

# Scaling limits for the critical Fortuin-Kasteleyn model on a random planar map I: cone times

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## Abstract

Sheffield (2011) introduced a discrete inventory accumulation model which encodes a random planar map decorated by a collection of loops sampled from the critical Fortuin-Kasteleyn (FK) model and showed that a certain two-dimensional random walk associated with an infinite-volume version of the model converges in the scaling limit to a correlated planar Brownian motion. We improve on this scaling limit result by showing that the times corresponding to complementary connected components of FK loops (or “flexible orders”) in the discrete model converge to the  $\pi/2$ -cone times of this correlated Brownian motion. Our result can be used to obtain convergence of many interesting functionals of the FK loops (e.g. their lengths and areas) toward the corresponding “quantum” functionals of the loops of a conformal loop ensemble on a Liouville quantum gravity surface.

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# 1 Introduction

The *critical Fortuin-Kasteleyn (FK) cluster model with parameter  $q > 0$*  on a planar map  $M$  is a statistical physics model, first introduced in [FK72], in which one chooses a random subset  $S$  of the set of edges of  $M$ . This collection of edges gives rise to a collection  $\mathcal{L}$  of loops on  $M$  which form the interfaces between the edges in  $S$  and the edges not in  $S$ . The probability of any given realization of  $S$  is proportional to  $q^{\#\mathcal{L}}$ . The FK model is closely related to the critical  $q$ -state Potts model [BKW76] for general integer values of  $q$ ; to critical percolation for  $q = 1$ ; and to the Ising model for  $q = 2$ . See e.g. [KN04, Gri06] for more on the FK model and its relationship to other statistical physics models.

If the pair  $(M, \mathcal{L})$  is chosen according to the uniform measure on such pairs, weighted by  $q^K$ , the loop-decorated planar map thus obtained is conjectured to converge in the scaling limit to a *conformal loop ensemble* ( $\text{CLE}_\kappa$ ) [She09, SW12] with  $\kappa \in (4, 8)$  satisfying  $q = 2 + 2\cos(8\pi/\kappa)$  on top of a *Liouville quantum gravity surface* [DS11, She10, DMS14] with parameter  $\gamma = 4/\sqrt{\kappa}$ . We refer the reader to [KN04, She11] and the references therein for more details regarding this conjecture.

In [She11], Sheffield introduces a simple inventory accumulation model involving a word  $X$  in an alphabet of five symbols representing two types of “burgers” and three types of “orders”; and constructs a bijection between certain realizations of this model and rooted random planar maps  $M$  decorated by a collection  $\mathcal{L}$  of loops. There is a family of probability measures on realizations of this model, indexed by a parameter  $p \in (0, 1/2)$ , with the property that the law of the pair  $(M, \mathcal{L})$  when the inventory accumulation model is sampled according to the probability measure with parameter  $p$  is given by the uniform measure on such pairs weighted by  $q^{K/2}$ , where  $K$  is the number of loops and  $q = 4p^2/(1-p)^2$ . In [She11, Theorem 2.5], it is shown that a random walk which encodes an infinite-volume version of his model converges in the scaling limit to a pair of Brownian motions with correlation depending on  $p$ .

In [DMS14, Sections 9 and 10] (see also [MS13]), it is shown that a  $\text{CLE}_\kappa$  on a  $4/\sqrt{\kappa}$ -Liouville quantum gravity surface can be encoded by a pair of correlated Brownian motions via a procedure which is directly analogous to the bijection in [She11]. The correlation between this pair of Brownian motions is the same as the correlation between the pair of limiting Brownian motions in [She11, Theorem 2.5] provided

$$p = \frac{\sqrt{2 + 2\cos(8\pi/\kappa)}}{2 + \sqrt{2 + 2\cos(8\pi/\kappa)}}, \tag{1}$$

which is consistent with the conjectured relationship between the FK model and CLE described above.

In this paper, we will improve on the scaling limit result of [She11] by showing that the times corresponding to FK loops (or “flexible orders”) in the infinite-volume discrete model converge in the scaling limit to the  $\pi/2$ -cone times of the Brownian motion (see Section 1.2 below for a precise statement). As we will explain below, this result answers [DMS14, Question 13.3] (at least in the infinite-volume setting).

Along the way, we will also prove several other results regarding the model of [She11] which are of independent interest. We prove tail estimates for various quantities associated with this model, including a polynomial lower bound for the probability that a word of length  $2n$  in the discrete model reduces to the empty word (Proposition 2.13) which confirms a prediction of Sheffield in [She11, Section 4.2]. Several of the laws of these quantities are in fact shown to have regularly varying tails (see Section 5.1). We also obtain the scaling limit of the discrete path conditioned on the event that the reduced word contains no burgers, or equivalently the event that this path stays in the first quadrant until a certain time when run backward (Theorem 4.1). Scaling limit results for random walks with independent increments conditioned to stay in a cone are obtained in several places in the literature (see [Shi91, Gar11, DW11] and the references therein). Our Theorem 4.1 is an analogue of these results for a certain random walk with non-independent increments.

Although this paper is motivated by the relationships between the inventory accumulation model of [She11], the FK cluster model on a random planar map, and  $\text{CLE}_\kappa$  on a Liouville quantum gravity surface, our proofs use only basic properties of the inventory accumulation model, Brownian motion, and stable processes.

Our results can be viewed as a small step toward proving the convergence of the FK cluster model loops on an infinite-volume random planar map to CLE on a quantum gravity surface, in the sense that our main result implies the convergence of many quantities associated with the discrete loops to the corresponding “quantum” quantities associated with the CLE loops (e.g. the quantum areas and quantum boundary lengths of the complementary connected components of the macroscopic loops). In order to obtain a full convergence result, one must additionally embed each FK-weighted random planar map into the Riemann

sphere and show that the loops themselves converge in an appropriate sense to CLE loops. We expect that proving this convergence is a substantially more difficult problem than proving the convergence statements of this paper. Nevertheless, due to the correspondence between the bijection in [She11] and the encoding of CLE via Brownian motion in [DMS14], our results have direct applications to the study of CLE. For example, some the results of this paper will be used in the forthcoming paper [GM] to prove conformal invariance of whole-plane  $\text{CLE}_\kappa$  for  $\kappa \in (4, 8)$ .

We end by pointing out some forthcoming related works. Shortly before this paper was posted, we learned of an independent work [BLR15] which calculates tail exponents for several quantities related to a generic loop on an FK-weighted random planar map, and which has been posted to the ArXiv at the same time as this work. In the forthcoming paper [SW15], the third author and D. Wilson study unicycle-decorated random planar maps via the bijection of [She11] and obtain the joint distribution of the length and area of the unicycle in the infinite volume limit. The first and third authors are currently preparing a sequel [GS] to the present paper in which we prove various scaling limit results for the finite-volume version of the model of [She11] (which is the version of the model for which [She11] describes the bijection with FK-weighted random planar maps).

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## 1.1 Inventory accumulation model

In this paper, we will consider a discrete model first introduced by Sheffield [She11], which we describe in this section. The notation introduced in this section will remain fixed throughout the remainder of the paper.

Let  $\Theta$  be the collection of symbols  $\{\textcircled{\text{H}}, \textcircled{\text{C}}, \boxed{\text{H}}, \boxed{\text{C}}, \boxed{\text{F}}\}$ . We can think of these symbols as representing, respectively, a hamburger, a cheeseburger, and hamburger order, a cheeseburger order, and a flexible order. We view  $\Theta$  as the generating set of a semigroup, which consists of the set of all finite words in elements of  $\Theta$ , modulo the relations

$$\textcircled{\text{C}}\boxed{\text{C}} = \textcircled{\text{H}}\boxed{\text{H}} = \textcircled{\text{C}}\boxed{\text{F}} = \textcircled{\text{H}}\boxed{\text{F}} = \emptyset \quad (2)$$

(order fulfilment) and

$$\textcircled{\text{C}}\boxed{\text{H}} = \boxed{\text{H}}\textcircled{\text{C}}, \quad \textcircled{\text{H}}\boxed{\text{C}} = \boxed{\text{C}}\textcircled{\text{H}} \quad (3)$$

(commutativity). Given a word  $W$  in elements of  $\Theta$ , we denote by  $\mathcal{R}(W)$  the word reduced modulo the above relations, with all burgers to the right of all orders. In the burger interpretation,  $\mathcal{R}(W)$  represents the burgers which remain after all orders have been fulfilled along with the unfulfilled orders. We also write  $|W|$  for the number of symbols in  $W$  (regardless of whether or not  $W$  is reduced).

For  $p \in [0, 1]$  (in this paper we will in fact typically take  $p \in (0, 1/2)$ , for reasons which will become apparent just below), we define a probability measure on  $\Theta$  by

$$\mathbb{P}(\textcircled{\text{H}}) = \mathbb{P}(\textcircled{\text{C}}) = \frac{1}{4}, \quad \mathbb{P}(\boxed{\text{H}}) = \mathbb{P}(\boxed{\text{C}}) = \frac{1-p}{4}, \quad \mathbb{P}(\boxed{\text{F}}) = \frac{p}{2}. \quad (4)$$

Let  $X = \dots X_{-1}X_0X_1\dots$  be an infinite word with each symbol sampled independently according to the probabilities (4). For  $a \leq b \in \mathbb{R}$ , we set

$$X(a, b) := \mathcal{R}(X_{\lfloor a \rfloor} \dots X_{\lfloor b \rfloor}). \quad (5)$$

**Remark 1.1.** There is an explicit bijection between words  $W$  in  $\Theta$  with  $|W| = 2n$  and  $\mathcal{R}(W) = \emptyset$ ; and pairs  $(M, \mathcal{L})$ , where  $M$  is a rooted planar map with  $n$  edges and  $\mathcal{L}$  is a set of loops on  $M$  [She11, Section 4.1]. If  $W$  is chosen according to the law of  $X_1 \dots X_{2n}$  (as above) with  $p \in (0, 1/2)$ , conditioned on the event that  $X(1, 2n) = \emptyset$ , then the law of  $(M, \mathcal{L})$  is that of a collection of FK-loops on top of an FK-weighted random planar map, as described in the introduction. This latter model is conjectured to converge under an

appropriate scaling limit to a  $\text{CLE}_\kappa$  (with  $\kappa$  and  $p$  related as in (1)) on top of an independent  $4/\sqrt{\kappa}$ -quantum sphere [DMS14].

As explained in [She11, Section 4.2], the unconditioned word  $X$  corresponds to an infinite-volume limit of FK-weighted random planar maps decorated by FK loops (a more detailed description of the bijection in the infinite volume case will appear in the forthcoming work [BLR15]). This infinite-volume model can be viewed as a discrete analogue of a  $\text{CLE}_\kappa$  on top of a quantum cone, a certain type of infinite-volume Liouville quantum gravity surface [DMS14]. In this paper we focus on the infinite-volume case. The finite volume case will be treated in a subsequent paper [GS].

By [She11, Proposition 2.2], it is a.s. the case that each symbol  $X_i$  in the word  $X$  has a unique match which cancels it out in the reduced word (i.e. burgers are matched to orders and orders matched to burgers). Heuristically, the reduced word  $X(-\infty, \infty)$  is a.s. empty.

**Notation 1.2.** For  $i \in \mathbb{Z}$  we write  $\phi(i)$  for the index of the match of  $X_i$ .

**Notation 1.3.** For  $\theta \in \Theta$  and a word  $W$  consisting of elements of  $\Theta$ , we write  $\mathcal{N}_\theta(W)$  for the number of  $\theta$ -symbols in  $W$ . We also let

$$\begin{aligned}\mathcal{C}(W) &:= \mathcal{N}_{\textcircled{\text{H}}}(W) + \mathcal{N}_{\textcircled{\text{C}}}(W) - \mathcal{N}_{\textcircled{\text{H}}}(W) - \mathcal{N}_{\textcircled{\text{C}}}(W) - \mathcal{N}_{\textcircled{\text{F}}}(W) \\ \mathcal{D}(W) &:= \mathcal{N}_{\textcircled{\text{H}}}(W) + \mathcal{N}_{\textcircled{\text{C}}}(W) - \mathcal{N}_{\textcircled{\text{C}}}(W) - \mathcal{N}_{\textcircled{\text{H}}}(W) \\ d(W) &:= \mathcal{N}_{\textcircled{\text{H}}}(W) - \mathcal{N}_{\textcircled{\text{H}}}(W) \\ d^*(W) &:= \mathcal{N}_{\textcircled{\text{C}}}(W) - \mathcal{N}_{\textcircled{\text{C}}}(W).\end{aligned}$$

The notations for  $\mathcal{C}$  and  $\mathcal{D}$  are taken from [She11]. The reason for the notation  $d$  and  $d^*$  is that these functions give the distances in the tree and dual tree which encode the collection of loops in the bijection of [She11, Section 4.1].

For  $i \in \mathbb{Z}$ , we define  $Y_i = X_i$  if  $X_i \in \{\textcircled{\text{H}}, \textcircled{\text{C}}, \textcircled{\text{H}}, \textcircled{\text{C}}\}$ ;  $Y_i = \textcircled{\text{H}}$  if  $X_i = \textcircled{\text{F}}$  and  $X_{\phi(i)} = \textcircled{\text{H}}$ ; and  $Y_i = \textcircled{\text{C}}$  if  $X_i = \textcircled{\text{F}}$  and  $X_{\phi(i)} = \textcircled{\text{C}}$ . For  $a \leq b \in \mathbb{R}$ , define  $Y(a, b)$  as in (5) with  $Y$  in place of  $X$ .

For  $n \geq 0$ , define  $\mathcal{C}(n) = \mathcal{C}(Y(1, n))$  and for  $n < 0$ , define  $\mathcal{C}(n) = -\mathcal{C}(Y(n+1, 0))$ . Define  $\mathcal{D}(n)$ ,  $d(n)$ , and  $d^*(n)$  similarly. Extend each of these functions from  $\mathbb{Z}$  to  $\mathbb{R}$  by linear interpolation.

**Remark 1.4.** Note that we have inserted a minus sign in the definition of  $\mathcal{C}(n)$ , etc., when  $n < 0$ . This is done so that the definitions of  $\mathcal{C}(\cdot)$ ,  $\mathcal{D}(\cdot)$ ,  $d(\cdot)$ , and  $d^*(\cdot)$  are translation invariant.

Let

$$D(t) := (d(t), d^*(t)). \quad (6)$$

For  $n \in \mathbb{N}$  and  $t \in \mathbb{R}$ , let

$$U^n(t) := n^{-1/2}d(nt), \quad V^n(t) := n^{-1/2}d^*(nt), \quad Z_t^n := (U_t^n, V_t^n). \quad (7)$$

For  $p \in [0, 1/2)$ , we also let  $Z = (U, V)$  be a two-sided two-dimensional Brownian motion with  $Z(0) = 0$  and variances and covariances at each time  $t \in \mathbb{R}$  given by

$$\text{Var}(U(t)) = \frac{1-p}{2}|t| \quad \text{Var}(V(t)) = \frac{1-p}{2}|t| \quad \text{Cov}(U(t), V(t)) = \frac{p}{2}|t|. \quad (8)$$

It is shown in [She11, Theorem 2.5] that as  $n \rightarrow \infty$ , the random paths  $t \mapsto n^{-1/2}(\mathcal{C}(nt), \mathcal{D}(nt))$  converge in law in the topology of uniform convergence on compacts to a pair of independent Brownian motions, with respective variances 1 and  $(1-2p) \vee 0$ . The following result is an immediate consequence.

**Theorem 1.5** (Sheffield). For  $p \in (0, 1/2)$ , the random paths  $Z^n$  defined in (7) converge in law in the topology of uniform convergence on compacts to the random path  $Z$  of (8).

Throughout the remainder of this paper, we fix  $p \in (0, 1/2)$  and do not make dependence on  $p$  explicit.

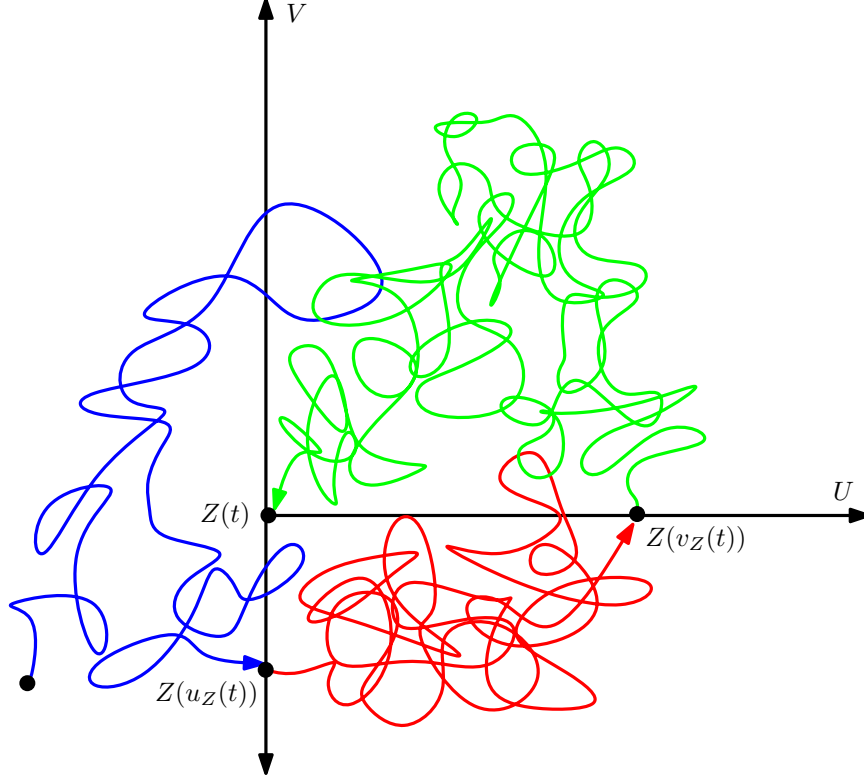


Figure 1: An illustration of a left  $\pi/2$ -cone time  $t$  for a path  $Z = (U, V)$ . The set  $Z([u_Z(t), v_Z(t)])$  is shown in red. The set  $Z([v_Z(t), t])$  is shown in green. We note that we may have  $U_{u_Z(t)} < U_t$  (as shown in the figure) or  $U_{u_Z(t)} \geq U_t$ .

## 1.2 Cone times

The main result of this paper is Theorem 1.9 below, which says that the times for which  $X_i = \boxed{\mathbf{F}}$  converge under a suitable scaling limit to the  $\pi/2$ -cone times of  $Z$ , defined as follows.

**Definition 1.6.** A time  $t$  is called a (weak)  $\pi/2$ -cone time for a function  $Z = (U, V) : \mathbb{R} \rightarrow \mathbb{R}^2$  if there exists  $t' < t$  such that  $U_s \geq U_t$  and  $V_s \geq V_t$  for  $s \in [t', t]$ . Equivalently,  $Z([t', t])$  is contained in the “cone”  $Z_t + \{z \in \mathbb{C} : \arg z \in [0, \pi/2]\}$ . We write  $v_Z(t)$  for the infimum of the times  $t'$  for which this condition is satisfied, i.e.  $v_Z(t)$  is the entrance time of the cone. We say that  $t$  is a *left* (resp. *right*)  $\pi/2$ -cone time if  $V_t = V_{v_Z(t)}$  (resp.  $U_t = U_{v_Z(t)}$ ). Two  $\pi/2$ -cone times for  $Z$  are said to be in the *same direction* if they are both left or both right  $\pi/2$ -cone times, and in the *opposite direction* otherwise. For a  $\pi/2$ -cone time  $t$ , we write  $u_Z(t)$  for the supremum of the times  $t^* < t$  such that

$$\inf_{s \in [t^*, t]} U_s < U_t \quad \text{and} \quad \inf_{s \in [t^*, t]} V_s < V_t.$$

That is,  $u_Z(t)$  is the last time before  $t$  that  $Z$  crosses the boundary line of the cone which it does not cross at time  $v_Z(t)$ .

See Figure 1 for an illustration of Definition 1.6. The reader may easily check that if  $i \in \mathbb{Z}$  is such that  $X_i = \boxed{\mathbf{F}}$ , then  $i/n$  is a (weak)  $\pi/2$ -cone time for  $Z^n$  with  $v_{Z^n}(i/n) = (\phi(i) - 1)/n$ , and the direction of this  $\pi/2$ -cone time is determined by what type of burger  $X_{\phi(i)}$  is (this assertion requires the minus sign discussed in Remark 1.4). We further note that a Brownian motion  $Z$  with variances and covariances as in (8) a.s. has uncountably many  $\pi/2$ -cone times [Shi85, Eva85]. There is a substantial literature concerning cone times of Brownian motion; we refer the reader to [LG92, Sections 3 and 4], [MP10, Section 10.4], and the references therein for more on this topic.

The following remark explains why  $\pi/2$ -cone times are of interest in the study of  $\text{CLE}_\kappa$  and why one should expect these times to be related to the times  $i \in \mathbb{Z}$  for which  $X_i = \boxed{\mathbb{F}}$ .

**Remark 1.7.** In the context of a whole-plane conformal loop ensemble ( $\text{CLE}_\kappa$ ) on a Liouville quantum gravity surface, the quantum lengths  $Z_t = (L_t, R_t)$  of the left and right boundaries of the space-filling  $\text{SLE}_\kappa$  process  $\eta'$  which traces the  $\text{CLE}_\kappa$  loops when it is parametrized by quantum mass has the same law as the Brownian motion (8) (see [DMS14, Theorem 9.1]). In this setting,  $\pi/2$ -cone times for  $Z$ <sup>1</sup> correspond to times at which  $\eta'$  finishes filling in “bubbles” which it disconnects from  $\infty$ , as explained in [DMS14, Figure 1.16]. Each bounded complementary connected component of a CLE loop is such a bubble. The time  $v_Z(t)$  corresponds to the time at which  $\eta'$  disconnects the bubble which it finishes tracing at time  $t$ , and the time  $u_Z(t)$  corresponds to the time at which  $\eta'$  begins tracing the boundary of this bubble. The quantum area and quantum boundary length of this bubble are given, respectively, by  $t - v_Z(t)$ ; and  $L_t - L_{v_Z(t)}$  or  $R_t - R_{v_Z(t)}$ , depending on the direction of the cone time  $t$ .

In the context of an infinite-volume FK-weighted random planar map  $M$ , making sense of “bounded complementary connected components” of FK loops requires a little more thought because these loops do not have self-intersections. One considers bounded complementary connected components of the union of an FK loop  $L$  and the set of quadrilaterals of the quadrangulation  $Q(M)$  associated with  $M$  (as in [She11, Section 4.1]) which intersect  $L$ . Each such complementary connected component  $C$  corresponds under Sheffield’s bijection to a discrete interval of times  $\{\phi(i), \dots, i\}$  with  $X_i = \boxed{\mathbb{F}}$ ; the discrete exploration path of [She11, Section 4.1] traces the last edge in  $C$  at time  $i$  and traces the first edge in  $C$  at time  $\phi(i)$ . It is natural to define the area of the component  $C$  to be the number of edges of  $Q(M)$  it contains, and the boundary length of  $C$  to be the number of edges of  $M$  or its dual graph which are adjacent to the outer boundary of  $C$ . Then the area of  $C$  is  $i - \phi(i)$  and its boundary length is either  $d^*(i) - d^*(\phi(i))$  or  $d(i) - d(\phi(i))$  (as defined in (6)), depending on whether  $X_{\phi(i)} = \textcircled{\mathbb{H}}$  or  $X_{\phi(i)} = \textcircled{\mathbb{C}}$ .

Thus “bounded complementary connected components” of FK loops are described by the times  $i$  for which  $X_i = \boxed{\mathbb{F}}$  in the same manner in which bonded complementary connected components of CLE loops are described by the  $\pi/2$ -cone times of  $Z$ . One can similarly describe other functionals of the CLE loops and the FK loops, respectively, in terms of the  $\pi/2$ -cone times for  $Z$  and the times for which  $X_i = \boxed{\mathbb{F}}$ , e.g. the boundary length of the unbounded complementary connected component of each loop, whether or not two given loops are nested, and whether or not two given loops intersect. For such functionals, one has a similar correspondence between the discrete and continuum descriptions.

A more detailed discussion of the relationship between Sheffield’s inventory accumulation model and loops in the FK model can be found in [BLR15]. More detailed descriptions of several functionals of CLE loops on a quantum gravity surface and the corresponding functionals of FK loops on a random planar map will appear in [GM].

In light of Remark 1.7, it is natural to expect that the times for which  $X_i = \boxed{\mathbb{F}}$  converge in the scaling limit to the  $\pi/2$ -cone times for  $Z$ . This is indeed the case, but one needs to be careful about the precise sense in which this convergence occurs. Indeed, there are uncountably many  $\pi/2$ -cone times for  $Z$ , but only countably many times for which  $X_i = \boxed{\mathbb{F}}$ . To get around this issue, we prove convergence of several large but countable sets of distinguished  $\pi/2$ -cone times which are dense enough to approximate most interesting functionals of the set of  $\pi/2$ -cone times for  $Z$ . One such set is defined as follows.

**Definition 1.8.** A  $\pi/2$ -cone time for  $Z$  is called a *maximal  $\pi/2$ -cone time* in an (open or closed) interval  $I \subset \mathbb{R}$  if  $[v_Z(t), t] \subset I$  and there is no  $\pi/2$ -cone time  $t'$  for  $Z$  such that  $[v_Z(t'), t'] \subset I$  and  $[v_Z(t), t] \subset (v_Z(t'), t')$ . An integer  $i \in \mathbb{Z}$  is called a *maximal flexible order time* in an interval  $I \subset \mathbb{R}$  if  $X_i = \boxed{\mathbb{F}}$ ,  $\{\phi(i), \dots, i\} \subset I$ , and there is no  $i' \in \mathbb{Z}$  with  $\phi(i') = \boxed{\mathbb{F}}$ ,  $\{\phi(i), \dots, i\} \subset \{\phi(i') + 1, \dots, i' - 1\}$ , and  $\{\phi(i'), \dots, i'\} \subset I$ .

We are now ready to state our main result.

**Theorem 1.9.** *Let  $\mathcal{T}$  be the set of  $\pi/2$ -cone times for  $Z$ . Let  $\mathcal{I}$  be the set of  $i \in \mathbb{Z}$  such that  $X_i = \boxed{\mathbb{F}}$  and for  $n \in \mathbb{N}$  let  $\mathcal{T}_n = \{n^{-1}i : i \in \mathcal{I}\}$ . There is a coupling of countably many instances  $(X^n)$  of the infinite word  $X$  described in Section 1.1 with the Brownian motion  $Z$  such that when  $Z^n$  and  $\mathcal{T}_n$  are constructed from  $X^n$ , the following holds a.s.*

<sup>1</sup>The definition of a  $\pi/2$ -cone time used in this paper corresponds to a  $\pi/2$ -cone time for the time reversal of  $Z$  in the terminology of [DMS14].

1.  $Z^n \rightarrow Z$  uniformly on compacts.
2.  $\mathcal{T}$  is precisely the set of limits of convergent sequences  $(t_{n_j}) \in \mathcal{T}_{n_j}$  satisfying  $\liminf_{j \rightarrow \infty} (t_{n_j} - v_{Z^{n_j}}(t_{n_j})) > 0$  as  $(n_j)$  ranges over all strictly increasing sequences of positive integers.
3. For each sequence of times  $t_{n_j} \in \mathcal{T}_{n_j}$  as in condition 2, we have  $\lim_{j \rightarrow \infty} v_{Z^{n_j}}(t_{n_j}) = v_Z(t)$ ,  $\lim_{j \rightarrow \infty} u_{Z^{n_j}}(t_{n_j}) = u_Z(t)$ , and the direction of the  $\pi/2$ -cone time  $t_{n_j}$  is the same as the direction of  $t$  for sufficiently large  $j$ .
4. Suppose given a bounded open interval  $I \subset \mathbb{R}$  with rational endpoints and  $a \in I \cap \mathbb{Q}$ . Let  $t$  be the maximal (Definition 1.8)  $\pi/2$ -cone time for  $Z$  in  $I$  with  $a \in [v_Z(t), t]$ . For  $n \in \mathbb{N}$ , let  $i_n$  be the maximal flexible order time (with respect to  $X^n$ )  $i$  in  $nI$  with  $an \in [\phi(i), i]$  (if such an  $i$  exists) and  $i_n = \lfloor an \rfloor$  otherwise; and let  $t_n = n^{-1}i_n$ . Then a.s.  $t_n \rightarrow t$ .
5. For  $r > 0$  and  $a \in \mathbb{R}$ , let  $\tau^{a,r}$  be the smallest  $\pi/2$ -cone time  $t$  for  $Z$  such that  $t \geq a$  and  $t - v_Z(t) \geq r$ . For  $n \in \mathbb{N}$ , let  $i_n^{a,r}$  be the smallest  $i \in \mathbb{N}$  such that  $X_i^n = \boxed{F}$ ,  $i \geq an$ , and  $i - \phi(i) \geq rn - 1$ ; and let  $\tau_n^{a,r} = n^{-1}i_n^{a,r}$ . We have  $\tau_n^{a,r} \rightarrow \tau^{a,r}$  for each  $(a, r) \in \mathbb{Q} \times (\mathbb{Q} \cap [0, \infty))$ .

Using Theorem 1.9, one can obtain the convergence in law of most reasonable functionals of the sets  $\mathcal{T}_n$  to the corresponding functionals of  $\mathcal{T}$ . By Remark 1.7, this implies the convergence of many quantities associated with the complementary connected components of  $\text{CLE}_\kappa$  loops on a Liouville quantum gravity surface, e.g. the quantum areas of these components, the quantum lengths of their boundaries, or the adjacency graph of the set of loops. Hence Theorem 1.9 provides a complete solution to [DMS14, Question 13.3].

The main difficulty in the proof of Theorem 1.9 is showing that there in fact exist “macroscopic  $\boxed{F}$ -excursions” in the discrete model with high probability when  $n$  is large. More precisely,

**Proposition 1.10.** *For  $\delta > 0$  and  $n \in \mathbb{N}$ , let  $\mathcal{E}_n(\delta)$  be the event that there is an  $i \in \{\lfloor \delta n \rfloor, \dots, n\}$  such that  $X_i = \boxed{F}$  and  $\phi(i) \leq 0$ . Then*

$$\lim_{\delta \rightarrow 0} \liminf_{n \rightarrow \infty} \mathbb{P}(\mathcal{E}_n(\delta)) = 1.$$

We will prove Proposition 1.10 in Section 5.1, via an argument which requires most of the results of the earlier sections of the paper.

**Remark 1.11.** Proposition 1.10 is not obvious from the results of [She11]. At first glance, it may appear that one should be able to obtain large  $\boxed{F}$ -excursions in the discrete model by applying [She11, Theorem 2.5] and considering times  $t$  which are “close” to being  $\pi/2$ -cone times for  $Z^n$ . However, this line of reasoning only yields times  $t$  at which  $U^n(t) \leq U^n(s) + \epsilon$  and  $V^n(t) \leq V^n(s) + \epsilon$  for each  $s \in [t', t]$  for some  $t' < t$ . One still needs Proposition 1.10 or something similar to clear out the remaining  $\epsilon n^{1/2}$  burgers on the stack at time  $\lfloor tn \rfloor$  and produce an actual  $\boxed{F}$ -excursion.

### 1.3 Basic notation

Throughout the remainder of the paper, we will use the following notation.

**Notation 1.12.** *If  $a$  and  $b$  are two quantities, we write  $a \preceq b$  (resp.  $a \succeq b$ ) if there is a constant  $C$  (independent of the parameters of interest) such that  $a \leq Cb$  (resp.  $a \geq Cb$ ). We write  $a \asymp b$  if  $a \preceq b$  and  $a \succeq b$ .*

**Notation 1.13.** *If  $a$  and  $b$  are two quantities which depend on a parameter  $x$ , we write  $a = o_x(b)$  (resp.  $a = O_x(b)$ ) if  $a/b \rightarrow 0$  (resp.  $a/b$  remains bounded) as  $x \rightarrow 0$  (or as  $x \rightarrow \infty$ , depending on context). We write  $a = o_x^\infty(b)$  if  $a = o_x(b^s)$  for each  $s \in \mathbb{R}$ .*

**Notation 1.14.** *For  $a < b \in \mathbb{R}$ , we define the discrete intervals  $[a, b]_{\mathbb{Z}} := [a, b] \cap \mathbb{Z}$  and  $(a, b)_{\mathbb{Z}} := (a, b) \cap \mathbb{Z}$ .*

Unless otherwise stated, all implicit constants in  $\asymp$ ,  $\preceq$ , and  $\succeq$  and  $O_x(\cdot)$  and  $o_x(\cdot)$  errors involved in the proof of a result are required to satisfy the same dependencies as described in the statement of the lemma.

## 1.4 Outline

The remainder of this paper is structured as follows. In Section 2, we prove a variety of probabilistic estimates. These include some estimates for Brownian motion, lower bounds for the probabilities of several rare events associated with the word  $X$  (including the probability that the reduced word is empty), and an upper bound for the number of flexible orders remaining on the stack at a given time. In Section 3, we prove a regularity result for the conditional law of the path  $Z^n$  given that the word  $X(-n, -1)$  contains no burgers. In Section 4, we use said regularity result to prove convergence of the conditional law of  $Z^n|_{[-1,0]}$  given that that  $X(-n, -1)$  has no burgers to the law of a correlated Brownian motion conditioned to stay in the third quadrant. In Section 5, we prove Theorem 1.9.

## 2 Probabilistic estimates

In this section we will prove a variety of probabilistic estimates. In Section 2.1, we will prove some estimates for Brownian motion, mostly using results from [Shi85], and make sense of the notion of a Brownian motion conditioned to stay in the first quadrant. In Section 2.2, we will use our estimates for Brownian motion prove lower bounds for various rare events associated with the word  $X$ . In Section 2.3, we will prove an upper bound for the number of  $\boxed{\mathbb{F}}$ -symbols in the reduced word  $X(1, n)$ , which is a sharper version of [She11, Lemma 3.7]. In Section 2.4, we will prove an explicit power-law lower bound for the probability that the reduced word  $X(1, 2n)$  is empty, thereby confirming a prediction made by Sheffield in [She11]. Several of the estimates in this section are not optimal, and will be improved upon later in the paper. However, the proofs of said improvements require the results of this section.

### 2.1 Brownian motion lemmas

In [Shi85, Theorem 2], the author constructs for each  $\theta \in (0, 2\pi)$  a probability measure on the space of continuous functions  $[0, 1] \rightarrow \mathbb{R}^2$  which can be viewed as the law of a standard two-dimensional Brownian motion (started from 0) conditioned to stay in the cone  $\{z \in \mathbb{C} : 0 \leq \arg z \leq \theta\}$  until time 1. By applying an appropriate linear transformation to a path with this law, we obtain a law on continuous paths in  $\mathbb{R}^2$  which we interpret as that of the correlated two-dimensional Brownian motion  $Z$  in (8) conditioned to stay in the first quadrant until time 1. For  $\alpha = 0$ , this law is uniquely characterized as follows.

**Lemma 2.1.** *Let  $\widehat{Z} = (\widehat{U}, \widehat{V}) : [0, 1] \rightarrow \mathbb{R}^2$  be sampled from the conditional law of  $Z|_{[0,1]}$  given that it stays in the first quadrant. Then  $\widehat{Z}$  is a.s. continuous and satisfies the following conditions.*

1. *For each  $t \in (0, 1]$ , a.s.  $\widehat{U}(t) > 0$  and  $\widehat{V}(t) > 0$ .*
2. *For each  $\zeta \in (0, 1)$ , the conditional law of  $\widehat{Z}|_{[\zeta, 1]}$  given  $\widehat{Z}|_{[0, \zeta]}$  is that of a Brownian motion with covariances as in (8), starting from  $\widehat{Z}(\zeta)$ , parametrized by  $[\zeta, 1]$ , and conditioned on the (a.s. positive probability) event that it stays in the first quadrant.*

*If  $\widetilde{Z} = (\widetilde{U}, \widetilde{V}) : [0, 1] \rightarrow \mathbb{R}^2$  is another random a.s. continuous path satisfying the above two conditions, then  $\widetilde{Z} \stackrel{d}{=} \widehat{Z}$ .*

*Proof of Lemma 2.1.* First we verify that  $\widehat{Z}$  satisfies the above two conditions. It is clear from the form of the density for  $\widehat{Z}_t$  given in [Shi85, Theorem 3] that condition 1 holds. To verify condition 2, fix  $\zeta > 0$ . We have that  $\widehat{Z}$  is the limit in law in the uniform topology as  $\delta \rightarrow 0$  of the law of  $Z|_{[0,1]}$  conditioned on the event  $E_\delta$  that  $U(t) \geq -\delta$  and  $V(t) \geq -\delta$  for each  $t \in [0, 1]$ . By the Markov property, for each  $\zeta > 0$ , the conditional law of  $Z|_{[\zeta, 1]}$  given  $Z|_{[0, \zeta]}$  and  $E_\delta$  is that of a Brownian motion with covariances as in (8), starting from  $Z(\zeta)$ , parametrized by  $[\zeta, 1]$ , and conditioned to stay in the  $\delta$ -neighborhood of the first quadrant. As  $\delta \rightarrow 0$ , this law converges to the law described in condition 2.

Now suppose that  $\widetilde{Z} = (\widetilde{U}, \widetilde{V}) : [0, 1] \rightarrow \mathbb{R}^2$  is another random continuous path satisfying the above two conditions. For  $\zeta > 0$ , let  $\widetilde{Z}^\zeta : [0, 1] \rightarrow \mathbb{R}^2$  be the random continuous path such that  $\widetilde{Z}^\zeta(t) = \widetilde{Z}(t + \zeta)$  for  $t \in [0, 1 - \zeta]$ ; and conditioned on  $\widetilde{Z}|_{[0, 1]}$ ,  $\widetilde{Z}^\zeta$  evolves as a Brownian motion with variances and covariances as in (8) started from  $\widetilde{Z}(1)$  and conditioned to stay in the first quadrant for  $t \in [1 - \zeta, 1]$ . By condition 2



and [Shi85, Theorem 2], we can find  $\epsilon \in (0, \alpha/2)$  such that the Prokhorov distance (in the uniform topology) between the conditional law of  $\tilde{Z}^\zeta$  given any realization of  $\tilde{Z}|_{[0, \zeta]}$  for which  $|\tilde{Z}(\zeta)| \leq \epsilon$  is at most  $\alpha/2$ . By continuity, we can find  $\zeta_0 > 0$  such that for  $\zeta \in (0, \zeta_0]$ , we have  $\mathbb{P}\left(\sup_{t \in [0, \zeta]} |\tilde{Z}(t)| \geq \alpha/2\right) \leq \alpha/2$ . Hence for  $\zeta \in (0, \zeta_0]$  the Prokhorov distance between the law of  $\tilde{Z}^\zeta$  and the law of  $\tilde{Z}$  is at most  $\alpha$ . Since  $\alpha$  is arbitrary we obtain  $\tilde{Z}^\zeta \rightarrow \tilde{Z}$  in law. By continuity,  $\tilde{Z}^\zeta$  converges to  $\tilde{Z}$  in law as  $\zeta \rightarrow 0$ . Hence  $\tilde{Z} \stackrel{d}{=} \hat{Z}$ .  $\square$

**Lemma 2.2.** *Let  $p \in (0, 1/2)$  and  $\kappa \in (4, 8)$  be related as in (1). Let*

$$\mu := \frac{\pi}{2\left(\pi - \arctan \frac{\sqrt{1-2p}}{p}\right)} = \frac{\kappa}{8}, \quad \mu' := \frac{\pi}{2\left(\pi + \arctan \frac{\sqrt{1-2p}}{p}\right)} = \frac{\kappa}{4(\kappa - 2)}. \quad (9)$$

Let  $Z = (U, V)$  be as in (8). For  $\delta > 0$  and  $C > 1$ , let

$$\begin{aligned} E_\delta &:= \left\{ \inf_{t \in [0, 1]} U(t) \geq -\delta^{1/2} \text{ and } \inf_{t \in [0, 1]} V(t) \geq -\delta^{1/2} \right\} \\ E'_\delta &:= \left\{ U(t) \geq -\delta^{1/2} \text{ or } V(t) \geq -\delta^{1/2} \text{ for each } t \in [0, 1] \right\} \\ G(C) &:= \left\{ \sup_{t \in [0, 1]} |Z(t)| \leq C \right\} \cap \{U(1) \geq C^{-1} \text{ and } V(1) \geq C^{-1}\}. \end{aligned}$$

For each  $C > 1$  we have

$$\mathbb{P}(E_\delta \cap G(C)) \asymp \mathbb{P}(E_\delta) \asymp \delta^\mu \quad (10)$$

and

$$\mathbb{P}(E'_\delta \cap G(C)) \asymp \mathbb{P}(E'_\delta) \asymp \delta^{\mu'} \quad (11)$$

with the implicit constants independent of  $\delta$ .

*Proof.* Let

$$A := \begin{pmatrix} 1 & -\frac{p}{\sqrt{1-2p}} \\ 0 & \frac{\sqrt{1-2p}}{1-p} \end{pmatrix}, \quad \tilde{Z} = (\tilde{U}, \tilde{V}) := AZ. \quad (12)$$

Then  $\tilde{Z}$  is a pair of independent Brownian motions. Note that  $A$  maps the first quadrant to the cone

$$F_p := \left\{ w \in \mathbb{C} : 0 < \arg w < \pi - \arctan \frac{\sqrt{1-2p}}{p} \right\} \quad (13)$$

and the complement of the third quadrant to the cone

$$F'_p := \left\{ w \in \mathbb{C} : \arg w \notin \left[ \pi, 2\pi - \arctan \frac{\sqrt{1-2p}}{p} \right] \right\}. \quad (14)$$

Let  $F_p^\delta$  be the  $\delta^{1/2}$ -neighborhood of  $F_p$  and let  $z := \exp\left(\frac{i}{2}\left(\pi - \arctan \frac{\sqrt{1-2p}}{p}\right)\right)$  be the unit vector pointing into  $F_p$ . We have

$$\{\tilde{Z}([0, 1]) \subset F_p^{c_1 \delta}\} \subset \tilde{E}_\delta \subset \{\tilde{Z}([0, 1]) \subset F_p^{c_2 \delta}\}$$

for positive constants  $c_1$  and  $c_2$  depending only on  $A$ . By scale invariance of Brownian motion, we have

$$\delta^\mu \mathbb{P}\left(\tilde{Z}([0, 1]) \subset F_p^\delta\right) = \delta^\mu \mathbb{P}\left(\tilde{Z}([0, \delta^{-1}]) + z \subset F_p\right).$$

By [Shi85, Equation 4.3] this quantity converges to a finite positive constant as  $\delta \rightarrow 0$ . We therefore obtain

$$\mathbb{P}(E_\delta) \asymp \delta^\mu.$$

Similarly, we have

$$\mathbb{P}(E'_\delta) \asymp \delta^{\mu'}.$$

This proves the second proportions in (10) and (11). By [Shi85, Theorem 2], the conditional law of  $\tilde{Z}|_{[0,1]}$  given  $\{\tilde{Z}([0,1]) \subset F_p^\delta\}$  converges in the uniform topology as  $\delta \rightarrow 0$  to the law  $\widehat{\mathbb{P}}$  of a continuous path  $\tilde{Z} : [0,1] \rightarrow \mathbb{C}$  satisfying (with  $G(C)$  as in the statement of the lemma)

$$\widehat{\mathbb{P}}(G(C)) > 0 \quad \forall C > 1, \quad \text{and} \quad \lim_{C \rightarrow \infty} \widehat{\mathbb{P}}(G(C)) = 1.$$

By combining this observation with our argument above, we obtain the first proportion in (10). We similarly obtain the first proportion in (11).  $\square$

## 2.2 Lower bounds for various probabilities

In this section we will prove lower bounds for the probabilities of various rare events associated with the word  $X$ . This will be accomplished by breaking up a segment of the word  $X$  of length  $n$  into sub-words of length approximately  $\delta^k n$  for  $\delta$  small but fixed and  $k \in \mathbb{N}$  such that  $\delta^k n \geq 1$ ; then estimating the probabilities of events for each sub-word using [She11, Theorem 2.5] and Lemma 2.2.

**Lemma 2.3.** *Let  $\mu$  be as in (9). For  $n \in \mathbb{N}$  and  $C > 1$ , let  $R_n(C)$  be the event that the following is true.*

1.  $X(1, n)$  contains no burgers.
2.  $X(1, n)$  contains at least  $C^{-1}n^{1/2}$  hamburger orders, at least  $C^{-1}n^{1/2}$  cheeseburger orders, and at most  $Cn^{1/2}$  total orders.

Also let  $R_n^*(C)$  be the event that the following is true.

1.  $X(1, n)$  contains no orders.
2.  $X(1, n)$  contains at least  $C^{-1}n^{1/2}$  burgers of each type and at most  $Cn^{1/2}$  total burgers.

If  $C$  is chosen sufficiently large, then

$$\mathbb{P}(R_n(C)) \geq n^{-\mu + o_n(1)} \tag{15}$$

and

$$\mathbb{P}(R_n^*(C)) \geq n^{-\mu + o_n(1)}. \tag{16}$$

**Remark 2.4.** We will prove a sharper version of the estimate (15) later, which also includes an upper bound (see Proposition 5.1 below).

**Remark 2.5.** As explained in [BLR15], Lemma 2.3 and the stronger Proposition 5.1 can be viewed as estimates for the area of the "envelope of a generic loop" in an FK-weighted random planar map  $M$ . The paper [BLR15] obtains asymptotics (including upper bounds, but not regular variation) for the area and length of a full generic loop.

*Proof of Lemma 2.3.* We will prove (15). We find it more convenient to do this with the word  $X(-n, -1)$  in place of the word  $X(1, n)$ . This suffices by translation invariance. The estimate (16) is proven similarly, but with the word  $X$  read in the forward rather than the reverse direction.

Fix  $C > 1$  and  $\delta < 1/4C^2$ , to be chosen later independently of  $n$ . Let

$$K_n := \left\lceil \frac{\log n}{\log \delta^{-1}} \right\rceil \tag{17}$$

be the smallest integer  $k$  such that  $\delta^k n \leq 1$  and for  $k \in [0, K_n]_{\mathbb{Z}}$  let  $m_n^k := \lfloor \delta^k n \rfloor$ . Also let  $E_k$  be the event that the following is true.

1.  $X(-m_n^{k-1}, -m_n^k - 1)$  has at most  $0 \vee (C^{-1}(\delta^k n)^{1/2} - 1)$  burgers of each type.
2.  $C^{-1}(\delta^{k-1} n)^{1/2} \leq \mathcal{N}_\theta(X(-m_n^{k-1}, -m_n^k - 1)) \leq C(\delta^{k-1} n)^{1/2}$  for  $\theta \in \{\overline{\mathbb{H}}, \overline{\mathbb{C}}\}$ .

Observe that on  $\bigcap_{k=1}^{K_n} E_k$ , the word  $X(-n, -1)$  has empty burger stack and contains at most

$$2Cn^{1/2} \sum_{k=1}^{\infty} \delta^{\frac{k-1}{2}} \leq 4Cn^{1/2}$$

total orders. Furthermore, since  $X(-n, -m_n^1)$  contains at least  $C^{-1}n^{1/2}$  hamburger orders and at least the same number of cheeseburger orders, so does  $X(-n, -1)$ . Consequently, if  $m$  is chosen sufficiently large then

$$\bigcap_{k=1}^{K_n} E_k \subset R_n^-(4C), \quad (18)$$

where  $R_n^-(4C)$  is defined in the same manner as  $R_n(4C)$  but with  $X_{-n} \dots X_{-1}$  in place of  $X_1 \dots X_n$ . The events  $E_k$  for  $k \in [1, K_n]_{\mathbb{Z}}$  are independent, and by translation invariance  $\mathbb{P}(R_n^-(4C)) = \mathbb{P}(R_n(4C))$ . So, to obtain (15) (with  $4C$  in place of  $C$ ) we just need to prove a suitable lower bound for  $\mathbb{P}(E_k)$ .

To this end, fix a deterministic sequence  $\xi = (\xi_j)$  with  $\xi_j = o_j(\sqrt{j})$  and for  $k \in [1, K_n]_{\mathbb{Z}}$  let  $\tilde{E}_k$  be the event that the following is true.

1.  $\inf_{j \in [m_n^k+1, m_n^{k-1}]_{\mathbb{Z}}} (d(-j) - d(-m_n^k - 1)) \geq -\left(0 \vee (C^{-1}(\delta^k n)^{1/2} - 1 - \xi_{m_n^{k-1}})\right)$  and similarly with  $d^*$  in place of  $d$ .
2.  $C^{-1}(\delta^{k-1} n)^{1/2} + \xi_{m_n^{k-1}} \leq d(-m_n^{k-1}) - d(-m_n^k - 1) \leq C(\delta^{k-1} n)^{1/2} - \xi_{m_n^{k-1}}$  and similarly with  $d^*$  in place of  $d$ .
3.  $\mathcal{N}_{\boxed{F}}(X(-m_n^{k-1}, -m_n^k - 1)) \leq \xi_{m_n^{k-1}}$ .

Observe that  $\tilde{E}_k \subset E_k$ . By [She11, Lemma 3.7], we can choose  $\xi$  in such a way that it holds with probability tending to 1 as  $m \rightarrow \infty$  that  $X(1, m)$  has at most  $\xi_m$  flexible orders. By [She11, Theorem 2.5], it follows that as  $n \rightarrow \infty$ , the probability of the event  $\tilde{E}_1$  converges to the probability of the event that  $Z$  stays within the  $C^{-1}\delta^{1/2}$ -neighborhood of the first quadrant in the time interval  $[0, 1 - \delta]$  and satisfies  $C^{-1} \leq -U(1) \leq C$  and  $C^{-1} \leq -V(1) \leq C$ . By Lemma 2.2 this latter event has probability  $\geq \delta^\mu$  with the implicit constant independent of  $\delta$ . Hence we can find  $b \in (0, 1)$ , independent of  $\delta$ , and  $m_* = m_*(\delta, C, \xi)$  such that whenever  $m_n^k \geq m_*$ , we have  $\mathbb{P}(\tilde{E}_k) \geq b\delta^\mu$  (here we use that  $E_k$  and  $\tilde{E}_k$  are defined in the same manner as  $E_1$  and  $\tilde{E}_1$  but with  $\delta^{k-1}n$  in place of  $n$ ).

Let  $k_*$  be the largest  $k \in [1, K_n]_{\mathbb{Z}}$  for which  $m_n^k \leq m_*$ . Then

$$\mathbb{P}\left(\bigcap_{k=1}^{k_*} E_k\right) \geq b^{k_*} \delta^{k_*\mu} \geq b^{K_n} \delta^{K_n\mu} \geq n^{-\mu + o_\delta(1)},$$

with the  $o_\delta(1)$  independent of  $n$ . Since the event  $\bigcap_{k=k_*+1}^{K_n} E_k$  involves only the word  $X_1 \dots X_{m_*}$ ,  $\mathbb{P}\left(\bigcap_{k=k_*+1}^{K_n} E_k\right)$  is at least a positive constant which does not depend on  $n$ . We infer from (18) that

$$\mathbb{P}(R_n^-(4C)) \geq n^{-\mu - \zeta + o_\delta(1)},$$

with the implicit constant depending on  $\delta$ , but not  $n$ . Since  $\delta$  is arbitrary, this implies (15).  $\square$

From Lemma 2.3, we obtain the following.

**Proposition 2.6.** *Almost surely, there are infinitely many  $i \in \mathbb{N}$  for which  $X(1, i)$  contains no burgers; infinitely many  $j \in \mathbb{N}$  for which  $X(-j, -1)$  contains no orders; and infinitely many  $\boxed{F}$ -symbols in  $X(1, \infty)$ .*

*Proof.* For  $m \in \mathbb{N}$ , let  $K_m$  be the  $m$ th smallest  $i \in \mathbb{N}$  for which  $X(1, i)$  contains no burgers (or  $K_m = \infty$  if there are fewer than  $m$  such  $i$ ). Observe that  $K_m$  can equivalently be described as the smallest  $i \geq K_{m-1} + 1$  for which  $X(K_{m-1} + 1, i)$  contains no burgers. Hence the words  $X_{K_{m-1}+1} \dots X_{K_m}$  are iid. It follows that

$\{K_m\}_{m \in \mathbb{N}}$  is a renewal process. Note that  $i \in \mathbb{N}$  is equal to one of the times  $K_m$  if and only if the word  $X(1, i)$  contains no burgers. By Lemma 2.3, we thus have

$$\sum_{i=1}^{\infty} \mathbb{P}(i = K_m \text{ for some } m \in \mathbb{N}) \geq \sum_{i=1}^{\infty} i^{\mu+o_i(1)} = \infty$$

since  $\mu < 1$ . By elementary renewal theory,  $K_1$  is a.s. finite, whence there are a.s. infinitely many  $i \in \mathbb{N}$  for which  $X(1, i)$  contains no burgers. We similarly deduce from (16) that there are a.s. infinitely many  $j \in \mathbb{N}$  for which  $X(-j, -1)$  contains no orders. To obtain the last statement, we note that for each  $m \in \mathbb{N}$ , we have  $\mathbb{P}(X_{K_{m+1}} = \boxed{\text{F}}) = p/2$ , so there are a.s. infinitely many  $m \in \mathbb{N}$  for which  $X_{K_{m+1}} = \boxed{\text{F}}$ . For each such  $m$ , a  $\boxed{\text{F}}$  symbol is added to the order stack at time  $K_{m+1}$ .  $\square$

Next we consider an analogue of Lemma 2.3 which involves  $3\pi/2$ -cone times instead of  $\pi/2$ -cone times.

**Lemma 2.7.** *For  $n \in \mathbb{N}$  and  $C > 4$ , let  $R'_n(C)$  be the event that the following is true.*

1.  $X(1, i)$  contains a burger for each  $i \in [1, n]_{\mathbb{Z}}$ .
2.  $X(1, n)$  contains at least  $C^{-1}n^{1/2}$  hamburger orders and at least  $C^{-1}n^{1/2}$  cheeseburger orders.
3.  $|X(1, n)| \leq Cn^{1/2}$ .

Also let  $(R'_n)^*(C)$  be the event that the following is true.

1.  $X(-j, -1)$  contains either a hamburger order or a cheeseburger order for each  $j \in [1, n]_{\mathbb{Z}}$ .
2.  $X(-n, -1)$  contains at least  $C^{-1}n^{1/2}$  burgers of each type and at most  $Cn^{1/2}$  total burgers.
3.  $|X(-n, -1)| \leq Cn^{1/2}$ .

For  $C > 4$  we have

$$\mathbb{P}(R'_n(C)) \geq n^{-\mu' + o_n(1)} \quad (19)$$

and

$$\mathbb{P}((R'_n)^*(C)) \geq n^{-\mu' + o_n(1)} \quad (20)$$

with  $\mu'$  as in (9).

*Proof.* We will prove (19). The estimate (20) is proven similarly, but with the word  $X$  read in the reverse, rather than the forward, direction.

Fix  $C > 4$ ,  $\delta \in (0, (8C)^{-2}]$ , and a deterministic sequence  $\xi = (\xi_j)$  with  $\xi_j = o_j(\sqrt{j})$  to be chosen later independently of  $n$ . We assume  $\xi_j \leq \delta j^{1/2}$  for each  $j \in \mathbb{N}$ . Let  $K_n$  be as in (17) and let  $m_n^k = \lfloor \delta^k n \rfloor$  for  $k \in [0, K_n]_{\mathbb{Z}}$  be defined in the discussion thereafter. For  $k \in [1, K_n]_{\mathbb{Z}}$ , let  $E'_k$  be the event that the following is true.

1. For each  $i \in [m_n^k + 1, m_n^{k-1}]_{\mathbb{Z}}$ , at least one of the following three conditions holds:  $\mathcal{N}_{\boxed{\text{H}}}(X(m_n^k + 1, i)) \leq 0 \vee (C^{-1}(\delta^k n)^{1/2} - \xi_{m_n^{k-1}})$ ;  $\mathcal{N}_{\boxed{\text{C}}}(X(m_n^k + 1, i)) \leq 0 \vee (C^{-1}(\delta^k n)^{1/2} - \xi_{m_n^{k-1}})$ ; or  $X(m_n^k + 1, i)$  contains a burger.
2.  $\mathcal{N}_{\theta}(X(m_n^k + 1, m_n^{k-1})) \geq C^{-1}(\delta^{k-1} n)^{1/2}$  for  $\theta \in \{\boxed{\text{H}}, \boxed{\text{C}}\}$ .
3.  $\mathcal{N}_{\theta}(X(m_n^k + 1, m_n^{k-1})) \geq C^{-1}(\delta^{k-1} n)^{1/2} - \xi_{m_n^{k-1}}$  for  $\theta \in \{\boxed{\text{H}}, \boxed{\text{C}}\}$ .
4.  $|X(m_n^k + 1, m_n^{k-1})| \leq C(\delta^{k-1} n)^{1/2}$ .
5.  $\mathcal{N}_{\boxed{\text{F}}}(X(m_n^k + 1, m_n^{k-1})) \leq \xi_{m_n^{k-1}}$ .

We claim that

$$\bigcap_{k=1}^{K_n} E'_k \subset R'_n(8C). \quad (21)$$

First we observe that conditions 1, 2, and 5 in the definition of  $E'_k$  imply that condition 1 in the definition of  $R'_n(8C)$  holds on  $\bigcap_{k=1}^{K_n} E'_k$ . From condition 3 and 4 in the definition of  $E'_k$ , we infer that on  $\bigcap_{k=1}^{K_n} E'_k$ , we have for  $\theta \in \{\overline{\mathbf{H}}, \overline{\mathbf{C}}\}$  that

$$\begin{aligned} \mathcal{N}_\theta(X(1, n)) &\geq C^{-1}n^{1/2} - \xi_{m_n^0} - Cn^{1/2} \sum_{k=2}^{K_n} \delta^{(k-1)/2} \\ &\geq \frac{1}{2}C^{-1}n^{1/2} - 2\delta^{1/2}Cn^{1/2} \geq \frac{1}{8}C^{-1} \end{aligned}$$

where the last inequality is by our choice of  $\delta$ . Thus condition 2 in the definition of  $R'_n(8C)$  holds. Finally, it is clear from condition 4 in the definition of  $E'_k$  that condition 3 in the definition of  $R'_n(8C)$  holds on  $\bigcap_{k=1}^{K_n} E'_k$ . This completes the proof of (21).

The events  $E'_k$  for  $k \in [1, K_n]_{\mathbb{Z}}$  are independent, so in light of (21), to obtain (19) (with  $8C$  in place of  $C$ ) we just need to prove a suitable lower bound for  $\mathbb{P}(E'_k)$ . To this end, for  $k \in [1, K_n]_{\mathbb{Z}}$  let  $\tilde{E}'_k$  be the event that the following is true.

1. For each  $i \in [m_n^k + 1, m_n^{k-1}]_{\mathbb{Z}}$ , either  $d(i) - d(m_n^k + 1) \geq 0 \wedge \left(-C^{-1}(\delta^k n)^{1/2} + \xi_{m_n^{k-1}}\right)$  or  $d^*(i) - d^*(m_n^k + 1) \geq 0 \wedge \left(-C^{-1}(\delta^k n)^{1/2} + \xi_{m_n^{k-1}}\right)$ .
2.  $d(m_n^{k-1}) - d(m_n^k + 1)$  and  $d^*(m_n^{k-1}) - d^*(m_n^k + 1)$  are each at least  $C^{-1}(\delta^{k-1}n)^{1/2}$ .
3.  $\inf_{i \in [m_n^k + 1, m_n^{k-1}]_{\mathbb{Z}}} (d(i) - d(m_n^k + 1)) \leq -C^{-1}(\delta^{k-1}n)^{1/2} - \xi_{m_n^{k-1}}$  and similarly with  $d^*$  in place of  $d$ .
4.  $\sup_{i \in [m_n^k + 1, m_n^{k-1}]_{\mathbb{Z}}} |D(i)| \leq (C/2)(\delta^{k-1}n)^{1/2} - \xi_{m_n^{k-1}}$ .
5.  $\mathcal{N}_{\overline{\mathbf{F}}}(X(m_n^k + 1, m_n^{k-1})) \leq \xi_{m_n^{k-1}}$ .

We claim that  $\tilde{E}'_k \subset E'_k$ . It is clear that conditions 2, and 5 in the definition of  $\tilde{E}'_k$  imply the corresponding conditions in the definition of  $E'_k$ . Since the running infima of  $d$  and  $d^*$  up to time  $n$  differ from  $\mathcal{N}_{\overline{\mathbf{H}}}(X(1, n))$  and  $\mathcal{N}_{\overline{\mathbf{C}}}(X(1, n))$ , respectively, by at most  $\mathcal{N}_{\overline{\mathbf{F}}}(X(1, n))$ , we find that conditions 3 and 4 imply the corresponding conditions in the definition of  $E'_k$ .

Suppose condition 1 in the definition of  $\tilde{E}'_k$  holds. If  $i \in [m_n^k + 1, m_n^{k-1}]_{\mathbb{Z}}$  and  $X(m_n^k + 1, i)$  contains no burgers, then the condition  $d(i) - d(m_n^k + 1) \geq 0 \wedge \left(-\epsilon(\delta^k n)^{1/2} + \xi_{m_n^{k-1}}\right)$  together with condition 5 in the definition of  $\tilde{E}'_k$  implies  $\mathcal{N}_{\overline{\mathbf{H}}}(X(m_n^k + 1, i)) \leq 0 \vee \left(\epsilon(\delta^k n)^{1/2} - \xi_{m_n^{k-1}}\right)$ . A similar statement holds if  $d^*(i) - d^*(m_n^k + 1) \geq 0 \wedge \left(-\epsilon(\delta^k n)^{1/2} + \xi_{m_n^{k-1}}\right)$ . This proves our claim.

It now follows from the results of [She11] together with Lemma 2.2 (c.f. the proof of Lemma 2.3) that if  $\xi$  is chosen appropriately (independently of  $n$ ) then there is a constant  $b \in (0, 1)$ , independent of  $n$  and  $\delta$ , and a constant  $m_* = m_*(\delta, \epsilon, \xi)$  such that whenever  $m_n^k \geq m_*$ , we have  $\mathbb{P}(E'_k) \geq b\delta^{\mu'}$ . We conclude exactly as in the proof of Lemma 2.3.  $\square$

### 2.3 Estimate for the number of flexible orders

The main goal of this section is to prove the following more quantitative version of [She11, Lemma 3.7], which will turn out to be a relatively straightforward consequence of Lemma 2.7.

**Lemma 2.8.** Let  $\mu'$  be as in (9). For each  $n \in \mathbb{N}$  and each  $\nu > \mu'$  we have

$$\mathbb{P} \left( \exists i \geq n \text{ with } \mathcal{N}_{\square}^{\mathbb{F}}(X(1, i)) \geq i^\nu \right) = o_n^\infty(n) \quad (22)$$

(recall notation 1.13). The same holds if we fix  $C > 1$  condition on the event  $\{X(1, n') \text{ has no burgers}\}$  for some  $n' \in [n, Cn]_{\mathbb{Z}}$ , with the  $o_n^\infty(n)$  depending on  $C$  but not  $n'$ .

**Remark 2.9.** Since  $\mu' \in (1/3, 1/2)$  for each  $p \in (0, 1/2)$ , we have in particular that (22) holds for some  $\nu < 1/2$ . In other words, with high probability the number of flexible orders in  $X(1, i)$  is of strictly smaller polynomial order than the length of  $X(1, i)$ , for each  $i \geq n$ .

**Remark 2.10.** The exponent  $\mu'$  in Lemma 2.8 is not optimal. We will show in Corollary 5.2 below that  $\mu'$  can be replaced by  $1 - \mu \leq \mu'$ . However, the proof of Corollary 5.2 indirectly uses Lemma 2.8.

**Lemma 2.11.** For  $i \in \mathbb{N}$ , let  $E_i$  be the event that  $X(1, i)$  contains no burgers. Let  $0 = i_0 < i_1 < \dots < i_n \in \mathbb{N}$ . Then we have

$$\mathbb{P} \left( \bigcap_{k=1}^n E_{i_k} \right) = \prod_{k=1}^n \mathbb{P}(E_{i_k - i_{k-1}}). \quad (23)$$

Furthermore, for each  $i < j \in \mathbb{N}$  we have

$$2^{i-j} \mathbb{P}(E_i) \leq \mathbb{P}(E_j) \leq \mathbb{P}(E_i). \quad (24)$$

*Proof.* Let  $i' > i$ . If  $E_i$  occurs, then  $E_{i'}$  occurs if and only if  $X(i+1, i')$  contains no burgers. By independence of  $X_1 \dots X_i$  and  $X_{i+1} \dots X_{i'}$  and translation invariance, we have

$$\mathbb{P}(E_{i'} | X_1 \dots X_i) = \mathbb{P}(X(i+1, i') \text{ contains no burgers}) = \mathbb{P}(X(1, i' - i) \text{ contains no burgers}).$$

Hence, in the setting of (23) we have

$$\mathbb{P} \left( \bigcap_{k=1}^n E_{i_k} \mid \bigcap_{k=1}^{n-1} E_{i_k} \right) = \mathbb{P}(E_{i_n - i_{n-1}}),$$

so

$$\mathbb{P} \left( \bigcap_{k=1}^n E_{i_k} \right) = \mathbb{P}(E_{i_n - i_{n-1}}) \mathbb{P} \left( \bigcap_{k=1}^{n-1} E_{i_k} \right).$$

We can now obtain (23) by induction on  $n$ .

The lower bound in (24) is immediate from (23) (note  $\mathbb{P}(E_1) = 1/2$ ). For the upper bound, let  $J$  be the smallest  $J \in \mathbb{N}$  for which  $X(-J, -1)$  contains a burger. Then we have

$$\mathbb{P}(E_i) = \mathbb{P}(X(-i, -1) \text{ contains no burgers}) = \mathbb{P}(J > i),$$

which is manifestly decreasing in  $i$ . □

**Lemma 2.12.** For  $n \in \mathbb{N}$ , let  $B_n$  be the number of  $i \in [1, n]_{\mathbb{Z}}$  for which  $X(1, n)$  has empty burger stack. Let  $\mu'$  be as in (9). Then for  $k \in \mathbb{N}$  we have

$$\mathbb{E}(B_n^k) \leq n^{k\mu' + o_n(1)}. \quad (25)$$

*Proof.* We first consider the case  $k = 1$ . Let  $K_0 = 0$  and for  $m \in \mathbb{N}$ , let  $K_m$  be the  $m$ th smallest  $i \in \mathbb{N}$  such that  $X(1, i)$  has empty burger stack. Note that the times  $K_m - K_{m-1}$  are iid and each has the same law as  $K_1$ . Furthermore, we have

$$B_n = \sup\{m \in \mathbb{N} : K_m \leq n\}. \quad (26)$$

Therefore,

$$\mathbb{P}(B_n > m) \leq \mathbb{P}(K_m \leq n) \leq \mathbb{P} \left( \max_{j \in [1, m]_{\mathbb{Z}}} (K_j - K_{j-1}) \leq n \right) = \mathbb{P}(K_1 \leq n)^m.$$

On the event  $\{K_1 > n\}$ ,  $X(1, i)$  contains a burger for each  $i \in [1, n]_{\mathbb{Z}}$ . By Lemma 2.7 we therefore have

$$\mathbb{P}(K_1 \leq n)^m \leq \left(1 - n^{-\mu' + o_n(1)}\right)^m.$$

Hence

$$\mathbb{E}(B_n) \leq \sum_{m=1}^n \left(1 - n^{-\mu' + o_n(1)}\right)^m = n^{\mu' + o_n(1)}.$$

Now consider the case  $k > 1$ . Define the events  $E_i$  as in Lemma 2.11. By Lemma 2.11 we have

$$\begin{aligned} \mathbb{E}(B_n^k) &\leq \sum_{i=1}^n \sum_{i \leq j_1, \dots, j_{k-1} \leq n} \mathbb{P}(E_i \cap E_{j_1} \cap \dots \cap E_{j_{k-1}}) \\ &\leq \sum_{i=1}^n \mathbb{P}(E_i) + \sum_{i=1}^n \sum_{m=1}^{k-1} \sum_{i < j_1 < \dots < j_m \leq n} \mathbb{P}(E_i \cap E_{j_1} \cap \dots \cap E_{j_m}) \\ &= \sum_{i=1}^n \mathbb{P}(E_i) + \sum_{i=1}^n \mathbb{P}(E_i) \sum_{m=1}^{k-1} \sum_{i < j_1 < \dots < j_m \leq n} \mathbb{P}(E_{j_1-i} \cap \dots \cap E_{j_m-i}) \\ &\leq \sum_{i=1}^n \mathbb{P}(E_i) + \sum_{i=1}^n \mathbb{P}(E_i) \sum_{m=1}^{k-1} \sum_{1 \leq j_1 < \dots < j_m \leq n} \mathbb{P}(E_{j_1} \cap \dots \cap E_{j_m}) \\ &\leq \mathbb{E}(B_n) \sum_{m=0}^{k-1} \mathbb{E}(B_n^m), \end{aligned} \tag{27}$$

with implicit constants depending on  $k$ , but not  $n$ . We can now obtain (25) by induction on  $k$ . □

*Proof of Lemma 2.8.* Let

$$A_n := \left\{ \exists i \geq n \text{ with } \mathcal{N}_{\boxed{\mathbb{F}}}(X(1, i)) \geq i^\nu \right\}$$

be the event of (22). By Lemma 2.12, for  $k \in \mathbb{N}$  and  $i \in \mathbb{N}$  we have (in the notation of that lemma)

$$\mathbb{E} \left( \mathcal{N}_{\boxed{\mathbb{F}}}(X(1, i))^k \right) \leq \mathbb{E}(B_i^k) \leq i^{k\mu' + o_i(1)}.$$

By the Chebyshev inequality,

$$\mathbb{P} \left( \mathcal{N}_{\boxed{\mathbb{F}}}(X(1, i)) \geq i^\nu \right) \leq i^{k(\mu' - \nu) + o_i(1)}.$$

By the union bound,

$$\mathbb{P}(A_n) \leq \sum_{i=1}^{\infty} i^{k(\mu' - \nu) + o_i(1)} \leq n^{1 + k(\mu' - \nu) + o_i(1)}.$$

Since  $k$  is arbitrary we obtain (22). For the statement about the conditional law, we use Lemma 2.3 to get that for each  $n \in \mathbb{N}$ ,

$$\mathbb{P}(A_n \mid X(1, n') \text{ has no burgers}) \leq \frac{\mathbb{P}(A_n)}{\mathbb{P}(X(1, n') \text{ has no burgers})} \leq n^{k(\mu' - \nu) + \mu + o_n(1)},$$

with the  $o_n(1)$  depending on  $n'/n$ . We then conclude as above. □

## 2.4 Probability that the reduced word is empty

In this section, we will prove a polynomial lower bound for the probability that the reduced word  $X(1, 2n)$  is empty. We expect, but do not prove, that this lower bound is sharp. The result of this section is not needed later in the paper, but confirms a prediction made by Sheffield in [She11, Section 4.2]. We will prove several additional estimates (using different methods) for events related to an empty reduced word in [GS].

**Proposition 2.13.** *Let  $\mu$  be as in (9). For each  $n \in \mathbb{N}$ ,*

$$\mathbb{P}(X(1, 2n) = \emptyset) \geq n^{-2\mu-1+o_n(1)}. \quad (28)$$

The proof of Proposition 2.13 uses a similar idea to the proofs of Lemmas 2.3 and 2.7. First we need to estimate an appropriate probability for Brownian motion.

**Lemma 2.14.** *Fix a constant  $C > 1$ . Let  $z \in [C^{-1}, C]^2$ . Let  $Z$  be a Brownian motion as in (8) (started from 0). For  $\delta > 0$ , let  $F_\delta^z$  be the event that  $U(t) \geq -\delta^{1/2}$  and  $V(t) \geq -\delta^{1/2}$  for each  $t \in [0, 1]$ ; and  $|Z(1) - z| \leq \delta^{1/2}$ . Then  $\mathbb{P}(F_\delta^z) \geq \delta^{\mu+1}$ , where  $\mu$  is as in (9) and the implicit constant depends on  $C$ , but not  $z$  or  $\delta$ .*

*Proof.* Let  $\tilde{E}_\delta$  be the event that  $U(t) \geq -\delta^{1/2}$  and  $V(t) \geq -\delta^{1/2}$  for each  $s \in [0, 1/2]$ ; and  $Z(1/2) \in [C^{-1}, C]^2$ . By Lemma 2.2 and scale invariance, we have  $\mathbb{P}(\tilde{E}_\delta) \asymp \delta^\mu$ . The conditional law of  $Z|_{[1/2, 1]}$  given  $Z|_{[0, 1/2]}$  is that of a Brownian motion with covariances as in (8) started from  $Z(1/2)$ . On the event  $\tilde{E}_\delta$ , the probability that such a Brownian motion stays in the first quadrant until time 1 and satisfies  $|Z(1) - z| \leq \delta^{1/2}$  is proportional to  $\delta$ . Hence  $\mathbb{P}(F_\delta^z | \tilde{E}_\delta) \asymp \delta$ . The statement of the lemma follows.  $\square$

*Proof of Proposition 2.13.* We find it more convenient to prove (28) with  $X(-2n, -1)$  in place of  $X(1, 2n)$  (which we can do by translation invariance). Fix  $\delta > 0$  and  $C > 1$ , to be chosen later. Let  $K_n$  be as in (17) and let  $m_n^k = \lfloor \delta^k n \rfloor$  for  $k \in [0, K_n]_{\mathbb{Z}}$  be defined in the discussion thereafter.

Let  $b_0^H$  (resp.  $b_0^C$ ) be the number of hamburgers (resp. cheeseburgers) in  $X(-2n, -n-1)$ . For  $k \in [0, K_n]_{\mathbb{Z}}$  let  $b_k^H$  (resp.  $b_k^C$ ) be the number of hamburgers (resp. cheeseburgers) in  $X(-2n, -m_n^k - 1)$ .

Let  $\mu'$  be as in (9) and fix  $\nu \in (\mu', 1/2)$ . Let  $G_0 = R_n^*(C/7)$  be the event of Lemma 2.3, but with  $X_{-2n} \dots X_{-n-1}$  in place of  $X_1 \dots X_n$  and  $C/7$  in place of  $C$ . For  $k \in [1, K_n]_{\mathbb{Z}}$ , let  $G_k$  be the event that the following is true.

1.  $\mathcal{N}_\theta(X(-m_n^{k-1}, -m_n^k - 1)) \leq 0 \vee (C^{-1}(\delta^k n)^{1/2} - (\delta n)^{\nu(k-1)} - 1)$  for  $\theta \in \{\textcircled{\text{H}}, \textcircled{\text{C}}\}$ .
2.  $\mathcal{N}_{\textcircled{\text{F}}}(X(-m_n^{k-1}, -m_n^k - 1)) \leq (\delta n)^{\nu(k-1)}$ .
3.  $b_{k-1}^H - 4C^{-1}(\delta^k n)^{1/2} - (\delta n)^{\nu(k-1)} \leq \mathcal{N}_\theta(X(m_n^{k-1} + 1, m_n^k)) \leq b_{k-1}^H - 3C^{-1}(\delta^k n)^{1/2} - (\delta n)^{\nu(k-1)}$  for  $\theta \in \{\textcircled{\text{H}}, \textcircled{\text{C}}\}$ .

By Lemma 2.3, if  $C$  is chosen sufficiently large then we have  $\mathbb{P}(G_0) \geq n^{-\mu+o_n(1)}$ . By inspection, if  $k \in [1, K_n]_{\mathbb{Z}}$  and  $m_n^k$  is sufficiently large then on  $G_k$ , the word  $X(-2n, -m_n^k - 1)$  contains no orders and

$$C^{-1}(\delta^k n)^{1/2} \leq b_k^H \leq 6C^{-1}(\delta^k n)^{1/2} \quad \text{and} \quad C^{-1}(\delta^k n)^{1/2} \leq b_k^C \leq 6C^{-1}(\delta^k n)^{1/2}. \quad (29)$$

By [She11, Theorem 2.5], Lemma 2.8, and Lemma 2.14, we can choose  $m_* \in \mathbb{N}$ , independent of  $n$ , in such a way that there is a deterministic constant  $c \in (0, 1)$ , independent of  $n$  and  $\delta$ , such that whenever  $k \in [1, K_n]_{\mathbb{Z}}$  with  $m_n^k \geq m_*$ , (29) holds on the event  $G_k$  and

$$\mathbb{P}(G_k | X_{-2n} \dots X_{-m_n^k}) \mathbb{1}_{G_{k-1}} \geq c\delta^{\mu+1} \mathbb{1}_{G_{k-1}}.$$

Let  $k_*$  be the largest  $k \in [1, K_n]_{\mathbb{Z}}$  for which  $m_n^k \geq m_*$ . Then

$$\mathbb{P}\left(\bigcap_{k=0}^{k_*} G_k\right) \geq c^{K_n} \delta^{K_n(\mu+1)} n^{-\mu+o_n(1)} \geq n^{-2\mu-1+o_n(1)+o_\delta(1)} \quad (30)$$



with the  $o_\delta(1)$  independent of  $n$ . It follows from (29) that on  $\bigcap_{k=0}^{k_*} G_k$ , the word  $X(-2n, -m_n^k - 1)$  contains no orders and fewer than  $m_n^{k_*}$  burgers. Since  $m_*$  does not depend on  $n$  and  $m_n^{k_*} \leq \delta^{-1}m_*$ , we have

$$\mathbb{P} \left( X(1, 2n) = \emptyset \mid \bigcap_{k=0}^{k_*} G_k \right) \geq 1, \quad (31)$$

with the implicit constant independent of  $n$ . By combining (30) and (31) and using that  $\delta$  is arbitrary, we obtain the statement of the proposition.  $\square$

### 3 Regularity conditioned on no burgers

#### 3.1 Statement and overview of the proof

The goal of this section is to prove a regularity statement for the conditional law of the word  $X(1, n)$  given the event that it contains no burgers. It will be convenient to read the word backwards, rather than forward, so we will mostly work with  $X(-n, -1)$  instead of  $X(1, n)$ .

We will use the following notation. Let  $J$  be the smallest  $j \in \mathbb{N}$  for which  $X(-j, -1)$  contains a burger. Note that  $\{J > n\}$  is the same as the event that  $X(-n, -1)$  contains no burgers. Let  $\mu'$  be as in Lemma 2.8 and fix  $\nu \in (\mu', 1/2)$ . Let  $F_n$  be the event that  $\mathcal{N}_{\mathbb{F}}(X(-n, -1)) \leq n^\nu$ , so that by Lemma 2.8 we have  $\mathbb{P}(F_n) \geq 1 - o_n^\infty(n)$ . For  $\epsilon > 0$  and  $n \in \mathbb{N}$ , let  $E_n(\epsilon)$  be the event that  $J > n$  and  $X(1, n)$  contains at least  $\epsilon n^{1/2}$  hamburger orders and at least  $\epsilon n^{1/2}$  cheeseburger orders. Let

$$a_n(\epsilon) := \mathbb{P}(E_n(\epsilon) \mid J > n). \quad (32)$$

The main result of this section is the following.

**Proposition 3.1.** *In the above setting,*

$$\lim_{\epsilon \rightarrow 0} \liminf_{n \rightarrow \infty} a_n(\epsilon) = 1. \quad (33)$$

It will take quite a bit of work to prove Proposition 3.1. We give a brief overview. We will start by reading the word  $X$  forward. For  $n \in \mathbb{N}$ , let  $K_n$  be the last time  $i \leq n$  for which  $X(1, i)$  contains no burgers. We will argue (via an argument based on translation invariance of the word  $X$ ) that  $X(1, K_n)$  has uniformly positive probability to contain at least  $\epsilon n^{1/2}$  hamburger orders and at least  $\epsilon n^{1/2}$  cheeseburger orders if  $\epsilon$  is chosen sufficiently small. By taking  $n$  large and conditioning on  $X(1, K_n)$ , this will allow us to extract a (possibly very sparse) sequence  $m_j \rightarrow \infty$  for which  $\liminf_{j \rightarrow \infty} a_{m_j}(\epsilon) > 0$ . This is accomplished in Section 3.2.

In Section 3.3, we will prove a general result which, for  $s \in (0, 1)$ , allows us to compare the conditional law of  $Z^n(\cdot) - Z^n(-s)$  given  $\{J > n\}$  and a realization of  $X_{-[ns]} \dots X_{-1}$  to the law of  $Z(\cdot) - Z(-s)$  conditioned to stay in a neighborhood of the third quadrant.

In Section 3.4, we will use the result of Section 3.3 to show that if  $a_m(\epsilon)$  is bounded below for some small  $\epsilon > 0$  and  $m$  is very large, then  $a_n(\epsilon)$  is close to 1 for  $n \geq m$  such that  $m/n$  is of constant order. The intuitive reason why this is the case is that if  $\epsilon$  is very small and  $E_m(\epsilon)$  fails to occur, then it is unlikely that  $J > n$ ; and if  $E_m(\epsilon) \cap \{J > n\}$  occurs, then (by [She11, Theorem 2.5])  $E_n(\epsilon)$  is likely to occur for small  $\epsilon$ . We will then complete the proof of Proposition 3.1 using an induction argument and the results of Section 3.2. See Figure 2 for an illustration of the basic idea of this argument.

#### 3.2 Regularity along a subsequence

In this section we will prove the following result, which is a much weaker version of Proposition 3.1.

**Lemma 3.2.** *In the notation of (32), there is a  $\epsilon_0 > 0$  and a  $q_0 \in (0, 1)$  such that for  $\epsilon \in (0, \epsilon_0]$  there exists a sequence of positive integers  $m_j \rightarrow \infty$  (depending on  $\epsilon$ ) such that for each  $j \in \mathbb{N}$ ,*

$$a_{m_j}(\epsilon) \geq q_0. \quad (34)$$

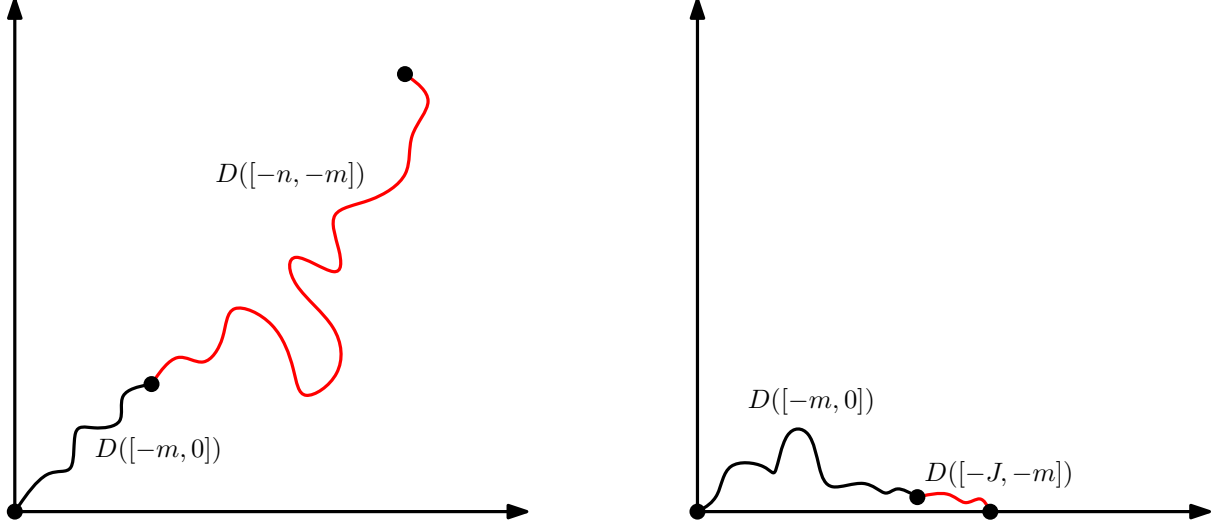


Figure 2: An illustration of the main ideas of the proof of Proposition 3.1. Suppose  $m < n \in \mathbb{N}$  with  $m$  at least a constant times  $n$ . Left figure: if  $E_m(\epsilon)$  occurs, i.e. the path  $D$  (defined as in (6)) is at uniformly positive distance from the boundary of the first quadrant at time  $m$ . By Lemma 3.6, if  $m$  is very large then it holds with uniformly positive probability that  $J > n$  and  $E_n(\epsilon)$  occurs, i.e.  $D$  stays in the first quadrant until time  $n$  and ends up at uniformly positive distance away from the boundary. Right side: if  $E_m(\epsilon)$  fails to occur and  $n$  is very large, then it is unlikely that  $J > n$ . Hence if we start from a suitably large value of  $m$  for which  $a_m(\epsilon)$  is uniformly positive, then Bayes' rule and an induction argument imply that  $a_n(\epsilon)$  is close to 1 for  $n > 2m$ , say. We prove the existence of arbitrarily large values of  $m$  for which  $a_m(\epsilon)$  is uniformly positive in Section 3.2.

The proof of Lemma 3.2 will require several further lemmas. First we need a result to the effect that the  $\overline{\mathbb{F}}$ -excursions around 0, i.e. the discrete interval  $[\phi(i), i]_{\mathbb{Z}}$  containing 0 with  $X_i = \overline{\mathbb{F}}$ , are not extremely likely to have all of their mass on the right side of 0.

**Lemma 3.3.** *For  $n \in \mathbb{N}$ , let  $K_n$  be the largest  $i \in [1, n]_{\mathbb{Z}}$  for which  $X_i = \overline{\mathbb{F}}$  and  $\phi(i) \leq 0$  (or  $K_n = n + 1$  if no such  $k$  exists). For  $\epsilon \geq 0$ , let  $A_n(\epsilon)$  be the event that  $K_n < n + 1$  and  $K_n \leq (1 - \epsilon)(K_n - \phi(K_n))$ . There exists  $\epsilon_0 > 0$ ,  $n_0 \in \mathbb{N}$ , and  $q_0 \in (0, 1/3)$  such that for each  $\epsilon \in (0, \epsilon_0]$  and  $n \geq n_0$ ,*

$$\mathbb{P}(A_n(\epsilon)) \geq 3q_0.$$

*Proof.* The idea of the proof is as follow. We look at a carefully chosen collection of disjoint discrete intervals  $I = [\phi(j), j]_{\mathbb{Z}}$  with  $X_j = \overline{\mathbb{F}}$ . We will choose these intervals in such a way that for each such interval  $I$ , the event  $A_n(\epsilon)$  occurs (with  $i$  rather than 0 playing the role of the starting point of the word  $X$ ) whenever  $i \in I$  with  $i \geq \epsilon(j - \phi(j)) + \phi(j)$  (i.e., for “most” points of  $I$ ). We then use translation invariance to conclude the statement of the lemma. See Figure 3 for an illustration.

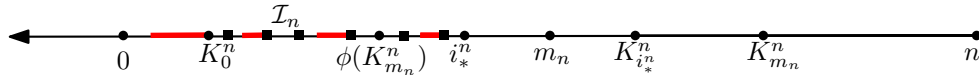


Figure 3: An illustration of the proof of Lemma 3.3. On the event  $Q_n$  defined in the proof, we have  $0 \leq K_0^n \leq \phi(K_{i_*^n}^n) \leq i_*^n \leq m_n$ . Intervals belonging to  $\mathcal{I}_n$  are shown with square endpoints. Points shown in red are those for which we know  $A_i^n(\epsilon)$  occurs. If we make  $\epsilon > 0$  small enough, the red points occupy a uniformly positive fraction of  $[0, n]$  with uniformly positive probability. Since  $\mathbb{P}(A_i^n(\epsilon))$  does not depend on  $i$ , this yields a lower bound for  $\mathbb{P}(A_i^n(\epsilon))$ .

For  $n \in \mathbb{N}$  and  $i \in \mathbb{Z}$ , let  $K_i^n$  be the largest  $j \in [i, i+n]_{\mathbb{Z}}$  for which  $X_j = \boxed{\mathbb{F}}$  and  $\phi(j) \leq i$  (if such a  $j$  exists) and otherwise let  $K_i^n = i$ . For  $\epsilon \geq 0$ , let  $A_i^n(\epsilon)$  be the event that  $K_i^n \neq i$  and  $K_i^n - i \leq (1 - \epsilon)(K_i^n - \phi(K_i^n))$ , so in particular  $A_i^n(0) = \{K_i^n \neq i\}$ . Note that  $A_0^n(\epsilon) = A_n(\epsilon)$ , and on the event  $A_n(0)$  we have  $K_n = K_0^n$ . By translation invariance,

$$\mathbb{P}(A_i^n(0)) = \mathbb{P}(A_n(\epsilon)), \quad \forall i \in \mathbb{Z}, \quad \forall \epsilon \geq 0. \quad (35)$$

Let  $m_n := \lfloor n/2 \rfloor$ . Let  $Q_n$  be the event that the following is true.

1. For each  $t \in [1, 2]$  we have (in the notation (7)) either  $U^n(t) \geq U^n(m_n/n) + 1$  or  $V^n(t) \geq V^n(m_n/n) + 1$ .
2. For each  $t \in [m_n/n, 1]$ , either  $U^n(t) \geq U^n(0) + 1$  or  $V^n(t) \geq V^n(0) + 1$ .

By [She11, Theorem 2.5], there exists  $\tilde{q}_0 \in (0, 1)$ , independent of  $n$ , such that  $\mathbb{P}(Q_n) \geq \tilde{q}_0$  for each  $n \in \mathbb{N}$  with  $n \geq 100$  (say). We observe that for each  $i \in \mathbb{Z}$ ,  $n^{-1}K_i^n$  is a  $\pi/2$ -cone time for  $Z^n$  (Definition 1.6) with  $v_{Z^n}(n^{-1}K_i^n) \leq n^{-1}i$ . Consequently, condition 1 in the definition of  $Q_n$  implies  $K_i^n \leq n$  for each  $i \in [1, m_n]_{\mathbb{Z}}$ . Similarly, condition 2 in the definition of  $Q_n$  implies  $K_0^n < m_n$ .

We claim that on  $Q_n$ , each  $i \in [1, K_0^n]_{\mathbb{Z}}$  satisfies  $K_i^n = K_0^n$ . Since  $K_i^n \leq n$ , it follows from the definition of  $K_0^n$  that either  $K_i^n = K_0^n$  or  $\phi(K_i^n) > 0$ . Since two  $\boxed{\mathbb{F}}$ -excursions are either nested or disjoint, if  $\phi(K_i^n) > 0$ , then  $[\phi(K_i^n), K_i^n]_{\mathbb{Z}} \subset (\phi(K_0^n), K_0^n)_{\mathbb{Z}}$ , which contradicts maximality of  $K_i^n$ . Therefore we in fact have  $K_i^n = K_0^n$ .

Let  $i_*^n$  be the smallest  $i \in [1, m_n]_{\mathbb{Z}}$  such that  $K_i^n \geq m_n$ . By maximality of  $K_{m_n}^n$ , we have  $\phi(K_{i_*^n}^n) \leq m_n \leq K_{i_*^n}^n \leq K_{m_n}^n$ , whence  $[\phi(K_{i_*^n}^n), K_{i_*^n}^n]_{\mathbb{Z}} \subset [\phi(K_{m_n}^n), K_{m_n}^n]_{\mathbb{Z}}$  and

$$m_n - i_*^n \leq m_n - \phi(K_{m_n}^n). \quad (36)$$

Furthermore, on the event  $Q_n$  we have  $i_*^n > K_0^n$ .

Let  $\mathcal{I}_n$  be the set of maximal  $\boxed{\mathbb{F}}$ -excursions in  $[K_0^n, i_*^n - 1]_{\mathbb{Z}}$ , i.e. the set of discrete intervals  $I = [\phi(j), j]_{\mathbb{Z}} \subset [K_0^n, i_*^n - 1]_{\mathbb{Z}}$  with  $X_j = \boxed{\mathbb{F}}$  which are not contained in any larger such discrete interval. For  $I = [\phi(j), j]_{\mathbb{Z}} \in \mathcal{I}_n$ , we write  $\text{len}(I) = j - \phi(j)$ .

Observe that if  $i \in I$  for some  $I = [\phi(j), j]_{\mathbb{Z}} \in \mathcal{I}_n$ , then  $K_i^n = j$ . Indeed, we have  $K_i^n \leq m_n$  and  $i \in [\phi(j), j]_{\mathbb{Z}}$ , so the claim follows from maximality of  $I$  and  $K_i^n$ . Conversely, suppose  $i \in (K_0^n, i_*^n)_{\mathbb{Z}}$  and  $A_i^n(0)$  occurs. Then  $K_i^n \in (K_0^n, i_*^n)_{\mathbb{Z}}$ . By maximality of  $K_0^n$ , we must have  $\phi(K_i^n) > 0$ , so by maximality of  $K_i^n$  we have  $[\phi(K_i^n), K_i^n]_{\mathbb{Z}} \in \mathcal{I}_n$ . Thus  $\mathcal{I}$  can alternatively be described as the set of discrete intervals  $[\phi(K_i^n), K_i^n]_{\mathbb{Z}}$  for  $i \in [\phi(j), j]_{\mathbb{Z}}$ .

On  $Q_n$ , we therefore have

$$\sum_{i=1}^{m_n} \mathbb{1}_{A_i^n(0)} \leq \sum_{I \in \mathcal{I}_n} \text{len}(I) + K_0^n + m_n - i_*^n. \quad (37)$$

On the other hand, if  $i \in I$  for some  $I = [\phi(j), j]_{\mathbb{Z}} \in \mathcal{I}_n$  and  $i \geq \epsilon(\phi(j) - j) + j$ , then since  $K_i^n = j$ , we have that  $A_i^n(\epsilon)$  occurs. Therefore, on  $Q_n$  we have

$$\sum_{i=1}^{m_n} \mathbb{1}_{A_i^n(\epsilon)} \geq (1 - \epsilon) \sum_{I \in \mathcal{I}_n} \text{len}(I) + (1 - \epsilon)K_0^n. \quad (38)$$

By Proposition 2.6, we have  $\mathbb{P}(A_i^n(0)) \rightarrow 1$  as  $n \rightarrow \infty$  (uniformly in  $i$  by translation invariance) so for sufficiently large  $n$  we have

$$\mathbb{E} \left( \mathbb{1}_{Q_n} \sum_{i=1}^{m_n} \mathbb{1}_{A_i^n(0)} \right) = \sum_{i=1}^{m_n} \mathbb{P}(A_i^n(0) \cap Q_n) \geq (\mathbb{P}(Q_n) - o_n(1)) m_n \geq \frac{\tilde{q}_0}{2} m_n.$$

By (37),

$$\mathbb{E} \left( \mathbb{1}_{Q_n} \sum_{I \in \mathcal{I}_n} \text{len}(I) \right) + \mathbb{E}(\mathbb{1}_{Q_n} K_0^n) + \mathbb{E}(\mathbb{1}_{Q_n} (m_n - i_*^n)) \geq \frac{\tilde{q}_0}{2} m_n.$$

By (38),

$$\mathbb{E} \left( \mathbb{1}_{Q_n} \sum_{i=1}^{m_n} \mathbb{1}_{A_i^n(\epsilon)} \right) \geq (1 - \epsilon) \frac{\tilde{q}_0}{2} m_n - \mathbb{E}(\mathbb{1}_{Q_n}(m_n - i_*^n)) - \epsilon \mathbb{E}(\mathbb{1}_{Q_n} \iota_0^n). \quad (39)$$

On the event  $A_{m_n}^n(\epsilon)^c$ , we have  $m_n - \phi(K_{m_n}^n) \leq \epsilon n$ , whence (36) implies  $m_n - i_*^n \leq \epsilon n$ . On  $Q_n$ , we always have  $m_n - i_*^n \leq m_n$ . Therefore,

$$\begin{aligned} \mathbb{E}(\mathbb{1}_{Q_n}(m_n - i_*^n)) &\leq m_n \mathbb{P}(A_{m_n}^n(\epsilon) \cap Q_n) + \epsilon n \mathbb{P}(A_{m_n}^n(\epsilon)^c \cap Q_n) \\ &\leq m_n \mathbb{P}(A_{m_n}^n(\epsilon)) + \epsilon n. \end{aligned}$$

It is clear that  $\mathbb{E}(\mathbb{1}_{Q_n} \iota_0^n) \leq \epsilon n$ , so (39) implies that for sufficiently large  $n$ ,

$$\mathbb{E} \left( \mathbb{1}_{Q_n} \sum_{i=1}^{m_n} \mathbb{1}_{A_i^n(\epsilon)} \right) + m_n \mathbb{P}(A_{m_n}^n(\epsilon)) \geq (1 - \epsilon) \frac{\tilde{q}_0}{2} m_n - 2\epsilon n.$$

By (35),

$$2m_n \mathbb{P}(A_n(\epsilon)) \geq (1 - \epsilon) \frac{\tilde{q}_0}{2} m_n - 2\epsilon n.$$

Re-arranging this inequality implies the statement of the lemma for appropriate  $\epsilon_0 > 0$  and  $q_0 \in (0, 1/3)$ .  $\square$

**Lemma 3.4.** *Let  $K_n$  be defined as in the statement of Lemma 3.3. For  $\epsilon > 0$ , let  $G_n(\epsilon)$  be the event that  $X(1, K_n)$  contains at least  $\epsilon\sqrt{K_n}$  hamburger orders and at least  $\epsilon\sqrt{K_n}$  cheeseburger orders. Let  $q_0$  be as in Lemma 3.3. There exists  $\epsilon_0 > 0$  and  $n_0 \in \mathbb{N}$  (depending only on  $q_0$ ) such that for  $\epsilon \in (0, \epsilon_0]$  and  $n \geq n_0$ ,*

$$\mathbb{P}(G_n(\epsilon)) \geq 2q_0.$$

*Proof.* Let  $\tilde{\epsilon}_0 > 0$  and  $\tilde{n}_0 \in \mathbb{N}$  be chosen so that the conclusion of Lemma 3.3 holds (with  $\tilde{\epsilon}_0$  in place of  $\epsilon_0$  and  $\tilde{n}_0$  in place of  $n_0$ ). For  $n \in \mathbb{N}$  let  $A_n(\tilde{\epsilon}_0)$  be the event of that lemma (with  $\epsilon = \tilde{\epsilon}_0$ ). Then for  $n \geq \tilde{n}_0$ , we have  $\mathbb{P}(A_n(\tilde{\epsilon}_0)) \geq 3q_0$ .

Fix  $\alpha \in (0, 1)$ . Let  $F_{K_n}$  be defined as in Section 3.1 with  $K_n$  in place of  $n$  and  $X(1, K_n)$  in place of  $X(-K_n, -1)$ . By Lemma 2.8, we can find  $m \in \mathbb{N}$  such that the probability that there is even one  $k \geq m$  such that  $X(1, k)$  contains more than  $k^\nu$   $\mathbb{F}$ -symbols is at most  $\alpha/2$ . By Proposition 2.6, we can find  $n'_0 \geq \tilde{n}_0$  such that for  $n \geq n'_0$ , we have  $\mathbb{P}(K_n \geq m) \geq 1 - \alpha/2$ . For  $n \geq n'_0$ , we therefore have

$$\mathbb{P}(F_{K_n}) \geq 1 - \alpha. \quad (40)$$

For  $\epsilon > 0$  and  $k \in \mathbb{N}$ , let  $J_k^H(\epsilon)$  (resp.  $J_k^C(\epsilon)$ ) be the smallest  $j \in \mathbb{N}$  for which the word  $X(-j, 0)$  contains at least  $\epsilon k^{1/2} + k^\nu + 1$  hamburgers (resp. cheeseburgers). By [She11, Theorem 2.5], the times  $J_k^H(\epsilon)$  and  $J_k^C(\epsilon)$  are typically of order  $\epsilon^2 k$ . More precisely, we can find  $\epsilon_0 \in (0, \tilde{\epsilon}_0]$  and  $k_0 \in \mathbb{N}$  such that for  $k \geq k_0$  and  $\epsilon \in (0, \epsilon_0]$ ,

$$\mathbb{P}(J_k^H(\epsilon) \vee J_k^C(\epsilon) \geq \tilde{\epsilon}_0^2 k) \leq \alpha.$$

By Proposition 2.6, we can find  $n_0 \geq n'_0$  such that for  $n \geq n_0$ , we have  $\mathbb{P}(K_n \leq k_0) \leq \alpha$ .

On the event  $G_n(\epsilon)^c \cap F_{K_n}$ , we have  $-\phi(K_n) \leq J_{K_n}^H(\epsilon) \vee J_{K_n}^C(\epsilon)$ . Since  $G_n(\epsilon)^c \cap F_n \cap \{K_n \geq k_0\}$  is independent from  $\dots X_{-2} X_{-1}$ , it follows that for  $n \geq n'_0$  we have

$$\begin{aligned} \mathbb{P}(G_n(\epsilon)^c \cap F_{K_n} \cap \{-\phi(K_n) \geq \tilde{\epsilon}_0^2 K_n\}) \\ \leq \mathbb{P}(K_n \leq k_0) + \mathbb{E}(\mathbb{P}(G_n(\epsilon)^c \cap F_{K_n} \cap \{-\phi(K_n) \geq \tilde{\epsilon}_0^2 K_n\} | X_1 X_2 \dots) \mathbb{1}_{(K_n \geq k_0)}) \\ \leq \alpha + \mathbb{E}(\mathbb{P}(J_{K_n}^H(\epsilon) \vee J_{K_n}^C(\epsilon) \geq \tilde{\epsilon}_0^2 K_n | K_n) \mathbb{1}_{(K_n \geq k_0)}) \leq 2\alpha. \end{aligned}$$

By definition, on the event  $A_n(\tilde{\epsilon}_0)$  we have  $-\phi(K_n) \geq \tilde{\epsilon}_0^2 K_n$ , so we have

$$\mathbb{P}(-\phi(K_n) \geq \tilde{\epsilon}_0^2 K_n) \geq 3q_0.$$

Therefore,

$$\mathbb{P}(G_n(\epsilon)^c \cap F_{K_n}) \leq 1 - 3q_0 + 2\alpha.$$

By combining this with (40) we obtain

$$\mathbb{P}(G_n(\epsilon)) \geq 3q_0 - 3\alpha.$$

Since  $\alpha$  is arbitrary this implies the statement of the lemma.  $\square$

*Proof of Lemma 3.2.* Let  $q_0$  be as in Lemma 3.3. For  $n \in \mathbb{N}$ , define the time  $K_n$  as in Lemma 3.3. Choose  $\epsilon_0 > 0$  and  $n_0 \in \mathbb{N}$  such that the conclusion of Lemma 3.4 holds, and fix  $\epsilon \in (0, \epsilon_0]$ . By Proposition 2.6, if we are given  $j \in \mathbb{N}$ , we can choose  $n \geq n_0$  such that  $\mathbb{P}(j+1 \leq K_n \leq n) \geq 1 - q_0/2$ . Henceforth fix such an  $n$ . Then with  $G_n(\epsilon)$  as in the statement of Lemma 3.4, we have

$$\mathbb{P}(G_n(\epsilon) \cap \{j+1 \leq K_n \leq n\}) \geq \frac{3}{2}q_0.$$

We therefore have

$$\frac{3}{2}q_0 \leq \sum_{k=j+1}^n \mathbb{P}(G_n(\epsilon) | K_n = k) \mathbb{P}(K_n = k).$$

Hence we can find some  $m_j \in [j, n-1]_{\mathbb{Z}}$  for which

$$\mathbb{P}(G_n(\epsilon) | K_n = m_j + 1) \geq \frac{3}{2}q_0.$$

We can write  $\{K_n = m_j + 1\}$  as the intersection of the event that  $X(1, m_j)$  contains no burgers; and the event that  $X_{m_j+1} = \boxed{\mathbf{F}}$  and  $\mathcal{N}_{\boxed{\mathbf{F}}}(X(m_j + 2, n)) = 0$ . The latter event is independent of  $X_1 \dots X_{m_j}$ , so the conditional law of  $X_1 \dots X_{m_j}$  given  $\{K_n = m_j + 1\}$  is the same as its conditional law given that  $X(1, m_j)$  contains no burgers. The event  $G_n(\epsilon) \cap \{K_n = m_j + 1\}$  is the same as the event that  $K_n = m_j + 1$  and  $X(1, m_j)$  contains at least  $\epsilon(m_j + 1)^{1/2}$  hamburger orders and at least  $\epsilon(m_j + 1)^{1/2}$  cheeseburger orders. By Lemma 2.8 and translation invariance, (34) holds for this choice of  $m_j$  (with a slightly smaller choice of  $\epsilon$ ) provided  $j$  is chosen sufficiently large. Since  $m_j \geq j$  and  $j \in \mathbb{N}$  was arbitrary, we conclude.  $\square$

### 3.3 Conditioning on an initial segment of the word

**Notation 3.5.** For  $t_1 < t_2 \in \mathbb{R}$ , we write

$$Z_{[t_1, t_2]}^n := (Z^n - Z^n(n^{-1}\lfloor nt_2 - 1 \rfloor))|_{[t_1, n^{-1}\lfloor nt_2 - 1 \rfloor]} \quad \text{and} \quad Z_{[t_1, t_2]} := (Z - Z(t_2))|_{[t_1, t_2]}. \quad (41)$$

We extend the definition of  $Z_{[t_1, t_2]}^n$  to  $[t_1, t_2]$  by defining it to be identically zero for  $t \in [n^{-1}\lfloor nt_2 - 1 \rfloor, t_2]$ .

The reason why we use  $n^{-1}\lfloor nt_2 - 1 \rfloor$  instead of just  $t_2$  in the definition of  $Z_{[t_1, t_2]}^n$  is that this choice implies that  $Z_{[t_1, t_2]}^n$  is independent of  $X_{\lfloor t_2 n \rfloor} X_{\lfloor t_2 n \rfloor + 1} \dots$ .

In this section we will prove a lemma which allows us to estimate the conditional law of  $Z_{[-1, -s]}^n$  for  $s \in (0, 1)$  given  $\{J > n\}$  and a realization of  $X_{-\lfloor ns \rfloor} \dots X_{-1}$ .

**Lemma 3.6.** Fix  $\lambda \in (0, 1/2)$ . For  $n \in \mathbb{N}$  and  $s \in [\lambda, 1 - \lambda]$  define

$$h_n^s := (sn)^{-1/2} \mathcal{N}_{\boxed{\mathbf{H}}}(X(-\lfloor sn \rfloor, -1)) \quad \text{and} \quad c_n^s := (sn)^{-1/2} \mathcal{N}_{\boxed{\mathbf{C}}}(X(-\lfloor sn \rfloor, -1)). \quad (42)$$

For  $\epsilon_1, \epsilon_2 > 0$ , let

$$\tilde{G}^s(\epsilon_1, \epsilon_2) := \left\{ \inf_{t \in [-1, -s]} (U_t - U_{-s}) \geq -s^{1/2} \epsilon_1 \right\} \cap \left\{ \inf_{t \in [-1, -s]} (V_t - V_{-s}) \geq -s^{1/2} \epsilon_2 \right\}. \quad (43)$$

Suppose given  $\epsilon > 0$  and  $\alpha > 0$ . There is an  $n_* \in \mathbb{N}$  and a  $\zeta > 0$  (depending only on  $\lambda, \epsilon$ , and  $\alpha$ ) such that the following holds. Suppose  $n \geq n_*$ ;  $s \in [\lambda, 1 - \lambda]$ ;  $\epsilon_1, \epsilon_2 \geq \epsilon$ ; and  $x$  is a realization of  $X_{-\lfloor sn \rfloor} \dots X_{-1}$  for which  $X(-\lfloor sn \rfloor, -1)$  contains no burgers,  $|h_n^s - \epsilon_1| \leq \zeta$ ,  $|c_n^s - \epsilon_2| \leq \zeta$ , and  $F_{\lfloor sn \rfloor}$  (as defined in Section 3.1) occurs. Then the Prokhorov distance (in the uniform metric) between the conditional law of  $Z_{[-1, -s]}^n$  given  $\{J > n\} \cap \{X_{-m_n} \dots X_{-1} = x\}$  and the conditional law of  $Z_{[-1, -s]}$  given  $\tilde{G}^s(\epsilon_1, \epsilon_2)$  is at most  $\alpha$ . Moreover, we can arrange that the same holds if we instead condition on  $\{J > n\} \cap \{X_{-m_n} \dots X_{-1} = x\} \cap F_n$ .

*Proof.* Let  $\nu$  be as in Section 3.1. For  $\epsilon_1, \epsilon_2 > 0$ , let  $\underline{G}_n^s(\epsilon_1, \epsilon_2)$  be the event that  $\mathcal{N}_{\textcircled{H}}(X(-n, -\lfloor sn \rfloor - 1)) \leq \epsilon_1 (sn)^{1/2}$  and  $\mathcal{N}_{\textcircled{C}}(X(-n, -\lfloor sn \rfloor - 1)) \leq \epsilon_2 (sn)^{1/2}$ . Let  $\overline{G}_n^s(\epsilon_1, \epsilon_2)$  be the event that  $\mathcal{N}_{\textcircled{H}}(X(-n, -\lfloor sn \rfloor - 1)) \leq \epsilon_1 (sn)^{1/2} + (sn)^\nu$  and  $\mathcal{N}_{\textcircled{C}}(X(-n, -\lfloor sn \rfloor - 1)) \leq \epsilon_2 (sn)^{1/2} + (sn)^\nu$ .

For any realization  $x$  of  $X_{-\lfloor sn \rfloor} \dots X_{-1}$  for which  $F_{\lfloor sn \rfloor}$  occurs, we have

$$\begin{aligned} \{X_{-\lfloor sn \rfloor} \dots X_{-1} = x\} \cap \underline{G}_n^s(h_n^s, c_n^s) &\subset \{X_{-\lfloor sn \rfloor} \dots X_{-1} = x\} \cap \{J > n\} \\ &\subset \{X_{-\lfloor sn \rfloor} \dots X_{-1} = x\} \cap \overline{G}_n^s(h_n^s, c_n^s). \end{aligned} \quad (44)$$

By (44) and independence of  $X_{-\lfloor sn \rfloor} \dots X_{-1}$  from  $Z_{[-1, -s]}^n$ , we obtain that for any open subset  $\mathcal{U}$  of the space of continuous functions  $[-1, -s] \rightarrow \mathbb{R}^2$  in the uniform topology,

$$\begin{aligned} \frac{\mathbb{P}\left(Z_{[-1, -s]}^n \in \mathcal{U}, \underline{G}_n^s(h_n^s, c_n^s) \mid (h_n^s, c_n^s)\right)}{\mathbb{P}\left(\overline{G}_n^s(h_n^s, c_n^s) \mid (h_n^s, c_n^s)\right)} &\leq \mathbb{P}\left(Z_{[-1, -s]}^n \in \mathcal{U} \mid J > n, X_{-\lfloor sn \rfloor} \dots X_{-1} = x\right) \\ &\leq \frac{\mathbb{P}\left(Z_{[-1, -s]}^n \in \mathcal{U}, \overline{G}_n^s(h_n^s, c_n^s) \mid (h_n^s, c_n^s)\right)}{\mathbb{P}\left(\underline{G}_n^s(h_n^s, c_n^s) \mid (h_n^s, c_n^s)\right)}. \end{aligned} \quad (45)$$

Let

$$r := \inf_{\substack{\epsilon_1, \epsilon_2 \geq \epsilon \\ s \in [\lambda, 1-\lambda]}} \mathbb{P}\left(\tilde{G}^s(\epsilon_1, \epsilon_2)\right).$$

Then  $r$  is a positive constant depending only on  $\epsilon$  and  $\lambda$ . We can find  $\zeta \in (0, \alpha)$  depending only on  $r$  and  $\alpha$  such that for  $\epsilon_1, \epsilon_2 \geq \epsilon$  and  $s \in [\lambda, 1-\lambda]$ ,

$$\left| \mathbb{P}\left(\tilde{G}^s(\epsilon_1 + \zeta, \epsilon_2 + \zeta)\right) - \mathbb{P}\left(\tilde{G}^s(\epsilon_1 - \zeta, \epsilon_2 - \zeta)\right) \right| \leq r\alpha. \quad (46)$$

By [She11, Theorem 2.5], we can find an  $n_* \in \mathbb{N}$  depending only on  $r$  and  $\alpha$  such that for  $n \geq n_*$ , the Prokhorov distance between the unconditional law of  $Z^n|_{[-1, 0]}$  and the law of  $Z|_{[-1, 0]}$  is at most a constant (depending only on  $\epsilon$ ) times  $r\alpha$ . The same holds with the laws of  $Z^n|_{[-1, 0]}$  and  $Z|_{[-1, 0]}$  replaced by the laws of  $Z_{[-1, -s]}^n$  and  $Z_{[-1, -s]}$  for each choice of  $s \in [\lambda, 1-\lambda]$ . By Lemma 2.8, by possibly further increasing  $n_*$ , we can arrange that the same holds with the law of  $Z^n|_{[-1, -s]}$  replaced by the conditional law of  $Z_{[-1, -s]}^n$  given  $F_n$  for each choice of  $s \in [\lambda, 1-\lambda]$ . By combining this with our choice of  $\zeta$  in (46), we obtain that whenever  $n \geq n_*$  and  $\epsilon_1^n, \epsilon_2^n > 0$  with  $|\epsilon_1^n - \epsilon_1|$  and  $|\epsilon_2^n - \epsilon_2|$  each smaller than  $\zeta$ ,

$$\left| \mathbb{P}\left(\overline{G}_n^s(\epsilon_1^n, \epsilon_2^n)\right) - \mathbb{P}\left(\tilde{G}^s(\epsilon_1, \epsilon_2)\right) \right| \leq r\alpha,$$

with the implicit constant depending only on  $\epsilon$ , and similarly with  $\underline{G}_n^s(\epsilon_1^n, \epsilon_2^n)$  in place of  $\overline{G}_n^s(\epsilon_1^n, \epsilon_2^n)$ . Since  $\alpha$  is arbitrary the statement of the lemma now follows from (45).  $\square$

### 3.4 Regularity at all sufficiently large times

In this section we will deduce Proposition 3.1 from Lemma 3.2 and an induction argument.

**Lemma 3.7.** *Let  $q \in (0, 1)$  and  $\lambda \in (0, 1)$ . There is a  $\delta_0 > 0$  (depending only on  $q$  and  $\lambda$ ) such that for each  $\delta \in (0, \delta_0]$  and each  $\epsilon > 0$ , there exists  $n_* = n_*(\lambda, \delta, \epsilon) \in \mathbb{N}$  such that for  $n \geq n_*$  and  $m \in \mathbb{N}$  with  $\lambda \leq m/n \leq 1 - \lambda$ , the following holds. Let  $x = x_{-m} \dots x_{-1}$  be any realization of  $X_{-m} \dots X_{-1}$  for which  $E_m(\epsilon) \cap F_m$  occurs. Then*

$$\mathbb{P}(E_n(\delta) \mid X_{-m} \dots X_{-1} = x, J > n) \geq 1 - q.$$

**Remark 3.8.** The main point of Lemma 3.7 is that  $\delta_0$  does not depend on  $\epsilon$  (indeed, the lemma is a trivial consequence of [She11, Theorem 2.5] without this requirement).

*Proof of Lemma 3.7.* For  $s \in [0, 1]$  and  $\delta > 0$ , let

$$G^s(\delta) := \{U_1 - U_s \geq \delta \text{ and } V_1 - V_s \geq \delta\}.$$

For  $\epsilon_1, \epsilon_2 > 0$  define the event  $\tilde{G}^s(\epsilon_1, \epsilon_2)$  as in (43). By Lemma 3.6, for each choice of  $\delta > 0$  we can find  $n_* \in \mathbb{N}$  (depending on  $\epsilon, \delta, q$ , and  $\lambda$ ) such that the following holds. Suppose  $n \geq n_*$ ;  $s \in [\lambda, 1 - \lambda]$ ; and  $x$  is a realization of  $X_{-\lfloor sn \rfloor} \dots X_{-1}$  for which  $F_{\lfloor sn \rfloor}$  occurs,  $h_n^s \geq \epsilon$ , and  $c_n^s \geq \epsilon$ , with  $h_n^s$  and  $c_n^s$  as in (42). Then

$$\mathbb{P}(E_n(\delta) \mid X_{-\lfloor sn \rfloor} \dots X_{-1} = x, J > n) \geq \mathbb{P}(G^s(2\delta) \mid \tilde{G}^s(s^{1/2}h_n^s, s^{1/2}c_n^s)) - \frac{q}{2}.$$

Hence it suffices to prove that for sufficiently small  $\delta > 0$ , we have

$$\inf_{\substack{\epsilon_1, \epsilon_2 > 0 \\ s \in [\lambda, 1 - \lambda]}} \mathbb{P}(G^s(\delta) \mid \tilde{G}^s(\epsilon_1, \epsilon_2)) \geq 1 - \frac{q}{2}. \quad (47)$$

By [Shi85, Theorem 2] (c.f. the proof of Lemma (2.2)) the conditional laws  $\mathbb{P}(\cdot \mid \tilde{G}^s(\epsilon_1, \epsilon_2))$  converge weakly as  $(\epsilon_1, \epsilon_2) \rightarrow 0$  to a non-degenerate limiting distribution. Hence we can find  $\tilde{\delta}_0 > 0$  and  $\tilde{\epsilon}_0 > 0$  depending only on  $q$  and  $\lambda$  such that whenever  $\delta \in (0, \tilde{\delta}_0]$  and  $\epsilon_1, \epsilon_2 \in (0, \tilde{\epsilon}_0]$ , we have

$$\inf_{s \in [\lambda, 1 - \lambda]} \mathbb{P}(G^s(\delta) \mid \tilde{G}^s(\epsilon_1, \epsilon_2)) \geq 1 - q. \quad (48)$$

Moreover, by taking the opening angle of the cone in [Shi85, Theorem 2] to be  $\pi$  and applying a linear transformation, we find that the conditional laws  $\mathbb{P}(\cdot \mid \tilde{G}^s(\epsilon_1, \epsilon_2))$  also converge weakly to a (different) non-degenerate limiting distribution if we send one of  $\epsilon_1$  or  $\epsilon_2$  to 0 and leave the other fixed. Hence we can find  $\delta_0 \in (0, \tilde{\delta}_0]$  depending only on  $q, \lambda$ , and  $\epsilon_0$  such that (48) holds whenever  $\delta \in (0, \delta_0]$  and one of  $\epsilon_1$  or  $\epsilon_2$  is at least  $\epsilon_0$ . Hence if  $\delta \in (0, \delta_0]$ , (48) holds for every choice of  $\epsilon_1, \epsilon_2 > 0$ . This completes the proof of the lemma.  $\square$

**Lemma 3.9.** *Fix  $\lambda \in (0, 1/2)$ ,  $q_0 \in (0, 1)$ , and  $\epsilon > 0$ . Suppose we are given  $m_0 \in \mathbb{N}$  such that  $a_{m_0}(\epsilon) \geq q_0$ . Then for  $m \in \mathbb{N}$  with  $\lambda \leq m_0/m \leq 1 - \lambda$ ,  $n \in \mathbb{N}$  with  $\lambda \leq m/n \leq 1 - \lambda$ , and  $\zeta > 0$  we have*

$$\frac{\mathbb{P}(J > n \mid E_m(\zeta))}{\mathbb{P}(J > n \mid E_m(\zeta)^c, J > m)} \gtrsim \frac{1}{\zeta + o_{m_0}(1)}, \quad (49)$$

where the implicit constant depends only on  $q_0, \lambda$ , and  $\epsilon$ ; and the  $o_{m_0}(1)$  depends only on  $\lambda, \epsilon$ , and  $\zeta$ .

*Proof.* Let  $\delta_0$  be chosen so that the conclusion of Lemma 3.7 holds with given  $\lambda$  and  $q = 1/2$ . Let  $n_* = n_*(\lambda, \delta_0, \epsilon)$  be as in that lemma. For  $m_0 \geq n_*$  and  $m$  as in the statement of the lemma,

$$\mathbb{P}(E_m(\delta_0) \mid E_{m_0}(\epsilon), J > m) \geq \frac{1}{2}.$$

Hence if  $m_0 \geq n_*$ , then

$$\begin{aligned} \mathbb{P}(E_m(\delta_0) \mid E_m(\zeta)) &\geq \mathbb{P}(E_m(\delta_0) \mid J > m) \\ &\geq \mathbb{P}(E_m(\delta_0) \mid E_{m_0}(\epsilon), J > m) \mathbb{P}(E_{m_0}(\epsilon) \mid J > m) \\ &\geq \frac{1}{2} \mathbb{P}(E_{m_0}(\epsilon) \mid J > m). \end{aligned} \quad (50)$$

By Bayes' rule,

$$\begin{aligned} \mathbb{P}(E_{m_0}(\epsilon) \mid J > m) &= \frac{\mathbb{P}(J > m \mid E_{m_0}(\epsilon)) \mathbb{P}(E_{m_0}(\epsilon) \mid J > m_0)}{\mathbb{P}(J > m \mid J > m_0)} \\ &\geq \frac{\mathbb{P}(J > m \mid E_{m_0}(\epsilon)) a_{m_0}(\epsilon)}{\mathbb{P}(J > m \mid E_{m_0}(\epsilon)) a_{m_0}(\epsilon) + \mathbb{P}(J > m, E_{m_0}(\epsilon)^c \mid J > m_0)}. \end{aligned} \quad (51)$$

By [She11, Theorem 2.5] and our hypothesis on  $a_{m_0}(\epsilon)$ , this quantity is bounded below by a constant depending only on  $q_0$ ,  $\lambda$ , and  $\epsilon$  (not on  $\zeta$ ). By (50), we arrive at

$$\mathbb{P}(E_m(\delta_0) | E_m(\zeta)) \geq 1.$$

By combining this with [She11, Theorem 2.5] we obtain

$$\mathbb{P}(J > n | E_m(\zeta)) \geq \mathbb{P}(J > n | E_m(\delta_0)) \mathbb{P}(E_m(\delta_0) | E_m(\zeta)) \geq 1. \quad (52)$$

Next we consider the denominator in (49). By Lemma 2.8, we have

$$\begin{aligned} \mathbb{P}(J > n | E_m(\zeta)^c, J > m) &= \frac{\mathbb{P}(J > n, E_m(\zeta)^c | J > m)}{\mathbb{P}(E_m(\zeta)^c | J > m)} \\ &\leq \frac{\mathbb{P}(J > n, F_m, E_m(\zeta)^c | J > m) + o_{m_0}^\infty(m_0)}{\mathbb{P}(E_m(\zeta)^c \cap F_m | J > m)}. \end{aligned} \quad (53)$$

We have

$$\begin{aligned} \mathbb{P}(E_m(\zeta)^c \cap F_m | J > m) &\geq \mathbb{P}(E_m(\zeta)^c \cap F_m | E_{m_0}(\epsilon)) \mathbb{P}(E_{m_0}(\epsilon) | J > m) \\ &\geq \mathbb{P}(E_m(\zeta)^c \cap F_m | E_{m_0}(\epsilon)) \frac{\mathbb{P}(E_{m_0}(\epsilon))}{\mathbb{P}(J > m)}. \end{aligned}$$

By [She11, Theorem 2.5],  $\mathbb{P}(E_m(\zeta)^c | E_{m_0}(\epsilon))$  is at least a positive constant depending on  $\epsilon$  and  $\lambda$  but not on  $\zeta$  or  $m_0$ . By Lemma 2.3,  $\frac{\mathbb{P}(E_{m_0}(\epsilon))}{\mathbb{P}(J > m)}$  is bounded below by a constant (depending only on  $\epsilon$  and  $\lambda$ ) times a power of  $m_0$ . Hence (53) implies

$$\mathbb{P}(J > n | E_m(\zeta)^c, J > m) \leq \mathbb{P}(J > n | E_m(\zeta)^c, F_m, J > m) + o_{m_0}^\infty(m_0).$$

Observe that if  $E_m(\zeta)^c \cap F_m$  occurs and  $J > n$ , then  $X(-n, -m-1)$  contains either at most  $\zeta m^{1/2} + O_n(n^\nu)$  hamburgers or at most  $\zeta m^{1/2} + O_n(n^\nu)$  cheeseburgers. By [She11, Theorem 2.5], we therefore have

$$\mathbb{P}(J > n | E_m(\zeta)^c, J > m) \leq \zeta + o_{m_0}(1). \quad (54)$$

We conclude by combining (52) and (54).  $\square$

**Lemma 3.10.** *Let  $q, q_0 \in (0, 1)$  and  $\lambda \in (0, 1/2)$ . There is a  $\epsilon_0 > 0$  (depending only on  $q, q_0$ , and  $\lambda$ ) such that for each  $\epsilon \in (0, \epsilon_0]$  we can find  $m_* = m_*(q, q_0, \lambda, \epsilon) \in \mathbb{N}$  with the following property. Suppose  $m < n \in \mathbb{N}$  with  $m \geq m_*$  and*

$$\lambda \leq m/n \leq 1 - \lambda. \quad (55)$$

*Suppose further that  $a_m(\epsilon) \geq q_0$ . Then  $a_n(\epsilon) \geq 1 - q$ .*

*Proof.* Fix  $q \in (0, 1)$ . Let  $\tilde{m} := \frac{m+n}{2}$ . By Lemma 3.7 we can find  $\epsilon_0 > 0$  (depending only on  $q$  and  $\lambda$ ) such that for  $\epsilon \in (0, \epsilon_0]$  and  $\zeta \in (0, \epsilon]$ , there exists  $\tilde{m}_* = \tilde{m}_*(\zeta, \epsilon, q, \lambda) \in \mathbb{N}$  such that if  $m \geq \tilde{m}_*$  and (55) holds, then

$$\mathbb{P}(E_n(\epsilon) | E_{\tilde{m}}(\zeta), J > n) \geq 1 - q \quad \text{and} \quad \mathbb{P}(E_{\tilde{m}}(\zeta) | E_m(\epsilon), J > \tilde{m}) \geq 1 - q. \quad (56)$$

Henceforth fix  $\epsilon \in (0, \epsilon_0]$ .

Fix  $\alpha \in (0, 1)$  to be chosen later (depending on  $q, q_0, \lambda$ , and  $\epsilon$ ). By Lemma 3.9, we can find  $\zeta \in (0, \epsilon]$  (depending on  $\lambda, \alpha, q_0$ , and  $\epsilon$ ) and  $m_* \geq \tilde{m}_*$  (depending on  $\lambda, \alpha, q_0, \epsilon$ , and  $\zeta$ ) for which the following holds. If  $m \geq m_*$ , (55) holds, and  $a_m(\epsilon) \geq q_0$ , then

$$\mathbb{P}(J > n | E_{\tilde{m}}(\zeta)^c, J > \tilde{m}) \leq \alpha \mathbb{P}(J > n | E_{\tilde{m}}(\zeta)). \quad (57)$$

Hence if  $m \geq m_*$ , (55) holds, and  $a_m(\epsilon) \geq q_0$  then

$$\begin{aligned} a_n(\epsilon) &= \frac{\mathbb{P}(E_n(\epsilon))}{\mathbb{P}(J > n)} \geq \frac{\mathbb{P}(E_n(\epsilon) | E_{\tilde{m}}(\zeta)) a_{\tilde{m}}(\zeta)}{\mathbb{P}(J > n | E_{\tilde{m}}(\zeta)) a_{\tilde{m}}(\zeta) + \mathbb{P}(J > n | E_{\tilde{m}}(\zeta)^c, J > \tilde{m}) (1 - a_{\tilde{m}}(\zeta))} \\ &\geq \frac{\mathbb{P}(E_n(\epsilon) | E_{\tilde{m}}(\zeta))}{\mathbb{P}(J > n | E_{\tilde{m}}(\zeta))} \times \frac{a_{\tilde{m}}(\zeta)}{a_{\tilde{m}}(\zeta) + \alpha(1 - a_{\tilde{m}}(\zeta))}. \end{aligned} \quad (58)$$



By (56),

$$\frac{\mathbb{P}(E_n(\epsilon) | E_{\tilde{m}}(\zeta))}{\mathbb{P}(J > n | E_{\tilde{m}}(\zeta))} = \mathbb{P}(E_n(\epsilon) | E_{\tilde{m}}(\zeta), J > n) \geq 1 - q.$$

Furthermore,

$$a_{\tilde{m}}(\zeta) \geq \mathbb{P}(E_{\tilde{m}}(\zeta) | E_m(\epsilon), J > \tilde{m}) \mathbb{P}(E_m(\epsilon) | J > \tilde{m}) \geq (1 - q) \mathbb{P}(E_m(\epsilon) | J > \tilde{m}). \quad (59)$$

By Bayes' rule,

$$\begin{aligned} \mathbb{P}(E_m(\epsilon) | J > \tilde{m}) &= \frac{\mathbb{P}(J > \tilde{m} | E_m(\epsilon)) \mathbb{P}(E_m(\epsilon) | J > m)}{\mathbb{P}(J > \tilde{m} | J > m)} \\ &\geq \frac{\mathbb{P}(J > \tilde{m} | E_m(\epsilon)) a_m(\epsilon)}{\mathbb{P}(J > \tilde{m} | E_m(\epsilon)) a_m(\epsilon) + \mathbb{P}(J > \tilde{m}, E_m(\epsilon)^c | J > m)}. \end{aligned} \quad (60)$$

By [She11, Theorem 2.5] and our assumption on  $a_m(\epsilon)$ , this quantity is at least a positive constant  $c$  depending on  $q_0$ ,  $\lambda$  and  $\epsilon$  (but not on  $\zeta$ ). Therefore, (59) implies  $a_{\tilde{m}}(\zeta) \geq (1 - q)c$ , so (58) implies

$$a_n(\epsilon) \geq \frac{(1 - q)^2 c}{(1 - q)c + \alpha}.$$

If we choose  $\alpha$  sufficiently small relative to  $c$  (and hence  $\zeta$  sufficiently small and  $m$  sufficiently large), we can make this quantity as close to  $1 - q$  as we like.  $\square$

*Proof of Proposition 3.1.* Let  $q_0$  be as in the conclusion of Lemma 3.2. Also fix  $q \in (0, 1 - q_0]$  and  $\lambda \in (0, 1/2)$ . Let  $\epsilon_0 > 0$  and  $m_* = m_*(q, q_0, \lambda, \epsilon_0) \in \mathbb{N}$  be chosen so that the conclusion of Lemma 3.10 holds with this choice of  $q_0$ . By Lemma 3.2 we can find  $m \geq m_*$  such that  $a_m(\epsilon_0) \geq q_0$ . It therefore follows from Lemma 3.10 that  $a_n(\epsilon_0) \geq 1 - q$  for each  $n \in \mathbb{N}$  with  $(1 - \lambda)^{-1}m \leq n \leq \lambda^{-1}m$ . By induction, for each  $k \in \mathbb{N}$  and each  $n \in \mathbb{N}$  with  $(1 - \lambda)^{-k}m \leq n \leq \lambda^{-k}m$ , we have  $a_n(\epsilon_0) \geq 1 - q \geq q_0$ . For sufficiently large  $k \in \mathbb{N}$ , the intervals  $[(1 - \lambda)^{-k}m, \lambda^{-k}m]$  and  $[(1 - \lambda)^{-k-1}m, \lambda^{-k-1}m]$  overlap, so it follows that for sufficiently large  $n \in \mathbb{N}$ , we have  $[n, \infty) \subset \bigcup_{k \in \mathbb{N}} [(1 - \lambda)^{-k}m, \lambda^{-k}m]$ . Hence  $a_n(\epsilon_0) \geq 1 - q$  for each such  $n$ . Thus (33) holds.  $\square$

## 4 Convergence conditioned on no burgers

### 4.1 Statement and overview of the proof

In this section we will prove the following theorem.

**Theorem 4.1.** *As  $n \rightarrow \infty$ , the conditional law of  $Z^n|_{[-1,0]}$  given the event that  $X(-n, -1)$  contains no burgers converges to the law of  $\underline{Z}$ , where  $\underline{Z}(\cdot) = \widehat{Z}(\cdot)$  and  $\widehat{Z}$  has the law of  $Z|_{[0,1]}$  conditioned to stay in the first quadrant (as defined just above Lemma 2.1).*

Throughout, we continue to use the notation of Section 3.1, so in particular  $J$  is the smallest  $j \in \mathbb{N}$  for which  $X(-j, -1)$  contains a burger.

The basic outline of the proof of Theorem 4.1 is as follows. First, in Section 4.2, we will prove a result to the effect that when  $N \in \mathbb{N}$  is large, it holds with uniformly positive probability that there is an  $i \in [n, Nn]_{\mathbb{Z}}$  such that  $X(1, i)$  contains no burgers. Using this and an argument similar to the proof of Lemma 3.2, in Section 4.3 we will prove several results to the effect that  $X(-n, -1)$  is unlikely to have too many orders when we condition on  $\{J > n\}$  (complementing Proposition 3.1, which says that it is unlikely to have too few orders under this conditioning). In Section 4.4, we will use these results to prove tightness of the conditional laws of  $Z^n|_{[-1,0]}$  given  $\{J > n\}$ . In Section 4.5, we will complete the proof of Theorem 4.1 using Lemma 2.1.

## 4.2 Times with empty burger stack

In this section, we will prove the following straightforward consequence of Lemma 3.1, which is a weaker version of Proposition 1.10 (but which is indirectly needed for the proof of Proposition 1.10).

**Lemma 4.2.** *Fix  $N \in \mathbb{N}$  and for  $n \in \mathbb{N}$ , let  $\mathcal{E}_n = \mathcal{E}_n(N)$  be the event that there is an  $i \in [n, Nn]_{\mathbb{Z}}$  such that  $X(1, i)$  has empty burger stack. There is a constant  $b > 0$  and an  $N_* \in \mathbb{N}$  (independent of  $n$ ) such that for  $N \geq N_*$  and  $n \in \mathbb{N}$ ,*

$$\mathbb{P}(\mathcal{E}_n) \geq b, \quad \forall n \in \mathbb{N}. \quad (61)$$

First we need the following lemma.

**Lemma 4.3.** *Let  $J$  be as in Section 3.1 and let  $\mu$  be as in (9). For each  $N \in \mathbb{N}$ , we have*

$$\mathbb{P}(J > Nn \mid J > n) \succeq N^{-\mu} + o_n(1),$$

with the implicit constant independent of  $n$  and  $N$ .

*Proof.* By Proposition 3.1, we can find  $\epsilon > 0$ , independent of  $n$ , such that (in the notation of that lemma) we have  $a_n(\epsilon) \geq \frac{1}{2} + o_n(1)$ . By [She11, Theorem 2.5] and Lemma 2.2 we have  $\mathbb{P}(J > Nn \mid E_n(\epsilon)) \succeq N^{-\mu} + o_n(1)$ , with the implicit constant depending on  $\epsilon$  but not on  $n$ . Therefore,

$$\mathbb{P}(J > Nn \mid J > n) \geq \mathbb{P}(J > Nn \mid E_n(\epsilon)) a_n(\epsilon) \succeq N^{-\mu} + o_n(1).$$

□

*Proof of Lemma 4.2.* For  $j_1 \leq j_2 \in \mathbb{N}$ , let  $B(j_1, j_2)$  be the number of  $i \in [j_1 + 1, j_2]_{\mathbb{Z}}$  such that  $X(1, i)$  has empty burger stack. Set  $B_n := B(n, Nn)$ . Also define the events  $E_i$  as in Lemma 2.11. By Lemma 2.11 we have

$$\begin{aligned} \mathbb{E}(B_n^2) &= \sum_{i=n}^{Nn} \mathbb{P}(E_i) + 2 \sum_{i=n}^{Nn} \sum_{j=i+1}^{Nn} \mathbb{P}(E_i \cap E_j) \\ &= \mathbb{E}(B_n) + 2 \sum_{i=n}^{Nn} \sum_{j=i+1}^{Nn} \mathbb{P}(E_i) \mathbb{P}(E_{j-i}) \\ &= \mathbb{E}(B_n) + 2 \sum_{i=n}^{Nn} \mathbb{P}(E_i) \sum_{j=1}^{N(n-1)-i} \mathbb{P}(E_j) \\ &= \mathbb{E}(B_n) + 2 \sum_{i=n}^{Nn} \mathbb{P}(E_i) \mathbb{E}(B(1, N(n-1) - i)) \\ &\leq \mathbb{E}(B_n) + 2\mathbb{E}(B_n) \mathbb{E}(B(1, N(n-1))). \end{aligned} \quad (62)$$

By Lemma 4.3, we can find a constant  $c > 0$ , independent from  $N$  and  $n$ , such that for sufficiently large  $i \in \mathbb{N}$  we have (with  $J$  as in that lemma) that

$$\mathbb{P}(E_{Ni}) = \mathbb{P}(J > Ni) \geq cN^{-\mu} \mathbb{P}(J > i) = cN^{-\mu} \mathbb{P}(E_i).$$

Therefore,

$$\mathbb{E}(B(1, N(n-1))) = \sum_{i=1}^{N(n-1)} \mathbb{P}(E_i) \leq c^{-1} N^{\mu} \sum_{i=1}^{N(n-1)} \mathbb{P}(E_{Ni}) + O_n(1).$$

By (24) this quantity is at most  $c^{-1} N^{\mu-1} \mathbb{E}(B(1, N^2(n-1))) + O_n(1)$ . On the other hand,

$$\mathbb{E}(B(1, N^2(n-1))) = \mathbb{E}(B(1, N(n-1))) + \sum_{k=2}^N \mathbb{E}(B((k-1)N(n-1) + 1, kN(n-1))). \quad (63)$$

By (24) each term in the big sum in (63) is at most  $\mathbb{E}(B_n)$ . Hence

$$(cN^{1-\mu}) \mathbb{E}(B(1, N(n-1))) \leq \mathbb{E}(B(1, N(n-1))) + (N-1)\mathbb{E}(B_n) + O_n(1).$$

Upon re-arranging we get that for  $N$  sufficiently large,

$$\mathbb{E}(B(1, N(n-1))) \leq \frac{N-1}{cN^{1-\mu}-1} \mathbb{E}(B_n) + O_n(1) \preceq N^\mu \mathbb{E}(B_n) + O_n(1).$$

By combining this with (62), we obtain

$$\mathbb{E}(B_n^2) \preceq \mathbb{E}(B_n) + N^\mu \mathbb{E}(B_n)^2.$$

Hence the Paley-Zygmund inequality implies

$$\mathbb{P}(\mathcal{E}_n) = \mathbb{P}(B_n > 0) \succeq N^{-\mu}.$$

It is clear that  $\mathbb{P}(\mathcal{E}_n)$  is increasing in  $N$ , so we obtain the statement of the lemma.  $\square$

### 4.3 Upper bound on the number of orders

Proposition 3.1 tells us that it is unlikely that there are fewer than  $O_n(n^{1/2})$  hamburger orders or cheeseburgers orders in  $X(-n, -1)$  when we condition on  $\{J > n\}$ . In this section, we will prove some results to the effect that it is unlikely that there are more than  $O_n(n^{1/2})$  orders in  $X(-n, -1)$  under this conditioning. These results are needed to prove tightness of the conditional law of  $Z^n|_{[-1,0]}$  given  $\{J > n\}$ .

We first need an elementary lemma which allows us to compare the lengths of the reduced words which we get when we read a given word forward to the lengths when we read the same word backward.

**Lemma 4.4.** *For  $n \in \mathbb{N}$  and  $j \in [2, n]_{\mathbb{Z}}$ , we have*

$$|X(j, n)| \leq |X(1, n)| + |X(1, j-1)|.$$

*Proof.* For  $j \in [1, n]_{\mathbb{Z}}$ , let  $A_j$  denote the set of  $k \in [j, n]_{\mathbb{Z}}$  with  $\phi(k) \in [1, j-1]_{\mathbb{Z}}$  and let  $B_j$  denote the set of  $k \in [j, n]_{\mathbb{Z}}$  with  $\phi(j) \leq 0$  or  $\phi(j) \geq n+1$ . Since every symbol in  $X$  a.s. has a match, it follows that  $|X(j, n)| = |A_j| + |B_j|$ . On the other hand, for  $k \in A_j$  we have that  $X_{\phi(k)}$  appears in  $X(1, j-1)$  and for  $k \in B_j$  we have that  $X_k$  appears in  $X(1, n)$ . The statement of the lemma follows.  $\square$

**Lemma 4.5.** *For  $C > 1$  and  $m \in \mathbb{N}$ , let*

$$\widehat{G}_m(C) := \left\{ \sup_{j \in [1, m]_{\mathbb{Z}}} |X(-j, -1)| \leq Cn^{1/2} \right\}.$$

*There is an  $N_* \in \mathbb{N}$  such that for each  $N \geq N_*$ , there is a constant  $c_*(N) > 0$  (depending only on  $N$ ) such that the following is true. For each  $q \in (0, 1)$ , there exists  $k_* = k_*(q, N)$  such that for  $k \geq k_*$ , we can find  $m \in [N^{k-1} + 1, N^k]_{\mathbb{Z}}$  satisfying*

$$\mathbb{P}\left(\widehat{G}_m(c_*(N) \log q^{-1}) \mid J > m\right) \geq 1 - q. \quad (64)$$

*Proof.* The proof is similar to that of Lemma 3.2. For  $k \in \mathbb{N}$ , define the time  $K_{N^k}$  as in Lemma 3.3 with  $n = N^k$ . Let  $A_k$  be the event that  $K_{N^k} \in [N^{k-1} + 1, N^k]_{\mathbb{Z}}$ . For  $C > 1$ , let  $\widehat{G}'_k(C)$  be the event that

$$\sup_{i \in [1, K_{N^k}]_{\mathbb{Z}}} |X(1, i)| \leq CK_{N^k}^{1/2}.$$

By Lemma 4.2, there is an  $N_* \in \mathbb{N}$ , a  $k_* \in \mathbb{N}$ , and a constant  $c_0 > 0$  such that for  $N \geq N_*$  and  $k \geq k_*$  we have  $\mathbb{P}(A_k) \geq c_0 N^{-\mu}$ . By the proof of [She11, Lemma 3.13], there are constants  $c_1 > 0$  and  $c_2 > 0$  (depending only on  $p$ ) such that for each  $C > 1$ ,

$$\mathbb{P}\left(\sup_{i \in [1, \dots, N^k]_{\mathbb{Z}}} |X(1, i)| \geq CN^{k/2}\right) \leq c_1 e^{-c_2 C}.$$

Hence

$$\mathbb{P}\left(\widehat{G}'_k(C)^c \cap A_k\right) \leq c_1 e^{-c_2 N^{-1/2} C}.$$

The right side of this inequality is  $\leq qc_0 N^{-\mu} \leq q\mathbb{P}(A_k)$  provided we take

$$C \geq 2c_*(N) \log q^{-1},$$

for an appropriate choice of  $c_*(N) > 0$  depending only on  $N$ . With this value of  $c_*(N)$  we therefore have

$$\mathbb{P}\left(\widehat{G}'_k(2c_*(N) \log q^{-1})\right) \geq (1-q)\mathbb{P}(A_k).$$

That is,

$$\sum_{n=N^{k-1}+1}^{N^k} \mathbb{P}\left(\widehat{G}'_k(2c_*(N) \log q^{-1}) \mid K_{N^k} = n\right) \mathbb{P}(K_{N^k} = n) \geq (1-q) \sum_{n=N^{k-1}+1}^{N^k} \mathbb{P}(K_{N^k} = n).$$

Hence we can find some  $m \in [N^{k-1}, N^k - 1]_{\mathbb{Z}}$  for which

$$\mathbb{P}\left(\sup_{i \in [1, m]_{\mathbb{Z}}} |X(1, i)| \leq 2c_*(N) \log q^{-1} m^{1/2} \mid K_{N^k} = m+1\right) \geq 1-q.$$

By taking a supremum over all  $j$  in the inequality of Lemma 4.4, we also have

$$\mathbb{P}\left(\sup_{j \in [1, m]_{\mathbb{Z}}} |X(j, m)| \leq c_*(N) \log q^{-1} m^{1/2} \mid K_{N^k} = m+1\right) \geq 1-q.$$

By the argument at the end of Lemma 3.2, for this choice of  $m$  we have

$$\mathbb{P}\left(\widehat{G}_m(c_*(N) \log q^{-1}) \mid J > m\right) \geq 1-q.$$

□

**Lemma 4.6.** *Let  $q \in (0, 1)$  and  $\zeta > 0$ . There exists  $\lambda_0, \lambda_1 \in (0, 1)$  and  $n_* \in \mathbb{N}$  (depending on  $\zeta$  and  $q$ ) such that for each  $n \geq n_*$ , we can find a deterministic  $m_n = m_n(\zeta, q) \in [\lambda_0 n, \lambda_1 n]_{\mathbb{Z}}$  such that the following is true. Let  $G_{m_n}(\zeta)$  be the event that  $J > m_n$  and  $|X(-j, -1)| \leq \zeta n^{1/2}$  for each  $j \in [1, m_n]_{\mathbb{Z}}$ . Then we have*

$$\mathbb{P}(G_{m_n}(\zeta) \mid J > n) \geq 1-q. \quad (65)$$

*Proof.* Fix  $\alpha \in (0, 1/4)$  to be chosen later (depending on  $\zeta$  and  $q$ ). Let  $N_* \in \mathbb{N}$  be chosen sufficiently large that the conclusion of Lemma 4.5 holds. Fix  $N \geq N_*$  and let  $c_*(N)$  be as in that lemma. Given  $\zeta > 0$ , let  $k_n$  be the largest  $k \in \mathbb{N}$  for which  $c_*(N) \log \alpha^{-1} N^{k/2} \leq \zeta n^{1/2}$ . If  $n$  is chosen sufficiently large, then by Lemma 4.5 we can find  $m_n \in [N^{k_n-1}, N^{k_n}]_{\mathbb{Z}}$  such that (64) holds with  $\alpha$  in place of  $q$ . In the notation of (64) we have

$$\widehat{G}_{m_n}(c_*(N) \log \alpha^{-1}) \cap \{J > m_n\} \subset G_{m_n}(\zeta).$$

Let

$$\rho(\alpha) := (c_*(N) \log \alpha^{-1})^{-1}.$$

Then we have  $\lambda_0(\alpha)n \leq m_n \leq \lambda_1(\alpha)n$  for  $\lambda_0(\alpha) = N^{-2}\rho(\alpha)^2\zeta^2$  and  $\lambda_1(\alpha) = \rho(\alpha)^2\zeta^2$ .

We have  $\mathbb{P}(G_{m_n}(\zeta) \mid J > m_n) \geq 1-\alpha$ . We need to show that if  $\alpha$  is chosen sufficiently small and  $n$  is chosen sufficiently large (depending on  $\zeta$  and  $q$ ), then we can transfer this to a lower bound when we further condition on  $\{J > n\}$ .

By Proposition 3.1, we can find  $\epsilon > 0$  (independent of  $\alpha$ ,  $N$ , and  $\zeta$ ) and  $\tilde{n}_* \in \mathbb{N}$  (depending on  $\epsilon$ ,  $\alpha$ ,  $N$ , and  $\zeta$ ) such that (in the notation of that proposition) we have  $a_{m_n}(\epsilon) \geq 1/2$  for each  $n \geq \tilde{n}_*$ . For this choice of  $\epsilon$ , we have for  $n \geq \tilde{n}_*$  that

$$\begin{aligned} \mathbb{P}(J > n \mid G_{m_n}(\zeta)) &\geq \mathbb{P}(J > n \mid E_{m_n}(\epsilon) \cap G_{m_n}(\zeta)) \mathbb{P}(E_{m_n}(\epsilon) \cap G_{m_n}(\zeta) \mid J > m_n) \\ &\geq \frac{1}{4} \mathbb{P}(J > n \mid E_{m_n}(\epsilon) \cap G_{m_n}(\zeta)). \end{aligned}$$

By [She11, Theorem 2.5] and Lemma 2.2, there is an  $n_* \geq \tilde{n}_*$  (depending on  $\epsilon$  and  $\lambda_0$ ) and a constant  $c_0 > 0$  (independent of all of the other parameters) such that for  $n \geq n_*$ ,

$$\mathbb{P}(J > n \mid E_{m_n}(\epsilon) \cap G_{m_n}(\zeta)) \geq c_0 \epsilon^\mu (N^{-1} \rho(\alpha) \zeta)^\mu.$$

Hence for  $n \geq n_*$ ,

$$\mathbb{P}(J > n \mid G_{m_n}(\zeta)) \geq \hat{\rho}(\alpha) := \frac{c_0 \epsilon^\mu}{4} (N^{-1} \rho(\alpha) \zeta)^{2\mu}.$$

By Bayes' rule,

$$\begin{aligned} \mathbb{P}(G_{m_n}(\zeta) \mid J > n) &\geq \frac{\mathbb{P}(J > n \mid G_{m_n}(\zeta)) \mathbb{P}(G_{m_n}(\zeta) \mid J > m_n)}{\mathbb{P}(J > n \mid G_{m_n}(\zeta)) \mathbb{P}(G_{m_n}(\zeta) \mid J > m_n) + \mathbb{P}(G_{m_n}(\zeta)^c \mid J > m_n)} \\ &\geq \frac{(1-\alpha)\hat{\rho}(\alpha)}{(1-\alpha)\hat{\rho}(\alpha) + \alpha} = \frac{(1-\alpha)}{(1-\alpha) + \alpha\hat{\rho}(\alpha)^{-1}}. \end{aligned}$$

As  $\alpha \rightarrow 0$ , we have  $\alpha\hat{\rho}(\alpha)^{-1} \rightarrow 0$ , so if  $\alpha$  is chosen sufficiently small (depending on  $\zeta$  and  $q$ ), and hence  $n_*$  is chosen sufficiently large (depending on  $\zeta$ ,  $q$ , and  $\alpha$ ) this quantity is at least  $1 - q$ .  $\square$

#### 4.4 Proof of tightness

In this section we will prove tightness of the conditional laws of  $Z^n|_{[-1,0]}$  given  $\{J > n\}$ . We first need the following basic consequence of the results of Section 3.

**Lemma 4.7.** *Suppose we are in the setting of Section 3.1. Let  $\lambda \in (0, 1/2)$  and  $q \in (0, 1)$ . There exists  $\epsilon > 0$  and  $n_* \in \mathbb{N}$ , depending only on  $q$  and  $\lambda$ , such that for each  $n \geq n_*$  and  $m \in \mathbb{N}$  with  $\lambda \leq m/n \leq 1 - \lambda$ ,*

$$\mathbb{P}(E_m(\epsilon) \mid J > n) \geq 1 - q.$$

*Proof.* Fix  $\alpha \in (0, 1)$  to be determined later, depending only on  $q$ . By Proposition 3.1, we can find  $\epsilon_0 > 0$  and  $m_* \in \mathbb{N}$  such that (in the notation of Section 3.1) it holds for each  $m \geq m_*$  and  $\epsilon \in (0, \epsilon_0]$  that  $a_m(\epsilon) \geq 1 - \alpha$ . By Proposition 3.9, we can find  $\epsilon \in (0, \epsilon_0]$  and  $n_* \in \mathbb{N}$  with  $n_* \geq \lambda^{-2} m_*$  such that for  $n \geq n_*$  and  $m$  as in the statement of the lemma, we have

$$\mathbb{P}(J > n \mid E_m(\epsilon)^c, J > m) \leq \alpha \mathbb{P}(J > n \mid E_m(\epsilon)).$$

By Bayes' rule,

$$\begin{aligned} \mathbb{P}(E_m(\epsilon) \mid J > n) &= \frac{\mathbb{P}(J > n \mid E_m(\epsilon)) a_m(\epsilon)}{\mathbb{P}(J > n \mid E_m(\epsilon)) a_m(\epsilon) + \mathbb{P}(J > n \mid E_m(\epsilon)^c, J > m) (1 - a_m(\epsilon))} \\ &\geq \frac{1 - \alpha}{1 - \alpha + \alpha^2}. \end{aligned}$$

By choosing  $\alpha$  sufficiently small, in a manner which depends only on  $q$ , we can make this last quantity greater than or equal to  $1 - q$ .  $\square$

**Lemma 4.8.** *The conditional laws of  $Z^n|_{[-1,0]}$  given  $\{J > n\}$  for  $n \in \mathbb{N}$  are a tight family of probability measures on the set of continuous functions on  $[-1, 0]$  in the topology of uniform convergence.*

*Proof.* For  $\delta, \zeta > 0$  and  $n \in \mathbb{N}$ , let  $\tilde{G}_n(\zeta, \delta)$  be the event that the following is true. Whenever  $t_1, t_2 \in [-1, 0]$  with  $|t_1 - t_2| \leq \delta$ , we have  $|Z^n(t_1) - Z^n(t_2)| \leq \zeta$ . For a continuous non-decreasing function  $\rho : (0, \infty) \rightarrow (0, \infty)$  with  $\lim_{\zeta \rightarrow 0} \rho(\zeta) = 0$ , let

$$\mathcal{G}_n(\rho) := \bigcap_{\zeta > 0} \tilde{G}_n(\zeta, \rho(\zeta)).$$

By the Arzela-Ascoli theorem (note that equicontinuity implies uniform boundedness in this case since each  $Z^n$  vanishes at the origin), we must show that for each given  $q \in (0, 1)$ , we can find  $\rho$  as above, independent of  $n$ , such that

$$\mathbb{P}(\mathcal{G}_n(\rho) \mid J > n) \geq 1 - q, \quad \forall n \in \mathbb{N}. \tag{66}$$

First suppose that we are given  $\zeta > 0$  and  $\alpha \in (0, 1)$ . By Lemma 4.6, we can find  $n_1 \in \mathbb{N}$  and  $\lambda_0, \lambda_1 \in (0, 1)$  (depending on  $\zeta$  and  $\alpha$ ) such that for each  $n \geq n_1$  there exists  $m_n \in [\lambda_0 n, \lambda_1 n]_{\mathbb{Z}}$  such that (in the notation of Lemma 4.6), we have that (65) holds with  $1 - \alpha/2$  in place of  $1 - q$ .

By Lemma 4.7, we can find  $\epsilon > 0$  and  $n_2 \geq n_1$  (depending on  $\zeta$  and  $\alpha$ ) such that for  $n \geq n_2$ , we have  $\mathbb{P}(E_{m_n}(\epsilon) | J > n) \geq 1 - \alpha/2$ . By Lemma 3.6 that we can find  $n_3 \geq n_2$  and  $\delta_0 = \delta_0(\alpha, \zeta) > 0$  such that if  $n \geq n_3$ , then with conditional probability at least  $1 - \alpha$  given  $E_{m_n}(\epsilon) \cap G_{m_n}(\zeta) \cap \{J > n\}$ , it holds that whenever  $t_1, t_2 \in [-1, -m_n/n]$  with  $|t_1 - t_2| \leq \delta_0$ , we have  $|Z^n(t_1) - Z^n(t_2)| \leq \zeta$ . Call this last event  $A$ . If  $A$  occurs and  $G_{m_n}(\zeta)$  occurs then  $\tilde{G}_n(\zeta, \delta_0)$  occurs. Therefore, if  $n \geq n_3$ , then

$$\begin{aligned} \mathbb{P}\left(\tilde{G}_n(2\zeta, \delta_0) | J > n\right) &\geq \mathbb{P}(A \cap G_{m_n}(\zeta) | J > n) \\ &\geq \mathbb{P}(A | E_{m_n}(\epsilon) \cap G_{m_n}(\zeta), J > n) \mathbb{P}(E_{m_n}(\epsilon) \cap G_{m_n}(\zeta) | J > n) \\ &\geq (1 - \alpha)^2. \end{aligned}$$

Clearly, there exists a deterministic  $\delta \in (0, \delta_0]$  and  $C > C_1$  depending only on  $n_3$  such that  $\tilde{G}_n(2\zeta, \delta_0) \cap \left\{\sup_{t \in [-1, 0]} |Z^n(t)| \leq C\right\}$  occurs a.s. for each  $n \in [1, n_3]_{\mathbb{Z}}$ . Therefore,

$$\mathbb{P}\left(\tilde{G}_n(2\zeta, \delta) | J > n\right) \geq (1 - \alpha)^2, \quad \forall n \in \mathbb{N}. \quad (67)$$

Now fix  $q \in (0, 1)$ . For  $j \in \mathbb{N}$ , choose  $\delta_j > 0$  for which (67) holds with  $\delta = \delta_j$ ,  $\zeta = 2^{-j-1}$ , and  $\alpha$  chosen so that  $(1 - \alpha)^2 = 1 - q2^{-j-1}$ . Let

$$\rho(\zeta) := C\mathbb{1}_{[1, \infty)}(\zeta) + \sum_{j=1}^{\infty} \delta_j \mathbb{1}_{[2^{-j}, 2^{-j+1})}(\zeta).$$

Then (66) holds for this choice of  $\rho$ . □

## 4.5 Identifying the limiting law

To identify the law of a subsequential limit of the laws of  $Z^n|_{[-1, 0]}$  given  $\{J > n\}$ , we need the following fact from elementary probability theory.

**Lemma 4.9.** *Let  $(X_n, Y_n)$  be a sequence of pairs of random variables taking values in a product of separable metric spaces  $\Omega_X \times \Omega_Y$  and let  $(X, Y)$  be another such pair of random variables. Suppose  $(X_n, Y_n) \rightarrow (X, Y)$  in law. Suppose further that there is a family of probability measures  $\mu_y$  on  $\Omega_X$ , indexed by  $\Omega_Y$ , and a family of  $Y_n$ -measurable events  $E_n$  with  $\lim_{n \rightarrow \infty} \mathbb{P}(E_n) = 1$  such that for each bounded continuous function  $f : \Omega_X \rightarrow \mathbb{R}$ , we have*

$$\mathbb{E}(f(X_n) | Y_n) \mathbb{1}_{E_n} \rightarrow \mathbb{E}_{\mu_Y}(f) \quad \text{in law.}$$

Then  $\mu_Y$  is the regular conditional law of  $X$  given  $Y$ .

*Proof.* Let  $g : \Omega_Y \rightarrow \mathbb{R}$  be a bounded continuous function. Then

$$\begin{aligned} \mathbb{E}(f(X)g(Y)) &= \lim_{n \rightarrow \infty} \mathbb{E}(f(X_n)g(Y_n)) \\ &= \lim_{n \rightarrow \infty} \mathbb{E}(f(X_n)g(Y_n)\mathbb{1}_{E_n}) \\ &= \lim_{n \rightarrow \infty} \mathbb{E}(\mathbb{E}(f(X_n) | Y_n) \mathbb{1}_{E_n} g(Y_n)) \\ &= \mathbb{E}(\mathbb{E}_{\mu_Y}(f)g(Y)). \end{aligned}$$

By the functional monotone class theorem, we have  $\mathbb{E}(F(X, Y)) = \mathbb{E}(\mathbb{E}_{\mu_Y}(F(\cdot, Y)))$  for every bounded Borel-measurable function  $F$  on  $\Omega_X \times \Omega_Y$ . This implies the statement of the lemma. □

*Proof of Theorem 4.1.* By Lemma 4.8 and the Prokhorov theorem, from any sequence of integers tending to  $\infty$ , we can extract a subsequence along which the conditional laws of  $Z^n$  given  $J > n$  converge to the law of

some random continuous function  $\tilde{Z} = (\tilde{U}, \tilde{V}) : [-1, 0] \rightarrow \mathbb{R}^2$  as  $n \rightarrow \infty$ , restricted to this subsequence. We must show that  $\tilde{Z} \stackrel{d}{=} \underline{Z}$ , with  $\underline{Z}$  as defined in the statement of the theorem.

By Lemma 4.7, we a.s. have  $\tilde{U}(s) > 0$  and  $\tilde{V}(s) > 0$  for each  $s \in (0, 1)$ . By Lemma 2.1, it therefore suffices to show that for each  $\zeta \in (0, 1)$ , the conditional law of  $\tilde{Z}|_{[-1, -\zeta]}$  given  $\tilde{Z}|_{[-\zeta, 0]}$  is that of a Brownian motion with covariances as in (8), starting from  $\tilde{Z}(-\zeta)$ , parametrized by  $[-1, -\zeta]$ , and conditioned to stay in the first quadrant.

To lighten notation we henceforth consider only values of  $n$  in our subsequence and implicitly assume that all statements involving  $n$  are for  $n$  restricted to this subsequence.

Fix  $\zeta \in (0, 1)$  and let  $\lfloor \zeta n \rfloor$ . Also let  $\widehat{D}_\zeta$  be the path defined in the same manner as the path  $D$  of (6) of Section 1.1 but with the following modification: if  $j \in [-\zeta n, -1]_{\mathbb{Z}}$ ,  $X_j = \boxed{\mathbf{F}}$ , and  $-\phi(-j) > \zeta n$ , then  $\widehat{D}_\zeta(-j) - \widehat{D}_\zeta(-j+1)$  is equal to zero rather than  $(1, 0)$  or  $(0, 1)$ . Extend  $\widehat{D}_\zeta$  to  $[-\zeta n, 0]$  by linear interpolation. For  $t \in [-\zeta, 0]$ , let  $\widehat{Z}_\zeta^n(t) := n^{-1/2} \widehat{D}_\zeta(nt)$ . It follows from Lemma 2.8 that  $\sup_{t \in [-\zeta, 0]} |\widehat{Z}_\zeta^n(t) - Z^n(t)| \rightarrow 0$  in law, even if we condition on  $\{J > n\}$ , whence  $\widehat{Z}_\zeta^n \rightarrow \tilde{Z}|_{[-\zeta, 0]}$  in law. We note that  $\widehat{Z}_\zeta^n$  determines and is determined by  $X_{-\lfloor \zeta n \rfloor} \dots X_{-1}$ , so is independent from  $\dots X_{-\lfloor \zeta n \rfloor - 2} X_{-\lfloor \zeta n \rfloor - 1}$  and hence also from  $Z^n|_{[-\zeta, 0]}$  (notation 3.5).

Let  $(\underline{X}^n)$  be a sequence of random words distributed according to the conditional law of  $X_{-n} \dots X_{-1}$  given  $\{J > n\}$ . Let  $(\underline{Z}^n)$  be the corresponding paths, so that each  $\underline{Z}^n$  has the conditional law of  $Z^n$  given  $\{J > n\}$ . Let  $\widehat{Z}_\zeta^n$  be the corresponding random paths  $\widehat{Z}_\zeta^n$ . By the Skorokhod theorem, we can couple  $(\underline{X}^n)$  with  $\tilde{Z}$  (with  $n$  restricted to our subsequence) in such a way that a.s.  $\widehat{Z}_\zeta^n \rightarrow \tilde{Z}|_{[-\zeta, 0]}$  uniformly.

For  $\epsilon_1, \epsilon_2 > 0$ , define  $\tilde{G}^\zeta(\epsilon_1, \epsilon_2)$  as in (43) with  $s = \zeta$ . By Lemma 3.6, for each fixed  $\epsilon > 0$ , the Prokhorov distance between the conditional law of  $Z^n|_{[-1, -\zeta]}$  given  $J > n$  and any realization of  $\widehat{Z}_\zeta^n$  for which  $E_{\lfloor \zeta n \rfloor}(\epsilon) \cap F_{\lfloor \zeta n \rfloor}$  occurs; and the conditional law of  $Z^n|_{[-1, -\zeta]}$  given the event  $\tilde{G}^\zeta(\tilde{U}(-\zeta), \tilde{V}(-\zeta))$  of Lemma 3.6 converges to zero as  $n \rightarrow \infty$ . By combining this with Lemma 4.7, we obtain that for any bounded continuous function  $f$  from the space of continuous functions on  $[-\zeta, -1]$  (in the uniform topology) to  $\mathbb{R}$ , we have

$$\mathbb{E} \left( f \left( Z^n|_{[-1, -\zeta]} \right) \mid J > n, \widehat{Z}_\zeta^n \right) \mathbb{1}_{F_{\lfloor \zeta n \rfloor}} \rightarrow \mathbb{E} \left( f \left( Z^n|_{[-1, -\zeta]} \right) \mid \tilde{G}^\zeta \left( \tilde{U}(-\zeta), \tilde{V}(-\zeta) \right) \right) \quad (68)$$

in law. We now conclude by applying Lemma 4.9 with  $X_n = \underline{Z}^n|_{[-1, -\zeta]}$ ,  $Y_n = \widehat{Z}_\zeta^n$ ,  $X = \tilde{Z}|_{[-1, -\zeta]}$ , and  $Y = \tilde{Z}|_{[-\zeta, 0]}$ .  $\square$

## 5 Convergence of the cone times

### 5.1 Regular variation

We say that the law of a random variable  $A$  is *regularly varying* with exponent  $\alpha$  if for each  $c > 1$ ,

$$\lim_{a \rightarrow \infty} \frac{\mathbb{P}(A > ca)}{\mathbb{P}(A > a)} = c^{-\alpha}.$$

In this section we will prove that the laws of several quantities associated with the word  $X$  are regularly varying. In doing so, we will obtain Proposition 1.10.

**Proposition 5.1.** *Let  $J$  be the smallest  $j \in \mathbb{N}$  for which  $X(-j, -1)$  contains a burger. The law of  $J$  is regularly varying with exponent  $\mu$ , as defined in (9). If  $\tilde{J}$  denotes the smallest  $j \in \mathbb{N}$  for which  $X(-j, -1)$  contains no  $\boxed{\mathbf{F}}$ -symbols, then  $\tilde{J}$  is also regularly varying with exponent  $\mu$ .*

We note that Proposition 5.1 can be viewed as an analogue for the random path  $D = (d, d^*)$  studied in this paper of the tail asymptotics for the exit time from a cone of a random walk with independent increments obtained in [DW11, Theorem 1]. However, unlike the estimate which is implicit in Proposition 5.1, the estimate of [DW11] does not involve a slowly varying correction.

*Proof of Proposition 5.1.* Fix  $c > 1$ . For  $z \in (0, \infty)^2$ , write  $\Phi_c(z)$  for the probability that a two-dimensional Brownian motion with covariances (8) started from  $z$  stays in the first quadrant until time  $c - 1$ . Note that  $\Phi_c$  is a bounded continuous function of  $z$ .

Let  $\tilde{Z} = (\tilde{U}, \tilde{V})$  have the law of  $Z|_{[-1,0]}$  conditioned to stay in the first quadrant. For  $n \in \mathbb{N}$ , let  $\hat{Z}^n$  be defined in the same manner as the path  $\tilde{Z}_\zeta^n$  used in the proof of Theorem 4.1, but with 1 in place of  $\zeta$ , so that  $\hat{Z}^n$  determines and is determined by  $X_{-n} \dots X_{-1}$  and is independent from  $\dots X_{-n-2} X_{-n-1}$  and hence also from  $Z_{[-c,-1]}^n$ .

By the same argument used to obtain (68) in the proof of Theorem 4.1, we have that

$$\mathbb{P}\left(J > cn \mid J > n, \hat{Z}^n\right) \mathbf{1}_{F_n} \rightarrow \Phi_c(\tilde{Z}(1)) \quad (69)$$

in law, where here (as usual)  $F_n$  is the event that  $X(-n, -1)$  contains at most  $n^\nu$  flexible orders for some  $\nu \in (\mu', 1/2)$ .

Since the conditional law of  $\hat{Z}^n$  given  $J > n$  converges to the law of  $\tilde{Z}$  and  $\lim_{n \rightarrow \infty} \mathbb{P}(F_n) = 1$ , we can take expectations to get

$$\mathbb{P}(J > cn \mid J > n) = \frac{\mathbb{P}(J > cn)}{\mathbb{P}(J > n)} \rightarrow f(c),$$

where  $f(c) := \mathbb{E}\left(\Phi_c(\tilde{Z}(1))\right)$ .

We have  $f(1) = 1$ ,  $f(c) \in (0, 1)$  for each  $c > 1$ , and

$$f(c)f(c') = \lim_{n \rightarrow \infty} \frac{\mathbb{P}(J > cn)}{\mathbb{P}(J > n)} \times \frac{\mathbb{P}(J > cc'n)}{\mathbb{P}(J > cn)} = f(cc').$$

We infer that  $f(c) = c^{-\alpha}$  for some  $\alpha > 0$ .

To identify  $\alpha$ , we need only consider the asymptotics of  $\mathbb{E}\left(\Phi_c(\tilde{Z}(1))\right)$  as  $c \rightarrow \infty$ . To this end, we apply [Shi85, Equation 4.3] (c.f. the proof of Lemma 2.2) to get that for fixed  $z \in (0, \infty)^2$ , we have

$$\lim_{c \rightarrow \infty} c^\mu \Phi_c(z) = \Psi(z)$$

for some positive continuous function  $\Psi$  on  $(0, \infty)^2$  which is bounded in every neighborhood of the origin. By the formula [Shi85, Equation 3.2] for the density of the law of  $\tilde{Z}(1)$ , it follows that  $\mathbb{P}\left(|\tilde{Z}(1)| > A\right)$  decays quadratic-exponentially in  $A$ . By Brownian scaling and [Shi85, Equation 4.2],

$$\sup_{z \in B_A(0) \cap (0, \infty)^2} |\Phi_c(z)| \preceq c^{-\mu} A^{2\mu}$$

with the implicit constant depending only on  $p$ . Hence

$$\mathbb{E}\left(|c^\mu \Phi_c(\tilde{Z}(1))| \mathbf{1}_{\{|c^\mu \Phi_c(\tilde{Z}(1))| \geq A\}}\right) \rightarrow 0$$

as  $A \rightarrow \infty$ , uniformly in  $c$ . By the Vitali convergence theorem,  $c^\mu f(c) = \mathbb{E}\left(c^\mu \Phi_c(\tilde{Z}(1))\right)$  converges to a finite constant as  $c \rightarrow \infty$ , whence we must have  $\alpha = \mu$ .

For the last statement, we note that with probability  $1 - p/2$  we have  $\tilde{J} = 1$ , and with probability  $p/2$ ,  $\tilde{J}$  is equal to the smallest  $j \in \mathbb{N}$  for which  $X(-j, -2)$  contains a burger. It follows that for  $n \geq 2$  we have  $\mathbb{P}\left(\tilde{J} > n\right) = \frac{p}{2} \mathbb{P}(J > n - 1)$ . Hence

$$\lim_{n \rightarrow \infty} \frac{\mathbb{P}(J > cn)}{\mathbb{P}(J > n)} = \lim_{n \rightarrow \infty} \frac{\mathbb{P}\left(\tilde{J} > cn\right)}{\mathbb{P}\left(\tilde{J} > n\right)}.$$

□



From Proposition 5.1, we can deduce that there a.s. exist macroscopic  $\boxed{\mathbb{F}}$ -excursions, which is the key input in our proof of Theorem 1.9 in the next section.

*Proof of Proposition 1.10.* For  $m \in \mathbb{N}$ , let  $\tilde{J}_m$  be the  $m$ th smallest  $j \in \mathbb{N}$  for which  $X(-j, -1)$  contains no  $\boxed{\mathbb{F}}$  symbols. Then the words  $X_{-\tilde{J}_m} \dots X_{-\tilde{J}_{m-1}-1}$  are iid. By Corollary 5.1,  $\tilde{J}_1$  is regularly varying with exponent  $-\mu \in (-1, 0)$ . For  $n \in \mathbb{N}$  let  $M_n$  be the largest  $m \in \mathbb{N}$  for which  $\tilde{J}_m \leq n$ . By the Dynkin-Lamperti theorem [Dyn55, Lam62],  $n^{-1}(n - J_{M_n})$  converges in law to a generalized arcsine distribution with parameter  $\mu$ . Since this distribution does not have a point mass at the origin we obtain the statement of the proposition.  $\square$

Although it will not be needed for the proof of Theorem 1.9, for the sake of completeness we end by recording some consequences of Proposition 5.1.

**Corollary 5.2.** *The statement of Lemma 2.8 holds, exactly as stated, with  $1 - \mu$  in place of  $\mu'$ .*

*Proof.* Define the events  $E_i$  as in Lemma 2.11. Then  $\mathbb{P}(E_i) = \mathbb{P}(J > i) = i^{-\mu+o_i(1)}$ , where the last inequality is by Proposition 5.1. Hence, with  $B_n$  as in Lemma 2.12, we have

$$\mathbb{E}(B_n) = \sum_{i=1}^n \mathbb{P}(E_i) = n^{1-\mu+o_n(1)}.$$

The last part of the proof of Lemma 2.12 now implies that that (25) holds with  $1 - \mu$  in place of  $\mu'$ . We conclude exactly as in the proof of Lemma 2.8.  $\square$

**Corollary 5.3.** *Let  $K^F$  be the smallest  $i \in \mathbb{N}$  for which  $X(1, i)$  contains a flexible order. The law of  $K^F$  is regularly varying with exponent  $1 - \mu$ .*

*Proof.* For  $m \in \mathbb{N}$ , let  $K_m^F$  be the smallest  $i \in \mathbb{N}$  for which  $X(1, i)$  contains at least  $m$  flexible orders. The words  $X_{K_{m-1}^F+1} \dots X_{K_m^F}$  are iid. For  $n \in \mathbb{N}$ , let  $M_n^*$  be the largest  $m \in \mathbb{N}$  for which  $K_m^F \leq n$ . Equivalently,  $K_{M_n^*}^F$  is the greatest integer  $i \in [1, n]_{\mathbb{Z}}$  such that  $X_i = \boxed{\mathbb{F}}$  and  $\phi(i) \leq 0$ . By translation invariance, we have  $K_{M_n^*}^F \stackrel{d}{=} n - \tilde{J}_{M_n^*}$ , with the latter defined in the proof of Lemma 1.10. Hence the law of  $n^{-1}K_{M_n^*}^F$  converges to that of a generalized arcsine distribution with parameter  $\mu$ . Therefore  $n^{-1}(n - K_{M_n^*}^F)$  converges in law to a generalized arcsine distribution with parameter  $1 - \mu$ . By the converse to the Dynkin-Lamperti theorem,  $K_{M_n^*}^F$  is regularly varying with exponent  $1 - \mu$ .  $\square$

**Remark 5.4.** In the terminology of [BLR15], Corollary 5.3 states that the law of the area of the part traced after time 0 of the “envelope” of the smallest loop surrounding the root vertex in the infinite-volume model is regularly varying with exponent  $1 - \mu$ . In [BLR15, Section 1.2], the authors conjecture that the tail exponent for the law of the area of this loop itself is  $1 - \mu$ . We expect that this conjecture (plus a regular variation statement for the tail) can be deduced from Proposition 5.1 and Corollary 5.3 via arguments which are very similar to some of those given in Sections 3 and 4 of the present paper, but we do not carry this out here.

## 5.2 Proof of Theorem 1.9

In this section, we will complete the proof of Theorem 1.9.

To complement Definition 1.6, one has a notion of a strict  $\pi/2$ -cone time, which is defined in the same manner as a weak  $\pi/2$ -cone time but with weak inequalities replaced by strict inequalities. More precisely,

**Definition 5.5.** A time  $t$  is called a *strict  $\pi/2$ -cone time* for a function  $Z = (U, V) : \mathbb{R} \rightarrow \mathbb{R}^2$  if there exists  $t' < t$  such that  $U_s > U_t$  and  $V_s > V_t$  for  $s \in (t', t)$ . Equivalently,  $Z((t', t))$  is contained in the open “cone”  $Z_t + \{z \in \mathbb{C} : \arg z \in [0, \pi]\}$ . We write  $\tilde{v}_Z(t)$  for the infimum of the times  $t'$  for which this condition is satisfied.

**Remark 5.6.** If  $t$  is a strict  $\pi/2$ -cone time for  $Z$ , then  $t$  is also a weak  $\pi/2$ -cone time for  $Z$  and we have  $\tilde{v}_Z(t) \leq v_Z(t)$ . The reverse inequality need not hold. For example,  $Z$  might enter the close cone at time  $\tilde{v}_Z(t)$ , hit the boundary of the closed cone at time  $v_Z(t) \in (\tilde{v}_Z(t), t)$ , then stay in the open cone until time  $t$ .

To prove Theorem 1.9 we first need a general deterministic statement about the convergence of  $\pi/2$ -cone times.

**Lemma 5.7.** *Let  $Z = (U, V) : \mathbb{R} \rightarrow \mathbb{R}^2$  be a continuous path with the following properties.*

1. *Each weak  $\pi/2$ -cone time for  $Z$  is a strict  $\pi/2$ -cone time for  $Z$  and satisfies  $\tilde{v}_Z(t) = v_Z(t)$ .*
2.  *$Z$  has no weak  $\pi/2$ -cone times  $t$  with  $Z_{v_Z(t)} = Z_t$ .*
3.  *$\liminf_{t \rightarrow -\infty} U(t) = \liminf_{t \rightarrow -\infty} V(t) = -\infty$ .*

*Let  $Z^n = (U^n, V^n)$  be a sequence of continuous paths  $\mathbb{R} \rightarrow \mathbb{R}^2$  such that  $Z^n \rightarrow Z$  uniformly on compacts. Suppose that for each  $n \in \mathbb{N}$ ,  $t_n$  is a weak  $\pi/2$ -cone time for  $Z^n$ . Suppose further that almost surely  $\liminf_{n \rightarrow \infty} (t_n - v_{Z^n}(t_n)) > 0$ . If  $t_n \rightarrow t$  for some  $t \in \mathbb{R}$ , then  $t$  is a strict  $\pi/2$ -cone time for  $Z$ . Furthermore,  $\lim_{n \rightarrow \infty} v_{Z^n}(t_n) = v_Z(t)$ ,  $\lim_{n \rightarrow \infty} u_{Z^n}(t_n) = u_Z(t)$ , and the direction of the  $\pi/2$ -cone time  $t_n$  for  $Z^n$  is the same as the direction of the  $\pi/2$ -cone time  $t$  for  $Z$  for sufficiently large  $n$ .*

*Proof.* We can choose a compact interval  $[a_0, b] \subset \mathbb{R}$  such that  $t_n \in [a_0, b]$  for each  $n \in \mathbb{N}$ . By our assumption 3 on  $Z$ , we can find  $a_1 < a_0$  such that  $\inf_{s \in [a_1, a_0]} U(s) < \inf_{s \in [a_0, b]} U(s)$  and  $\inf_{s \in [a_1, a_0]} V(s) < \inf_{s \in [a_0, b]} V(s)$ . For sufficiently large  $n$ , the same is true with  $(U^n, V^n)$  in place of  $(U, V)$ . Therefore, we can find  $a \in (-\infty, a_1]$  such that  $t_n, v_{Z^n}(t_n)$ , and  $u_{Z^n}(t_n)$  belong to  $[a, b]$  for each  $n \in \mathbb{N}$ .

By uniform convergence, we can find  $\delta > 0$  such that  $U(s) \geq U(t)$  and  $V(s) \geq V(t)$  for each  $s \in [t - \delta, t]$ , so  $t$  is a weak  $\pi/2$ -cone time for  $Z$ . By assumption 1,  $t$  is in fact a strict  $\pi/2$ -cone time for  $Z$ .

Suppose without loss of generality that  $t$  is a left  $\pi/2$ -cone time for  $Z$ , i.e.  $V(v_Z(t)) = V(t)$ . Let  $v$  be any subsequential limit of the times  $v_{Z^n}(t_n)$ . Then with  $n$  restricted to our subsequence we have  $\lim_{n \rightarrow \infty} U^n(v_{Z^n}(t_n)) = U(v)$  and  $\lim_{n \rightarrow \infty} V^n(v_{Z^n}(t_n)) = V(v)$ . Furthermore,  $U(s) \geq U(t)$  and  $V(s) \geq V(t)$  for each  $s \in [v, t]$ . Therefore  $v \geq v_Z(t)$ . We clearly have  $v < t$ , so since  $t$  is not a right  $\pi/2$ -cone time for  $Z$  (assumption 2) we have  $U(v) > U(t)$ . Hence  $U^n(v_{Z^n}(t_n)) > U(t)$  for sufficiently large  $n$  in our subsequence. Since  $U^n(t_n) \rightarrow U(t)$ , we have  $U^n(v_{Z^n}(t_n)) > U^n(t_n)$  for sufficiently large  $n$  in our subsequence. Hence  $V^n(v_{Z^n}(t_n)) = V^n(t_n)$  for sufficiently large  $n$  in our subsequence. Since this holds for every choice of subsequence we infer  $V^n(v_{Z^n}(t_n)) = V^n(t_n)$  for sufficiently large  $n$ . Moreover, for every choice of subsequence we have  $V(v) = \lim_{n \rightarrow \infty} V^n(t_n) = V(t)$ , whence  $v = v_Z(t)$  and  $v_{Z^n}(t_n) \rightarrow v_Z(t)$ .

Finally, let  $u$  be any subsequential limit of the times  $u_{Z^n}(t_n)$ . Then along our subsequence we have  $U(u) = \lim_{n \rightarrow \infty} U^n(u_{Z^n}(t_n)) = \lim_{n \rightarrow \infty} U^n(t_n) = U(t)$ . Furthermore,  $U(s) \geq U(t)$  for each  $s \in [u, t]$ . Therefore  $u = u_Z(t)$ . Since this holds for every such subsequential limit we obtain  $\lim_{n \rightarrow \infty} u_{Z^n}(t_n) = u_Z(t)$ .  $\square$

The following is the main ingredient in the proof of Theorem 1.9. See Figure 4 for an illustration of the proof.

**Lemma 5.8.** *Fix  $a \in \mathbb{R}$  and  $r > 0$ . Define the times  $\tau^{a,r}$ ,  $\iota_n^{a,r}$ , and  $\tau_n^{a,r}$  as in the statement of Theorem 1.9. Suppose we have (using [She11, Theorem 2.5]) coupled countably many instances of the infinite word  $X$  with the Brownian motion  $Z$  in such a way that  $Z^n \rightarrow Z$  uniformly on compacts a.s., with  $Z^n$  constructed from the  $n$ th instance of the word  $X$ . There exists a sequence of random positive integers  $(\hat{\iota}_n^{a,r})$ , each measurable with respect to the  $n$ th instance of the discrete model, such that the following is true. With  $\hat{\tau}_n^{a,r} = n^{-1}\hat{\iota}_n^{a,r}$ , we have  $\hat{\tau}_n^{a,r} \rightarrow \tau^{a,r}$  a.s. as  $n \rightarrow \infty$ ; and with probability tending to 1 as  $n \rightarrow \infty$  we have  $\hat{\iota}_n^{a,r} = \iota_n^{a,r}$ .*

*Proof.* By translation invariance we can assume without loss of generality that  $a = 0$ . To lighten notation, in what follows we fix  $r$  and omit both  $a$  and  $r$  from the notation. Let  $\epsilon > 0$  be arbitrary.

We observe the following.

1. By Proposition 1.10, we can find  $\zeta_1 \in (0, \epsilon)$  (depending only on  $\epsilon$ ) and an  $\tilde{N} \in \mathbb{N}$  such that for each  $n \geq \tilde{N}$ , it holds with probability at least  $1 - \epsilon/2$  that there is an  $i \in [\zeta_1 n, \epsilon n]_{\mathbb{Z}}$  such that  $X_i = \boxed{\mathbf{F}}$  and  $\phi(i) \leq 0$ . Note that for such an  $i$ ,  $X(1, i)$  has no burgers. By [She11, Theorem 2.5], after possibly increasing  $\tilde{N}$  we can find  $\delta_1 > 0$  (depending only on  $\zeta_1$ ) such that for  $n \geq \tilde{N}$ , it holds with probability at least  $1 - \epsilon$  that  $X(1, \zeta_1 n)$  contains at least  $\delta_1 n^{1/2}$  hamburger orders and at least  $\delta_1 n^{1/2}$  cheeseburger orders. Hence with probability at least  $1 - \epsilon$ , there is an  $i \in [\zeta_1 n, \epsilon n]_{\mathbb{Z}}$  such that  $X_i = \boxed{\mathbf{F}}$ ,  $\phi(i) \leq 0$ , and  $X(1, i)$  contains at least  $\delta_1 n^{1/2}$  hamburger orders and at least  $\delta_1 n^{1/2}$  cheeseburger orders.



For  $n \in \mathbb{N}$  let  $\tilde{\iota}_n$  be the smallest integer  $i > 0$  such that  $V^n(n^{-1}i) \leq V^n(s) + \delta$  and  $U^n(n^{-1}i) \leq U^n(s) + \delta$  for each  $s \in [n^{-1}i - r, n^{-1}i]$  and let  $\tilde{\tau}_n = n^{-1}\tilde{\iota}_n$ . We note that the defining condition for  $\tilde{\iota}_n$  is satisfied with  $i = \iota_n$ , so we necessarily have  $\iota_n \geq \tilde{\iota}_n$ .

We claim that if  $n \geq N$ , then on  $E$  we have

$$\tau - \zeta_1 \leq \tilde{\tau}_n \leq \tau. \quad (71)$$

It is clear from our choice of  $\zeta_2$  in observation 4 and our choice of  $N$  in observation 5 that the condition in the definition of  $\tilde{\iota}_n$  is satisfied provided  $i$  is chosen such that  $n^{-1}i \in [\tau - \zeta_2, \tau]$  (such an  $i$  must exist since  $N \geq \zeta_2^{-1}$ ). Therefore  $\tilde{\tau}_n \leq \tau$ . By our choice of  $\delta$  in observation 4 and our choice of  $N$  in observation 5 we have on  $E$  (in the notation of (70))

$$\widehat{V}(\tilde{\tau}_n) \leq V^n(\tilde{\tau}_n) - \inf_{s \in [\tilde{\tau}_n - r, \tilde{\tau}_n]} V^n(s) + 2\delta \leq \delta_2,$$

and similarly with  $\widehat{U}$  in place of  $\widehat{V}$ . By observation 3 there exists  $s \in [0, b]$  such that  $|s - \tilde{\tau}_n| \leq \zeta_1$  and  $\widehat{Z}(s) = 0$ . This  $s$  is a  $\pi/2$ -cone time for  $Z$  with  $s - v_Z(s) \geq r$ . By definition,  $s \geq \tau$ , so  $\tilde{\tau}_n \geq s - \zeta_1 \geq \tau - \zeta_1$ . This proves (71).

Observe that each time  $\tilde{\iota}_n$  is a stopping time for the filtration generated by the word  $X$ . By translation invariance and observation 1, it holds with probability at least  $1 - \epsilon$  that there exists  $i \in [\tilde{\iota}_n + \zeta_1 n, \tilde{\iota}_n + \epsilon n]_{\mathbb{Z}}$  such that  $X_i = \boxed{\mathbf{F}}$ ,  $\phi(i) \leq \tilde{\iota}_n$ , and  $X(\tilde{\iota}_n + 1, i)$  contains at least  $\delta_1 n^{1/2}$  hamburger orders and at least  $\delta_1 n^{1/2}$  cheeseburger orders. Let  $\iota'_n$  denote the smallest such  $i$  (if such an  $i$  exists) and otherwise let  $\iota'_n = \tilde{\iota}_n$ . For  $n \in \mathbb{N}$  let  $G_n$  be the event that  $\iota'_n > \tilde{\iota}_n$ . Then for  $n \geq N$  we have  $\mathbb{P}(G_n \cap E) \geq 1 - 5\epsilon$ .

Let  $\tau'_n = n^{-1}\iota'_n$ . By (71), on the event  $G_n \cap E$  we have  $\tau'_n \geq \tilde{\tau}_n + \zeta_1 \geq \tau$  and  $0 \leq \tau'_n - \tau \leq |\tilde{\tau}_n - \tau| + \epsilon \leq 2\epsilon$ . By combining this with (71) we obtain that if  $E$  occurs (even if  $G_n$  does not occur) then

$$|\tau'_n - \tau| \leq 2\epsilon. \quad (72)$$

Since  $V^n(\tilde{\tau}_n) \leq V^n(s) + \delta$  and  $U^n(\tilde{\tau}_n) \leq U^n(s) + \delta$  for each  $s \in [\tilde{\tau}_n - r, \tilde{\tau}_n]$  on the event  $E \cap G_n$ , the word  $X(\tilde{\iota}_n - rn, \tilde{\iota}_n)$  contains at most  $\delta n^{1/2} \leq \delta_1 n^{1/2}$  burgers of each type. On  $G_n$ , the word  $X(\tilde{\iota}_n + 1, \iota'_n)$  contains at least  $\delta_1 n^{1/2}$  hamburger orders and at least  $\delta_1 n^{1/2}$  cheeseburger orders, so on  $G_n \cap E$  we necessarily have  $\phi(\iota'_n) \leq \tilde{\iota}_n - rn \leq \iota'_n - rn$ . It follows that on  $G_n \cap E$ , we have

$$\tilde{\iota}_n \leq \iota_n \leq \iota'_n. \quad (73)$$

We will now let  $\epsilon$  tend to zero in an appropriate manner and construct the sequence  $(\widehat{\iota}_n)$  in the statement of the lemma. For  $j \in \mathbb{N}$  let  $N_j \in \mathbb{N}$  be the integer in condition 5 corresponding to  $\epsilon = 2^{-j}$ . For each  $n \in [N_j, N_{j+1} - 1]_{\mathbb{Z}}$  let  $E_j$  be the event  $E$  above with  $\epsilon = 2^{-j}$  and let  $\tilde{\iota}_n, \tilde{\tau}_n, \iota'_n, \tau'_n$ , and  $G_n$  be as defined above with  $\epsilon = 2^{-j}$ . Also let  $\widehat{\iota}_n := \iota_n$  if  $G_n$  occurs, and otherwise  $\widehat{\iota}_n := \iota'_n$ . Define  $\widehat{\tau}_n$  as in the statement of the lemma.

Since  $\mathbb{P}(G_n) \rightarrow 1$  as  $n \rightarrow \infty$ , we have  $\mathbb{P}(\widehat{\iota}_n = \iota_n) \rightarrow 1$  as  $n \rightarrow \infty$ . By the Borel-Cantelli lemma, a.s.  $E_j$  occurs for all but finitely many  $j$ . By (72), on  $E_j$  we have  $|\tau'_n - \tau| \leq 2^{-j+1}$  for each  $n \in [N_j, N_{j+1} - 1]_{\mathbb{Z}}$ . Hence a.s.  $\tau'_n \rightarrow \tau$ . Furthermore, if  $n \in [N_j, N_{j+1} - 1]_{\mathbb{Z}}$  then on  $G_n \cap E_j$  we have by (71), (72), and (73) that  $|\widehat{\tau}_n - \tau| \leq 2^{-j+1}$ , so a.s.  $\widehat{\tau}_n \rightarrow \tau$ .  $\square$

*Proof of Theorem 1.9.* By [She11, Theorem 2.5] and the Skorokhod theorem we can couple countably many instances of  $X$  with  $Z$  in such a way that a.s.  $Z^n \rightarrow Z$  uniformly on compacts. Define the times  $\iota_n^{a,r}$  and  $\tau_n^{a,r}$  as in condition 5 and the times  $\widehat{\iota}_n^{a,r}$ , and  $\widehat{\tau}_n^{a,r}$  as in Lemma 5.8. Then as  $n \rightarrow \infty$ ,  $\widehat{\tau}_n^{a,r} \rightarrow \tau^{a,r}$  a.s. for each  $(a, r) \in \mathbb{Q} \times (\mathbb{Q} \cap [0, \infty))$  and  $\mathbb{P}(\widehat{\tau}_n^{a,r} = \tau_n^{a,r}) \rightarrow 1$ . Hence  $\tau_n^{a,r} \rightarrow \tau^{a,r}$  in probability. It follows that the finite-dimensional marginals of the law of

$$\{Z^n\} \cup \{\tau_n^{a,r} : (a, r) \in \mathbb{Q} \times (\mathbb{Q} \cap (0, \infty))\}$$

converge to those of

$$\{Z\} \cup \{\tau^{a,r} : (a, r) \in \mathbb{Q} \times (\mathbb{Q} \cap (0, \infty))\}$$

as  $n \rightarrow \infty$ . By the Skorokhod theorem, we can re-couple in such a way that  $Z^n \rightarrow Z$  uniformly on compacts and  $\tau_n^{a,r} \rightarrow \tau^{a,r}$  a.s. as  $n \rightarrow \infty$  for each  $a, r \in \mathbb{Q} \times (\mathbb{Q} \cap (0, \infty))$ . Henceforth fix such a coupling. By construction, conditions 1 and 5 in the theorem statement are satisfied. We will verify conditions 2, 3, and 4.

Since each element of  $\mathcal{T}_n$  is a weak  $\pi/2$ -cone time for  $Z^n$ , it follows from Lemma 5.7 that each sequence  $(t_{n_j})$  as in condition 2 converges to an element of  $\mathcal{T}$  and satisfies condition 3.

Next we verify that every element of  $\mathcal{T}$  is in fact the limit of a sequence  $(t_{n_j})$  as in condition 2. Suppose we are given a  $\pi/2$ -cone time  $t$  for  $Z$ . Choose  $r \in \mathbb{Q} \cap (0, \infty)$  with  $r$  slightly less than  $t - v_Z(t)$  and a sequence  $(a_k) \in \mathbb{Q}$  increasing to  $t$ . It is almost surely the case that for each  $t \in \mathcal{T}$  and each choice of  $r$  and  $(a_k)$  as above we have  $\tau^{a_k, r} \rightarrow t$  as  $k \rightarrow \infty$ . For each  $j \in \mathbb{N}$ , we can choose  $k_j \in \mathbb{N}$  such that  $|\tau^{a_{k_j}, r} - t| \leq 2^{-j}$ . Since  $\tau_n^{a_{k_j}, r} \rightarrow \tau^{a_{k_j}, r}$  as  $n \rightarrow \infty$ , we can find  $n_j \in \mathbb{N}$  such that  $|\tau_{n_j}^{k_j, r} - \tau^{k_j, r}| \leq 2^{-j}$ . Set  $t_{n_j} = \tau_{n_j}^{k_j, r}$ . Then  $t_{n_j} \in \mathcal{T}_{n_j}$ ,  $t_{n_j} - v_{Z^{n_j}}(t_{n_j}) \geq n^{-1}(t_{n_j}^{k_j, r} - \phi(t_{n_j}^{k_j, r})) \geq r$  for each  $j \in \mathbb{N}$ , and  $t_{n_j} \rightarrow t$ . We conclude that condition 2 holds.

It remains to verify item 4. Fix a bounded open interval  $I \subset \mathbb{R}$  with rational endpoints and  $a \in I \cap \mathbb{Q}$  and  $\epsilon > 0$ . We can a.s. find a rational  $r > 0$  (random and depending on  $\delta$ ) such that  $t \in [\tau^{a,r}, \tau^{a,r} + \epsilon]$  and  $v_Z(t) \in [v_Z(\tau^{a,r}) - \epsilon, v_Z(\tau^{a,r})]$ . By condition 5, we have a.s.  $\tau_n^{a,r} \rightarrow \tau^{a,r}$  as  $n \rightarrow \infty$ . By condition 3, we a.s. have  $v_{Z^n}(\tau_n^{a,r}) \rightarrow v_Z(\tau^{a,r})$  as  $n \rightarrow \infty$ . Hence it is a.s. the case that for sufficiently large  $n \in \mathbb{N}$ ,  $[v_{Z^n}(\tau_n^{a,r}), \tau_n^{a,r}] \subset I$ . Hence for sufficiently large  $n \in \mathbb{N}$ , we have  $t_n \geq \tau_n^{a,r} \geq t - \epsilon$ . Since  $\epsilon$  is arbitrary, a.s.  $\liminf_{n \rightarrow \infty} t_n \geq t$ . Similarly  $\limsup_{n \rightarrow \infty} v_{Z^n}(t_n) \leq v_Z(t)$ .

To show that  $\lim_{n \rightarrow \infty} t_n = t$ , we note that from any sequence of integers tending to  $\infty$ , we can extract a subsequence  $n_j \rightarrow \infty$  and a  $t' \in I \cap [t, \infty)$  such that  $t_{n_j} \rightarrow t'$ . Our result above implies that  $[v_Z(t), t] \subset [v_Z(t'), t']$ . Since  $\liminf_{j \rightarrow \infty} (t_{n_j} - v_{Z^{n_j}}(t_{n_j})) \geq t - v_Z(t)$ , condition 2 implies that  $t'$  is a  $\pi/2$ -cone time for  $Z$  with  $[v_Z(t'), t'] \subset I$ . Since  $I$  has rational endpoints it is a.s. the case that neither of these endpoints is a  $\pi/2$ -cone time for  $Z$  or  $v_Z$  of a  $\pi/2$ -cone time for  $Z$ , simultaneously for all choices of  $I$ . Hence in fact  $[v_Z(t'), t'] \subset I$  for every such choice of subsequence. By maximality  $t' = t$ .  $\square$

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