A REMARK ON APPROXIMATION WITH POLYNOMIALS AND GREEDY BASES

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ABSTRACT. We investigate properties of the $m$-th error of approximation by polynomials with constant coefficients $D_m(x)$ and with modulus-constant coefficients $D^*_m(x)$ introduced by Berná and Blasco (2016) to study greedy bases in Banach spaces. We characterize when $\liminf_m D_m(x)$ and $\liminf_m D^*_m(x)$ are equivalent to $\|x\|$ in terms of the democracy and superdemocracy functions, and provide sufficient conditions ensuring that $\lim_m D^*_m(x) = \lim_m D_m(x) = \|x\|$, extending previous very particular results.

1. INTRODUCTION

Let $(X, \|\cdot\|)$ be a real Banach space and let $B = (e_n)_{n=1}^\infty$ be a semi-normalized (Schauder) basis of $X$ with biorthogonal functionals $(e^*_n)_{n=1}^\infty$, that is:

(i) There exist $a, b > 0$ such that $a \leq \|e_n\|, \|e_n^*\| \leq b$ for every $n \in \mathbb{N}$,

(ii) $e^*_k(e_n) = \delta_{kn}$ for every $k, n \in \mathbb{N}$,

(iii) The sequence of projections $P_m : X \rightarrow X$ given by

$$P_m(x) = \sum_{n=1}^m e^*_n(x) e_n, \quad x \in X$$

satisfy $\lim_n \|P_m(x) - x\| = 0$ for every $x \in X$. In this case, the basis constant of $B$ is

$$K_b := \sup_{m \in \mathbb{N}} \|P_m\| < \infty.$$ 

We say that $B$ is monotone whether $K_b = 1$.

Along the paper we will refer to every such $B$ simply as a basis. Of course, as $m$ increases $P_m(x)$ offers a good approximation of $x$ by linear combinations of $m$-elements of the basis, but it is natural to ask whether a suitable (and systematic) rearrangement can provide better convergence rates. A natural proposal is the Thresholding Greedy Algorithm (TGA) introduced by S. V. Konyagin and V. N. Temlyakov ([10]): given $x \in X$ we first consider the rearranging function $\rho : \mathbb{N} \rightarrow \mathbb{N}$ satisfying that if $j < k$ then either $|e^*_\rho(j)(x)| > |e^*_\rho(k)(x)|$ or $|e^*_\rho(j)(x)| = |e^*_\rho(k)(x)|$ and $\rho(j) < \rho(k)$. The $m$-th
greedy sum of $x$ is then

$$G_m(x) = \sum_{j=1}^{m} e_{\rho(j)}^*(x) e_{\rho(j)} = \sum_{k \in \Lambda_m(x)} e_k^*(x) e_k,$$

where $\Lambda_m(x) = \{ \rho(n) : n \leq m \}$ is the greedy set of $x$ with cardinality $m$. Related to this, S. V. Konyagin and V. N. Temlyakov defined in [10] the concepts of greedy and quasi-greedy bases.

**Definition 1.1.** We say that $\mathcal{B}$ is quasi-greedy if there exists a positive constant $C_q$ such that

$$\| x - G_m(x) \| \leq C_q \| x \|, \forall x \in X, \forall m \in \mathbb{N}.$$

P. Wojtaszczyk proved in [12] that quasi-greediness is equivalent to the convergence of the algorithm, that is, $\mathcal{B}$ is quasi-greedy if and only if

$$\lim_{m \to +\infty} \| x - G_m(x) \| = 0, \forall x \in X.$$

**Definition 1.2.** We say that $\mathcal{B}$ is greedy if there exists a positive constant $C$ such that

$$\| x - G_m(x) \| \leq C \sigma_m(x), \forall x \in X, \forall m \in \mathbb{N},$$

where

$$\sigma_m(x, \mathcal{B}) = \sigma_m(x) := \inf \left\{ \| x - \sum_{n \in A} a_n e_n \| : a_n \in \mathbb{R}, A \subset \mathbb{N}, |A| = m \right\}.$$

Konyagin and Temlyakov [10] showed that, although every greedy basis is quasigreedy, the converse does not hold (see also [11, Section 10.2]). They also characterize greedy bases as those which are unconditional and democratic. To define the last notion we have to introduce some notation. For each finite subset $A \subset \mathbb{N}$ and every scalar sequence $\varepsilon = (\varepsilon_n)$ with $|\varepsilon_n| = 1$ for each $n \in \mathbb{N}$ (from now on we will write $|\varepsilon| = 1$, for simplicity) let us denote

$$1_A := \sum_{n \in A} \varepsilon_n \quad \text{and} \quad 1_{\varepsilon A} := \sum_{n \in A} \varepsilon_n e_n.$$

As usual, $|A|$ stands for the cardinal of $A$. We then define the democracy functions as

$$h_l(m) = \inf_{|A|=m, |\varepsilon|=1} \| 1_{\varepsilon A} \|, \quad h_r(m) = \sup_{|A|=m, |\varepsilon|=1} \| 1_{\varepsilon A} \| \quad (m \in \mathbb{N}).$$

and the superdemocracy functions as

$$h^*_l(m) = \inf_{|A|=m, |\varepsilon|=1} \| 1_{\varepsilon A} \|, \quad h^*_r(m) = \sup_{|A|=m, |\varepsilon|=1} \| 1_{\varepsilon A} \| \quad (m \in \mathbb{N}).$$

**Definition 1.3.** We say that $\mathcal{B}$ is democratic (resp. superdemocratic) if there exists $C > 0$ such that $h_l(m) \leq C h^*_l(m)$ (resp. $h^*_r(m) \leq C h^*_r(m)$) for every $m \in \mathbb{N}$.

Another characterization of greedy bases was more recently provided by Ó. Blasco and the first author by means of the best $m$-th error in the approximation using polynomials of constant (resp. modulus-constant) coefficients:

$$\mathcal{D}_m(x, \mathcal{B}) = \mathcal{D}_m(x) = \inf \{ \| x - \alpha 1_A \| : \alpha \in \mathbb{R}, A \subset \mathbb{N}, |A| = m \}$$

$$\mathcal{D}_m^*(x, \mathcal{B}) = \mathcal{D}_m^*(x) = \inf \{ \| x - \alpha 1_{\varepsilon A} \| : \alpha \in \mathbb{R}, A \subset \mathbb{N}, |A| = m, |\varepsilon| = 1 \}$$

**Theorem 1.4.** [2 Corollary 1.8] Let $\mathcal{B}$ be a basis of a Banach space $X$. The following assertions are equivalent:

(i) $\mathcal{B}$ is greedy;

(ii) There is $C > 0$ such that $\| x - G_m(x) \| \leq C \mathcal{D}_m(x)$ for every $x \in X$ and $m \in \mathbb{N}$. 

(iii) There is $C > 0$ such that $\|x - D_m(x)\| \leq C D_m^*(x)$ for every $x \in \mathbb{X}$ and $m \in \mathbb{N}$.

The striking feature of this theorem compared to (1) is that, while $\lim_m \sigma_m(x) = 0$ for every $x \in \mathbb{X}$, the terms $D_m^*(x)$ and $D_m(x)$ do not necessarily converge to zero if $x \neq 0$. Indeed, we have the following examples:

$\triangleright$ [2, Theorem 3.2],[3, Theorem 1.4] If $\mathbb{X} = \mathbb{H}$ is a (separable) Hilbert space and $\mathcal{B}$ is an orthonormal basis, then

$$\lim_{m \to \infty} D_m(x) = \lim_{m \to \infty} D_m^*(x) = \|x\|, \quad \text{for every } x \in \mathbb{H}. \quad (2)$$

$\triangleright$ [2, Proposition 3.4] If $\mathbb{X} = l^p (1 < p < \infty)$ and $\mathcal{B}$ is the canonical basis, then

$$\lim_{m \to +\infty} D_m(1_B) = \lim_{m \to +\infty} D_m^*(1_B) = \|1_B\|, \quad \text{for every finite } B \subset \mathbb{N}. \quad (3)$$

In the present paper, we aim to delve into this aspect. Let us briefly explain the structure of the paper. In Section 2, we show that $D_m^*(x)$ and $D_m(x)$ do not converge to zero as $m \to +\infty$ for any $x \neq 0$. In Section 3, we prove the main result of the paper (Theorem 3.2), namely a characterization of those bases $\mathcal{B}$ for which there is a positive constant $c > 0$ such that

$$c \|x\| \leq \liminf_{m \to +\infty} D_m^*(x) \leq \limsup_{m \to +\infty} D_m^*(x) \leq \|x\| \quad \text{for every } x \in \mathbb{X},$$

in terms of the democracy and superdemocracy functions. We also provide a quite general condition ensuring that

$$\lim_{m \to +\infty} D_m^*(x) = \|x\| \quad \text{for every } x \in \mathbb{X}.$$

In Section 4, we deal with the notion of almost-greedy bases. We study how this property can be also characterized in terms of polynomials of constant or modulus-constant coefficients, extending a recent result of S. J. Dilworth and D. Khurana in [6].

Let us point out [11] as our basic reference for notation and fundamental results on greedy basis.

2. THE LIMIT OF ERRORS $D_m^*(x)$ AND $D_m(x)$ IS NONZERO

Since $D_m^*(x) \leq D_m(x) \leq \|x\|$ for every $m \in \mathbb{N}$ and every $x \in \mathbb{X}$, it is only necessary to study lower bounds of $D_m^*(x)$.

Proposition 2.1. Let $\mathcal{B} = (e_n)_{n=1}^\infty$ be a basis of a Banach space $\mathbb{X}$. Then, for every $x \in \mathbb{X}$

$$\frac{1}{4K_b} \sup_{n \in \mathbb{N}} |e_n^*(x)| \leq \liminf_{m \to +\infty} D_m^*(x).$$

Proof. Let $x \in \mathbb{X}$. Note that for every finite set $A \subset \mathbb{N}$, $\alpha \in \mathbb{R}$ and $|e| = 1$ it holds that

$$\|x - \alpha 1_{\mathbb{N}}\| \geq \sup_{n \in \mathbb{N}} \frac{|e_n^*(x - \alpha 1_{\mathbb{N}})|}{\|e_n^*\|} \geq \sup_{n \in \mathbb{N}} \frac{|e_n^*(x) - \alpha 1_{\mathbb{N}}|}{2K_b} \geq \sup_{n \in \mathbb{N}} \frac{|e_n^*(x) - \alpha|}{2K_b}.$$

Let us also fix $\delta > 0$ and $n_0 \in \mathbb{N}$ with the property that

$$|e_n^*(x)| \leq \delta \quad \text{for every } n \geq n_0.$$

If $A$ satisfies $|A| > n_0$, then there is $j \in A$ with $j > n_0$, and so

$$\|x - \alpha 1_{\mathbb{N}}\| \geq \frac{|e_j^*(x) - |\alpha|}{2K_b} \geq \frac{|\alpha| - \delta}{2K_b}.$$
In particular, combining both lower estimations we get that for $|A| > n_0$
\[
\|x - \alpha 1_{EA}\| \geq \frac{|\alpha - \delta| + \sup_{n \in \mathbb{N}} |e_n^*(x) - |\alpha||}{4K_b} \geq \sup_{n \in \mathbb{N}} \frac{|e_n^*(x) - \delta|}{4K_b}.
\]
Therefore, for $m > n_0$
\[
D_m^*(x) \geq \sup_{n \in \mathbb{N}} \frac{|e_n^*(x) - \delta|}{4K_b}.
\]

3. Main result: Equivalence with the norm

The issue of when $\liminf_m D_m^*(x)$ (resp. $\liminf_m D_m(x)$) is equivalent to $\|x\|$ is going to be determined by the behaviour of the superdemocracy functions (resp. democracy functions), see Section 1 for the definitions. Along the present section we are going to focus on proving the results for superstability case, namely for $h_i^*(m)$, $h_r^*(m)$ and the error $D_m^*(x)$. The arguments for the case $h_i(m)$, $h_r$ and the error $D_m(x)$ are completely analogous. First of all, we recall a trivial estimates of the superdemocracy functions for any basis:
\[
h_i^*(k) \leq K_b h_i^*(m), \quad h_r^*(k) \leq K_b h_r^*(m) \quad \text{for every } k \leq m.
\]
These relations together with the trivial inequality $h_i^*(m) \leq h_i^*(m)$ ($m \in \mathbb{N}$) yield that there are three possible cases:
\[
\begin{align*}
\triangleright & \ h_i^*(m) \text{ and } h_r^*(m) \text{ are bounded}. \\
\triangleright & \ h_i^*(m) \text{ is bounded and } h_r^*(m) \rightarrow +\infty \text{ as } m \rightarrow +\infty. \\
\triangleright & \ h_i^*(m), h_r^*(m) \rightarrow +\infty \text{ as } m \rightarrow +\infty.
\end{align*}
\]

**Definition 3.1.** The functions $h_i^*(m)$ and $h_r^*(m)$ (resp. $h_i(m)$ and $h_r(m)$) are said to be comparable if they are both bounded or divergent to infinity.

The main result of the section is the following theorem.

**Theorem 3.2.** Let $\mathcal{B}$ be a basis of a Banach space $\mathcal{X}$. The following assertions are equivalent:

(i) There is a positive constant $c > 0$ such that
\[
c \|x\| \leq \liminf_m D_m^*(x) \leq \limsup_m D_m^*(x) \leq \|x\| \quad \text{for every } x \in \mathcal{X}.
\]

(ii) $h_i^*(m)$ and $h_r^*(m)$ are comparable.

Moreover, if $\mathcal{B}$ is monotone and $h_i^*(m) \rightarrow +\infty$ as $m \rightarrow +\infty$, then
\[
\lim_{m \rightarrow +\infty} D_m^*(x) = \|x\|.
\]

(The theorem also holds if we replace $D_m^*(x)$, $h_i^*(m)$, $h_r^*(m)$ respectively by $D_m(x)$, $h_i(m)$, $h_r(m)$.)

Before going into the proof let us make a few observations:

\[
\begin{align*}
\triangleright & \ \text{From Theorem 3.2 we recover (2) and (3). Indeed, if } \mathcal{H} \text{ is a (separable) Hilbert space and } \\
& \text{ and } \mathcal{B} \text{ is an orthonormal basis of } \mathcal{H} \text{ then } h_i(m) = h_i^*(m) = m^{1/2}. \ \text{On the other hand, for } \mathcal{X} = \ell_p \\text{ with } 1 \leq p < \infty \text{ and } \\
& \mathcal{B} \text{ is the canonical basis, it holds that } h_i(m) = h_i^*(m) = m^{1/p}. \\
\triangleright & \ \text{For } \mathcal{X} = L_p[0,1] \text{ we have that the Haar basis } \mathcal{B} \text{ is monotone (see 7 Theorem 5.18)) and } \\
& \text{satisfies } h_i^*(m) = h_i(m) \approx m^{1-1/p} \text{ for } 1 \leq p < \infty. \ \text{Hence, it satisfies that } \lim_m D_m^*(x) = \\
& \lim_m D_m(x) = \|x\| \text{ for every } x \in \mathcal{X}.
\end{align*}
\]
If $\mathcal{B}$ is superdemocratic (resp. democratic), then it satisfies Theorem 3.2(ii) (resp. Theorem 3.2(ii) for $h_r(m)$ and $h_l(m)$). However, there are easy examples showing that converse is not true. For instance, the canonical basis of $\ell^2 \oplus_1 \ell^4$ satisfies that $h_l(m) = h_r^*(m) \approx m^{1/4}$ and $h_r(m) = h_r^*(m) \approx m^{1/2}$.

Example of basis not satisfying Theorem 3.2(ii): Let us consider $\mathbb{X} = \ell_1$ and let $\mathcal{B} = (x_n)_{n=1}^\infty$ be the difference basis, which in terms of the canonical basis $(e_n)_{n=1}^\infty$ is given by

$$x_1 = e_1, \quad x_n = e_n - e_{n-1}, \quad n = 2, 3, \ldots$$

By [4, Lemma 8.1], it holds that $h_l^*(m) = h_l(m) = 1$ and $h_r^*(m) = h_r(m) = 2m$.

Example of basis satisfying $\lim_m \mathcal{D}_m(x) = \|x\|$ for every $x \in \mathbb{X}$, but $\liminf_m \mathcal{D}^*(x)$ is not even equivalent to $\|x\|$: Let $\mathbb{X} = c$ be the space of convergent sequences and let $\mathcal{B} = (s_n)_{n=1}^\infty$ be the summing basis, defined as

$$s_n := (0, \ldots, 0, 1, 1, \ldots), \quad n \in \mathbb{N}.$$ 

By [4, Lemma 8.1] we know that $h_r^*(m) \approx 1$ and $h_r^*(m) \approx m$, so Theorem 3.2(ii) does not hold. On the other hand, $\mathcal{B}$ is monotone and $h_l(m) \approx h_r(m) \approx m$ by the same reference. Thus, $\lim_m \mathcal{D}_m(x) = \|x\|$ for every $x \in \mathbb{X}$.

Condition Theorem 3.2(ii) is not preserved for dual bases: If $(e_n)_{n=1}^\infty$ is the canonical basis of $\ell_1$, let us consider the sequence $x_n = e_n - (e_{2n+1} + e_{2n+2})/2, n \in \mathbb{N}$ and the space

$$\mathbb{X} := \overline{\text{span}}\{x_n : n \in \mathbb{N}\}.$$ 

This is known as the Lindenstrauss space [8] and the sequence $\mathcal{B} = (x_n)_{n=1}^\infty$ is actually a monotone basis for $\mathbb{X}$ (see [11, pg 457]). In [4, Section 8.2] it is shown that $h_l^*(m) \approx m$. On the other hand, in the same reference it is proved that the dual space $\mathbb{X}^*$ with the corresponding dual basis $\mathcal{B}^*$ satisfies $h_l^*(m) \approx 1$ and $h_r^*(m) \approx \ln(m)$.

3.1. Proof of the main result.

**Proposition 3.3.** Let $\mathcal{B}$ be a basis of a Banach space $\mathbb{X}$. Then,

$$\sup_{A \subset \mathbb{N}, \{\eta_A\} \in \{\pm 1\}^A} \liminf_{m \to +\infty} \mathcal{D}_m^*(1_{\eta A}) \leq (1 + K_h) \liminf_{m \to +\infty} h^*_l(m) \leq \infty, \quad (5)$$

$$\sup_{A \subset \mathbb{N}, \{\eta_A\} \in \{\pm 1\}^A} \liminf_{m \to +\infty} \mathcal{D}_m(1_A) \leq (1 + K_h) \liminf_{m \to +\infty} h_l(m) \leq \infty. \quad (6)$$

**Proof.** We explain the argument for (5), as the proof of (6) is completely analogous with the obvious replacements. Let us fix a finite set $A \subset \mathbb{N}$ and $\eta \in \{\pm 1\}^A$, and let us take $\lambda \in \mathbb{R}$ satisfying

$$\lambda < \liminf_{m \to +\infty} \mathcal{D}_m^*(1_{\eta A}). \quad (7)$$

We can then find $m_0, n_0 \in \mathbb{N}$ with the following properties:

- $\lambda < \|1_{\eta A} - \alpha 1_{EB}\|$ for every $\alpha \in \mathbb{R}, |\varepsilon| = 1$ and $B \subset \mathbb{N}$ with $|B| \geq m_0$,
- $A \subset \{1, \ldots, n_0\}$.
Let $C \subset \mathbb{N}$ be a finite set with $|C| \geq m_0 + n_0$. Then,

$$1_{\varepsilon C} - P_{m_0}(1_{\varepsilon C}) = 1_{\varepsilon C}$$

where $C' := C \setminus \{1, \ldots, n_0\}$. Notice that $|C'| \geq m_0$, so in particular

$$\lambda \leq \|1_{\eta A} - 1_{(\eta A) \cup (\varepsilon C')}\| = \|1_{\varepsilon C'}\| \leq \|\text{Id} - P_{m_0}\| \|1_{\varepsilon C}\| \leq (1 + K_b) \|1_{\varepsilon C}\|.$$

Thus, we have the relation

$$\lambda \leq (1 + K_b) \liminf_{m \to +\infty} h_f^*(m).$$

Taking supremums on $\lambda$ according to (7) we conclude that

$$\liminf_{m \to +\infty} \mathcal{D}_m^*(1_{\eta A}) \leq (1 + K_b) \liminf_{m \to +\infty} h_f^*(m).$$

\[\square\]

**Theorem 3.4.** Let $\mathcal{B}$ be a basis of a Banach space $\mathbb{X}$. Assume that there is a constant $C > 0$ satisfying

$$\sup_{n \in \mathbb{N}} h_r^*(n) \leq C \sup_{n \in \mathbb{N}} h_t^*(n) \leq \infty.$$

Then, for every $x \in \mathbb{X}$

$$\frac{1}{C + K_b(1 + C)} \| x \| \leq \liminf_{m \to +\infty} \mathcal{D}_m(x) \leq \limsup_{m \to +\infty} \mathcal{D}_m(x) \leq \| x \|.$$  \[8\]

**Proof.** Let us fix $x \in \mathbb{X}$. We just have to show that the left hand-side of (8) holds. For, let $0 < \delta < 1$ and $m_0, n_0 \in \mathbb{N}$ such that

$$\| P_n(x) - x \| \leq \delta \| x \| \quad \text{for every } n \geq n_0,$$

$$h_t^*(n_0) \leq C(1 - \delta) h_t^*(n_0).$$

Given $\alpha \in \mathbb{R}$, $A \subset \mathbb{N}$ with $|A| \geq m_0 + n_0$ and $\varepsilon \in \{\pm 1\}^A$, we are going to establish two lower bounds for $\|x - \alpha 1_{\varepsilon A}\|$.

\[\triangleleft\] Since $|A \cap (n_0, +\infty)| \geq m_0$ we can find $n \geq n_0$ such that $|A \cap (n, +\infty)| = m_0$. Thus, applying the operator $\text{Id} - P_n$ to $x - \alpha 1_{\varepsilon A}$ we have that

$$\| x - \alpha 1_{\varepsilon A} \| \geq \frac{1}{K_b + 1} \| (\text{Id} - P_n)(x) - \alpha 1_{\varepsilon A}(\{\varepsilon A \cap (n, +\infty)\}) \| \geq \frac{1}{K_b + 1} (|\alpha| h_t^*(m_0) - \delta \| x \|) \quad \text{(9)}$$

\[\triangleleft\] As $|A| \geq n_0$ we can find $n \geq n_0$ with $|A \cap [1, n]| = n_0$, so that

$$\| x - \alpha 1_{\varepsilon A} \| \geq \frac{1}{K_b} \left( \| P_n(x) - \alpha 1_{\varepsilon A}(\{\varepsilon A \cap [1, n]\}) \| \right) \geq \frac{1}{K_b} \left( \| x \| (1 - \delta) - |\alpha| h_t^*(n_0) \right) \quad \text{(10)}$$

$$\geq \frac{1 - \delta}{K_b} \left( \| x \| - C |\alpha| h_t^*(m_0) \right) \quad \text{(11)}$$

Note that the lower estimations (9) and (11) are respectively increasing and decreasing linear functions $f(t)$ and $g(t)$ on $t = |\alpha|$. Moreover these functions have a unique point of intersection $t_0 > 0$ which can be easily checked to satisfy

$$t_0 = \frac{\| x \|}{h_t^*(m_0)} \cdot \frac{(1 - \delta)(1 + K_b) + \delta K_b}{C(1 - \delta)(1 + K_b) + K_b}.$$

Thus

$$\| x - \alpha 1_{\varepsilon A} \| \geq \max \{ f(|\alpha|), g(|\alpha|) \} \geq f(t_0) = g(t_0) = \frac{\| x \|}{1 + K_b} \left[ \frac{(1 - \delta)(1 + K_b) + \delta K_b}{C(1 - \delta)(1 + K_b) + K_b} - \delta \right].$$
Taking the infimum of \(|x - \alpha I_{eA}|\) on \(\alpha \in \mathbb{R}\) and \(A\) satisfying the conditions above, we deduce that
\[
\liminf_{k \to +\infty} D_k^*(x) \geq \inf_{k \geq m_0 + n_0} D_k^*(x) \geq \frac{\|x\|}{1 + K_b} \left( \frac{(1 - \delta)(1 + K_b) + \delta K_b}{C(1 - \delta)(1 + K_b) + K_b} - \delta \right).
\]
Finally, making \(\delta \to 0^+\) we get the desired conclusion. \(\square\)

**Proof of Theorem 3.2.** To check (i) \(\Rightarrow\) (ii), note that using Proposition 3.3 we then deduce that
\[
\sup_{m \in \mathbb{N}} h^*_m(m) = \sup_{\|A\| \leq 1} \|1_{\eta A}\| \leq \sup_{m \to +\infty} \liminf_{m \to +\infty} D_m^*(1_{\eta A}) \leq (1 + K_b) \liminf_{m \to +1} h^*_m(m) \leq \infty.
\]
It is clear from this inequality that \(h^*_m(m)\) and \(h^*_m(m)\) are then comparable. To see the converse (ii) \(\Rightarrow\) (i), note first that if \(h^*_m(m)\) and \(h^*_m(m)\) are comparable, then there exists \(C > 0\) such that
\[
\sup_{m \in \mathbb{N}} h^*_m(m) \leq \sup_{m \in \mathbb{N}} Ch^*_m(m)
\]
and so Theorem 3.4 applies. The second statement of the theorem follows also from Theorem 3.4 since \(B\) being monotone means that \(K_b = 1\), and condition \(\lim_{m \to +\infty} h^*_m(m) = +\infty\) means that (13) holds for every \(C > 0\). \(\square\)

4. Almost-Greedyness and Polynomials with Constant Coefficients

**Definition 4.1.** Let \(B = (e_n^\infty)_{n=1}^\infty\) be a basis of a Banach space \(\mathbb{X}\). We say that \(B\) is *almost-greedy* if there exists a constant \(C > 0\) such that
\[
\|x - \mathcal{G}_m(x)\| \leq C \sigma_m(x)
\]
where
\[
\sigma_m(x, \mathcal{B})_\mathbb{X} = \sigma_m(x) := \inf\{\|x - \sum_{n \in A} e_n^* e_n\| : A \subseteq \mathbb{N}, |A| = m\}.
\]

This notion was introduced by S. J. Dilworth, N. J. Kalton, D. Kutzarova and V. N. Temlyakov in [5], together with two characterizations. First, that a basis is almost-greedy if and only if it is quasi-greedy and democratic. The second characterization is given in the next theorem.

**Theorem 4.2 (5 Theorem 3.3).** Let \(B\) be a basis of a Banach space \(\mathbb{X}\). Then, \(B\) is almost-greedy if and only if for some (resp. every) \(\lambda > 1\), there exists a positive constant \(C_\lambda\) such that
\[
\|x - \mathcal{G}_{[\lambda]}(x)\| \leq C_\lambda \sigma_m(x), \text{ for every } x \in \mathbb{X}, m \in \mathbb{N}.
\]
Indeed, \(C_\lambda \approx \frac{1}{\lambda - 1}\).

As in the case of greedy basis, we can replace the error \(\sigma_m(x)\) by the \(m\)-th error of approximation by polynomials with constant (resp. modulus-constant) coefficients.

**Theorem 4.3.** Let \(B\) be a basis of a Banach space \(\mathbb{X}\) and let \(\lambda > 1\). The following assertions are equivalent:

(i) \(B\) is almost-greedy.

(ii) There is \(C > 0\) such that \(\|x - \mathcal{G}_{[\lambda]}(x)\| \leq C_\lambda \sigma_m(x)\) for every \(x \in \mathbb{X}\) and every \(m \in \mathbb{N}\).

(iii) There is \(C > 0\) such that \(\|x - \mathcal{G}_{[\lambda]}(x)\| \leq C_\lambda \sigma_m(x)\) for every \(x \in \mathbb{X}\) and every \(m \in \mathbb{N}\).
Proof. Implication (i) ⇒ (iii) ⇒ (ii) are clear using Theorem 4.2 and the inequalities $\sigma_m(x) \leq D_m(x) \leq D_m(x)$. To show that (ii) ⇒ (i) we follow the ideas from the proof of Theorem 4.2 using the hypothesis, we argue that $B$ is democratic and quasi-greedy.

To see that it is democratic, let $m \in \mathbb{N}$ and $A, B \subset \mathbb{N}$ with $|A| = m$ and $|B| = \lfloor A \rfloor$. Let us consider a set $E \supset A, B$ with $|E| = m + \lfloor A \rfloor$, let $\delta > 0$ and consider the element $x = 1_A + (1 + \delta)1_{E \setminus A}$. Then,

$$
\|1_A\| = \|x - C_{\lfloor A \rfloor}(x)\| \leq C_{\lfloor A \rfloor} + (1 + \delta)1_{B \setminus A}.
$$

As $\delta > 0$ is arbitrary, taking supremum over $A$ and infimum over $B$ we deduce that

$$
h_r(m) \leq C_{\lfloor A \rfloor} h_1(\lfloor A \rfloor) \leq C_{\lfloor A \rfloor} K_b h_l(m),
$$

where in the last inequality we have used the estimations mentioned at the beginning of Section 2.

Let show now that the basis $B$ is quasi-greedy. For, take $m \in \mathbb{N}$ and $r \in \mathbb{N} \cup \{0\}$ such that $\lfloor r \rfloor \leq m < \lfloor r + 1 \rfloor$. Then,

$$
\|x - G_m(x)\| \leq \|x - G_{\lfloor r \rfloor}(x)\| + \|G_{\lfloor r \rfloor}(x) - G_m(x)\|.
$$

Note that $G_{\lfloor r \rfloor}(x) - G_m(x)$ contains at most $m - \lfloor r \rfloor \leq \lambda$ summands of the form $e_n(x) e_n$, so that

$$
\|G_{\lfloor r \rfloor}(x) - G_m(x)\| \leq (\lambda \sup_{n \in \mathbb{N}} \|e_n\| \sup_{n \in \mathbb{N}} \|e_n(x)\|) \|x\|.
$$

On the other hand, using the hypothesis

$$
\|x - G_{\lfloor r \rfloor}(x)\| \leq C_{\lfloor A \rfloor} D_m(x) \leq C_{\lfloor A \rfloor} \|x\|.
$$

Thus, the basis is quasi-greedy. \qed

Recently, S. J. Dilworth and D. Khurana provided the following characterization of almost-greedy bases in the same spirit of Theorem 1.4. In order to present it we have to introduce some notation: if $A, B \subset \mathbb{N}$ are finite sets, we will write $A < B$ if $\max A < \min B$.

$$
\mathcal{H}_m(x) := \inf \{\|x - \alpha 1_A\| : \alpha \in \mathbb{R}, |A| = m \text{ and either } A < \Lambda_m(x) \text{ or } A > \Lambda_m(x)\}
$$

where recall that $\Lambda_m(x)$ is the $m$-th greedy set associated to $x$ introduced in Section 1.

**Theorem 4.4.** ([6]) Let $B$ be a basis of a Banach space $X$. Then, $B$ is almost-greedy if and only if there exists $C > 0$ such that

$$
\|x - G_m(x)\| \leq C \inf_{1 \leq n \leq m} \mathcal{H}_n(x) \text{ for every } x \in X \text{ and } m \in \mathbb{N}.
$$

Inspiring on the previous theorem, we can prove the following result which is again strinking as $D_m(x) \leq \mathcal{H}_m(x)$ and so $\liminf \mathcal{H}_m(x) \approx \|x\|$ when $h_l(m)$ and $h_r(m)$ are comparable by Theorem 3.2.

**Corollary 4.5.** Let $B$ be a basis of a Banach space $X$. Then, $B$ is almost-greedy if and only if there exists $C > 0$ such that

$$
\|x - G_m(x)\| \leq C \mathcal{H}_m(x) \text{ for every } x \in X \text{ and } m \in \mathbb{N}. \quad (14)
$$

**Proof.** If $B$ is quasi-greedy then (14) holds by Theorem 4.4. To see the converse we use the aforementioned characterization of almost-greedy bases as those being quasi-greedy and democratic. The fact that $B$ is quasi-greedy follows from the hypothesis and the trivial inequality $\mathcal{H}_m(x) \leq \|x\|$. Let us show that $B$ is democratic. Let $A, B \subset \mathbb{N}$ be finite subsets of cardinality $m$, and take $E \subset \mathbb{N}$ also with $|E| = m$ and moreover $A < E$ and $B < E$. Fixed $\delta > 0$ consider the elements $x = 1_A + (1 + \delta)1_E$ and $y = 1_E + (1 + \delta)1_B$. Then,

$$
\|1_A\| = \|x - 1_E\| = \|x - G_m(x)\| \leq C \mathcal{H}_m(x) \leq C \|x - 1_A\| = C (1 + \delta) \|1_E\|.
$$
Analogously, \[ \|1_E\| = \|y - 1_B\| = \|y - \mathcal{H}_m(y)\| \leq C\mathcal{H}_m(y) \leq C\|y - 1_E\| = C(1 + \delta)\|1_B\|. \]
Since \(\delta > 0\) was arbitrary, we conclude that \(h_r(m) \leq C^2 h_l(m)\) for every \(m \in \mathbb{N}\), and so the basis is democratic. \(\square\)

REFERENCES


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