

ON THE STABILITY OF GEODESICS IN THE BROWNIAN MAP

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ABSTRACT. The Brownian map is a random geodesic metric space arising as the scaling limit of random planar maps. We strengthen the so-called *confluence of geodesics* phenomenon observed at the root of the map, and with this, reveal several properties of its rich geodesic structure.

Our main result is the continuity of the cut locus at typical points. A small shift from such a point results in a small, local modification to the cut locus. Moreover, the cut locus is uniformly stable, in the sense that any two cut loci coincide outside a nowhere dense set.

We obtain similar stability results for the set of points inside geodesics to a fixed point. Furthermore, we show that the set of points inside geodesics of the map is of first Baire category. Hence, most points in the Brownian map are endpoints.

Finally, we classify the types of geodesic networks which are dense. For each $k \in \{1, 2, 3, 4, 6, 9\}$, there is a dense set of pairs of points which are joined by networks of exactly k geodesics and of a specific topological form. We find the Hausdorff dimension of the set of pairs joined by each type of network. All other geodesic networks are nowhere dense.

1. INTRODUCTION

A universal scaling limit of random planar maps has recently been identified by Le Gall [27] (triangulations and $2k$ -angulations, $k > 1$) and Miermont [32] (quadrangulations) as a random geodesic metric space called the *Brownian map* (M, d) . In this work, we establish properties of the Brownian map which are a step towards a complete understanding of its geodesic structure.

The works of Cori and Vauquelin [15] and Schaeffer [36] describe a bijection from well-labelled plane trees to rooted planar maps. The Brownian map is obtained as a quotient of Aldous' [3, 4] *continuum random tree*, or CRT, by assigning Brownian labels to the CRT and then identifying some of its non-cut-points, or *leaves*, according to a continuum analogue of the *CVS-bijection* (see Section 2.1). The resulting object is homeomorphic to the sphere \mathbb{S}^2 (Le Gall and Paulin [29] and Miermont [30]) and of Hausdorff dimension four (Le Gall [25]), and is thus in a sense a random, fractal, spherical surface.

Le Gall [26] classifies the geodesics to the root of the Brownian map in terms of the label process on the CRT (see Section 2.2). Moreover, the Brownian map is shown to be invariant in distribution under uniform re-rooting from the volume measure λ on M (see Section 2.1). Hence, geodesics to typical points exhibit a similar structure as those to the root. It thus remains to investigate geodesics from special points of the Brownian map.

1.1. Geodesic nets. A striking consequence of Le Gall’s description of geodesics to the root is that any two such geodesics are bound to meet and then coalesce before reaching the root, a phenomenon referred to as the *confluence of geodesics* (see Section 2.3). Moreover, geodesics from nearby points to the root coalesce quickly. As a result, the set of points in the relative interior of a geodesic to the root is a small subset which is homeomorphic to an \mathbb{R} -tree and of Hausdorff dimension one (see [26]).

Definition. For $x \in M$, the *geodesic net* of x , denoted $G(x)$, is the set of points $y \in M$ that are contained in the relative interior of a geodesic to x .

We remark that it is not clear, nor is it shown in this work, whether $G(x)$ is an \mathbb{R} -tree in general, or a union thereof. That being said, we find that the Brownian map has a relatively uniform geodesic structure, providing further evidence that it is, to quote Le Gall [24], ‘very regular in its irregularity.’

The geodesic net of x is stable under perturbations of x .

For sets $A, B \subset M$, let $A \Delta B$ denote their symmetric difference.

Theorem 1. *Almost surely, for all $x, y \in M$, $G(x) \Delta G(y)$ is nowhere dense in M .*

Furthermore, for typical points $x \in M$, the effect of small perturbations of x on $G(x)$ is localized.

Theorem 2. *Almost surely, the function $x \mapsto G(x)$ is continuous almost everywhere in the following sense.*

For λ -almost every $x \in M$, for any neighbourhood N of x , there is a sub-neighbourhood $N' \subset N$ so that $G(x') \cap N$ is the same for all $x' \in N'$.

The *uniform infinite planar triangulation*, or UIPT, due to Angel and Schramm [5] is a random lattice which arises as the *local limit* of random triangulations of the sphere. The case of quadrangulations, giving rise to the UIPQ, was later investigated by Krikun [22]. We remark that Theorem 2 is in a sense a continuum analogue to a result of Krikun [23] (see also Curien, Ménard, and Miermont [19]), which shows that the ‘Schaeffer’s tree’ of the UIPQ only changes locally after relocating its root.

Next, we find that the union over of all geodesic nets is relatively small.

For a set $A \subset M$, let A^c denote its complement.

Definition. Let $F = \bigcup_{x \in M} G(x)$ denote the set of points in the relative interior of a geodesic in (M, d) . We refer to F as the *geodesic framework* and $E = F^c$ as the *endpoints* of the Brownian map.

Theorem 3. *Almost surely, the geodesic framework of the Brownian map is of first Baire category.*

Hence E is a residual subset. This property of the Brownian map is reminiscent of a result of Zamfirescu [38], which states that for most convex surfaces—that is, for all surfaces in a residual subset of the Baire space of convex surfaces in \mathbb{R}^n endowed with the Hausdorff metric—the endpoints of a surface form a residual set.

1.2. Cut loci. The construction of the Brownian map as a quotient of the CRT gives a natural mapping from the CRT to the map. Cut-points of the CRT correspond to a dense subset $S \subset M$ of Hausdorff dimension two (see [26]). Le Gall’s description of geodesics reveals that S is exactly the set of points with multiple geodesics to the root (see Section 2.2). More specifically, for any $y \in M$, the number of connected components of $S - \{y\}$ is precisely the number of geodesics from y to the root. This is similar to the case of a complete, analytic Riemannian surface homeomorphic to the sphere (see Poincaré [35] and Myers [34]), where the cut locus S is homeomorphic to an \mathbb{R} -tree and the number of ‘branches’ emanating from a point in S is exactly the number of geodesics to the root.

Recall that the cut locus of a point p in a Riemannian manifold—first examined by Poincaré [35]—is the set of points $q \neq p$ which are endpoints of maximal (minimizing) geodesics from p . This collection of points is more subtle than merely the set of points with multiple geodesics to p , and in fact, is generally the closure thereof (see Klingenberg [21, Section 2.1.14]). In the Brownian map this equivalence breaks completely. Indeed, almost all points are the end of a maximal geodesic (see Theorem 3), and every point is joined by multiple geodesics to a dense set of points (see the note after the proof of Proposition 22).

We introduce the following notion of cut locus for the Brownian map.

Definition. For $x \in M$, the *strong cut locus* of x , denoted $C(x)$, is the set of points $y \in M$ to which there are at least two geodesics from x that are disjoint in a neighbourhood of y .

For a discussion on our choice of definition, and a study of another weaker version of cut locus, see Section 4.2 below.

Since the strong cut locus of the root of the Brownian map corresponds to the CRT minus its leaves—that is, $S = C(\rho)$, where ρ is the root (see Section 2.2)—it is a fundamental subset of the map. For this reason, it is of interest to study the strong cut locus of general points in the Brownian map.

Similarly to the geodesic net, the strong cut locus of x is stable under perturbations of x .

Corollary 4. *Almost surely, the following hold.*

- (i) *For all $x, y \in M$, $C(x) \Delta C(y)$ is nowhere dense in M .*
- (ii) *The function $x \mapsto C(x)$ is continuous almost everywhere in the following sense. For λ -almost every $x \in M$, for any neighbourhood N of x , there is a sub-neighbourhood $N' \subset N$ so that $C(x') - N$ is the same for all $x' \in N'$.*

This result follows directly by Theorems 1,2. Indeed, if for some $x, y \in M$ and open set $U \subset M$ we have that $G(x)$ and $G(y)$ coincide in U , then it follows immediately by the definition of the strong cut locus that also $C(x)$ and $C(y)$ coincide in U .

Corollary 4(ii) brings to mind the results of Buchner [13] and Wall [37], which show that the cut locus of a fixed point in a compact manifold is continuously stable under perturbations of the metric on an open, dense subset of its Riemannian metrics (endowed with the Whitney topology).

1.3. Geodesic networks. Next, we investigate the structure of geodesics between pairs of points in the Brownian map.

Definition. For $x, y \in M$, the *geodesic network* between x and y , denoted $G(x, y)$, is the set of points in a geodesic segment from x to y .

Geodesic networks with at least one typical endpoint are well-understood. As discussed in Section 1.2, for any $y \in M$, the number of connected components in $S - \{y\}$ gives the number of geodesics from y to the root of the Brownian map. Hence, by properties of the CRT, almost surely there is a dense, two-dimensional set of points with exactly two geodesics to the root; a dense, countable set of points with exactly three geodesics to the root; and no points connected to the root by more than three geodesics. By invariance under re-rooting, it follows that the set of pairs that are joined by multiple geodesics is a zero-volume subset of $(M^2, \lambda \otimes \lambda)$ (see also Miermont [31]). Hence, the vast majority of networks in the Brownian map consist of a single geodesic segment. Furthermore, by Le Gall's description of geodesics to the root and invariance under re-rooting, geodesics from a typical point of the Brownian map have a specific topological structure (see Section 2.2).

Definition. We say that a geodesic network $G(x, y)$ is *regular* if any two distinct geodesic segments from x to y are disjoint inside, and coincide outside, a punctured ball centred at y of radius less than $d(x, y)$.

For typical points x , all networks $G(x, y)$ are regular. We note that $G(x, y)$ and $G(y, x)$ are regular if and only if there is a unique geodesic from x to y .

We find that most geodesic networks in the Brownian map are, in the following sense, a concatenation of two regular networks.

Definition. For $(x, y) \in M^2$ and $j, k \in \mathbb{N}$, we say that (x, y) induces a *normal (j, k) -network*, and write $(x, y) \in N(j, k)$, if for some z in all geodesic segments from x to y , the geodesic networks $G(z, x)$ and $G(z, y)$ are both regular and contain exactly j and k geodesic segments, respectively.

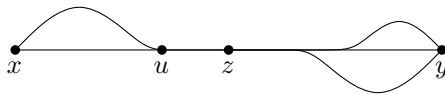


FIGURE 1. As depicted, $(x, y) \in N(2, 3)$. Note that (u, x) does not induce a normal (j, k) -network.

Not all networks are normal (j, k) -networks. For instance, if $(x, y) \in N(j, k)$ and $j > 1$, then there is a point $u \in G(x, y)$ so that u is joined to x by two

geodesics with disjoint relative interiors. See Figure 1. That being said, most pairs induce normal (j, k) -networks. Moreover, for each $(j, k) \in \{1, 2, 3\}^2$, there are many normal (j, k) -networks in the map. Hence, in particular, we establish the existence of atypical networks comprised of more than three geodesics (and up to nine).

Theorem 5. *The following hold almost surely.*

- (i) *For any $(j, k) \in \{1, 2, 3\}^2$, $N(j, k)$ is dense in M^2 .*
- (ii) *$M^2 - \bigcup_{(j,k) \in \{1,2,3\}^2} N(j, k)$ is nowhere dense in M^2 .*

By Theorem 5, there are essentially only six types of geodesic networks which are dense in the Brownian map. See Figure 2.

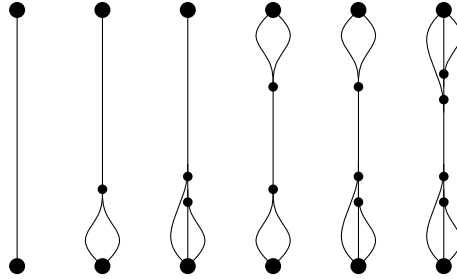


FIGURE 2. Classification of networks which are dense in the Brownian map (up to symmetries).

Additionally, we obtain the dimension of the sets $N(j, k)$, $j, k \leq 3$. For a set $A \subset M$, let $\dim A$ denote its Hausdorff dimension.

Theorem 6. *Almost surely, we have that $\dim N(j, k) = 2(6 - j - k)$, for all $j, k \in \{1, 2, 3\}$. Moreover, $N(3, 3)$ is countable.*

Definition. For each $k \in \mathbb{N}$, let $P(k) \subset M^2$ denote the set of pairs of points that are connected by exactly k geodesics.

Theorems 5,6 immediately imply the following results.

Corollary 7. *Put $K = \{1, 2, 3, 4, 6, 9\}$. The following hold almost surely.*

- (i) *For each $k \in K$, $P(k)$ is dense in M^2 .*
- (ii) *$M^2 - \bigcup_{k \in K} P(k)$ is nowhere dense in M^2 .*

Corollary 8. *Almost surely, we have that $\dim P(2) \geq 6$, $\dim P(3) \geq 4$, $\dim P(4) \geq 4$, and $\dim P(6) \geq 2$.*

We expect the lower bounds in Corollary 8 to give the correct dimensions of the sets $P(k)$, $k \in K - \{1, 9\}$. As discussed in Section 1.2, $P(1)$ is of full volume, and hence $\dim P(1) = 8$. We suspect that $P(9)$ is countable. It would be interesting to determine if the set $P(k)$ is non-empty for some $k \notin K$, and whether there is any $k \notin K$ for which it has positive dimension. We hope to address these issues in future work.

1.4. Confluence points. Our key tool is a strengthening of the confluence of geodesics phenomenon of Le Gall [26] (see Section 2.3). Specifically, we find that for any neighbourhood N of a typical point in the Brownian map, there is a *confluence point* x_0 between a sub-neighbourhood $N' \subset N$ and the complement of N . See Figure 3.

Proposition 9. *Almost surely, for λ -almost every $x \in M$, the following holds. For any neighbourhood N of x , there is a sub-neighbourhood $N' \subset N$ and some $x_0 \in N$ so that all geodesics between any points $x' \in N'$ and $y \in N^c$ pass through x_0 .*

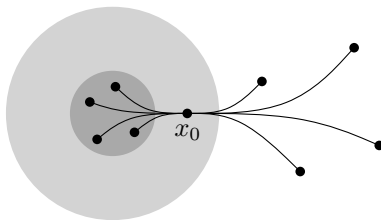


FIGURE 3. All geodesics from points in N' to points in the complement of $N \supset N'$ pass through a confluence point x_0 .

Recall that a subset $\gamma \subset M$ is a *geodesic segment* if (γ, d) is isometric to an interval. We will often denote a particular geodesic segment between $x, y \in M$ as $[x, y]$, and denote its relative interior by $(x, y) = [x, y] - \{x, y\}$. We define $[x, y)$ and $(x, y]$ similarly.

Definition. We say that a sequence of geodesic segments γ_n converges to a geodesic segment γ , and write $\gamma_n \rightarrow \gamma$, if γ_n converges to γ with respect to the Hausdorff topology.

Since (M, d) is almost surely homeomorphic to \mathbb{S}^2 , and hence almost surely compact, the following lemma is a straightforward consequence of the Arzelà-Ascoli theorem (see Bridson and Haefliger [12, Corollary 3.11]).

Lemma 10. *Almost surely, the set of geodesics in (M, d) is compact.*

Our key result, Proposition 9, is related to the fact that many sequences of geodesic segments in the Brownian map converge in a stronger sense.

Definition. We say that a sequence of geodesic segments $[x_n, y_n]$ *converges strongly* to $[x, y]$, and write $[x_n, y_n] \rightrightarrows [x, y]$, if for any neighbourhood N of $\{x, y\}$, we have that $[x_n, y_n] \Delta [x, y] \subset N$ eventually.

We show that all sequences of geodesic segments which converge to a geodesic segment with at least one typical endpoint converge strongly. (In fact, we prove slightly more, see Lemma 19 below.)

Proposition 11. *Almost surely, for λ -almost every $x \in M$, for every $y \in M$, the following holds. If $[x_n, y_n] \rightarrow [x, y]$, then we have that $[x_n, y_n] \rightrightarrows [x, y]$.*

Proposition 9 follows by Proposition 11, the confluence of geodesics phenomena, and the fact that (M, d) is almost surely compact.

In closing, we remark that it would be interesting to know if Proposition 11 holds for all $x \in M$. This question is related to the possible existence of *ghost geodesics* in the Brownian map which pass through all other geodesics at at most one point. It would be quite surprising if such geodesics exist, and we hope to rule them out in future work. We thus expect an analogue of Proposition 9 to hold for all $x \in M$. If so, then in particular, we would obtain the following result.

Conjecture. *Almost surely, the geodesic framework of the Brownian map is of Hausdorff dimension one.*

In this way, we suspect that although the Brownian map is a complicated, four-dimensional object, it has a relatively simple geodesic framework which is of first category and one-dimensional.

2. PRELIMINARIES

In this section, we briefly recount the construction of the Brownian map and what is known regarding its geodesics.

2.1. The Brownian map. Fix $q \in \{3\} \cup 2(\mathbb{N} + 1)$ and set c_q equal to $6^{1/4}$ if $q = 3$ or $(9/q(q - 2))^{1/4}$ if $q > 3$. Let M_n denote a uniform q -angulation of the sphere (see Le Gall and Miermont [28]) with n faces, and d_n the graph distance on M_n scaled by $c_q n^{-1/4}$. The works of Le Gall [27] and Miermont [32] (for $q = 4$) show that in the Gromov-Hausdorff topology on isometry classes of compact metric spaces (see Burago, Burago, and Ivanov [14]), (M_n, d_n) converges in distribution to a random metric space called the *Brownian map* (M, d) .

The Brownian map has also been identified as the scaling limit of bipartite planar maps with n edges, simple triangulations and quadrangulations, quadrangulations with n faces and no pendant vertices, planar maps with n edges, and bipartite planar maps with Boltzmann weights (see [1, 2, 6, 10, 27]).

The construction of the Brownian map involves a normalized Brownian excursion $\mathbf{e} = \{\mathbf{e}_t : 0 \leq t \leq 1\}$, a random \mathbb{R} -tree $(\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}})$ indexed by \mathbf{e} , and a Brownian label process $Z = \{Z_a : a \in \mathcal{T}_{\mathbf{e}}\}$. More specifically, define $\mathcal{T}_{\mathbf{e}} = [0, 1]/\{d_{\mathbf{e}} = 0\}$ as the quotient under the pseudo-distance

$$d_{\mathbf{e}}(s, t) = \mathbf{e}_s + \mathbf{e}_t - 2 \cdot \min_{s \wedge t \leq u \leq s \vee t} \mathbf{e}_u, \quad s, t \in [0, 1]$$

and equip it with the quotient distance, again denoted by $d_{\mathbf{e}}$. The random metric space $(\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}})$ is Aldous' *continuum random tree*, or CRT. Let $p_{\mathbf{e}} : [0, 1] \rightarrow \mathcal{T}_{\mathbf{e}}$ denote the canonical projection. Conditionally given \mathbf{e} , Z is a centred Gaussian process satisfying $\mathbf{E}[(Z_s - Z_t)^2] = d_{\mathbf{e}}(s, t)$ for all $s, t \in [0, 1]$. The random process Z is the so-called *head of the Brownian snake* (see [28]).

Note that Z is constant on each equivalence class $p_{\mathbf{e}}^{-1}(a)$, $a \in \mathcal{T}_{\mathbf{e}}$. In this sense, Z is Brownian motion indexed by the CRT.

Analogously to the definition of $d_{\mathbf{e}}$, we put

$$d_Z(s, t) = Z_s + Z_t - 2 \cdot \max \left\{ \inf_{u \in [s, t]} Z_u, \inf_{u \in [t, s]} Z_u \right\}, \quad s, t \in [0, 1]$$

where we set $[s, t] = [0, t] \cup [s, 1]$ in the case that $s > t$. Then, to obtain a pseudo-distance on $[0, 1]$, we define

$$D^*(s, t) = \inf \left\{ \sum_{i=1}^k d_Z(s_i, t_i) : s_1 = s, t_k = t, d_{\mathbf{e}}(t_i, s_{i+1}) = 0 \right\}, \quad s, t \in [0, 1].$$

Finally, we set $M = [0, 1] / \{D^* = 0\}$ and endow it with the quotient distance induced by D^* , which we denote by d . Let $\Pi : \mathcal{T}_{\mathbf{e}} \rightarrow M$ denote the canonical projection, and put $\mathbf{p} = \Pi \circ p_{\mathbf{e}}$. Almost surely, the process Z attains a unique minimum on $[0, 1]$, say at t_* . We set $\rho = \mathbf{p}(t_*)$. The random metric space $(M, d) = (M, d, \rho)$ is called the *Brownian map* and we call ρ its *root*. Being the Gromov-Hausdorff limit of geodesic spaces, (M, d) is almost surely a geodesic space (see [14]).

Almost surely, for every pair of distinct points $s \neq t \in [0, 1]$, at most one of $d_{\mathbf{e}}(s, t) = 0$ or $d_Z(s, t) = 0$ holds (see [29, Lemma 3.2]). Hence, only leaves of $\mathcal{T}_{\mathbf{e}}$ are identified in the construction of the Brownian map; and this occurs if and only if they have the same label and along either the clockwise or counter-clockwise, contour-ordered path about $\mathcal{T}_{\mathbf{e}}$ between them, one only finds vertices of larger label. Thus, as mentioned at the beginning of Section 1, in the construction of the Brownian map, $(\mathcal{T}_{\mathbf{e}}, Z)$ is a continuum analogue for a well-labelled plane tree, and the quotient by $\{D^* = 0\}$ for the CVS-bijection (which recall identifies well-labelled plane trees with rooted planar maps).

Lastly, we note that although the Brownian map is a rooted metric space, it is not so dependent on its root. The volume measure λ on M is defined as the push-forward of Lebesgue measure on $[0, 1]$ via \mathbf{p} . Le Gall [26] shows that the Brownian map is *invariant under re-rooting* in the sense that if U is uniformly distributed over $[0, 1]$ and independent of (M, d) , then (M, d, ρ) and $(M, d, \mathbf{p}(U))$ are equal in law. Hence, to a considerable extent, the root of the map is but an artifact of its construction.

2.2. Simple geodesics. Put $Z_* = Z_{t_*}$. As it turns out, $d(\rho, \mathbf{p}(t)) = Z_t - Z_*$ for all $t \in [0, 1]$ (see [25]). In other words, up to a shift by the minimum label Z_* , the Brownian label of a point in $\mathcal{T}_{\mathbf{e}}$ is precisely the distance to ρ from the corresponding point in the Brownian map.

All geodesics to ρ are *simple geodesics*, constructed as follows. In a natural way, each $t \in [0, 1]$ corresponds to a *corner* of $\mathcal{T}_{\mathbf{e}}$ with label Z_t . For $t \in [0, 1]$ and $\ell \in [0, Z_t - Z_*]$, let $s_t(\ell)$ denote the point in $[0, 1]$ corresponding to the first corner with label $Z_t - \ell$ in the clockwise, contour-ordered path around $\mathcal{T}_{\mathbf{e}}$ beginning at the corner corresponding to t . For each such t , the

function $\Gamma_t : [0, Z_t - Z_*] \rightarrow M$ taking ℓ to $\mathbf{p}(s_t(\ell))$ is a geodesic from $\mathbf{p}(t)$ to ρ . Moreover, the main result of [26] shows that all geodesics to ρ are of this form. Hence, the geodesic net of the root, $G(\rho)$, is precisely the set of cut-points of the \mathbb{R} -tree $\mathcal{T}_Z = [0, 1]/\{d_Z = 0\}$ projected into M .

Since the cut-points of \mathcal{T}_e are its vertices with multiple corners, we see that S is exactly the set of points with multiple geodesics to the root. Furthermore, the order of a cut-point of \mathcal{T}_e —that is, the number of its corners—is precisely the number of geodesics from the corresponding point in the map to the root.

Points in S correspond to leaves of \mathcal{T}_Z (see [29, Lemma 3.2]), and thus geodesics to the root of the map have a particular topological structure. For any $y \in M$, each pair of distinct geodesics from y to ρ are disjoint inside, and coincide outside, a punctured ball centred at y of radius less than $d(\rho, y)$. Hence, as mentioned in Section 1.2, we have that $S = C(\rho)$.

These results mirror the fact that from each corner of a labelled discrete tree, the CVS-bijection draws geodesics to the root of the resulting map in such a way that the label of a vertex visited by any such geodesic equals the distance to the root. See [24, 26] for more details.

2.3. Confluence at the root. As discussed in Section 1.1, a *confluence of geodesics* is observed at the root of the Brownian map. Combining this with invariance under re-rooting, the following result is obtained.

For $x \in M$, let $B(x, \delta)$ denote the ball of radius δ centred at x .

Lemma 12 (Le Gall [26, Corollary 7.7]). *Almost surely, for λ -almost every $x \in M$, the following holds. For every $\epsilon > 0$ there is an $\eta \in (0, \epsilon)$ so that if $y, y' \in B(x, \epsilon)^c$, then any pair of geodesics from x to y and y' coincide inside of $B(x, \eta)$.*

Moreover, geodesics to the root of the map coalesce quickly.

For $t \in [0, 1]$, let γ_t denote the image of the simple geodesic Γ_t from $\mathbf{p}(t)$ to the root of the map ρ (see Section 2.2).

Lemma 13 (Miermont [32, Lemma 5]). *Almost surely, for all $s, t \in [0, 1]$, γ_s and γ_t coincide outside of $B(\mathbf{p}(s), d_Z(s, t))$.*

We require the following lemma, which follows directly by Lemma 13, invariance under re-rooting, the continuity of \mathbf{p} (see [26]), and Le Gall's classification of geodesics to the root.

Lemma 14. *Almost surely, for λ -almost every $x \in M$, the following holds. For any $y \in M$ and neighbourhood N of y , there is a sub-neighbourhood $N' \subset N$ so that if $y' \in N'$, then any geodesic from x to y' coincides with a geodesic from x to y outside of N .*

Proof. Let ρ denote the root of the map. Let $y \in M$ and a neighbourhood N of y be given. Select $\epsilon > 0$ so that $B(y, \epsilon) \subset N$. Let N_ϵ denote the set of points $y' \in M$ with the property that for all $t' \in [0, 1]$ for which $\mathbf{p}(t') = y'$, there exists some $t \in [0, 1]$ so that $\mathbf{p}(t) = y$ and $d_Z(t, t') < \epsilon$. As discussed

in Section 2.2, Le Gall [26] shows that all geodesics to ρ are simple geodesics. Hence, by Lemma 13, any geodesic from ρ to a point $y' \in N_\epsilon$ coincides with some geodesic from ρ to y outside of N . We claim that N_ϵ is a neighbourhood of y . To see this, note that if $\mathbf{p}(t_n) = y_n \rightarrow y$ in (M, d) , then there is a subsequence t_{n_k} so that $t_{n_k} \rightarrow t_* \in [0, 1]$ as $k \rightarrow \infty$. Since \mathbf{p} is continuous, $\mathbf{p}(t_*) = y$, and hence $d_Z(t_*, t_{n_k}) < \epsilon$ for all large k . We conclude that for any $y_n \rightarrow y$ in (M, d) , $y_n \notin B_\epsilon$ for at most finitely many n . Hence, the lemma follows by invariance under re-rooting. \square

We remark that the value of η in Lemma 14 depends strongly on x and y . For instance, for a fixed $\epsilon > 0$ and sequences $x_n \rightarrow x$ and $y_n \rightarrow y$ in (M, d) , let $\eta_n \in (0, \epsilon)$ be as guaranteed by the lemma (assuming the x_n , say, are typical) for the pair x_n, y_n . It is quite possible that $\eta_n \rightarrow 0$ as $n \rightarrow \infty$.

3. CONFLUENCE NEAR THE ROOT

We show that a confluence of geodesics is observed near the root of the Brownian map, strengthening the results discussed in Section 2.3. Specifically, we establish the following result.

Lemma 15. *Almost surely, for λ -almost every $x \in M$, the following holds. For any $y \in M$ and neighbourhoods N_x of x and N_y of y , there are sub-neighbourhoods N'_x and N'_y so that if $x' \in N'_x$ and $y' \in N'_y$, then any geodesic from x' to y' coincides with a geodesic from x to y outside of $N_x \cup N_y$.*

We note that Lemma 15 strengthens Lemma 14 in that it allows for perturbations of both endpoints of a geodesic.

Once Lemma 15 is established, our key result follows easily by Lemma 12 and the fact that the Brownian map is almost surely compact.

Proof of Proposition 9. Let x denote the root of the map. Let a neighbourhood N of x be given. By Lemma 12, there is a point $x_0 \in N$ which is contained in all geodesic segments between x and points $y \in N^c$. Hence, by Lemma 15, for each $y \in N^c$ there is an $\eta_y > 0$ so that x_0 is contained in all geodesic segments between points $x' \in B(x, \eta_y)$ and $y' \in B(y, \eta_y)$. Since M is compact, there is a finite set $Y \subset N^c$ for which $\bigcup_{y \in Y} B(y, \eta_y) \supset N^c$. Put $N' = B(x, \min_{y \in Y} \eta_y)$. If $y_0 \in N^c$, then $y_0 \in B(y, \eta_y)$ for some $y \in Y$, and thus all geodesics from points $x' \in N' \subset B(x, \eta_y)$ to y_0 pass through x_0 . See Figure 4. Hence, by invariance under re-rooting, we obtain the result. \square

Hence, we turn to the proof of Lemma 15.

To establish Lemma 15 we must rule out the existence of a geodesic segment $[x, y]$ from the root of the map x , and a sequence of geodesic segments $[x_n, y_n]$ converging to $[x, y]$ in such a way that $[x_n, y_n]$ and $[x, y]$ are disjoint inside a fixed neighbourhood of x for all n .

For the remainder of this section we fix a realization of the Brownian map exhibiting the almost sure properties of the random metric space (M, d) . Slightly abusing notation, let us refer to this realization as (M, d) .

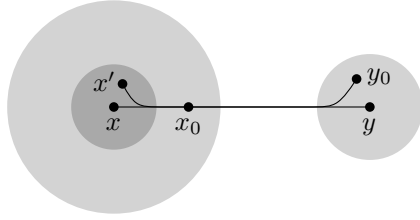


FIGURE 4. Given a neighbourhood N of x , we select a sub-neighbourhood N' so that x_0 is contained in all geodesic segments between any points $x' \in N'$ and $y_0 \in N^c$.

We shall refer to a dense subset of typical points $T \subset M$ for which Lemma 14 holds. Such a set exists almost surely. Next, we fix a point $y \in M$ and a geodesic segment $\gamma = [x, y]$ oriented from the root of the map x to y .

In what follows, we will at times shift our attention to the homeomorphic image of a neighbourhood of γ in which our arguments are more transparent. Whenever doing so, we will appeal only to topological properties of the map.

Fix a homeomorphism τ from M to $\hat{\mathbb{C}}$. Complete the image of γ under τ to a Jordan curve $J \subset \hat{\mathbb{C}}$ and fix a homeomorphism ϕ taking J to the extended real-axis $\{w \in \mathbb{C} : \text{Im } w = 0\} \cup \{\infty\}$ in such a way that $\phi \circ \tau$ sends γ to the unit interval $I = \{w \in \mathbb{C} : \text{Im } w = 0, \text{Re } w \in [0, 1]\}$ oriented from 0 to 1. By the Jordan-Schönflies theorem, $\phi \circ \tau$ can be extended to a homeomorphism from M to $\hat{\mathbb{C}}$ (see Mohar and Thomassen [33, Chapter 2.2]). We fix such a homeomorphism, and denote it by ψ .

Definition. Let d_E denote the Euclidean distance on \mathbb{C} . For $A \subset M$ and $\delta > 0$, let

- (i) $[A]_\delta = \{z \in M : d(z, A) \leq \delta\}$;
- (ii) $[A]_{\psi, \delta}$ denote the largest set

$$\{w \in \mathbb{C} : d_E(w, \psi(A)) \leq \xi\} \subset \psi([A]_\delta).$$

For a set $A \subset \mathbb{C}$, let $\text{Int } A$ denote its interior.

Definition. Let $\mathbb{H}_+ = \{w \in \mathbb{C} : \text{Im } w \geq 0\}$ (resp. $\mathbb{H}_- = \{w \in \mathbb{C} : \text{Im } w \leq 0\}$) denote the closed upper (resp. lower) half-plane of \mathbb{C} . We refer to $L = \psi^{-1}(\text{Int } \mathbb{H}_+)$ (resp. $R = \psi^{-1}(\text{Int } \mathbb{H}_-)$) as the *left* (resp. *right*) *side* of γ .

Definition. We define the *left* (resp. *right*) δ -*side* of $\psi(\gamma)$ as $[\gamma]_{\psi, \delta}^\ell = [\gamma]_{\psi, \delta} \cap \mathbb{H}_+$ (resp. $[\gamma]_{\psi, \delta}^r = [\gamma]_{\psi, \delta} \cap \mathbb{H}_-$). See Figure 5.

Lemma 16. *For any $\delta > 0$, we have that*

- (i) $[\gamma]_{\psi, \delta}$ is a closed, simply-connected neighbourhood of $\psi(\gamma)$;
- (ii) $\text{Int } [\gamma]_{\psi, \delta}^\ell$ and $\text{Int } [\gamma]_{\psi, \delta}^r$ are non-empty, simply-connected, and disjoint;
- (iii) $\psi(\gamma) \subset [\gamma]_{\psi, \delta}^\ell \cap [\gamma]_{\psi, \delta}^r$;
- (iv) $\psi^{-1}(\text{Int } [\gamma]_{\psi, \delta}^\ell) \subset [\gamma]_\delta \cap L$ and $\psi^{-1}(\text{Int } [\gamma]_{\psi, \delta}^r) \subset [\gamma]_\delta \cap R$.

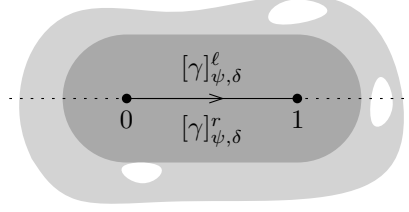


FIGURE 5. We select a closed, simply-connected sub-neighbourhood $[\gamma]_{\psi, \delta}$ of $\psi([\gamma]_\delta)$, and put $[\gamma]_{\psi, \delta}^\ell = [\gamma]_{\psi, \delta} \cap \mathbb{H}_+$ and $[\gamma]_{\psi, \delta}^r = [\gamma]_{\psi, \delta} \cap \mathbb{H}_-$.

Proof. For part (i), observe that $[\gamma]_{\psi, \delta} \neq \emptyset$, as otherwise, by the compactness of γ we would find $z \in \gamma$ with $d(z, \gamma) = \delta$. (In other words, although typical open balls in the Brownian map have infinitely many ‘holes’ (see [24, Proposition 3.7]), one does not find ‘holes’ in $[\gamma]_\delta$ arbitrarily close to γ .) Note that $[\gamma]_{\psi, \delta}$ is in fact a convex neighbourhood of $\psi(\gamma)$ (although $\psi^{-1}([\gamma]_{\psi, \delta})$ may not be a convex neighbourhood of γ). Hence, part (ii) follows by part (i) and the fact that $[\gamma]_{\psi, \delta}^\ell \subset \mathbb{H}_+$ and $[\gamma]_{\psi, \delta}^r \subset \mathbb{H}_-$. We obtain part (iii) by part (i) and observing that $\psi(\gamma) \subset \mathbb{H}_+ \cap \mathbb{H}_-$. Part (iv) is a direct consequence of the fact that $[\gamma]_{\psi, \delta}^\ell \subset \mathbb{H}_+$, $[\gamma]_{\psi, \delta}^r \subset \mathbb{H}_-$, and $\psi^{-1}([\gamma]_{\psi, \delta}) \subset [\gamma]_\delta$. \square

Lemma 17. *Let $u, v \in \gamma$. For all $\delta > 0$, there are typical points $u_\ell \in [u]_\delta \cap L \cap T$ and $v_\ell \in [v]_\delta \cap L \cap T$ so that $[u_\ell, v_\ell] - \gamma$ is contained in $[\{u, v\}]_\delta \cap L$. An analogous statement holds replacing L with R .*

Proof. Let $\delta > 0$ and $u, v \in \gamma$ be given. We discuss the argument for L , since the two cases are symmetrical. Moreover, we assume that $u \neq v$ and $\{u, v\} \cap \{x, y\} = \emptyset$; as in these cases, the following argument is easily adapted by replacing u (resp. v) with a point in γ sufficiently close to u (resp. v). Thus, we may assume that $d(x, u) < d(x, v)$ and $\delta < d(\{u, v\}, \{x, y\})$.

Let v' be the point in $[v, y] \subset \gamma$ at distance δ from v . Since $G(x, y)$ is regular, note that $[x, v'] \subset \gamma$ is the unique geodesic segment from x to v' . Hence, by considering a sequence of points in $L \cap T$ converging to v , we see by Lemmas 10, 14, and 16 that there is a point $v_\ell \in L \cap T$ such that $\psi([x, v_\ell] - [x, v]) \subset [v]_{\psi, \delta}$. Moreover, by the uniqueness of $[x, v']$ and since $v_\ell \in L$ and $v' \notin \text{Int}[v]_{\psi, \delta}$, we see that in fact $\psi([x, v_\ell] - \gamma) \subset [v]_{\psi, \delta} \cap \mathbb{H}_+$. Therefore, by Lemma 16, we find that $[x, v_\ell] - \gamma \subset [v]_\delta \cap L$.

By a similar argument, in which v_ℓ assumes the role of x , we find a point $u_\ell \in L \cap T$ so that $[v_\ell, u_\ell] - [v_\ell, x] \subset [u]_\delta \cap L$, and hence $[u_\ell, v_\ell] - \gamma \subset [\{u, v\}]_\delta \cap L$, as required. \square

Lemma 18. *Suppose that $[x', y'] \subset \gamma$ and $[x_n, y_n] \rightarrow [x', y']$ as $n \rightarrow \infty$. Then, for any $\epsilon > 0$, $[x_n, y_n] \Delta [x', y'] \subset [\{x', y'\}]_\epsilon$ for all large n .*

Proof. Let $\epsilon > 0$, $\gamma' = [x', y']$, and $\gamma_n = [x_n, y_n]$ as in the lemma be given. We may assume that $\epsilon < 2^{-1}d(x', y')$. Let u (resp. v) denote the point in γ' at

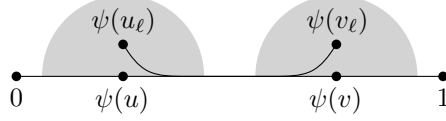


FIGURE 6. The image of $[u_\ell, v_\ell] - \gamma$ under ψ is contained in $[\{u, v\}]_{\psi, \delta} \cap \mathbb{H}_+$.

distance $\epsilon/2$ from x' (resp. y'). By Lemma 17, there are points $u_\ell, v_\ell \in L \cap T$ and $u_r, v_r \in R \cap T$ such that $[u_\ell, v_\ell] - \gamma$ (resp. $[u_r, v_r] - \gamma$) is contained in $[\{u, v\}]_{\epsilon/4} \cap L$ (resp. $[\{u, v\}]_{\epsilon/4} \cap R$). Put

$$\delta_1 = d(\gamma, \{u_\ell, u_r, v_\ell, v_r\})$$

and

$$\delta_2 = d(\{x', y'\}, [u_\ell, v_\ell] \cup [u_r, v_r]).$$

Let $\delta = 2^{-1}(\delta_1 \wedge \delta_2)$. Let H_ℓ (resp. H_r) denote the connected component of $[\gamma]_{\psi, \delta} - \psi([u_\ell, v_\ell])$ (resp. $[\gamma]_{\psi, \delta} - \psi([u_r, v_r])$) that is contained in $\text{Int } \mathbb{H}_+$ (resp. $\text{Int } \mathbb{H}_-$). Since $\gamma_n \rightarrow \gamma'$, we have that $\psi(\gamma_n) \subset [\gamma']_{\psi, \delta}$, $\psi(x_n) \in [x']_{\psi, \delta}$, and $\psi(y_n) \in [y']_{\psi, \delta}$ for all large n . Thus, by the uniqueness of $[u_\ell, v_\ell]$ and $[u_r, v_r]$ and the choice of δ , observe that for all large n , $\gamma_n \cap \psi^{-1}(H_\ell \cup H_r) = \emptyset$, and hence $\gamma_n \Delta \gamma' \subset [\{x', y'\}]_\epsilon$. See Figure 7. \square

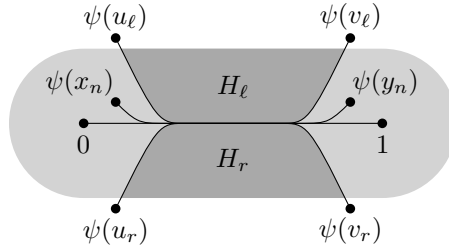


FIGURE 7. The image of $[u_\ell, v_\ell]$ and $[u_r, v_r]$ under ψ , and the sets $[\gamma]_{\psi, \delta}$, H_ℓ and H_r . For all large n , $\psi(\gamma_n)$ and $H_\ell \cup H_r$ are disjoint.

Since $\gamma = [x, y]$ is a general geodesic segment from the root of the map, we obtain the following result immediately by Lemma 18 and invariance under re-rooting.

Lemma 19. *For λ -almost every $x \in M$ the following holds almost surely. If $[x_n, y_n]$ and $[x', y'] \subset [x, y]$ are geodesic segments for which $[x_n, y_n] \rightarrow [x', y']$, then we have that $[x_n, y_n] \rightrightarrows [x', y']$.*

Thus, in a sense, there are no ‘parallel’ geodesic segments converging to a sub-segment of a geodesic segment with at least one typical endpoint.

With Lemma 19 at hand, Lemma 15 follows easily by Lemma 10.

Proof of Lemma 15. Let x denote the root of the Brownian map. Let $y \in M$ and neighbourhoods N_x of x and N_y of y be given. Suppose that $[x_n, y_n]$ is a sequence of geodesic segments with $x_n \rightarrow x$ and $y_n \rightarrow y$ in (M, d) . If $[x_{n_k}, y_{n_k}]$ is a convergent subsequence of $[x_n, y_n]$, then by Lemma 10, $[x_{n_k}, y_{n_k}]$ converges to some $[x, y]$, and hence by Lemma 19, we have that $[x_{n_k}, y_{n_k}] \Delta [x, y] \subset N_x \cup N_y$ eventually. We conclude that for any sequence $[x_n, y_n]$ as above, $[x_n, y_n] - G(x, y) \not\subset N_x \cup N_y$ for at most finitely many n . Hence, the result follows by invariance under re-rooting. \square

4. PROOF OF MAIN RESULTS

In this section, we use Proposition 9 to establish the properties of the Brownian map discussed in Section 1.

As in the previous section, we shall refer to a subset of typical points $T \subset M$, but with the following additional properties:

- (i) $\lambda(T^c) = 0$;
- (ii) Proposition 9 holds for all $x \in T$;
- (iii) for all $x \in T$ and $y \in M$, $G(x, y)$ is regular;
- (iv) for each $x \in T$, there is a dense, two-dimensional set of points with exactly two geodesics to x ; a dense, countable set of points with exactly three geodesics to x ; and no points with more than three geodesics to x ;
- (v) for each $x, y \in T$, there is a unique geodesic from x to y .

The existence of a set T satisfying (i)–(iv) follows by Le Gall’s description of geodesics to the root and invariance under re-rooting. Property (v) then follows by (iii), since as noted in Section 1.3, $G(x, y)$ and $G(y, x)$ are both regular if and only if there is a unique geodesic from x to y .

We remark in passing that since (M, d) is separable, and thus strongly Lindelöf (that is, all open subspaces of (M, d) are Lindelöf), it follows easily by Theorem 2 and Corollary 4 that $\dim \bigcup_{x \in T} G(x) = 1$ and $\dim \bigcup_{x \in T} C(x) = 2$. It would be interesting to know if the same is true if the union is taken over all $x \in M$.

4.1. Geodesic nets. Theorems 1,2 follow directly by Proposition 9.

Proof of Theorem 1. Let $x, y \in M$. We show that for any $u \in T - \{x, y\}$, there is a neighbourhood of u where geodesics to x and y agree. Since T is dense in M , this implies the theorem. Given such a u , Proposition 9 provides an $\eta > 0$ and $u_0 \notin B(u, \eta)$ so that all geodesics from $u' \in B(u, \eta)$ to x or y pass through u_0 . We claim that $U = B(u, \eta/2)$ is the required neighbourhood. Indeed, any geodesic from any $u' \in U$ to x or y cannot re-enter U after visiting u_0 . This is since $d(u', u_0) \geq \eta/2$ and $d(u_0, U) \geq \eta/2$, and so returning to U via u_0 is longer than the diameter of U . \square

Proof of Theorem 2. Let $x \in T$ and a neighbourhood N of x be given. Select $\epsilon > 0$ so that $B(x, \epsilon)$ is strictly contained in N . Let $N' \subset B(x, \epsilon)$ and $x_0 \in B(x, \epsilon)$ be as in Proposition 9. For any $y_0 \in N^c$ and $x' \in N'$, observe

that $y_0 \in G(x')$ if and only if there is some $y \in B(x, \epsilon)^c$ and geodesic segment $[x_0, y]$ so that $y_0 \in [x_0, y]$. Hence, all $G(x')$, $x \in N'$, coincide on N^c . \square

Next, we show that the geodesic framework of the Brownian map, F , is of first Baire category.

A *geodesic star* is a formation of geodesics which share a common endpoint and are otherwise pairwise disjoint (see [32]).

Definition. For $\epsilon > 0$, let $\mathcal{G}(\epsilon)$ denote the set of points $x \in M$ such that for some $y, y' \in B(x, \epsilon)^c$ and geodesic segments $[x, y]$ and $[x, y']$, we have that $(x, y] \cap (x, y'] = \emptyset$. We call a point in $\mathcal{G}(\epsilon)$ the *centre of a geodesic ϵ -star*.

Proposition 20. *Almost surely, for any $\epsilon > 0$, $\mathcal{G}(\epsilon)$ is nowhere dense in M .*

Proof. Let $\epsilon > 0$ and $x \in T$ be given. Put $N = B(x, \epsilon/2)$. Let $N' \subset N$ and $x_0 \in N$ be as in Proposition 9. Since $N \subset B(x', \epsilon)$ for all $x' \in N'$, note that x_0 is contained in all geodesic segments of length ϵ from points $x' \in N'$. Hence, $\mathcal{G}(\epsilon) \cap N' = \emptyset$. The result thus follows by the density of T . \square

Proof of Theorem 3. Since a point lying in the relative interior of a geodesic is contained in $\mathcal{G}(\epsilon)$ for some $\epsilon > 0$, we have that $F \subset \bigcup_{\epsilon > 0} \mathcal{G}(\epsilon)$. Hence, Proposition 20 implies Theorem 3. \square

4.2. Cut loci. As discussed in Section 1.2, Le Gall's study of geodesics to the root shows that the set $S \subset M$ corresponding to cut-points of the CRT is precisely the set of points with multiple geodesics to the root. Hence, Le Gall [26] states that S 'exactly corresponds to the cut locus of [the Brownian map] relative to the root.' In light of this, we make the following definition.

Definition. For $x \in M$, the *weak cut locus* of x , denoted $S(x)$, is the set of points $y \in M$ with multiple geodesics to x .

As it turns out however, $S(x)$ does not capture the essence of a cut locus of a general point $x \in M$ most effectively. The reason for this is the presence of a dense set of atypical points $D \subset M$ with the property that geodesics from $x \in D$ pass through some points in $S(x)$; and thus, a point in $S(x)$ is not necessarily an endpoint relative to x . For this reason, we also define a strong cut locus for the Brownian map, see Section 1.2.

By Le Gall's description of geodesics to the root, it follows immediately that typically the two notions of cut locus coincide.

Proposition 21. *Almost surely, for λ -almost every $x \in M$, $S(x) = C(x)$.*

We remark that the strong cut locus, as opposed to the weak cut locus, is not symmetric in x and y ; that is, $y \in C(x)$ does not imply that $x \in C(y)$. See Figure 8.

Although more in tune with the singular geometry of the Brownian map, the strong cut locus is not analogous to the cut locus of a smooth manifold in many respects. For instance, $C(x)$ is much smaller than the *closure* of all points with multiple geodesics to x , since the set of such points is dense



FIGURE 8. As depicted, $y \in C(x)$, however $x \notin C(y)$, since all geodesics from y to x coincide near x .

M . Also, it is not necessarily the case that all points in $C(x)$ are endpoints relative to x . Despite such differences, we propose that the set $C(x)$ is a more interesting notion of cut locus in our setting than, say, the set of all endpoints relative to x , which is a four-dimensional set of full volume.

We compare the weak and strong cut loci of the Brownian map. As noted in Section 1.2, analogues of Theorems 1,2 hold for the strong cut locus, see Corollary 4. On the other hand, due to a dense set of atypical points, the weak cut locus oscillates in volume and dimension near typical points.

Proposition 22. *Almost surely, for λ -almost every $x \in M$, for any neighbourhood N of x , there is a sub-neighbourhood $N' \subset N$ and a dense subset $D \subset N'$ with $\dim D = 2$ so that $S(x') \supset N^c$, for all $x' \in D$.*

Proof. Let $x \in T$ and a neighbourhood N of x be given. Let $N' \subset N$ and $x_0 \in N$ be as in Proposition 9. Fix some $u \in N^c \cap T$, and put $D = N' \cap C(u)$ so that D is dense in N' and satisfies $\dim D = 2$. Since $u, x \in T$, note that there is a unique geodesic from u to x_0 . Hence, by the choice of D , we see that there are multiple geodesics from each point $x' \in D$ to x_0 . We conclude that $N^c \subset S(x')$, for all $x' \in D$. \square

Since the weak cut locus is symmetric—that is, $y \in S(x)$ if and only if $x \in S(y)$ —we note that it follows immediately by Proposition 22 that almost surely, for all $x \in M$, $S(x)$ is dense in M and $\dim S(x) \geq 2$.

4.3. Geodesic networks. Our next result classifies the types of geodesic networks which are dense in the Brownian map.

Proof of Theorem 5. Let $u \neq v \in T$ be given. Put $\epsilon = 3^{-1}d(u, v)$. By Lemma 15, there is an $\eta > 0$ so that if $U = B(u, \eta)$ and $V = B(v, \eta)$, then for any $u' \in U$ and $v' \in V$, any geodesic segment $[u', v']$ coincides with $[u, v]$ outside of $[\{u, v\}]_\epsilon$. Let z denote the point in $[u, v]$ equidistant from u and v .

By the choice of η and since $u \in T$, we have that for all $v' \in V$, the geodesic network $G(z, v')$ is regular and consists of at most three geodesics. We split $V = V_1 \cup V_2 \cup V_3$, where V_k consists of those $v' \in V$ for which $(z, v') \in N(1, k)$. Similarly, we decompose $U = U_1 \cup U_2 \cup U_3$ according to the type of $G(z, u')$, $u' \in U$. Note that since $u, v \in T$, all U_j, V_k are dense in U, V . By the choice of η , observe that $U_j \times V_k \subset N(j, k)$, for all $j, k \in \{1, 2, 3\}$. Hence, parts (i),(ii) follow by the density of T . \square

To calculate the Hausdorff dimensions of the sets of points joined by normal (j, k) -networks, we require the following result, which is implicit in

Le Gall's [25] proof that $\dim M = 4$. For the record, we include a proof via the uniform volume estimates of balls in the Brownian map.

For a set $A \subset M$, let $\underline{\text{Dim}} A$ (resp. $\overline{\text{Dim}} A$) denote its lower (resp. upper) Minkowski dimension. If the lower and upper Minkowski dimensions coincide, we denote the common value by $\text{Dim} A$.

Proposition 23. *Almost surely, $\text{Dim} M = 4$.*

Proof. For $\epsilon > 0$, let $N(\epsilon)$ denote the number of balls of radius ϵ required to cover M . Let $\eta > 0$ be given. By [32, Lemma 15], there is a constant $c \in (0, \infty)$ and $\epsilon_0 > 0$ so that for all $\epsilon \in (0, \epsilon_0)$ and $x \in M$, we have that $\lambda(B(x, \epsilon)) \geq c\epsilon^{4+\eta}$. Hence, there is a constant $c_\eta \in (0, \infty)$ so that for all $\epsilon \in (0, \epsilon_0)$, $N(\epsilon) \leq c_\eta \epsilon^{-(4+\eta)}$, and so we find that $\overline{\text{Dim}} M \leq 4 + \eta$. Similarly, using [32, Lemma 14] (a consequence of [26, Corollary 6.2]), we see that $\underline{\text{Dim}} M \geq 4 - \eta$. (Although, of course, since $\dim M = 4$, it follows immediately that $\underline{\text{Dim}} M \geq 4$.) Taking $\eta \rightarrow 0$, we obtain the result. \square

Howroyd [20] shows that for any subsets A, B of a metric space, $\dim(A \times B)$ is at most the sum of the Hausdorff dimension of A and the packing dimension of B . The packing dimension of a set is at most its Minkowski dimension. Hence, by Proposition 23, for any subset $A \subset M$, $\dim(A \times M) = \dim A + 4$.

Proof of Theorem 6. Let $u \neq v \in T$ and U_j, V_k , $j, k \in \{1, 2, 3\}$, be as in the proof of Theorem 5. Since $u, v \in T$, $\dim(U_1) = \dim(V_1) = 4$, $\dim(U_2) = \dim(V_2) = 2$, and the U_3, V_3 are countable. Since $U_j \times V_k \subset N(j, k)$, for all $j, k \in \{1, 2, 3\}$, we obtain the lower bounds $\dim N(j, k) \geq 2(6 - j - k)$. Since $\dim M^2 = 8$, it follows immediately that $\dim N(1, 1) = 8$.

Towards the required upper bounds, fix a countable, dense subset $T_0 \subset T$. For each $x \in T_0$, we split $M - \{x\} = S_1(x) \cup S_2(x) \cup S_3(x)$, where $S_j(x)$ consists of those $y \in M$ for which $(x, y) \in N(1, j)$. For $j \in \{1, 2, 3\}$, let $S_j = \bigcup_{x \in T_0} S_j(x)$, and then for $j, k \in \{1, 2, 3\}$, put $S_{j,k} = S_j \times S_k$.

We claim that $\dim S_{j,k} = 2(6 - j - k)$ for all $j, k \in \{1, 2, 3\}$, and that $S_{3,3}$ is countable. Indeed, note that for each $x \in T_0$, $\dim S_1(x) = 4$, $\dim S_2(x) = 2$, and $S_3(x)$ is countable. Since T_0 is countable, we also have that $\dim S_1 = 4$, $\dim S_2 = 2$, and S_3 is countable. Hence $\dim S_{1,1} = 8$, $\dim S_{2,2} = 4$, and $S_{3,3}$ is countable. Since S_3 is countable, we have that $\dim S_{j,k} = 2(6 - j - k)$ if at least one of $j = 3$ or $k = 3$. Finally, to see that $\dim S_{1,2} = \dim S_{2,1} = 6$, we use the observation after the proof of Proposition 23, taking $A = S_2$.

Next, we claim that $N(j, k) \subset S_{j,k}$ for all $(j, k) \in \{1, 2, 3\}^2 - \{(1, 1)\}$, from which the required upper bounds follow. To see this, let $(x, y) \in N(j, k)$, with (j, k) as above. Let $[x_0, y_0] \subset [x, y]$ be the sub-segment of all geodesic segments between x and y which is maximal with respect to inclusion. We note that if $j = 1$ (resp. $k = 1$) then $x_0 = x$ (resp. $y_0 = y$). Let U be a connected component of $G(x, y)^c$ whose closure is disjoint from the relative interior of $[x_0, y_0]$. Fix $z \in U \cap T$. By interchanging labels if necessary, we may assume that $d(z, x_0) < d(z, y_0)$. Then, by the choice of $[x_0, y_0]$ and since $z \in T$, there is a unique geodesic segment $[z, y_0]$, and we have that

$[x_0, y_0] \subset [z, y_0]$. Therefore, since $z \in T$, we see by Lemma 19 that for some $z_0 \in T_0$ sufficiently close to the midpoint of x_0 and y_0 , it holds that $(z_0, x) \in N(1, j)$ and $(z_0, y) \in N(1, k)$. Hence, $(x, y) \in S_{j,k}$. See Figure 9.

Altogether, we conclude that $\dim N(j, k) = 2(6 - j - k)$ for all $j, k \in \{1, 2, 3\}$, and also that $N(3, 3)$ is countable. \square

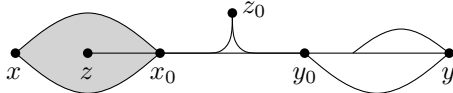


FIGURE 9. As depicted, $(x, y) \in N(2, 3)$. We select a typical point $z \in U$ and note that $[z, y_0] \supset [x_0, y_0]$. For some $z_0 \in T_0$ sufficiently close to $[x_0, y_0]$, we have that $(z_0, x) \in N(1, 2)$ and $(z_0, y) \in N(1, 3)$.

Proof of Corollaries 7, 8. Noting that $N(j, k) \subset P(jk)$, for all $j, k \in \mathbb{N}$, we observe that Theorems 5, 6 immediately yield Corollaries 7, 8. \square

5. RELATED MODELS

An infinite volume version of the Brownian map, the *Brownian plane* (P, D) , has been introduced by Curien and Le Gall [18]. The random metric space (P, D) is homeomorphic to the plane \mathbb{R}^2 and arises as the local Gromov-Hausdorff scaling limit of the UIPQ. The Brownian plane has a additional scale invariance property, which makes it more amenable to analysis, see the recent works of Curien and Le Gall [16, 17]. As discussed in [24], almost surely, there are isometric neighbourhoods of the roots of (M, d) and (P, D) . Using this fact and scale invariance, properties of the Brownian plane can be deduced from those of the Brownian map.

In a series of works, Bettinelli [8, 9, 7] investigates *Brownian surfaces* of positive genus. In [8], only subsequential Gromov-Hausdorff convergence of uniform random bipartite quadrangulations of the g -torus \mathbb{T}_g is established (also, general orientable surfaces with a boundary are analyzed in [7]), and it is an ongoing work of Bettinelli and Miermont [11] to lift this constraint. Some properties hold independently of which subsequence is extracted. For instance, the scaling limit of bipartite quadrangulations of \mathbb{T}_g is homeomorphic to \mathbb{T}_g (see [9]) and of Hausdorff dimension four (see [8]). Also, a confluence of geodesics is observed at typical points of the surface (see [7]). Our results imply further properties of the geodesic structure of such surfaces, although in this setting there are more technicalities to be addressed.

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REFERENCES

- [1] ABRAHAM, C. (2014). Rescaled bipartite planar maps converge to the Brownian map. Preprint available at arXiv:1312.5959.
- [2] ADDARIO-BERRY, L. and ALBENQUE, M. (2014). The scaling limit of random simple triangulations and random simple quadrangulations. Preprint available at arXiv:1306.5227.
- [3] ALDOUS, D. (1991). The continuum random tree. I. *Ann. Probab.* **19** 1–28. MR1085326
- [4] ALDOUS, D. (1993). The continuum random tree. III. *Ann. Probab.* **21** 248–289. MR1207226
- [5] ANGEL, O. and SCHRAMM, O. (2003). Uniform infinite planar triangulations. *Comm. Math. Phys.* **241** 191–213. MR2013797
- [6] BELTRAN, J. and LE GALL, J.-F. (2013). Quadrangulations with no pendant vertices. *Bernoulli* **19** 1150–1175. MR3102547
- [7] BETTINELLI, J. (2014). Geodesics in Brownian surfaces (Brownian maps). *Ann. Inst. H. Poincaré*, to appear. Preprint available at arXiv:1401.3602.
- [8] BETTINELLI, J. (2010). Scaling limits for random quadrangulations of positive genus. *Electron. J. Probab.* **15** 1594–1644. MR2735376
- [9] BETTINELLI, J. (2012). The topology of scaling limits of positive genus random quadrangulations. *Ann. Probab.* **40** 1897–1944. MR3025705
- [10] BETTINELLI, J., JACOB, E. and MIERMONT, G. (2014). The scaling limit of uniform random plane maps, *via* the Ambjørn-Budd bijection. *Electron. J. Probab.* **19** 1–16. MR3256874
- [11] BETTINELLI, J. and MIERMONT, G. Uniqueness of Brownian surfaces by a surgical approach. In preparation.
- [12] BRIDSON, M. R. and HAEFLIGER, A. (1999). *Metric spaces of non-positive curvature. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **319**. Springer-Verlag, Berlin. MR1744486
- [13] BUCHNER, M. A. (1977). Stability of the cut locus in dimensions less than or equal to 6. *Invent. Math.* **43** 199–231. MR0482816
- [14] BURAGO, D., BURAGO, Y. and IVANOV, S. (2001). *A course in metric geometry. Graduate Studies in Mathematics* **33**. Amer. Math. Soc., Providence, RI. MR1835418
- [15] CORI, R. and VAUQUELIN, B. (1981). Planar maps are well labeled trees. *Canad. J. Math.* **33** 1023–1042. MR638363
- [16] CURIEN, N. and LE GALL, J.-F. (2014). The hull process of the Brownian plane. Preprint available at arXiv:1409.4026.
- [17] CURIEN, N. and LE GALL, J.-F. (2014). Scaling limits for the peeling process on random maps. Preprint available at arXiv:1412.5509.
- [18] CURIEN, N. and LE GALL, J.-F. (2014). The Brownian Plane. *J. Theoret. Probab.* **27** 1249–1291. MR3278940
- [19] CURIEN, N., MÉNARD, L. and MIERMONT, G. (2013). A view from infinity of the uniform infinite planar quadrangulation. *ALEA Lat. Am. J. Probab. Math. Stat.* **10** 45–88. MR3083919
- [20] HOWROYD, J. D. (1996). On Hausdorff and packing dimension of product spaces. *Math. Proc. Cambridge Philos. Soc.* **119** 715–727. MR1362951

- [21] KLINGENBERG, W. P. A. (1995). *Riemannian geometry. Second edition. de Gruyter Studies in Mathematics* **1**. Walter de Gruyter, Berlin. MR1330918
- [22] KRIKUN, M. (2006). Local structure of random quadrangulations. Preprint available at arXiv:math/0512304.
- [23] KRIKUN, M. (2008). On one property of distances in the infinite random quadrangulation. Preprint available at arXiv:0805.1907.
- [24] LE GALL, J.-F. (2014). Random geometry of the sphere. *Proceedings of the International Congress of Mathematicians, Seoul, 2014*, to appear. Preprint available at arXiv:1403.7943.
- [25] LE GALL, J.-F. (2007). The topological structure of scaling limits of large planar maps. *Invent. Math.* **169** 621–670. MR2336042
- [26] LE GALL, J.-F. (2010). Geodesics in large planar maps and in the Brownian map. *Acta Math.* **205** 287–360. MR2746349
- [27] LE GALL, J.-F. (2013). Uniqueness and universality of the Brownian map. *Ann. Probab.* **41** 2880–2960. MR3112934
- [28] LE GALL, J.-F. and MIERMONT, G. (2012) Scaling limits of random trees and planar maps. In *Probability and statistical physics in two and more dimensions, Clay Math. Proc.* **15** 155–211. Amer. Math. Soc., Providence, RI. MR3025391
- [29] LE GALL, J.-F. and PAULIN, F. (2008). Scaling limits of bipartite planar maps are homeomorphic to the 2-sphere. *Geom. Funct. Anal.* **18** 893–918. MR2438999
- [30] MIERMONT, G. (2008). On the sphericity of scaling limits of random planar quadrangulations. *Electron. Commun. Probab.* **13** 248–257. MR2399286
- [31] MIERMONT, G. (2009). Tessellations of random maps of arbitrary genus. *Ann. Sci. Éc. Norm. Supér. (4)* **42** 725–781. MR2571957
- [32] MIERMONT, G. (2013). The Brownian map is the scaling limit of uniform random plane quadrangulations. *Acta Math.* **210** 319–401. MR3070569
- [33] MOHAR, B. and THOMASSEN, C. 2001. *Graphs on surfaces*. Johns Hopkins Univ. Press, Baltimore, MD. MR1844449
- [34] MYERS, S. B. (1935). Connections between differential geometry and topology. I. Simply connected surfaces. *Duke Math. J.* **1** 376–391. MR1545884
- [35] POINCARÉ, H. (1905). Sur les lignes géodésiques des surfaces convexes. *Trans. Amer. Math. Soc.* **6** 237–274. MR1500710
- [36] SCHAEFFER, G. (1998). *Conjugaison d'arbres et cartes combinatoires aléatoires*. Ph.D. thesis, Université Bordeaux I.
- [37] WALL, C. T. C. (1977). Geometric properties of generic differentiable manifolds. In *Geometry and topology, Proc. III Latin Amer. School of Math., Inst. Mat. Pura Aplicada CNPq, Rio de Janeiro, 1976, Lecture Notes in Math.* **59** 707–774. Springer, Berlin. MR0494233
- [38] ZAMFIRESCU, T. (1982). Many endpoints and few interior points of geodesics. *Invent. Math.* **69** 253–257. MR674405

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