

EULER TOTIENT OF SUBFACTOR PLANAR ALGEBRAS

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ABSTRACT. We define a notion of Euler totient for any irreducible subfactor planar algebra, using the Möbius function of the biprojection lattice. We prove that if it is nonzero then there is a minimal 2-box projection generating the identity biprojection. We deduce a bridge between combinatorics and representations in finite groups theory. We also get an alternative result at depth 2.

1. INTRODUCTION

The usual Euler's totient function $\varphi(n)$ counts the number of positive integers up to n that are relatively prime to n . For any finite group G , let $\mathcal{L}(G)$ be its subgroup lattice and μ the Möbius function of $\mathcal{L}(G)$. By Crosscut Theorem and inclusion-exclusion principle,

$$\varphi(G) := \sum_{H \in \mathcal{L}(G)} \mu(H, G) |H|$$

is the cardinal of $\{g \in G \mid \langle g \rangle = G\}$. Then $\varphi(G)$ is nonzero iff G is cyclic, and $\varphi(C_n) = \varphi(n)$. This paper generalizes one way of this equivalence to the irreducible subfactor planar algebras. Let \mathcal{P} be an irreducible subfactor planar algebra and μ the Möbius function of its biprojection lattice $[e_1, id]$. The Euler totient of \mathcal{P} is

$$\varphi(\mathcal{P}) := \sum_{b \in [e_1, id]} \mu(b, id) |b : e_1|$$

Theorem 1.1. *If $\varphi(\mathcal{P})$ is nonzero then \mathcal{P} is w -cyclic (i.e. there is a minimal 2-box projection generating the identity biprojection).*

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By applying the above theorem to $\mathcal{P} = \mathcal{P}(R^G \subset R)$ for any finite group G , we get that if the “dual” Euler totient

$$\hat{\varphi}(G) := \sum_{H \in \mathcal{L}(G)} \mu(1, H) |G : H|$$

is nonzero then G has a faithful irreducible complex representation. It is a weak dual version of the initial group result. As a general application, we get a non-trivial upper-bound for the minimal number of minimal central projections generating the identity biprojection. By applying this result to any finite group G , we deduce a non-trivial upper-bound for the minimal number of irreducible complex representation generating (for \oplus and \otimes) the left regular representation. It is a bridge between combinatorics and representations in finite groups theory. We finally prove an alternative equivalence for the irreducible subfactor planar algebras of depth 2, involving the central biprojection lattice, and so the normal subgroup lattice for the dual group case.

Because this paper is mainly intended to people in subfactors theory we will start by some basics on lattice theory, and we just refer to [4] for the basics on subfactor planar algebras.

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2. BASICS ON LATTICE THEORY

A lattice (L, \wedge, \vee) is a poset L in which every two elements a, b have a unique supremum (or join) $a \vee b$ and a unique infimum (or meet) $a \wedge b$. Let G be a finite group. The set of subgroups $K \subseteq G$ forms a lattice, denoted by $\mathcal{L}(G)$, ordered by \subseteq , with $K_1 \vee K_2 = \langle K_1, K_2 \rangle$ and $K_1 \wedge K_2 = K_1 \cap K_2$. A sublattice of (L, \wedge, \vee) is a subset $L' \subseteq L$ such that (L', \wedge, \vee) is also a lattice. Let $a, b \in L$ with $a \leq b$, then the interval $[a, b]$ is the sublattice $\{c \in L \mid a \leq c \leq b\}$. Any finite lattice admits a minimum and a maximum, denoted by $\hat{0}$ and $\hat{1}$. Atoms (resp.

coatoms) are minimum (resp. maximum) elements in $L \setminus \{\hat{0}\}$ (resp. $L \setminus \{\hat{1}\}$). The top interval of a finite lattice L is the interval $[t, \hat{1}]$ with t the meet of all the coatoms. The height of a finite lattice L is the greatest length of a (strict) chain. A lattice is distributive if the join and meet operations distribute over each other. A distributive lattice is called boolean if any element b admits a unique complement b^c (i.e. $b \wedge b^c = \hat{0}$ and $b \vee b^c = \hat{1}$). The subset lattice of $\{1, 2, \dots, n\}$, with union and intersection, is called the boolean lattice \mathcal{B}_n of rank n . Any finite boolean lattice is isomorphic to some \mathcal{B}_n .

Lemma 2.1. *The top interval of a finite distributive lattice is boolean.*

Proof. See [6, items a-i p254-255] which uses Birkhoff's representation theorem (a finite lattice is distributive iff it embeds into some \mathcal{B}_n). \square

Remark 2.2. *A finite lattice is boolean if and only if it is uniquely atomistic, i.e. every element can be written uniquely as a join of atoms. It follows that if $[a, b]$ and $[c, d]$ are intervals in a boolean lattice, then*

$$[a, b] \vee [c, d] := \{k \vee k' \mid k \in [a, b], k' \in [c, d]\},$$

is the interval $[a \vee c, b \vee d]$.

See [6] for more details on lattice basics.

3. EULER TOTIENT

We define a notion of Euler totient on the irreducible subfactor planar algebras as an extension of the usual Euler's totient function on the natural numbers.

Definition 3.1. *The Möbius function μ of a finite poset P is defined inductively as follows. For $a \leq b$*

$$\mu(a, b) = \begin{cases} 1 & \text{if } a = b, \\ -\sum_{c \in (a, b)} \mu(c, b) & \text{otherwise.} \end{cases}$$

The following result can be seen as a boolean representation for the Möbius function of a finite lattice.

Theorem 3.2 (Crosscut Theorem). *Let L be a finite lattice and a_1, \dots, a_n its coatoms. Consider the (order-reversing) map $m : \mathcal{B}_n \rightarrow L$*

$$m(I) = \begin{cases} \hat{1} & \text{if } I = \emptyset, \\ \bigwedge_{i \in I} a_i & \text{otherwise.} \end{cases}$$

Then for any $a \in L$,

$$\mu(a, \hat{1}) = \sum_{I \in m^{-1}(\{a\})} (-1)^{|I|}$$

Proof. Immediate from [6, Corollary 3.9.4]. \square

Definition 3.3. *The Euler totient of an irreducible subfactor planar algebra \mathcal{P} , with biprojection lattice $[e_1, id]$ and Möbius function μ , is*

$$\varphi(\mathcal{P}) := \varphi(e_1, id) := \sum_{b \in [e_1, id]} \mu(b, id) |b : e_1|$$

Proposition 3.4. *The Euler totient $\varphi(\mathcal{P})$ is equal to*

$$|t : e_1| \cdot \varphi(t, id)$$

with $[t, id]$ the top interval of $[e_1, id]$.

Proof. If $b \notin [t, id]$ then $\mu(b, id) = 0$ by Crosscut Theorem 3.2 because $m^{-1}(\{b\}) = \emptyset$. Finally, for $b \in [t, id]$, $|b : e_1| = |b : t| \cdot |t : e_1|$. \square

Remark 3.5. *For $n = \prod_i p_i^{n_i}$ then $\varphi(\mathcal{P}(R \subseteq R \rtimes C_n))$ is equal to*

$$\prod_i p_i^{n_i - 1} \cdot \prod_i (p_i - 1)$$

which is the usual Euler's totient $\varphi(n)$. Thus, we can see $\varphi(\mathcal{P})$ as an extension from the natural numbers to the subfactor planar algebras.

Lemma 3.6. *The Euler totient of a finite group G ,*

$$\varphi(G) := \sum_{H \in \mathcal{L}(G)} \mu(H, G) |H| = \varphi(\mathcal{P}(R \subseteq R \rtimes G)),$$

is the cardinal of $\{g \in G \mid \langle g \rangle = G\}$.

Proof. By Crosscut Theorem 3.2 and inclusion-exclusion principle, $\varphi(G) = |G \setminus \bigcup M_i|$, with M_1, \dots, M_n the maximal subgroups of G . \square

Corollary 3.7. *A finite group G is cyclic iff $\varphi(G)$ is nonzero.*

We will generalize one way of this result to subfactor planar algebras, and deduce a weak dual version involving irreducible representations.

4. MAIN RESULT

In this section, we prove that an irreducible subfactor planar algebra with a nonzero Euler totient is w-cyclic.

Definition 4.1 ([5]). *A planar algebra \mathcal{P} is weakly cyclic (or w-cyclic) if it satisfies one of the following equivalent assertion:*

- $\exists u \in \mathcal{P}_{2,+}$ minimal projection such that $\langle u \rangle = id$,
- $\exists p \in \mathcal{P}_{2,+}$ minimal central projection such that $\langle p \rangle = id$.

The notation $\langle a \rangle$ means the biprojection generating by $a > 0$.

Theorem 4.2. *Let \mathcal{P} be an irreducible subfactor planar algebra. If the Euler totient $\varphi(\mathcal{P})$ is nonzero, then \mathcal{P} is w-cyclic.*

Proof. Let p_1, \dots, p_r be the minimal central projections of $\mathcal{P}_{2,+}$. Consider the sum

$$S(i) := \sum_{b \in [e_1, id]} \mu(b, id) tr(bp_i).$$

Let b_1, \dots, b_n be the coatoms of $[e_1, id]$, by Crosscut Theorem 3.2

$$S(i) = \sum_{b \in [e_1, id]} \sum_{\beta \in m^{-1}(\{b\})} (-1)^{|\beta|} tr(bp_i) = \sum_{\beta \in \mathcal{B}_n} (-1)^{|\beta|} tr(m(\beta)p_i)$$

Recall that the map m (defined in Theorem 3.2) is order-reversing and the image of the atoms of \mathcal{B}_n are the coatoms of $[e_1, id]$. Let A_i be the set of atoms α of \mathcal{B}_n satisfying $p_i \leq m(\alpha)$, and B_i the set of atoms not in A_i . Let α_i (resp. β_i) be the join of all the elements of A_i (resp. B_i).

Claim: For $\alpha \in \mathcal{B}_n$, $p_i \leq m(\alpha) \Leftrightarrow \alpha \in [\alpha_i, \hat{1}]$.

Proof: Just observe that $p_i \leq \bigwedge_{j \in \alpha} b_j$ if and only if $\forall j \in \alpha$, $p_i \leq b_j$. ■
Now by Remark 2.2, we have

$$\mathcal{B}_n = [\emptyset, \alpha_i] \vee [\emptyset, \beta_i] = \bigsqcup_{\alpha \in [\emptyset, \alpha_i]} \alpha \vee [\emptyset, \beta_i],$$

Let the following sum

$$T(i) := \sum_{\beta \in [\emptyset, \beta_i]} (-1)^{|\beta|} tr(m(\beta)p_i)$$

For any $\alpha \in [\emptyset, \alpha_i]$ and $\beta \in [\emptyset, \beta_i]$, then $(-1)^{|\alpha \vee \beta|} = (-1)^{|\alpha|} (-1)^{|\beta|}$ and $m(\alpha \vee \beta)p_i = m(\alpha)p_i \wedge m(\beta)p_i = m(\beta)p_i$. So we get that

$$S(i) = \sum_{\alpha \in [\emptyset, \alpha_i]} (-1)^{|\alpha|} T(i) = T(i) \cdot (1 - 1)^{|A_i|}.$$

Claim: \mathcal{P} is w-cyclic if and only if $\exists i$ with $|A_i| = 0$.

Proof: First if $\exists i$ such that $|A_i| = 0$, then $p_i \not\leq b$ (and so $\langle p_i \rangle \not\leq b$) for any coatom b of $[e_1, id]$, hence $\langle p_i \rangle = id$. Next if \mathcal{P} is w-cyclic, $\exists i$ such that $\langle p_i \rangle = id$, then for any coatom b of $[e_1, id]$, $b \not\geq p_i$, so $|A_i| = 0$. ■
If \mathcal{P} is not w-cyclic, then $\forall i$ $|A_i| \neq 0$, so $S(i) = 0$; but $|b : e_1| = tr(b)/tr(e_1)$, $tr(b) = \sum_i tr(bp_i)$ and $tr(e_1) = \delta^{-2}$, so $\varphi(e_1, id) = \delta^2 \sum_{i=1}^r S(i) = 0$; the result follows. □

It is an extended generalization of [1, Theorem 3.21] and a purely combinatorial criterion for a subfactor planar algebra to be w-cyclic.

Remark 4.3. *The converse is false. For $G = M_2(4)$ the modular maximal-cyclic group (of order 16), the planar algebra $\mathcal{P} = \mathcal{P}(R^G \subset R)$ is w-cyclic, whereas $\varphi(\mathcal{P}) = 0$. This is not surprising because according to Proposition 3.4, $\varphi(e_1, id) \neq 0$ iff $\varphi(t, id) \neq 0$ with $[t, id]$ the top interval of $[e_1, id]$; and the bottom interval of $[1, M_2(4)]$ is $[1, C_2^2]$. Even if we assume that $t = e_1$, then the converse is still false: there are exactly two counter-examples of the form $\mathcal{P}(R^G \subset R)$ and index ≤ 100 , given by $G = D_8 \rtimes C_2^2$ or $D_8 \rtimes S_3$ (of order 64 and 96 respectively).*

Proposition 4.4. *Assume all the biprojections to be central. Then*

$$\varphi(\mathcal{P}) = \delta^2 \sum_{\langle p_i \rangle = id} tr(p_i)$$

with p_1, \dots, p_r be the minimal central projections of $\mathcal{P}_{2,+}$. It follows that the converse of Theorem 4.2 is obviously true in this case.

Proof. By Crosscut Theorem 3.2 and inclusion-exclusion principle. \square

By [5, Theorem 4.24], if all the biprojection are central and form a distributive lattice, then \mathcal{P} is w-cyclic, so by Proposition 4.4, $\varphi(\mathcal{P})$ is nonzero. We believe that the central assumption is unnecessary:

Conjecture 4.5. *If $[e_1, id]$ is distributive, then $\varphi(\mathcal{P}) \neq 0$.*

This conjecture reduces to the boolean case, and we expect more:

Question 4.6. *Assume $[e_1, id]$ to be boolean of rank $n + 1$. Is it true that $\varphi(\mathcal{P}) \geq \phi^n$ (with ϕ the golden ratio)?*

If this lower bound is correct, then it is optimal because it is realized by $\mathcal{T}\mathcal{L}\mathcal{J}(\sqrt{2}) \otimes \mathcal{T}\mathcal{L}\mathcal{J}(\phi)^{\otimes n}$.

5. APPLICATIONS

As for [5], we give many group theoretic translations of Theorem 4.2, and a non-trivial upper-bound, giving a bridge between combinatorics and representations in finite groups theory.

Definition 5.1. *The Euler totient of an interval of finite groups is*

$$\varphi(H, G) := \sum_{K \in [H, G]} \mu(K, G) |K : H| = \varphi(\mathcal{P}(R \rtimes H \subseteq R \rtimes G)).$$

Corollary 5.2. *There is $g \in G$ with $\langle Hg \rangle = G$ iff $\varphi(H, G)$ is nonzero.*

Proof. By Proposition 4.4, or directly by observing that $\varphi(H, G) |H| = |G \setminus \bigcup M_i|$, with M_1, \dots, M_n the coatoms of $[H, G]$, so that $\varphi(H, G)$ is the cardinal of $\{Hg \mid g \in G \text{ and } \langle Hg \rangle = G\}$. \square

Corollary 5.3. *The minimal cardinal for a generating set of a finite group G , is the minimal length ℓ for an ordered chain of subgroups*

$$\{e\} = H_0 < H_1 < \cdots < H_\ell = G$$

such that $\varphi(H_i, H_{i+1})$ is nonzero.

Proof. Immediate from Corollary 5.2 and $\langle Hg \rangle = \langle H, g \rangle$. \square

We generalize to planar algebras by a non-trivial upper-bound:

Theorem 5.4. *The minimal number r of minimal projections generating the identity biprojection (i.e. $\langle u_1, \dots, u_r \rangle = id$) is less than the minimal length ℓ for an ordered chain of biprojections*

$$e_1 = b_0 < b_1 < \cdots < b_\ell = id$$

such that $\varphi(b_i, b_{i+1})$ is nonzero.

Proof. Immediate from Theorem 4.2 and [5, Lemma 6.1]. \square

We deduce (weak) dual versions of Corollaries 5.2 and 5.3, giving the bridge between combinatorics and representations theory:

Definition 5.5. *The dual Euler totient of the interval $[H, G]$ is*

$$\hat{\varphi}(H, G) := \sum_{K \in [H, G]} \mu(H, K) |G : K| = \varphi(\mathcal{P}(R^G \subseteq R^H)).$$

Corollary 5.6. *For an interval of finite groups $[H, G]$, if the dual Euler totient $\hat{\varphi}(H, G)$ is nonzero then there is an irreducible complex representation of G such that $G_{(V^H)} = H$.*

Proof. It is the group theoretic reformulation of Theorem 4.2 for $\mathcal{P}(R^G \subseteq R^H)$, using [5, Theorem 6.10]. \square

In particular, if $H = 1$ and for $\hat{\varphi}(G) := \hat{\varphi}(1, G)$, we have:

Corollary 5.7. *A finite group G admits a faithful irreducible complex representation if its dual Euler totient $\hat{\varphi}(G)$ is nonzero.*

Corollary 5.8. *The minimal number of irreducible complex representations of G generating (with \oplus and \otimes) the left regular representation, is less than the minimal length ℓ for an ordered chain of subgroups*

$$\{e\} = H_0 < H_1 < \cdots < H_\ell = G$$

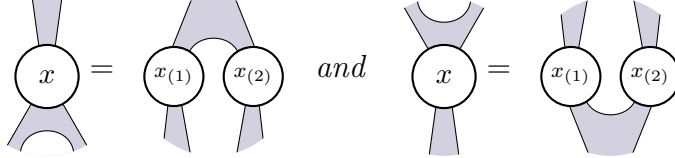
such that $\hat{\varphi}(H_i, H_{i+1})$ is nonzero.

Proof. By Theorem 5.4 and $\hat{\varphi}(H, G) = \varphi(\mathcal{P}(R^G \subseteq R^H))$. \square

6. ALTERNATIVE RESULT FOR THE DEPTH 2

We will prove an alternative equivalence in the irreducible depth 2 case, involving the central biprojection lattice.

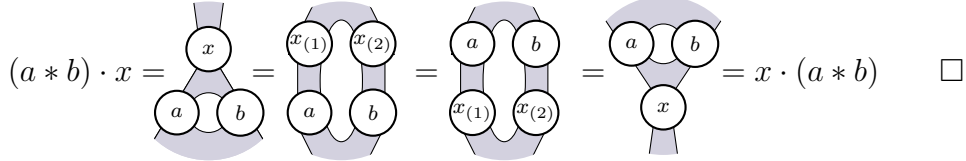
Theorem 6.1 (Splitting, [2] p39). *Let \mathcal{P} be an irreducible depth 2 subfactor planar algebra. Any element $x \in \mathcal{P}_{2,+}$ splits as follows:*



Note that $\Delta(x) = x_{(1)} \otimes x_{(2)}$ is the sumless Sweedler notation for the comultiplication of the corresponding Kac algebra.

Corollary 6.2. *If $a, b \in \mathcal{P}_{2,+}$ are central, then so is the coproduct $a * b$.*

Proof. This diagrammatic proof by splitting is due to Vijay Kodiyalam.



Corollary 6.3. *The set of central biprojections is a sublattice of the biprojection lattice.*

Proof. Let b_1 and b_2 be central biprojections. Then, $b_1 \wedge b_2$ is obviously central. Next, $b_1 \vee b_2$ is the range projection of $(b_1 * b_2)^{*k}$ for k sufficiently large, so is central by Corollary 6.2. \square

Let \mathcal{C} be the central biprojection lattice and $\mu_{\mathcal{C}}$ its Möbius function. Let the central Euler totient be

$$\varphi_{\mathcal{C}}(\mathcal{P}) := \sum_{b \in \mathcal{C}} \mu_{\mathcal{C}}(b, id) |b : e_1|.$$

By Crosscut Theorem 3.2 and inclusion-exclusion principle:

$$\varphi_{\mathcal{C}}(\mathcal{P}) = \delta^2 \sum_{\langle p_i \rangle = id} tr(p_i)$$

with p_1, \dots, p_r be the minimal central projections of $\mathcal{P}_{2,+}$.

Corollary 6.4. *Let \mathcal{P} be an irreducible subfactor planar algebra of depth 2. Then \mathcal{P} is w -cyclic if and only if $\varphi_{\mathcal{C}}(\mathcal{P})$ is nonzero.*

Let G be a finite group, $\mathcal{N}(G)$ its normal subgroup lattice and $\mu_{\mathcal{N}}$ the Möbius function of $\mathcal{N}(G)$. Let the dual normal Euler totient be

$$\hat{\varphi}_{\mathcal{N}}(G) = \sum_{H \in \mathcal{N}(G)} \mu_{\mathcal{N}}(1, H) |G : H|$$

Let V_1, \dots, V_r be equivalent class representatives of the irreducible complex representations of G . As a group theoretic reformulation of the above paragraph, we recover a formula extracted from [3, p97].

$$\hat{\varphi}_N(G) = \sum_{V_i \text{ faithful}} \dim(V_i)^2.$$

Corollary 6.5. *A finite group G has a faithful irreducible complex representation if and only if $\hat{\varphi}_N(G)$ is nonzero.*

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