THEORY OF DARK ENERGY AND DARK MATTER

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Dedicated to Professor Louis Nirenberg on the occasion of his 90th birthday with great admiration and respect

Abstract. In (Ma and Wang, 2014b), a new set of gravitational field equations are derived based only on 1) the Einstein principle of general relativity, and 2) the principle of interaction dynamics, due to the the presence of dark energy and dark matter. With the field equations, we show that gravity can display both attractive and repulsive behavior, and the dark matter and dark energy are just a property of gravity caused by the nonlinear interactions of the gravitational potential $g_{\mu\nu}$ and its dual field. The main objectives of this paper are two-fold. The first is to study the PID-induced cosmological model, and to show explicitly, as addressed in (Ma and Wang, 2014a), that 1) dark matter is due to the curvature of space, and 2) dark energy corresponds to the negative pressure generated by the dual gravitational potential in the field equations, and maintains the stability of geometry and large scale structure of the Universe. Second, for the gravitational field outside of a ball of centrally symmetric matter field, there exist precisely two physical parameters dictating the two-dimensional stable manifold of asymptotically flat space-time geometry, such that, as the distance to the center of the ball of the matter field increases, gravity behaves as Newtonian gravity, then additional attraction due to the curvature of space (dark matter effect), and repulsive (dark energy effect). This also clearly demonstrates that both dark matter and dark energy are just a property of gravity. We note that the dark matter property of the gravity and the approximate gravitational interaction formula are consistent with the MOND theory proposed by (Milgrom, 1983); see also (Milgrom, 2014) and the references therein.

Contents

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1. Introduction

Gravity is one of the four fundamental interactions/forces of Nature, and is certainly the first interaction/force that people studied over centuries, dating back to Aristotle, Galileo, Johannes Kepler, Isaac Newton, and Albert Einstein. It was Albert Einstein who first derived the basic law of gravity — the Einstein gravitational field equations — by postulating the principle of equivalence and the principle of general relativity. In mathematical terms, the principle of equivalence says that the space-time is a 4-dimensional Riemannian manifold $\{\mathcal{M}, g_{\mu\nu}\}$ with metric tensor $\{g_{\mu\nu}\}$ of $\mathcal{M}$ being the gravitational potential. The principle of general relativity requires that the law of gravity be independent of general coordinate transformations, and dictates the Einstein-Hilbert functional. The Einstein gravitational field equations are then derived using the least action principle, also called the principle of Lagrangian dynamics.

Dark matter and dark energy phenomena are two important phenomena, which requires a more fundamental examination of the law of gravity (Riess and et al., 1989; Perlmutter and et al., 1999; Zwicky, 1937; Rubin and Ford, 1970). Recently, we have shown in (Ma and Wang, 2014b) that the presence of dark matter and dark energy implies that the variation of the Einstein-Hilbert functional must be taken under energy-momentum conservation constraint, and we call such variation the principle of interaction dynamics (PID). With PID, we have derived the new gravitational field equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{8\pi G}{c^4}T_{\mu\nu} - \nabla_\mu\nabla_\nu\varphi, \tag{1.1}$$

supplemented by the energy-momentum conservation:

$$\nabla^\mu\left[\frac{8\pi G}{c^4}T_{\mu\nu} + \nabla_\mu\nabla_\nu\varphi\right] = 0. \tag{1.2}$$

Here ϕ is a scalar field defined on M, and needs to be solved together with the Riemannian metric $g_{\mu\nu}$, representing the gravitational potential. Also ∇^μ is the gradient operator on M, $R_{\mu\nu}$ and R are the Ricci and scalar curvatures, and $T_{\mu\nu}$ is the energy-momentum of the baryonic matter in the universe.

With the new gravitational field equations, we have shown in (Ma and Wang, 2014b) that gravity can display both attractive and repulsive effect, caused by the duality between the *attracting* gravitational field $\{g_{\mu\nu}\}$ and the *repulsive* dual vector field $\{\Phi_\mu\}$, together with their nonlinear interactions governed by the field equations. Consequently, dark energy and dark matter phenomena are simply a property of gravity.

The main objective of this article is to further explore the nature of dark matter and dark energy, in connection with

1) the geometric structure of our Universe derived in (Ma and Wang, 2014a), and
2) the gravitational force formula and large distance asymptotic flatness of gravity in a central gravitational field.

We proceed as follows.

FIRST, we have shown in (Ma and Wang, 2014a) a new cosmology theorem that our Universe is a three dimensional sphere and is static, assuming the Einstein general relativity and the cosmological principle that our Universe is homogeneous and isotropic. By using the new gravitational field equations applied to homogeneous spherical universe, we explicitly demonstrate that 1) dark matter is caused by the space curvature, and 2) dark energy represents the negative pressure in the PID cosmological model:

$$p = -\frac{c^4}{8\pi G R^2}, \tag{1.3}$$

caused by the dual gravitational field $\Phi_\mu = \nabla_\mu \varphi$ in the new gravitational field equations. Here R is the cosmic radius.

SECOND, consider a central gravitational field generated by a ball B_{r_0} with radius r_0 and mass M. It is known that the metric of the central field at $r > r_0$ can be written in the form

$$ds^2 = -e^u c^2 dt^2 + e^v dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{1.4}$$

and $u = u(r), v = v(r)$. Then the fields equations (1.1) take the form

$$\begin{aligned} & v' + \frac{1}{r}(e^v - 1) = -\frac{r}{2}u'\phi', \\ & u' - \frac{1}{r}(e^v - 1) = r(\phi'' - \frac{1}{2}v'\phi'), \\ & u'' + \left(\frac{1}{2}u' + \frac{1}{r}\right)(u' - v') = -\frac{2}{r}\phi'. \end{aligned} \tag{1.5}$$

We have derived in (Ma and Wang, 2014b) an approximate gravitational force formula

$$\begin{aligned} & F = mMG\left(-\frac{1}{r^2} - \frac{k_0}{r} + k_1 r\right) \qquad \text{for } r > r_0, \\ & k_0 = 4 \times 10^{-18}\text{Km}^{-1}, \quad k_1 = 10^{-57}\text{Km}^{-3}, \end{aligned} \tag{1.6}$$

demonstrating the presence of both dark matter and dark energy.

THIRD, we discover in this paper that under the following transformation

$$\begin{aligned} & (r, w) = (e^s, e^v - 1), \\ & x(s) \overset{\text{def}}{=} (x_1(s), x_2(s), x_3(s)) = \left(e^s u'(e^s), w(e^s), e^s \phi'(e^s)\right), \end{aligned} \tag{1.7}$$

the *non-autonomous* gravitational field equations (1.5) are amazingly becoming an *autonomous system*:

$$\begin{aligned} & x_1' = -x_2 + 2x_3 - \frac{1}{2}x_1^2 - \frac{1}{2}x_1 x_3 - \frac{1}{2}x_1 x_2 - \frac{1}{4}x_1^2 x_3, \\ & x_2' = -x_2 - \frac{1}{2}x_1 x_3 - x_2^2 - \frac{1}{2}x_1 x_2 x_3, \\ & x_3' = x_1 - x_2 + x_3 - \frac{1}{2}x_2 x_3 - \frac{1}{4}x_1 x_3^2. \end{aligned} \tag{1.8}$$

Then we rigorously show the following conclusions:

1) The asymptotically flat space-time geometry is represented by $x = 0$, which is a fixed point of the system (1.8). There is a two-dimensional stable manifold E^s near $x = 0$, and consequently there are exactly two free parameters, to be determined by experiments (or by astronomical measurements), which give rise to asymptotical flatness at $r = \infty$.
2) The gravitational force F induced by the centrally symmetric matter is given by (4.5), and is asymptotically zero as the distance tends to infinity.
3) For the initial date near the Schwarzschild solution (4.7) satisfied by all physically meaningful central fields, there exists a sufficiently large r_1 such that the gravitational force F is repulsive for $r > r_1$.

These asymptotic properties of gravity, displaying the key features of dark energy, plays the role to stabilize the large scale homogeneous structure of the Universe.

The paper is organized as follows. Section 2 recalls the dark energy and dark matter phenomena. Section 3 explores the nature of dark energy using the PID-induced cosmological model. Section 4 recalls the gravitational force formula deduced in (Ma and Wang, 2014b). In Section 5, we study the asymptotic flatness and rigorously prove the existence of dark energy at large distance outside of a spherically symmetric matter field. Section 6 addresses the two physical parameters in the gravitational interaction formula, and Section 7 addresses the nature of dark energy and dark matter.

2. Dark Energy and Dark Matter Phenomena

Dark matter and Rubin rotational curve. In astrophysics, dark matter is an unknown form of matter, which appears only participating in gravitational interaction, but does not emit nor absorb electromagnetic radiations.

Dark matter was first postulated in 1932 by Holland astronomer Jan Oort, who noted that the orbital velocities of stars in the Milky Way don't match their measured masses. Namely, the orbital velocity v and the gravity should satisfy the equilibrium relation

$$\frac{v^2}{r} = \frac{M_r G}{r^2}, \tag{2.1}$$

where M_r is the total mass in the ball B_r with radius r. But the observed mass M_0 was less than the theoretic mass M_r in (2.1), and the difference $M_r - M_0$ was explained as the presence of dark matter. The phenomenon was also discovered by Fritz Zwicky in 1933 for the missing mass in the orbital velocities of galaxies in clusters. Subsequently, other observations have manifested the existence of dark matter in the Universe, including the rotational velocities of galaxies, gravitational lensing, and the temperature distribution of hot gaseous.

A strong support to the existence of dark matter is the Rubin rotational curves for galactic rotational velocity. The rotational curve of a galaxy is the rotational velocity of visible stars or gases in the galaxy on their radial distance from the center of the galaxy. The Rubin rotational curve amounts to saying that most stars in spiral galaxies orbit at roughly the same speed. If a galaxy had a mass distribution as the observed distribution of visible astronomical objects, the rotational velocity would decrease at large distances. Hence, the Rubin curve demonstrates the existence of additional gravitational effect to the gravity by the visible matter in the galaxy.

More precisely, the orbital velocity $v(r)$ of the stars located at radius r from the center of galaxies is almost a constant:

$$v(r) \cong \text{a constant for a given galaxy,} \tag{2.2}$$

as illustrated typically by Figure 2.1 (a), where the vertical axis represents the velocity (Km/s), and the horizontal axis is the distance from the galaxy center (extending to the galaxy radius). However, the calculation from (2.1) gives a theoretic curve as shown in Figure f6.15(b), showing discrepancies between the mass determined from the gravitational effect and the mass calculated from the visible matter. The missing mass suggests the presence of dark matter in the Universe.

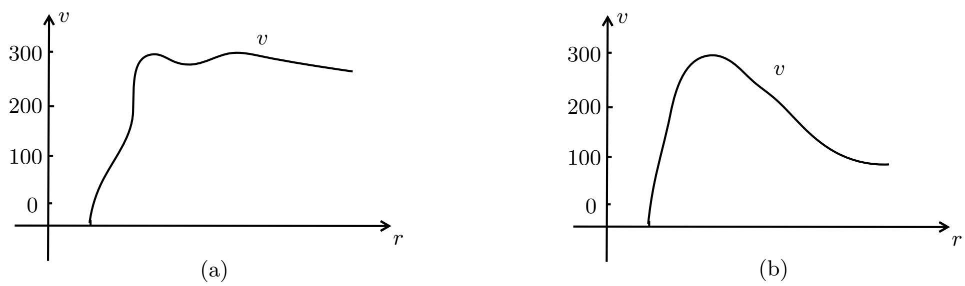

FIGURE 2.1. (a) Typical galactic rotational curve by Rubin, and (b) theoretic curve based on the Newtonian gravitational law.

In fact, we have seen in (Ma and Wang, 2014a, 2015) that the dark matter is a space curved energy, or equivalently a gravitational effect, which is also reflected in the revised gravitational force formula in which there is an additional attracting force to the classical Newtonian gravity.

Dark energy. Dark energy was first proposed in 1990's, which was based on the hypotheses that the Universe is expanding.

The High-z Supernova Search Team in 1998 and the Supernova Cosmology Project in 1999 published their precisely measured data of the distances of supernovas and the redshifts. The observations indicated that the measured and theoretical data have a deviation, which was explained, based on the Hubble Law and the Friedmann model, as the acceleration of the expanding universe. The accelerating expansion is widely accepted as an evidence of the existence of dark energy.

However, based on the new cosmology postulated in the last section, the dark energy is a field energy form of gravitation which balances the gravitational attraction to maintain the homogeneity and stability of the Universe.

3. PID Cosmological Model and Dark Energy

We have shown in (Ma and Wang, 2014b) that both dark matter and dark energy are a property of gravity. Dark matter and dark energy are reflected in a) the large scale space curved structure of the Universe caused by gravity, and b) the gravitational attracting and repelling aspects of gravity. In this section, we mainly explore the nature of dark energy in aspect a) using the PID-induced cosmological model.

PID cosmological model

The metric of a homogeneous spherical universe is of the form

$$ds^2 = -c^2dt^2 + R^2\left[\frac{dr^2}{1-r^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right], \tag{3.1}$$

where $R = R(t)$ is the cosmic radius. The PID induced gravitational field equations are given by

$$R_{\mu\nu} = -\frac{8\pi G}{c^4}(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T) - (\nabla_{\mu\nu}\phi - \frac{1}{2}g_{\mu\nu}\Phi), \tag{3.2}$$

where $\Phi = g^{\alpha\beta}D_{\alpha\beta}\phi$, and ϕ depends only on t.

The nonzero components of $R_{\mu\nu}$ read as

$$\begin{aligned} R_{00} &= \frac{3}{c^2}\frac{1}{R}R_{tt}, \\ R_{kk} &= -\frac{1}{c^2R^2}g_{kk}(RR''_{tt} + 2R_t^2 + 2c^2) \qquad \text{for } 1 \le k \le 3, \end{aligned}$$

and by $T_{\mu\nu} = \text{diag}(c^2\rho, g_{11}p, g_{22}p, g_{33}p)$, we have

$$\begin{aligned} T_{00} - \frac{1}{2}g_{00}T &= \frac{c^2}{2}\left(\rho + \frac{3p}{c^2}\right), \\ T_{kk} - \frac{1}{2}g_{kk}T &= \frac{c^2}{2}g_{kk}\left(\rho - \frac{p}{c^2}\right) \qquad \text{for } 1 \le k \le 3, \\ \phi_{00} - \frac{1}{2}g_{00}\Phi &= \frac{1}{2c^2}\left(\phi_{tt} - \frac{3R_t}{R}\phi_t\right), \\ \phi_{kk} - \frac{1}{2}g_{kk}\Phi &= \frac{1}{2c^2}g_{kk}\left(\phi_{tt} + \frac{R_t}{R}\phi_t\right) \qquad \text{for } 1 \le k \le 3. \end{aligned}$$

Thus, we derive from (3.2) two independent field equations as

$$R'' = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)R - \frac{1}{6}\phi''R + \frac{1}{2}R'\phi', \tag{3.3}$$

$$\frac{R''}{R} + 2\left(\frac{R'}{R}\right)^2 + \frac{2c^2}{R^2} = 4\pi G\left(\rho - \frac{p}{c^2}\right) + \frac{1}{2}\phi'' + \frac{1}{2}\frac{R'}{R}\phi'. \tag{3.4}$$

We infer from (3.3) and (3.4) that

$$(R')^2 = \frac{8\pi G}{3}R^2\rho + \frac{1}{3}R^2\phi'' - c^2. \tag{3.5}$$

By the Bianchi identity:

$$\nabla^\mu(\nabla_{\mu\nu}\phi + \frac{8\pi G}{c^4}T_{\mu\nu}) = 0,$$

we deduce that

$$\phi''' + \frac{3R'}{R}\phi'' = -8\pi G\left(\rho' + \frac{3R'}{R}\rho + \frac{3R'}{R}\frac{p}{c^2}\right). \tag{3.6}$$

It is known that the energy density ρ and the cosmic radius R (also called the scale factor) satisfy the relation:

$$\rho = \frac{\rho_0}{R^3}, \qquad \rho_0 \text{ the density at } R = 1. \tag{3.7}$$

Hence, it follows from (3.7) that

$$\rho' = -3\rho R'/R.$$

Thus, (3.6) is rewritten as

$$\phi''' + \frac{3R'}{R}\phi'' = -\frac{24\pi G}{c^2}\frac{R'}{R}p. \tag{3.8}$$

In addition, making the transformation

$$\phi'' = \frac{\psi}{R^3}, \tag{3.9}$$

then, from (3.3), (3.5) and (3.7)-(3.9) we can deduce that

$$(R')^2\phi' = 0. \tag{3.10}$$

Denote $\varphi = \phi''$, by (3.10), the equations (3.3), (3.5) and (3.8) can be rewritten in the form

$$\begin{aligned} R'' &= -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2} + \frac{\varphi}{8\pi G}\right)R, \\ (R')^2 &= \frac{1}{3}(8\pi G\rho + \varphi)R^2 - c^2, \\ \varphi' + \frac{3R'}{R}\varphi &= -\frac{24\pi G}{c^2}\frac{R'}{R}p. \end{aligned} \tag{3.11}$$

Only two equations in (3.11) are independent. However, there are three unknown functions R, φ, p in (3.11). Hence, we need to add an additional equation, the equation of state, as follows:

$$p = f(\rho, \varphi). \tag{3.12}$$

Based on (Ma and Wang, 2014a, Theorem 6.2), the model describing the static Universe is the equation (3.12) together with the stationary equations of (3.11), which are equivalent to the form

$$\begin{aligned} \varphi &= -8\pi G\left(\rho + \frac{3p}{c^2}\right), \\ p &= -\frac{c^4}{8\pi G R^2}. \end{aligned} \tag{3.13}$$

The equations (3.12) and (3.13) provide a theoretic basis for the static Universe, including the dark energy.

Now, we need to determine the explicit expression for the equation (3.12) of state. It is natural to postulate that the equation of state is linear. Hence, (3.12) can be written as

$$p = \frac{c^2}{G}(\alpha_1\varphi - \alpha_2 G\rho), \tag{3.14}$$

where α_1 and α_2 are nondimensional parameter, which will be determined by the observed data.

The equations (3.13) and (3.14) are the PID cosmological model, where the cosmological significants of R, p, φ, ρ are as follows:

(3.15)

- R the cosmic radius (of the 3D spherical universe),
- p the negative pressure, generated by the repulsive aspect of gravity,
- φ represents the dual gravitational potential,
- ρ the cosmic density, given by $\dfrac{3M}{4\pi R^3} = \dfrac{M_{\text{total}}}{\pi^2 R^3}$,

where M and M_{total} are the observed and total mass respectively; see also (Ma and Wang, 2014a, Remark 6.1).

Here, we remark that in the classical Einstein field equations where $\phi = 0$, the relation (3.7) still holds true, by which we deduce that $p = 0$. It implies that the Friedmann equations has no stationary solution for $k \neq 0$, and has only stationary $R = 0$ for $k = 0$. Therefore, the PID gravitational theory is essential for establishing a static cosmological model.

Theory of dark energy

In the static cosmology, dark energy is defined in the following manner. Let E_{ob} be the observed energy, and R be the cosmic radius. We define the observable mass and the total mass as follows:

$$M_{\text{ob}} = \frac{E_{\text{ob}}}{c^2}, \tag{3.16}$$

$$M_{\text{T}} = \frac{Rc^2}{2G}. \tag{3.17}$$

If $M_{\text{T}} > M_{\text{ob}}$, then the difference

$$\Delta E = E_{\text{T}} - E_{\text{ob}} \tag{3.18}$$

is called the dark energy.

The CMB measurement and the WMAP analysis indicate that the difference ΔE in (3.18) is positive,

$$\Delta E > 0,$$

which is considered as another evidence for the presence of dark energy.

From the PID cosmological model (3.13)-(3.15), we see that the dark energy ΔE in (3.18) is essentially due to the dual gravitational potential φ. In fact, we infer from (3.13) that

$$\begin{aligned} &\varphi = 0 \quad \Leftrightarrow \quad R = 2M_{\text{ob}}G/c^2 \quad \text{(i.e. } \Delta E = 0), \\ &\varphi > 0 \quad \Leftrightarrow \quad \Delta E > 0. \end{aligned} \tag{3.19}$$

Hence, dark energy is generated by the dual gravitational field. This fact is reflected in the PID gravitational force formula derived in sections hereafter.

If we can measure precisely, with astronomical observations, the energy (3.16) and the cosmic radius R (i.e. M_{T} of (3.17)), then we can obtain a relation between the parameters α_1 and α_2 in (3.14). In fact, we deduce from (3.13) and (3.14) that

$$\rho + \frac{\beta p}{c^2} = 0, \qquad \beta = \frac{1 + 24\pi\alpha_1}{\alpha_2 + 8\pi\alpha_1}. \tag{3.20}$$

As we get

$$\frac{\Delta M}{M_{\text{ob}}} = \frac{M_{\text{T}} - M_{\text{ob}}}{M_{\text{ob}}} = k \quad (k > 0). \tag{3.21}$$

Then by (3.20) and

$$\rho = \frac{3M_{\text{ob}}}{4\pi R^3}, \quad p = -\frac{c^4}{8\pi G R^2},$$

we obtain from (3.20) that

$$3\alpha_2 = 24k\pi\alpha_1 + k + 1. \tag{3.22}$$

By the relation (3.22) from (3.20), we can also derive, in the same fashion as above, the dark energy formula (3.21).

4. PID Gravitational Interaction Formula

Consider a central gravitational field generated by a ball B_{r_0} with radius r_0 and mass M. It is known that the metric of the central field at $r > r_0$ can be written in the form

$$ds^2 = -e^u c^2 dt^2 + e^v dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{4.1}$$

and $u = u(r), v = v(r)$.

In the exterior of B_{r_0}, the energy-momentum is zero, i.e.

$$T_{\mu\nu} = 0, \qquad \text{for } r > r_0.$$

Hence, the PID gravitational field equation for the metric (4.1) is given by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\nabla_{\mu\nu}\phi, \qquad r > r_0. \tag{4.2}$$

where $\phi = \phi(r)$ is a scalar function of r.

As in (Ma and Wang, 2014b), we have

$$\begin{aligned}
&R_{00} - \frac{1}{2} g_{00} R = -\frac{1}{r} e^{u-v} \left[v' + \frac{1}{r}(e^v - 1) \right], \\
&R_{11} - \frac{1}{2} g_{11} R = -\frac{1}{r} \left[u' - \frac{1}{r}(e^v - 1) \right], \\
&R_{22} - \frac{1}{2} g_{22} R = -\frac{r^2}{2} e^{-v} \left[u'' + \left(\frac{1}{2} u' + \frac{1}{r} \right)(u' - v') \right], \\
&\nabla_{00}\phi = -\frac{1}{2} e^{u-v} u' \phi', \\
&\nabla_{11}\phi = \phi'' - \frac{1}{2} v' \phi', \\
&\nabla_{22}\phi = -r e^{-v} \phi'.
\end{aligned}$$

Thus, the fields equations (4.2) are as follows

$$\begin{aligned}
&v' + \frac{1}{r}(e^v - 1) = -\frac{r}{2} u' \phi', \\
&u' - \frac{1}{r}(e^v - 1) = r(\phi'' - \frac{1}{2} v' \phi'), \\
&u'' + \left(\frac{1}{2} u' + \frac{1}{r} \right)(u' - v') = -\frac{2}{r} \phi'.
\end{aligned} \tag{4.3}$$

Now we are ready to deduce from (4.3) the PID gravitational interaction formula as follows.

First, we infer from (4.3) that

$$\begin{aligned}
&u' + v' = \frac{r\phi''}{1 + \frac{r}{2}\phi'}, \\
&u' - v' = \frac{1}{1 - \frac{r}{2}\phi'} \left[\frac{2}{r}(e^v - 1) + r\phi'' \right].
\end{aligned}$$

Consequently,

$$u' = \frac{1}{1 - \frac{r}{2}\phi'} \frac{1}{r}(e^v - 1) + \frac{r\phi''}{1 - (\frac{r}{2}\phi')^2}. \tag{4.4}$$

It is known that the interaction force F is given by

$$F = -m\nabla\psi, \qquad \psi = \frac{c^2}{2}(e^u - 1).$$

Then, it follows from (4.4) that

$$F = \frac{mc^2}{2}e^u\left[-\frac{1}{1-\frac{r}{2}\phi'}\frac{1}{r}(e^v - 1) - \frac{r\phi''}{1-\left(\frac{r}{2}\phi'\right)^2}\right]. \tag{4.5}$$

The formula (4.5) provides the precise gravitational interaction force exerted on an object with mass m in a spherically symmetric gravitation field.

In classical physics, the field functions u and v in (4.5) are taken by the Schwarzschild solution:

$$e^u = 1 - \frac{2GM}{c^2r}, \qquad e^v = \left(1 - \frac{2GM}{c^2r}\right)^{-1}, \tag{4.6}$$

and $\phi' = \phi'' = 0$, which leads to the Newton gravitation.

However, due to the presence of dark matter and dark energy, the field functions u, v, ϕ in (4.5) should be an approximation of the Schwarzschild solution (4.6). Hence we have

$$|r\phi'| \ll 1 \qquad \text{for } r > r_0. \tag{4.7}$$

Under the condition (4.7), formula (4.5) can be approximatively expressed as

$$F = \frac{mc^2}{2}e^u\left[-\frac{1}{r}(e^v - 1) - r\phi''\right]. \tag{4.8}$$

5. Asymptotic Repulsion of Gravity

In this section, we shall consider the asymptotic properties of gravity, and rigorously prove that the interaction force given by (4.8) is repulsive at very large distance.

To this end, we need to make the following transformation to convert the field equations (4.3) into a first order autonomous system:

$$\begin{aligned} &r = e^s, \\ &w = e^v - 1, \\ &x_1(s) = e^s u'(e^s), \\ &x_2(s) = w(e^s), \\ &x_3(s) = e^s\phi'(e^s). \end{aligned} \tag{5.1}$$

Then the equations (4.3) can be rewritten as

$$\begin{aligned} &x_1' = -x_2 + 2x_3 - \frac{1}{2}x_1^2 - \frac{1}{2}x_1x_3 - \frac{1}{2}x_1x_2 - \frac{1}{4}x_1^2x_3, \\ &x_2' = -x_2 - \frac{1}{2}x_1x_3 - x_2^2 - \frac{1}{2}x_1x_2x_3, \\ &x_3' = x_1 - x_2 + x_3 - \frac{1}{2}x_2x_3 - \frac{1}{4}x_1x_3^2. \end{aligned} \tag{5.2}$$

The system (5.2) is supplemented with an initial condition

$$(x_1, x_2, x_3)(s_0) = (\alpha_1, \alpha_2, \alpha_3) \qquad \text{with } r_0 = e^{s_0}. \tag{5.3}$$

We now study the problem (5.2)-(5.3) in a few steps as follows.

Step 1. Asymptotic flatness. For a globular matter distribution, its gravitational field should be asymptotically flat, i.e.

$$g_{00} \to -1, \quad g_{11} \to 1 \quad \text{if} \quad r \to \infty.$$

It implies that $x = 0$ is the uniquely physical equilibrium point of (5.2) and the following holds true:

$$x(s) \to 0 \quad \text{if} \quad s \to \infty \quad (\text{i.e. } r \to \infty). \tag{5.4}$$

Step 2. Physical initial values. The physically meaningful initial values $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ in (5.3) have to satisfy the following two conditions:

(a) The solutions $x(s,\alpha)$ of (5.2)-(5.3) are asymptotically flat in the sense of (5.4). Namely, the initial values α are in the stable manifold E^s of $x = 0$, defined by

$$E^s = \{\alpha \in \mathbb{R}^3 |\ x(s,\alpha) \to 0 \text{ for } s \to \infty\}; \tag{5.5}$$

(b) The solutions $x(s,\alpha)$ are near the Schwarzschild solution:

$$|x_1 - e^s u_0'|, \quad |x_2 + 1 - e^{v_0}|, \quad |x_3| \ll 1, \tag{5.6}$$

where u_0, v_0 are as in (4.7).

In fact, by (4.7) and (5.1) we can see that all Schwarzschild solutions lie on the line

$$L = \{(x_1, x_2, 0) |\ x_1 = x_2,\ x_1, x_2 > 0\}. \tag{5.7}$$

In particular, the line L is on the stable manifold E^s of (5.5).

Step 3. Stable manifold E^s. The equations (5.2) can be written as

$$\dot{x} = Ax + O(|x|^2),$$

where

$$A = \begin{pmatrix} 0 & -1 & 2 \\ 0 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}. \tag{5.8}$$

The dimension of the stable manifold E^s is the number of negative eigenvalues of the matrix A. It is easy to see that the eigenvalues of A are given by

$$\lambda_1 = -1, \quad \lambda_2 = -1, \quad \lambda_3 = 2.$$

Hence, the dimension of E^s is two:

$$\dim E^s = 2.$$

Consequently, the initial value α of an asymptotically flat solution has only two independent components due to $\alpha \in E^s$, which is of two dimensional. Namely, we arrive at the following conclusion.

Physical Conclusion 5.1. *In the gravitation formula (4.5) there are two free parameters to be determined by experiments (or by astronomical measurements).*

In fact, the two free parameters will be determined by the Rubin rotational curve and the repulsive property of gravity at large distance.

Step 4. Local expression of E^s. In order to derive the asymptotic property of the gravitational force F of (4.5), we need to derive the local expression of the stable manifold E^s near $x = 0$. Since the tangent space of E^s at $x = 0$ is spanned by the two eigenvectors $(1, 1, 0)^t$ and $(1, -1, -1)^t$ corresponding to the two negative eigenvalues $\lambda_1 = \lambda_2 = -1$, the coordinate vector $(0, 0, 1)$ of x_3 is not contained in E^s. This implies that the stable manifold can be expressed near $x = 0$ in the form

$$x_3 = h(x_1, x_2). \tag{5.9}$$

Inserting the Taylor expansion for (5.9) into (5.2), and comparing the coefficients, we derive the following local expression of (5.9) of the stable manifold function:

$$h(x_1, x_2) = -\frac{1}{2}x_1 + \frac{1}{2}x_2 + \frac{1}{16}x_1^2 - \frac{1}{16}x_2^2 + O(|x|^3). \tag{5.10}$$

Inserting (5.9)-(5.10) into the first two equations of (5.2), we deduce that

$$\begin{aligned} x_1' &= -x_1 - \frac{1}{8}x_1^2 - \frac{1}{8}x_2^2 - \frac{3}{4}x_1x_2 + O(|x|^3), \\ x_2' &= -x_2 + \frac{1}{4}x_1^2 - x_2^2 - \frac{1}{4}x_1x_2 + O(|x|^3). \end{aligned} \tag{5.11}$$

The system (5.11) is the system (5.2) restricted on the stable manifold E^s. Hence, its asymptotic behavior represents that of the interaction force F in (4.5).

Step 5. Phase diagram of system (5.11). In order to obtain the phase diagram of (5.11) near $x = 0$, we consider the ratio: $x_2'/x_1' = dx_2/dx_1$. Omitting the terms $O(|x|^3)$, we infer from (5.11) that

$$\frac{dx_2}{dx_1} = \frac{x_2 + x_2^2 + \frac{1}{4}x_2x_1 - \frac{1}{4}x_1^2}{x_1 + \frac{1}{8}x_2^2 + \frac{3}{4}x_2x_1 + \frac{1}{8}x_1^2}. \tag{5.12}$$

Let k be the slope of an orbit reaching to $x = 0$:

$$k = \frac{x_2}{x_1} \quad \text{as} \quad (x_2, x_1) \to 0.$$

Then (5.12) can be expressed as

$$k = \frac{k + k^2x_1 + \frac{1}{4}kx_1 - \frac{1}{4}x_1}{1 + \frac{1}{8}k^2x_1 + \frac{3}{4}kx_1 + \frac{1}{8}x_1},$$

which leads to the equation

$$k^3 - 2k^2 - k + 2 = 0.$$

This equation has three solutions:

$$k = \pm 1, \quad k = 2,$$

giving rise to three line orbits:

$$x_2 = x_1, \quad x_2 = 2x_1, \quad x_2 = -x_1,$$

which divide the neighborhood of $x = 0$ into six invariant regions. It is clear that all physically meaningful orbits can only be in the following three regions:

$$\begin{aligned} &\Omega_1 = \left\{ -x_2 < x_1 < \frac{1}{2}x_2, \ x_2 > 0 \right\}, \\ &\Omega_2 = \left\{ \frac{1}{2}x_2 \le x_1 \le x_2, \ x_2 > 0 \right\}, \\ &\Omega_3 = \{x_2 < x_1, \ x_2 > 0\}. \end{aligned} \tag{5.13}$$

On the positive x_2-axis (i.e. $x_1 = 0, x_2 > 0$), which lies in Ω_1, the equations (5.11) take the form

$$\begin{aligned} x_1' &= -\frac{1}{8}x_2^2 + O(|x|^3), \\ x_2' &= -x_2 - x_2^2 + O(|x|^3). \end{aligned}$$

It is easy to show that the orbits in Ω_1 with $x_1 > 0$ will eventually cross the x_2-axis. Thus, using the three invariant sets in (5.13), we obtain the phase diagram of (5.11) on $x_2 > 0$ as shown in Figure 5.1. In this diagram, we see that, the orbits in Ω_2 and Ω_3 will not cross the x_2-axis, but these in Ω_1 with $x_1 > 0$ will do.

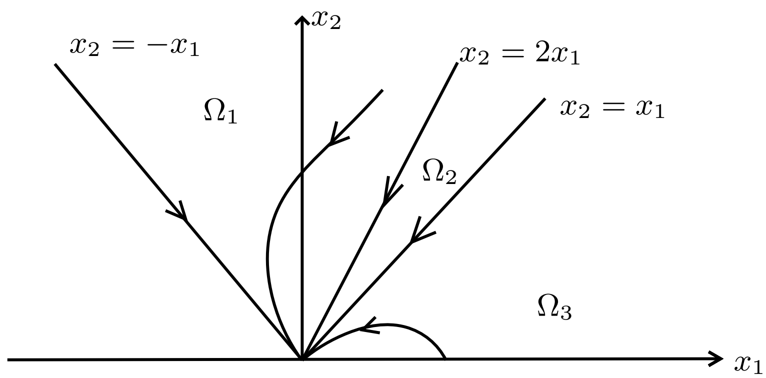

FIGURE 5.1. Only the orbits on Ω_1 with $x_1 > 0$ will eventually cross the x_2-axis, leading to the sign change of x_1, and to a repelling gravitational force corresponding to dark energy.

Step 6. Asymptotic repulsion theorem of gravity. We now derive an asymptotic repulsion theorem of gravity, based on the phase diagram in Figure 5.1. In fact, by (4.4) and (4.5), the gravitational force F reads as

$$F = -\frac{mc^2}{2}e^u u'. \tag{5.14}$$

It is known that

$$\begin{aligned} &F < 0 &&\text{represents attraction,} \\ &F > 0 &&\text{represents repelling.} \end{aligned}$$

Hence, by $x_1 = ru'(r)$ and (5.14), the phase diagram shows that an orbit in Ω_1, starting with $x_1 > 0$, will cross the x_2-axis, and the sign of x_1 changes from positive to negative, leading consequently to a repulsive gravitational force F. Namely, we have obtained the following theorem.

Theorem 5.2 (Asymptotic Repulsion of Gravitation)**.** *For a central gravitational field, the following assertions hold true:*

1) *The gravitational force F is given by (4.5), and is asymptotic zero:*

$$F \to 0 \quad \text{if} \quad r \to \infty. \tag{5.15}$$

2) *If the initial value α in (5.3) is near the Schwarzschild solution (4.7) with $0 < \alpha_1 < \alpha_2/2$, then there exists a sufficiently large r_1 such that the gravitational force F is repulsive for $r > r_1$:*

$$F > 0 \quad \text{for} \quad r > r_1. \tag{5.16}$$

We remark that Theorem 5.2 is valid provided the initial value α is small because the diagram given by Figure 5.1 is in a neighborhood of $x = 0$. However, all physically meaningful central fields satisfy the condition (note that any a black hole is enclosed by a huge quantity of matter with radius $r > 0 \gg 2MG/c^2$). In fact, the Schwarzschild initial values are as

$$x_1(r_0) = x_2(r_0) = \frac{\delta}{1-\delta}, \qquad \delta = \frac{2MG}{c^2 r_0}. \tag{5.17}$$

For example see (4.5), where the δ-factors are of the order $\delta \leq 10^{-1}$, sufficient for the requirements of Theorem 5.2.

The most important cases are for galaxies and clusters of galaxies. For these two types of astronomical objects, we have

$$\begin{aligned} &\text{galaxy}: && M = 10^{11} M_\odot, && r_0 = 3 \times 10^5 \text{ly},\\ &\text{cluster of galaxies}: && M = 10^{14} M_\odot, && r_0 = 3 \times 10^6 \text{ly}. \end{aligned}$$

Thus the δ-factors are

$$\text{galaxies } \delta = 10^{-7}, \qquad \text{cluster of galaxies } \delta = 10^{-5}. \tag{5.18}$$

In fact, the dark energy phenomenon is mainly evident between galaxies and between clusters of galaxies. Hence, (5.18) shows that Theorem 5.2 is valid for both central gravitational fields of galaxies and clusters of galaxies. The asymptotic repulsion of gravity plays the role to stabilize the large scale homogeneous structure of the Universe.

6. Simplified Gravitational Interaction Formula

We have shown that all four fundamental interactions are layered. Namely, each interaction has distinct attracting and repelling behaviors in different scales and levels. The dark matter and dark energy represent the layered property of gravity.

In this section, we simplify the gravitational formula (4.8) to clearly exhibit the layered phenomena of gravity.

In (4.8) the field functions u and v can be approximatively replaced by the Schwarzschild solution (4.6). Since $2MG/c^2 r$ is very small for $r > r_0$ as indicted in (4.5) and (5.18), the formula (4.8) can be expressed as

$$F = mMG\left[-\frac{1}{r^2} - \frac{r}{\delta r_0}\phi''\right], \qquad r > r_0. \tag{6.1}$$

By the field equation (4.2), we have

$$R = \Phi \quad \text{for } r > r_0, \tag{6.2}$$

where R is the scalar curvature, and

$$\Phi = g^{\mu\nu} D_{\mu\nu}\phi = e^{-v}\left[-\phi'' + \frac{1}{2}(u'-v')\phi' + \frac{2}{r}\varphi'\right].$$

In view of (6.2), we obtain that

$$\phi'' = -e^{v}R + \frac{2}{r}\phi' + \frac{1}{2}(u'-v')\phi'$$

Again by the Schwarzschild approximation, we have

$$\phi'' = \left(\frac{2}{r} + \frac{\delta r_0}{r^2}\right)\phi' - R. \tag{6.3}$$

Integrating (6.3) and omitting $e^{\pm\delta r_0/r}$, we derive that

$$\phi' = -r^2\left[\varepsilon + \int r^{-2} R dr\right],$$

where ε is a constant. Thus (6.1) can be rewritten as

$$F = mMG\left[-\frac{1}{r^2} + \frac{r}{\delta r_0}R + \left(1 + \frac{2r}{\delta r_0}\right)\left(\varepsilon r + r\int \frac{R}{r^2}dr\right)\right]. \tag{6.4}$$

The solutions of (5.2) can be Taylor expanded. Hence by (5.1) we see that

$$u'(r) = \frac{1}{r^2}\sum_{k=0}^{\infty} a_k (r-r_0)^k.$$

By (5.14), the gravitational force F takes the following form

$$F = \frac{1}{r^2}\sum_{k=0}^{\infty} b_k r^k, \qquad b_0 = -mMG.$$

In view of (6.4), it implies that R can be expanded as

$$R = \frac{\varepsilon_0}{r} - \varepsilon_1 + O(r),$$

and by Physical Conclusion 5.1, ε_0 and ε_1 are two to-be-determined free parameters. Inserting R into (6.4) we obtain that

$$F = mMG\left[-\frac{1}{r^2} - \frac{k_0}{r} + \varphi(r)\right] \qquad \text{for } r > r_0. \tag{6.5}$$

where $k_0 = \frac{1}{2}\varepsilon_0$, and

$$\varphi(r) = \varepsilon_1 + k_1 r + O(r), \qquad k_1 = \varepsilon + \frac{\varepsilon_1}{\delta r_0}.$$

The nature of dark matter and dark energy suggests that

$$k_0 > 0, \qquad k_1 > 0.$$

Based on Theorem 5.2, $\varphi(r) \to 0$ as $r \to \infty$, and (6.5) can be further simplified as in the form for $r_0 < r < r_1$,

$$F = mMG\left[-\frac{1}{r^2} - \frac{k_0}{r} + k_1 r\right], \tag{6.6}$$

where k_0 and k_1 will be determined by the Rubin rotational curve and the astronomical data for clusters of galaxies in the next section, where we obtain that

$$k_0 = 4 \times 10^{-18}\text{km}^{-2}, \qquad k_1 = 10^{-57}\text{km}^{-3}. \tag{6.7}$$

The formula (6.6) is valid only in the interval

$$r_0 < r < r_1,$$

and r_1 is the distance at which F changes its sign:

$$F(r_1) = 0.$$

Both observational evidence on dark energy and Theorem 5.2 show that the distance r_1 exists. The formula (6.6) with (6.7) clearly displays the layered property of gravity: attracting at short distance and repelling at large distance.

7. Nature of Dark Matter and Dark Energy

As mentioned in Section 3, both dark matter and dark energy are a property of gravitational effect, reflected in two aspects, which will be addressed in detail in this section:

a) spatially geometrical structure, and
b) gravitational attracting and repelling as in (6.6).

Space curved energy and negative pressure

Gravitational potential causes space curvature and the spherical structure of the Universe, and displays two types of energies: a) dark matter contributed by the curvature of space, and b) dark energy generated by the dual gravitational potential in (3.15). We address each type of energy as follows.

1) *Dark matter: the space curved energy.* In (Ma and Wang, 2014a, Section 6.3), we have introduced the space curved energy M_{total} for the 3D spherical Universe as follows:

$$M_{\text{total}} = \frac{3\pi}{2} M, \qquad M \text{ is the observed mass in the hemisphere.}$$

Now, we consider a galaxy with an observed mass M_Ω. In (Ma and Wang, 2014a, Section 6.3), we have shown that the space curved energy $M_{\text{total};\Omega}$ is

$$M_{\text{total};\Omega} = \frac{V_\Omega}{|\Omega|} M_\Omega, \tag{7.1}$$

where Ω is the domain occupied by the galaxy, V_Ω and $|\Omega|$ are the volumes of curved and flat Ω. V_Ω contain two parts:

$$V_\Omega = \text{cosmic spherical } V_\Omega^1 + \text{local bump } V_\Omega^2. \tag{7.2}$$

It is known that

$$V_\Omega^1 = \frac{3\pi}{4} |\Omega|.$$

For V_Ω^2, we propose that

$$V_\Omega^2 = \pi^2 r_0^3, \qquad r_0 \text{ the galaxy radius.}$$

In fact, the formula is precise for the galaxy nucleus.

By $|\Omega| = \frac{4}{3}\pi r_0^3$, we infer from (7.1) and (7.2) that

$$M_{\text{total};\Omega} = \frac{3\pi}{2} M_\Omega, \tag{7.3}$$

which gives rise to the relation between the masses of dark matter and observable matter.

2) *Dark energy: the dual gravitational potential.* The static universe is described by the stationary solution of (3.11)-(3.12), which is given by (3.13)-(3.14). In the solution a negative pressure presents, which prevents galaxies and clusters of galaxies from gravitational contraction to form a void universe, and maintains the homogeneous distribution of the Universe. The negative pressure contains two parts:

$$p = -\frac{1}{3}\rho c^2 - \frac{c^2}{24\pi G}\varphi \quad \text{(see (3.13))}, \tag{7.4}$$

where the first term is contributed by the observable energy, and the second term is the dark energy generated by the dual gravitational potential φ; see also (3.19).

By the Blackhole Theorem, (Ma and Wang, 2014a, Theorem 4.1), black holes are incompressible in their interiors. Hence, in (7.4) the negative pressure

$$p = -\frac{1}{3}\rho c^2, \tag{7.5}$$

is essentially the incompressible pressure of the black hole generated by the normal energy.

By the cosmology theorem, (Ma and Wang, 2014a, Theorem 6.2), the Universe is a $3D$ sphere with a blackhole radius. However, the CMB and the $WMAP$ measurements manifest that the cosmic radius R is greater than the blackhole radius of normal energy. By (3.19), the deficient energy is compensated by the dual gravitational potential, i.e. by the second term of (7.4).

Attraction and repulsion of gravity

Based on Theorem 5.2, gravity possesses additional attraction and repelling to the Newtonian gravity, as shown in the revised gravitational formula:

$$F = mMG\left(-\frac{1}{r^2} - \frac{k_0}{r} + k_1 r\right). \tag{7.6}$$

By using this formula we can explain the dark matter and dark energy phenomena. In particular, based on the Rubin rotational curve and astronomical data, we can determine an approximation of the parameters k_0 and k_1 in (7.6).

1) *Dark matter: the additional attracting.* Let M_r be the total mass in the ball with radius r of galaxy, and V be the constant galactic rotational velocity. By the force equilibrium, we infer from (7.6) that

$$\frac{V^2}{r} = M_r G\left(\frac{1}{r^2} + \frac{k_0}{r} - k_1 r\right), \tag{7.7}$$

which implies that

$$M_r = \frac{V^2}{G}\frac{r}{1 + k_0 r - k_1 r^3}. \tag{7.8}$$

The mass distribution (7.8) is derived based on both the Rubin rotational curve and the revised formula (7.6). In the following we show that the mass distribution M_r fits the observed data.

We know that the theoretic rotational curve given by Figure 2.1 (b) is derived by using the observed mass M_{ob} and the Newton formula

$$F = -\frac{mM_{ob}G}{r^2}.$$

Hence, to show that $M_r = M_{ob}$, we only need to calculate the rotational curve v_r by the Newton formula from the mass M_r, and to verify that v_r is consistent with the theoretic curve. To this end, we have

$$\frac{v_r^2}{r} = \frac{M_r G}{r^2},$$

which, by (7.8), leads to

$$v_r = \frac{V}{\sqrt{1 - k_0 r - k_1 r^2}}.$$

As $k_1 \ll k_0 \ll 1, v_r$ can be approximatively written as

$$v_r = V(1 - \frac{1}{2}k_0 r + \frac{1}{4}k_0^2 r^2). \tag{7.9}$$

It is clear that the rotational curve described by (7.9) is consistent with the theoretic rotational curve as illustrated by Figure 2.1 (b). Therefore, it implies that

$$M_r = M_{ob}. \tag{7.10}$$

The facts of (7.7) and (7.10) are strong evidence to show that the revised formula (7.6) is in agreement with the astronomical observations.

We now determine the constant k_0 in (7.6). According to astronomical data, the average mass M_{r_1} and radius r_1 of galaxies are about

$$\begin{aligned} & M_{r_1} = 10^{11} M_\odot \cong 2 \times 10^{41}\text{Kg}, \\ & r_1 = 10^4 \sim 10^5 \text{pc} \cong 10^{18}\text{Km}. \end{aligned} \tag{7.11}$$

The observations show that the constant velocity V in the Rubin rotational curve is about $V = 300km/s$. Then we have

$$\frac{V^2}{G} = 10^{24}\text{kg/km}$$

Based on physical considerations,

$$k_0 \gg k_1 r_1 \quad (r_1 \text{ as in } (7.11)). \tag{7.12}$$

Then, we deduce from (7.8) that

$$k_0 = \frac{V^2}{G}\frac{1}{M_{r_1}} - \frac{1}{r_1} = 4 \times 10^{-18} K_m^{-1}. \tag{7.13}$$

We can explain the dark matter by the revised formula (7.6). As the matter distribution M_r is in the form

$$M_r = \frac{V^2}{G}\frac{r}{1 + k_0 r},$$

then the Rubin law holds true. In addition, the revised gravitation produces an excessive mass $\widetilde{M}$ as

$$\widetilde{M} = M_{\text{T}} - M_{r_1} = \frac{V^2}{G} r_1 - \frac{V^2}{G}\frac{r_1}{1 + k_0 r_1},$$

where $M_1 = V^2 r_1 / G$ is the Newton theoretic value of the total mass. Hence we have

$$\frac{\widetilde{M}}{M_{\text{T}}} = \frac{k_0 r_1}{1 + k_0 r_1} = \frac{4}{5} \quad \text{or} \quad \frac{\widetilde{M}}{M_{r_1}} = 5.$$

Namely, the additional mass $\widetilde{M}$ is four time the visible matter $M_{r_1} = M_T - \widetilde{M}$. Thus, it gives an explanation for the dark matter.

We remark that the ratio $\widetilde{M}/M_{\text{ob}} = 5$ is essentially the same as in (7.3). It shows that the dark matter is a gravitational effect, reflected in both the space curvature and the additional gravitational attraction.

2) *Dark energy: asymptotic repulsion of gravity.* If gravity is always attracting as given by the Newton formula, then the cosmic homogeneity is unstable. In fact, It is known that the average mass M and distance for the clusters of galaxies are as

$$\begin{aligned} &M = 10^{14} M_{\odot} \cong 10^{44}\text{Kg} \\ &r \cong 10^8\text{pc} \cong 10^{20} \sim 10^{21}\text{Km}. \end{aligned} \tag{7.14}$$

Then the Newton gravitation between two clusters of galaxies is

$$F = -\frac{M^2 G}{r^2} \cong 10^{29} N = 10^{28}\text{Kg}. \tag{7.15}$$

However, astronomical observations indicate that no gravitational interaction between clusters of galaxies. The Universe is isotropic, therefore no rotation to balance the huge force of (7.15) in the clusters.

Thus, the new cosmology theorem, (Ma and Wang, 2014a, Theorem 6.2), suggests that gravity should be asymptotically repulsive. Theorem 5.2 offers a solid theoretic foundation for the property, based on which we derive the simplified gravitational force formula (7.6).

Now we consider the constant k_1 in (7.6). Due to the astronomical fact that no gravitational force between clusters of galaxies, we have

$$F(\bar{r}) = 0, \qquad \bar{r} = \text{the average distance between galactic clusters.}$$

By (7.14), we take

$$\bar{r} = \sqrt{\frac{2}{5}} \times 10^{20}\text{km}. \tag{7.16}$$

Then we deduce from (7.6) that

$$k_1 \bar{r} - \frac{k_0}{\bar{r}} - \frac{1}{\bar{r}^2} = 0,$$

which, by (7.13) and (7.16), leads to

$$k_1 = 10^{-57}\text{km}^{-3}. \tag{7.17}$$

In summary, we conclude that the dark matter and dark energy are essentially gravitational effect generated by the gravitational potential field $g_{\mu\nu}$, its dual field Φ_μ and their nonlinear interactions. They can be regarded as the gravitational field energy caused by $g_{\mu\nu}$ and Φ_μ.

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