g_{b\bar{r}}$ and $g_{g\bar{b}}$ have the same color. However, they represent different quantum states. In particular, in the weakton model [3, 4],

$$g_{r\bar{g}} = w_r^* w_{\bar{g}}^*, \qquad g_{b\bar{r}} = w_b^* w_{\bar{r}}^*, \qquad g_{g\bar{b}} = w_g^* w_{\bar{b}}^*,$$

which represent different quantum states.

2.2. **Color index formula for hadrons.** The main objective of this subsection is to study color neutral problem for hadrons and the radiation and absorption of gluons for quarks.

We start with color neutral problem for hadrons. Consider the constituents of a proton

$$p = u_{c_1} + u_{c_2} + d_{c_3} + \sum n_k g^k \in P, \qquad \sum n_k = N, \tag{2.10}$$

whose color index is given by

$$\text{Ind}_c(p) = c_1 c_2 c_3 \prod_{k=1}^{8} (\text{Ind}_c(g^k))^{n_k}, \tag{2.11}$$

where $\text{Ind}_c(g^k)$ is the color of the gluon g^k. The color neutral law requires that $\text{Ind}_c(p) = w$, which does not necessarily lead to $c_1c_2c_3 = w$. For example, for the following constituents of a proton p:

$$p = u_r + u_r + d_g + 2g_{r\bar{g}} + g_{r\bar{b}} \tag{2.12}$$

we have

$$\text{Ind}_c(p) = r^2 g y^2 \bar{y} = \bar{r} g y = \bar{r} r = w, \qquad c_1c_2c_3 = rrg = \bar{r} g = \bar{y} \neq w.$$

In summary, the hadron color quantum numbers based on color algebra is very different from the classical QCD theory.

For a baryon $\mathcal{B}$ with its constituents given by

$$\mathcal{B} = \sum_{i=1}^{3} q_{c_i} + \sum_{k=1}^{3} (n_k g^k + m_k \bar{g}^k) + N g^4 + M \bar{g}^4, \tag{2.13}$$

its quantum number distribution satisfies the following color index formula:

$$c_1c_2c_3 = \bar{y}^{N_1} y^{N_2}, \qquad N_1 = \sum_{k=1}^{3} n_k, \qquad N_2 = \sum_{k=1}^{3} m_k. \tag{2.14}$$

For a meson μ with constituents:

$$\mu = q_{c_1} + \bar{q}_{c_2} + \sum_{k=1}^{3} (n_k g^k + m_k \bar{g}^k) + N g^4 + M \bar{g}^4, \tag{2.15}$$

its neutral color requires that

$$c_1 \bar{c}_2 = \bar{y}^{N_1} y^{N_2}, \tag{2.16}$$

where N_1 and N_2 are given as in (2.14). In (2.13) and (2.15),

$$g^1 = g_{r\bar{g}}, \quad g^2 = g_{b\bar{r}}, \quad g^3 = g_{g\bar{b}}, \quad g^4 = \text{combination of neutral gluons},$$
$$\bar{g}^1 = g_{\bar{r}g}, \quad \bar{g}^2 = g_{\bar{b}r}, \quad \bar{g}^3 = g_{\bar{g}b}, \quad \bar{g}^4 = \text{the anti-gluon of } g^4.$$

2.3. **Color transformation of gluon radiation.** Consider the transformation of a quark q_c with color c to another quark q_{c_3} after emitting a gluon $g_{c_1\bar{c}_2}$:

$$q_c \to g_{c_1\bar{c}_2} + q_{c_3}, \tag{2.17}$$

then the color c_3 of the transformed quark q_{c_3} is given by

$$c_3 = cc_2\bar{c}_1. \tag{2.18}$$

Also, for the transformation of a quark q_c to another quark q_{c_4} after absorbing a gluon $g_{c_1\bar{c}_2}$:

$$q_c + g_{c_1\bar{c}_2} \to q_{c_4}, \tag{2.19}$$

then the color c_4 of the transformed quark q_{c_4} is given by

$$c_4 = cc_1\bar{c}_2. \tag{2.20}$$

3. w^*-Color Algebra

3.1. **Weakton model.** We first recall the weakton model introduced in [3]. The starting point of the model is the puzzling decay and reaction behavior of subatomic particles. For example, the electron radiations and the electron-positron annihilation into photons or quark-antiquark pair clearly shows that there must be interior structure of electrons, and the constituents of an electron contribute to the making of photon or the quark in the hadrons formed in the process. In fact, all sub-atomic decays and reactions show clearly that there must be interior structure of charged leptons, quarks and mediators. A careful examination of these subatomic decays and reactions leads us to propose six elementary particles, which we call weaktons, and their anti-particles:

$$\begin{aligned} &w^*, \quad w_1, \quad w_2, \quad \nu_e, \quad \nu_\mu, \quad \nu_\tau, \\ &\bar{w}^*, \quad \bar{w}_1, \quad \bar{w}_2, \quad \bar{\nu}_e, \quad \bar{\nu}_\mu, \quad \bar{\nu}_\tau, \end{aligned} \tag{3.1}$$

where ν_e, ν_μ, ν_τ are the three generation neutrinos, and w^*, w_1, w_2 are three new particles, which we call w-weaktons. These are massless, spin-$\frac{1}{2}$ particles with one unit of weak charge g_w. Both w^* and $\bar{w}^*$ are the only weaktons carrying strong charge g_s.

With these weaktons at our disposal, the weakton constituents of charged leptons and quarks are then given as follows:

$$\begin{aligned} &e = \nu_e w_1 w_2, \quad \mu = \nu_\mu w_1 w_2, \quad \tau = \nu_\tau w_1 w_2, \\ &u = w^* w_1 \bar{w}_1, \quad c = w^* w_2 \bar{w}_2, \quad t = w^* w_2 \bar{w}_2, \\ &d = w^* w_1 w_2, \quad s = w^* w_1 w_2, \quad b = w^* w_1 w_2, \end{aligned} \tag{3.2}$$

where c, t and d, s, b are distinguished by the spin arrangements; see [3] for details.

In the weakton model, the mediators include the massless spin-1 photon γ and its dual spin-0 massless particle ϕ_γ, the gluons g^k and their dual fields ϕ_g^k, and the

ν mediators:

$$
\begin{aligned}
&\gamma = \alpha w_1 \bar{w}_1 + \beta w_2 \bar{w}_2 (\upuparrows, \downdownarrows), \\
&\phi_\gamma = \alpha w_1 \bar{w}_1 + \beta w_2 \bar{w}_w (\uparrow\downarrow, \downarrow\uparrow), \qquad \alpha^2 + \beta^2 = 1, \\
&\phi_\nu = \alpha_1 \nu_e \bar{\nu}_e + \alpha_2 \nu_\mu \bar{\nu}_\mu + \alpha_3 \nu_\tau \bar{\nu}_\tau (\downarrow\uparrow), \quad \sum_{i=1}^{3} \alpha_i^2 = 1, \\
&g^k = w^* \bar{w}^* (\upuparrows, \downdownarrows), \\
&\phi_g^k = w^* \bar{w}^* (\uparrow\downarrow, \downarrow\uparrow).
\end{aligned} \tag{3.3}
$$

All mediators participate weak interaction through two weaktons carrying weak charges. Both the gluons g^k and their duals ϕ_g^k carry both strong charge and color charges, and participate strong interactions as well. Also, the color index for both g^k and ϕ_g^k are the same.

3.2. **w^*-color algebra.** In an abstract sense, a color algebra (CG, P^N, Ind_c) consists of:

(1) a finite group CG, called color group,
(2) an algebra with generators $e_1, \cdots, e_N$ defined on $\mathbb{Z}$:

$$
P^N = \left\{ \sum_{k=1}^{N} n_k e_k \;\middle|\; n_k \in \mathbb{Z} \right\}, \tag{3.4}
$$

(3) a homomorphism

$$
\text{Ind}_c : P^N \to CG, \tag{3.5}
$$

such that for each element $\omega = \sum_{k=1}^{N} n_k e_k \in P^N$, we can define a color index for ω by

$$
\text{Ind}_c(\omega) = \prod_{k=1}^{N} \text{Ind}_c(e_k)^{n_k}, \tag{3.6}
$$

where $\text{Ind}_c(e_k) \in CG$ is the image of e_k under the homomorphism Ind_c.

Based on the weakton model, the weakton constituents of a quark and a gluon are given by

$$
q = w^* w w, \qquad g = w^* \bar{w}^*.
$$

The only weakton that has colors is w^*, which has three colors: r, g, b:

$$
w_r^*, \qquad w_g^*, \qquad w_b^*, \tag{3.7}
$$

and their antiparticles have colors $\bar{r}, \bar{g}, \bar{b}$.

We are now in position to define the w^*-color algebra as follows:

Definition 3.1. *The w^*-color algebra (CG, P^3, Ind_c) is defined as follows:*

(1) *The color group*

$$
CG = \{r, g, b, y, \bar{r}, \bar{g}, \bar{b}, \bar{y}, w\} \tag{3.8}
$$

with group operation given in Definition 2.1;

(2) *an algebra P^3 on the integer group $\mathbb{Z}$:*

$$
P^3 = \left\{ \sum_{k=r,g,b} n_k w_k^* \;\middle|\; n_k \in \mathbb{Z} \right\}; \tag{3.9}
$$

(3) *an homomorphism* $\text{Ind}_c : P^3 \to CG$ *defined by*

$$\text{Ind}_c(w_k^*) = k, \qquad \text{Ind}_c(-w_k^*) = \bar{k} \quad \text{for } k = r, g, b. \tag{3.10}$$

Consider a particle system ω, which consists of n_k quark q_k, $\bar{n}_k$ antiquark $\bar{q}_k$, m_1 gluons $g^1 = g_{r\bar{g}}$, m_2 gluons $g^2 = g_{b\bar{r}}$, m_3 gluons $g^3 = g_{g\bar{b}}$, m_4 color-neutral gluons g^4, $\bar{m}_1$ gluons $\bar{g}^1$, $\bar{m}_2$ gluons $\bar{g}^2$, $\bar{m}_3$ gluons $\bar{g}^3$, and $\bar{m}_4$ gluons $\bar{g}^4$:

$$\omega = \sum_{k=r,g,b} (n_k q_k + \bar{n}_k \bar{q}_k) + \sum_{i=1}^{4} (m_i g^i + \bar{m}_i \bar{g}^i). \tag{3.11}$$

Then ω corresponds to an element $X_\omega \in P^3$ expressed by

$$X_\omega = \sum_{k=r,g,b} N_k w_k^*, \tag{3.12}$$

where

$$\begin{aligned} N_r &= (n_r - \bar{n}_r) + (m_1 - m_2) - (\bar{m}_1 - \bar{m}_2), \\ N_g &= (n_g - \bar{n}_g) + (m_3 - m_1) - (\bar{m}_3 - \bar{m}_1), \\ N_b &= (n_b - \bar{n}_b) + (m_2 - m_3) - (\bar{m}_2 - \bar{m}_3), \end{aligned} \tag{3.13}$$

and, consequently, the color index for ω is defined by

$$\text{Ind}_c(\omega) = \text{Ind}_c(X_\omega) = r^{N_r} g^{N_g} b^{N_b}, \tag{3.14}$$

where for negative $N_k < 0$, we define

$$k^{N_k} = \bar{k}^{-N_k}.$$

It is then clear that

$$\text{Ind}_c(\omega_1 + \cdots + \omega_s) = \prod_{i=1}^{s} \text{Ind}_c(\omega_i). \tag{3.15}$$

The following is a basic theorem for w^*-color algebra, providing the needed foundation for the structure for charged leptons and quarks.

Theorem 3.1. *The following hold true for the w^*-color algebra.*

(1) *The color index of any gluon particle system with no quarks and no antiquarks*

$$\varepsilon = \sum_{i=1}^{4} (m_i g^i + \bar{m}_i \bar{g}^i) \tag{3.16}$$

obeys

$$\text{Ind}_c(\varepsilon) = \begin{cases} w & \text{for } \sum_{i=1}^{3} (m_i - \bar{m}_i) = \pm 3k, \\ y & \text{for } \sum_{i=1}^{3} (m_i - \bar{m}_i) = \pm 3k + 1, \\ \bar{y} & \text{for } \sum_{i=1}^{3} (m_i - \bar{m}_i) = \pm 3k + 2, \end{cases} \tag{3.17}$$

for some integer $k = 0, 1, 2, \cdots$.

(2) *The color index of any single quark system*

$$\omega = q + \varepsilon, \quad \omega' = \bar{q} + \varepsilon \qquad \text{with } \varepsilon \text{ given by (3.16)} \tag{3.18}$$

obeys

$$\text{Ind}_c(\omega) = r, q, b, \quad \text{Ind}_c(\omega') = \bar{r}, \bar{g}, \bar{b}. \tag{3.19}$$

(3) *The color indices of hadronic cloud systems*

$$\begin{aligned} &\mathcal{M} = q + \bar{q} + \varepsilon && \text{meson system} \\ &\mathcal{B} = q + q + q + \varepsilon && \text{baryon system} \end{aligned} \qquad \text{with } \varepsilon \text{ given by (3.16)} \tag{3.20}$$

must be given by

$$\text{Ind}_c(\mathcal{M}) = w, y, \bar{y}, \qquad \text{Ind}_c(\mathcal{B}) = w, y, \bar{y}. \tag{3.21}$$

Two remarks are now in order.

1. In particle physics, all basic and important particle systems are given by the particle systems in Assertions (1)-(3) in this theorem. System (3.16) represents a cloud system of gluons around charged leptons, (3.18) represents for a cloud system of gluons around a quark or an anti-quark, and (3.20) represents a cloud system of gluons around a hadron (a meson or a baryon).

2. Physically, $\sum_{i=1}^{3}(m_i - \bar{m}_i) = \pm 3k$ indicates that through exchange of weaktons, a cloud system (3.16) of gluons around charged leptons can become a cloud system of white gluons coupled with the same number of yellow and anti-yellow gluons.

Proof of Theorem 3.1. Step 1. By the basic properties (3.11)-(3.14) of color index,

$$\text{Ind}_c(\varepsilon) = r^{M_r} g^{M_g} b^{M_b} \tag{3.22}$$

where

$$\begin{aligned} M_r &= (m_1 - m_2) - (\bar{m}_1 - \bar{m}_2), \\ M_g &= (m_3 - m_1) - (\bar{m}_3 - \bar{m}_1), \\ M_b &= (m_2 - m_3) - (\bar{m}_2 - \bar{m}_3). \end{aligned}$$

Consequently, using $c^{-m} = \bar{c}^m$, we have

$$\begin{aligned} \text{Ind}_c(\varepsilon) =& (r\bar{g})^{m_1} (\bar{r}b)^{m_2} (g\bar{b})^{m_3} (\bar{r}g)^{\bar{m}_1} (r\bar{b})^{\bar{m}_2} (\bar{g}b)^{\bar{m}_3} \\ =& y^{m_1} y^{m_2} y^{m_3} \bar{y}^{\bar{m}_1} \bar{y}^{\bar{m}_2} \bar{y}^{\bar{m}_3} = y^M. \end{aligned}$$

Here

$$M = \sum_{i=1}^{3} (m_i - \bar{m}_i).$$

Notice that $y^2 = \bar{y}$ and $\bar{y}^2 = y$ and Assertion (1) follows.

Step 2. For Assertion (2), with the above argument, it is easy to see that

$$\begin{aligned} \text{Ind}_c(\omega) &= \text{Ind}_c(q)\text{Ind}_c(\varepsilon) = \text{Ind}_c(q) y^M, \\ \text{Ind}_c(\omega') &= \text{Ind}_c(\bar{q})\text{Ind}_c(\varepsilon) = \text{Ind}_c(\bar{q}) y^M. \end{aligned} \tag{3.23}$$

Then Assertion (2) follows from the fact that

$$\text{Ind}_c(q) = r, g, b, \qquad \text{Ind}_c(\bar{q}) = \bar{r}, \bar{q}, \bar{b}.$$

STEP 3. With the same arguments as above, we derive that

$$\mathrm{Ind}_c(\pi) = c_i\bar{c}_j y^M, \qquad \mathrm{Ind}_c(\mathcal{B}) = c_i c_j c_k y^M \qquad (1 \leq i, j, k \leq 3),$$

where $c_1 = r, c_2 = g, c_3 = b$. By the basic rules for the color operator given in (2.3) and (2.6), we have

$$c_i\bar{c}_j = w, y, \bar{y}, \qquad c_i c_j c_k = \bar{c}_l c_k = w, y, \bar{y},$$

and therefore Assertion (3) follows. □

4. Structure of Mediator Clouds Around Sub-Atomic Particles

Sub-atomic particles consist of charged leptons, quarks and hadrons. As demonstrated in [3], strong and weak interactions consist of different layers, leading to different particle structures in different layers: weakton layer (elementary particle layer), mediator layer, charged lepton and quark layer, hadron layer and nucleon layer.

In this section, we examine the structure of mediator clouds of sub-atomic particles, based on the layered properties of strong and weak interactions and the color algebra introduced in the previous sections, leading to explanations of sub-atomic decays and scatterings.

4.1. **Strong and weak interaction potentials.** In [3], we have derived both strong and weak interaction potentials based on the principle of interaction dynamics (PID) and the principle of representation invariance.

The strong interaction potential Φ_s for a particle with radius ρ and the strong interaction potential V_{12}^s between two particles, with radius ρ_1 and radius ρ_2 and carrying N_1 and N_2 strong charges respectively, are given by

$$\Phi_s = g_s(\rho)\left[\frac{1}{r} - \frac{A_s}{\rho}\left(1 + \frac{r}{R}\right)e^{-r/R}\right], \tag{4.1}$$

$$V_{12}^s = N_1 N_2 g_s(\rho_1) g_s(\rho_2)\left[\frac{1}{r} - \frac{\widetilde{A}_s}{\sqrt{\rho_1\rho_2}}\left(1 + \frac{r}{R}\right)e^{-r/R}\right], \tag{4.2}$$

where A_s and $\widetilde{A}_s$ are dimensionless constants,

$$g_s(\rho) = \left(\frac{\rho}{\rho_w}\right)^3 g_s,$$

$$g_s^2 = \frac{2}{3(8\sqrt{e} - e)}\left(\frac{\rho_n}{\rho_w}\right)^6 g^2,$$

$$R = \begin{cases} 10^{-13}\ \mathrm{cm} & \text{for hadrons and nucleons,} \\ 10^{-16}\ \mathrm{cm} & \text{for quarks,} \end{cases}$$

ρ_w and ρ_n are the radii of weaktons and nucleons, g_s is the strong charge, and $g^2 \sim 1 - 10\hbar c$ is the Yukawa coupling constant of strong interaction.

Also, the layered weak interaction potential Φ_w for a particle with radius ρ and the weak interaction potential V_{12}^w between two particles, with radius ρ_1 and radius

ρ_2 and carrying N_1 and N_2 weak charges respectively, are given by

$$\Phi_w = g_w(\rho)e^{-r/r_0}\left[\frac{1}{r} - \frac{A_w}{\rho}\left(1+\frac{2r}{r_0}\right)e^{-r/r_0}\right], \tag{4.3}$$

$$V_{12}^w = N_1N_2g_w(\rho_1)g_w(\rho_2)e^{-r/r_0}\left[\frac{1}{r} - \frac{\widetilde{A}_w}{\sqrt{\rho_1\rho_2}}\left(1+\frac{r}{r_0}\right)e^{-r/r_0}\right], \tag{4.4}$$

where A_w and $\widetilde{A}_w$ are dimensionless constants, $r_0 = 10^{-16}$cm,

$$g_w(\rho) = \left(\frac{\rho}{\rho_w}\right)^3 g_w, \qquad g_w^2 = 0.63\left(\frac{\rho_n}{\rho_w}\right)^6 \hbar c,$$

and g_w is the weak charge.

With the above layered properties of both strong and weak interaction potentials, we see that in the sub-atomic scale, there is always an attracting shell-region near a naked particle, such as the electron structure as shown in Figure 4.1; see also [3]. Mediators are then attracted to this attracting shell-region, forming a cloud of mediators. More precisely, by the weak interaction potential, there is an attracting shell region of weak force:

$$\rho_1 < r < \rho_2, \qquad \rho_1 = 10^{-16} \text{ cm} \tag{4.5}$$

with small weak force. Outside this region, the weak force is repelling:

$$F_w > 0 \qquad \text{for } r < \rho_1 \quad \text{and } r > \rho_2. \tag{4.6}$$

Since the mediators γ, ϕ_γ, g, ϕ_g and ϕ_ν contain two weak charges $2g_w$, they are attached to the electron in the attracting shell region (4.5), forming a cloud of mediators. The irregular triangle distribution of the weaktons ν_2, w_1 and w_2 generate a small moment of force on the mediators in the shell region, and there exist weak forces between them. Therefore the bosons will rotate at a speed lower than the speed of light, and generate a small mass attached to the naked electron $\nu_e w_1 w_2$.

The main purpose of this section is to study the structure of the mediator clouds near naked sub-atomic particles, using the color algebra introduced in the previous sections.

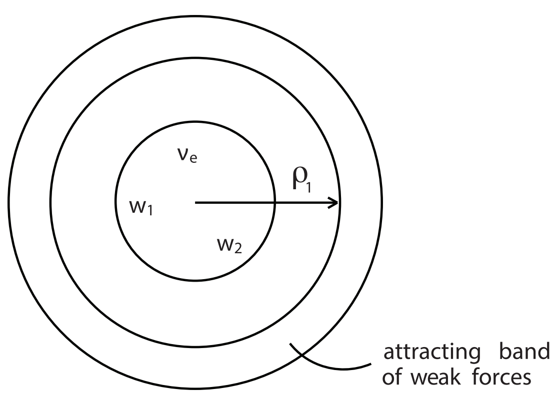

FIGURE 4.1. Electron structure: mediator cloud of an electron

4.2. **Mediator clouds for charged leptons.** For simplicity and due to similarities, we consider only the case for electrons.

First, we recall the electron structure presented in [3]. The weakton constituents of an electron are $\nu_e w_1 w_2$. Noting that

$$\begin{aligned} &\text{electric charge:} && Q_\nu = 0, && Q_{w_1} = -\frac{1}{3}, && Q_{w_2} = -\frac{2}{3}, \\ &\text{weak charge:} && C_w = 1, && C_{w_1} = 1, && C_{w_2} = 1, \end{aligned}$$

we see that the distribution of weaktons ν_e, w_1 and w_2 in an electron is in an irregular triangle due to the asymptotic forces on the weaktons by the electromagnetic and weak interactions.

It is known that an electron emits photons as its velocity changes. This is called bremsstrahlung, and the reasons why bremsstrahlung can occur is unknown in classical theories. As an electron is in an electromagnetic field, which exerts a Coulomb force on its naked electron $\nu_e w_1 w_2$, but not on the attached neutral mediators. Thus, the naked electron changes its velocity, which draws the mediator cloud to move as well, causing a perturbation to moment of force on the mediators. As the attracting weak force in the shell region (4.5) is small, under the perturbation, the centrifugal force makes some mediators in the cloud, such as photons, flying away from the attracting shell region, and further accelerated by the weak repelling force (4.6) to the speed of light, as shown in Figure 4.2.

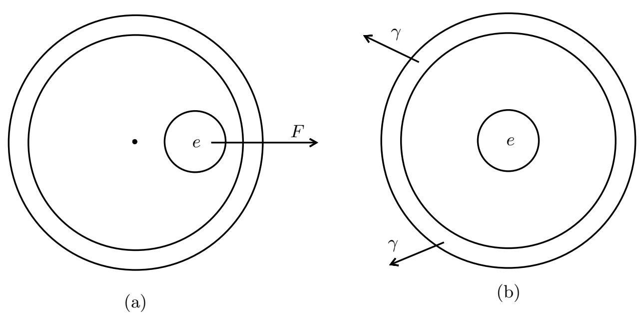

Figure 4.2. (a) The naked electron is accelerated in an electromagnetic field; (b) the mediators (photons) fly away from the attracting shell region under a perturbation of moment of force.

Second, due to the weak interaction between mediators, they must be maintain a distance r_0 between each other with r_0 being approximately the weak repelling radius of the mediators. Consequently, the total number N of particles in each layer of mediators satisfies that

$$N \leq 4\pi r^2 / r_0^2, \tag{4.7}$$

where r is the radius of the layer. For example, if $r = r_0$, then $N \leq 12$.

Third, an electron refers to the system consisting of the naked electron $\nu_e w_1 w_2$ and its mediator cloud. The mediator cloud consists of spin-1 photons and gluons. As the total spin of an electron is $J_e = \frac{1}{2}$, both the number of photons N_γ and the number of gluons N_g are even with half being left-handed and the other half being right-handed.

FOURTH, as an electron is white, the total color index of the gluons and the dual gluons is white. By Theorem 3.1, the numbers of yellow and anti-yellow gluons and dual gluons satisfy the following relationship in 3.17:

$$\sum_{i=1}^{3}(m_i - \bar{m}_i) = \pm 3k, \qquad k = 0, 1, \cdots . \tag{4.8}$$

Hence the colorless of an electron, as well as the balance between yellow and anti-yellow particles, can be achieved through the exchange of w^* and $\bar{w}^*$ between the gluons and the dual gluons.

4.3. **Mediators of quarks and gluon radiations.** Different from the weakton constituents of an electron, the weakton constituents of a quark include a weakton with strong charge, leading to different structure of mediator clouds.

The strong interaction between a quark and a gluon consists of three parts: the interaction between strong charges, the interaction between color charges and the interaction between a strong charge and a color charge. We have developed layered formulas for strong interaction potentials between strong charges, and it is clear that the other two interactions must have much shorter range. Consequently, the mediator cloud of a quark consists of two layers:

$$\begin{aligned} &\text{gluon layer:} && \text{gluons and dual gluons,} \\ &\text{photon layer:} && \text{photons, dual photons, and } \nu\text{-mediators.} \end{aligned} \tag{4.9}$$

Here the gluon layer is due to strong interaction, and the photon layer is due to the weak interaction. The radius of the gluon layer is about the radius ρ_0 of a quark:

$$\rho_0 \sim 10^{-23} - 10^{-21}\text{cm}.$$

The radius of photon layer is about the radius of an electron:

$$\rho_e = 10^{-18} - 10^{-16}\text{cm}.$$

By Theorem 3.1, the color of an quark system carrying gluon cloud can only be either r, g, b or $\bar{r}, \bar{g}, \bar{b}$. The gluons in the cloud layer of a quark are confined. However, gluons and dual gluons between quarks in a hadron can be exchanged.

The gluon exchange process between quarks in a hadron is called gluon radiation. In fact, a quark can emit gluons, which will be absorbed by other quarks in the hadron. For example, consider a quark q_1 is transformed to q_2 after emitting a gluon g_0, and the quark q_2 is transformed to q_4 after absorbing the gluon g_0:

$$q_1 \to q_3 + g_0, \qquad q_2 + g_0 \to q_4, \tag{4.10}$$

equivalently,

$$q_1 + q_2 \to q_3 + q_4.$$

The corresponding color transformation is given by

$$\begin{aligned} &\text{Ind}_c(q_1) = \text{Ind}_c(q_3)\text{Ind}_c(g_0), \\ &\text{Ind}_c(q_4) = \text{Ind}_c(q_2)\text{Ind}_c(g_0), \\ &\text{Ind}_c(q_1 + q_2) = \text{Ind}_c(q_3 + q_4). \end{aligned}$$

4.4. **Mediator clouds of hadrons.** Different from electrons and quarks, there is no photon cloud layer around a hadron, due to the fact that the radius of a naked hadron ρ_h is greater than the range of the weak interaction:

$$\rho_h \geq 10^{-16}\text{cm}.$$

Namely, a hadron can only have a strong attraction to gluons and dual gluons, forming a gluon cloud with radius about the same as the radius of a hadron:

$$r_1 \sim 10^{-16} - 10^{-14}\text{cm}.$$

Also, hadrons care colorless, for a baryon $\mathcal{B}$ and a meson $\mathcal{M}$ given by (3.20), we have

$$\begin{aligned} &\text{Ind}_c(\mathcal{M}) = c_i\bar{c}_j\text{Ind}_c(\varepsilon) = w, \\ &\text{Ind}_c(\mathcal{B}) = c_ic_jc_k\text{Ind}_c(\varepsilon) = w, \qquad 1 \leq i, j, k \leq 3. \end{aligned}$$

Consequently, if the naked hadron is white, then the gluon cloud is white as well.

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