A REMARK ON APPROXIMATION WITH POLYNOMIALS AND GREEDY BASES

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ABSTRACT. We investigate properties of the m-th error of approximation by polynomials with constant coefficients $\mathscr{D}_m(x)$ and with modulus-constant coefficients $\mathscr{D}^*_m(x)$ introduced by Berná and Blasco (2016) to study greedy bases in Banach spaces. We characterize when $\liminf_m \mathscr{D}_m(x)$ and $\liminf_m \mathscr{D}^*_m(x)$ are equivalent to $\|x\|$ in terms of the democracy and superdemocracy functions, and provide sufficient conditions ensuring that $\lim_m \mathscr{D}^*_m(x) = \lim_m \mathscr{D}_m(x) = \|x\|$, extending previous very particular results.

1. INTRODUCTION

Let $(\mathbb{X}, \|\cdot\|)$ be a real Banach space and let $\mathscr{B} = (e_n)_{n=1}^\infty$ be a semi-normalized (Schauder) basis of $\mathbb{X}$ with biorthogonal functionals $(e_n^*)_{n=1}^\infty$, that is:

(i) There exist $a, b > 0$ such that $a \leq \|e_n\|, \|e_n^*\| \leq b$ for every $n \in \mathbb{N}$,

(ii) $e_k^*(e_n) = \delta_{kn}$ for every $k, n \in \mathbb{N}$,

(iii) The sequence of projections $P_m : \mathbb{X} \longrightarrow \mathbb{X}$ given by

$$P_m(x) = \sum_{n=1}^m e_n^*(x)\, e_n\,, \quad x \in \mathbb{X}$$

satisfy $\lim_n \|P_m(x) - x\| = 0$ for every $x \in \mathbb{X}$. In this case, the *basis constant* of $\mathscr{B}$ is

$$K_b := \sup_{m \in \mathbb{N}} \|P_m\| < \infty.$$

We say that $\mathscr{B}$ is *monotone* whether $K_b = 1$.

Along the paper we will refer to every such $\mathscr{B}$ simply as a *basis*. Of course, as m increases $P_m(x)$ offers a good approximation of x by linear combinations of m-elements of the basis, but it is natural to ask whether a suitable (and systematic) rearrangement can provide better convergence rates. A natural proposal is the *Thresholding Greedy Algorithm* (TGA) introduced by S. V. Konyagin and V. N. Temlyakov ([10]): given $x \in \mathbb{X}$ we first consider the rearranging function $\rho : \mathbb{N} \longrightarrow \mathbb{N}$ satisfying that if $j < k$ then either $|e^*_{\rho(j)}(x)| > |e^*_{\rho(k)}(x)|$ or $|e^*_{\rho(j)}(x)| = |e^*_{\rho(k)}(x)|$ and $\rho(j) < \rho(k)$. The *m-th*

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greedy sum of x is then

$$\mathscr{G}_m(x) \;=\; \sum_{j=1}^{m} e^*_{\rho(j)}(x)\, e_{\rho(j)} \;=\; \sum_{k\in\Lambda_m(x)} e^*_k(x) e_k\,,$$

where $\Lambda_m(x) = \{\rho(n) : n \leq m\}$ is the *greedy set* of x with cardinality m. Related to this, S. V. Konyagin and V. N. Temlyakov defined in [10] the concepts of *greedy* and *quasi-greedy* bases.

Definition 1.1. We say that $\mathscr{B}$ is *quasi-greedy* if there exists a positive constant C_q such that

$$\|x-\mathscr{G}_m(x)\| \leq C_q\|x\|,\ \forall x\in\mathbb{X}, \forall m\in\mathbb{N}.$$

P. Wojtaszczyk proved in [12] that quasi-greediness is equivalent to the convergence of the algorithm, that is, $\mathscr{B}$ is quasi-greedy if and only if

$$\lim_{m\to+\infty}\|x-\mathscr{G}_m(x)\| = 0,\ \forall x\in\mathbb{X}.$$

Definition 1.2. We say that $\mathscr{B}$ is *greedy* if there exists a positive constant C such that

$$\|x-\mathscr{G}_m(x)\| \leq C\sigma_m(x),\ \forall x\in\mathbb{X}, \forall m\in\mathbb{N}, \tag{1}$$

where

$$\sigma_m(x,\mathscr{B})_{\mathbb{X}} = \sigma_m(x) := \inf\left\{\left\|x-\sum_{n\in A} a_n e_n\right\| : a_n\in\mathbb{F}, A\subset\mathbb{N}, |A| = m\right\}.$$

Konyagin and Temlykov [10] showed that, although every greedy basis is quasigreedy, the converse does not holds (see also [1, Section 10.2]). They also characterize greedy bases as those which are unconditional and democratic. To define the last notion we have to introduce some notation. For each finite subset $A\subset\mathbb{N}$ and every scalar sequence $\varepsilon = (\varepsilon_n)$ with $|\varepsilon_n| = 1$ for each $n\in\mathbb{N}$ (from now on we will write $|\varepsilon| = 1$, for simplicity) let us denote

$$\mathbf{1}_A := \sum_{n\in A} e_n \qquad \text{and} \qquad \mathbf{1}_{\varepsilon A} := \sum_{n\in A} \varepsilon_n e_n\,.$$

As usual, $|A|$ stands for the cardinal of A. We then define the *democracy functions* as

$$h_l(m) = \inf_{|A|=m, |\varepsilon|=1} \|\mathbf{1}_{\varepsilon A}\| \quad , \quad h_r(m) = \sup_{|A|=m, |\varepsilon|=1} \|\mathbf{1}_{\varepsilon A}\| \qquad (m\in\mathbb{N})\,.$$

and the *superdemocracy functions* as

$$h^*_l(m) = \inf_{|A|=m, |\varepsilon|=1} \|\mathbf{1}_{\varepsilon A}\| \quad , \quad h^*_r(m) = \sup_{|A|=m, |\varepsilon|=1} \|\mathbf{1}_{\varepsilon A}\| \qquad (m\in\mathbb{N})\,.$$

Definition 1.3. We say that $\mathscr{B}$ is *democratic* (resp. *superdemocratic*) if there exists $C>0$ such that $h_r(m) \leq C\, h_l(m)$ (resp. $h^*_r(m) \leq C\, h^*_l(m)$) for every $m\in\mathbb{N}$.

Another characterization of greedy bases was more recently provided by Ó. Blasco and the first author by means of the *best m-th error in the approximation using polynomials of constant* (*resp. modulus-constant*) *coefficients*:

$$\begin{aligned}\mathscr{D}_m(x,\mathscr{B})_{\mathbb{X}} &= \mathscr{D}_m(x) = \inf\{\|x-\alpha\mathbf{1}_A\| : \alpha\in\mathbb{R}, A\subset\mathbb{N},\ |A| = m\}\\ \mathscr{D}^*_m(x,\mathscr{B})_{\mathbb{X}} &= \mathscr{D}^*_m(x) = \inf\{\|x-\alpha\mathbf{1}_{\varepsilon A}\| : \alpha\in\mathbb{R}, A\subset\mathbb{N},\ |A| = m, |\varepsilon| = 1\}\end{aligned}$$

Theorem 1.4. [2, Corollary 1.8] *Let $\mathscr{B}$ be a basis of a Banach space $\mathbb{X}$. The following assertions are equivalent:*

(i) *$\mathscr{B}$ is greedy;*

(ii) *There is $C>0$ such that $\|x-\mathscr{G}_m(x)\| \leq C\,\mathscr{D}_m(x)$ for every $x\in\mathbb{X}$ and $m\in\mathbb{N}$.*

(iii) *There is* $C > 0$ *such that* $\|x - \mathcal{G}_m(x)\| \leq C\, \mathcal{D}_m^*(x)$ *for every* $x \in \mathbb{X}$ *and* $m \in \mathbb{N}$.

The striking feature of this theorem compared to (1) is that, while $\lim_m \sigma_m(x) = 0$ for every $x \in \mathbb{X}$, the terms $\mathcal{D}_m^*(x)$ and $\mathcal{D}_m(x)$ do not necessarily converge to zero if $x \neq 0$. Indeed, we have the following examples:

- ▷ [2, Theorem 3.2],[3, Theorem 1.4] If $\mathbb{X} = \mathbb{H}$ is a (separable) Hilbert space and $\mathcal{B}$ is an orthonormal basis, then

$$\lim_{m\to\infty} \mathcal{D}_m(x) = \lim_{m\to\infty} \mathcal{D}_m^*(x) = \|x\|, \qquad \text{for every } x \in \mathbb{H}. \tag{2}$$

- ▷ [2, Proposition 3.4] If $\mathbb{X} = \ell^p$ $(1 < p < \infty)$ and $\mathcal{B}$ is the canonical basis, then

$$\lim_{m\to+\infty} \mathcal{D}_m(\mathbf{1}_B) = \lim_{m\to+\infty} \mathcal{D}_m^*(\mathbf{1}_B) = \|\mathbf{1}_B\|, \qquad \text{for every finite } B \subset \mathbb{N}. \tag{3}$$

In the present paper, we aim to delve into this aspect. Let us briefly explain the structure of the paper. In Section 2 we show that $\mathcal{D}_m^*(x)$ and $\mathcal{D}_m(x)$ do not converge to zero as $m \to +\infty$ for any $x \neq 0$. In Section 3 we prove the main result of the paper (Theorem 3.2), namely a characterization of those bases $\mathcal{B}$ for which there is a positive constant $c > 0$ such that

$$c\|x\| \leq \liminf_{m\to+\infty} \mathcal{D}_m^*(x) \leq \limsup_{m\to+\infty} \mathcal{D}_m^*(x) \leq \|x\| \qquad \text{for every } x \in \mathbb{X},$$

in terms of the democracy and superdemocracy functions. We also provide a quite general condition ensuring that

$$\lim_{m\to+\infty} \mathcal{D}_m^*(x) = \|x\| \quad \text{ for every } x \in \mathbb{X}.$$

In Section 4 we deal with the notion of almost-greedy bases. We study how this property can be also characterized in terms of polynomials of constant or modulus-constant coefficients, extending a recent result of S. J. Dilworth and D. Khurana in [6].

Let us point out [1] as our basic reference for notation and fundamental results on greedy basis.

2. The limit of errors $\mathcal{D}_m^*(x)$ and $\mathcal{D}_m(x)$ is nonzero

Since $\mathcal{D}_m^*(x) \leq \mathcal{D}_m(x) \leq \|x\|$ for every $m \in \mathbb{N}$ and every $x \in \mathbb{X}$, it is only necessary to study lower bounds of $\mathcal{D}_m^*(x)$.

Proposition 2.1. *Let* $\mathcal{B} = (e_n)_{n=1}^\infty$ *be a basis of a Banach space* $\mathbb{X}$. *Then, for every* $x \in \mathbb{X}$

$$\frac{1}{4K_b} \sup_{n\in\mathbb{N}} |e_n^*(x)| \leq \liminf_{m\to\infty} \mathcal{D}_m^*(x).$$

Proof. Let $x \in \mathbb{X}$. Note that for every finite set $A \subset \mathbb{N}$, $\alpha \in \mathbb{R}$ and $|\varepsilon| = 1$ it holds that

$$\|x - \alpha \mathbf{1}_{\varepsilon A}\| \geq \sup_{n\in\mathbb{N}} \frac{|e_n^*(x - \alpha \mathbf{1}_{\eta A})|}{\|e_n^*\|} \geq \frac{\sup_{n\in\mathbb{N}} |e_n^*(x - \alpha \mathbf{1}_{\varepsilon A})|}{2K_b} \geq \frac{\sup_{n\in\mathbb{N}} \big||e_n^*(x)| - |\alpha|\big|}{2K_b}.$$

Let us also fix $\delta > 0$ and $n_0 \in \mathbb{N}$ with the property that

$$|e_n^*(x)| \leq \delta \quad \text{ for every } n \geq n_0.$$

If A satisfies $|A| > n_0$, then there is $j \in A$ with $j > n_0$, and so

$$\|x - \alpha \mathbf{1}_{\varepsilon A}\| \geq \frac{|e_j^*(x) - |\alpha||}{2K_b} \geq \frac{||\alpha| - \delta|}{2K_b}.$$

In particular, combining both lower estimations we get that for $|A| > n_0$

$$\|x-\alpha\mathbf{1}_{\varepsilon A}\| \geq \frac{||\alpha|-\delta| + \sup_{n\in\mathbb{N}} \big||e_n^*(x)|-|\alpha|\big|}{4K_b} \geq \sup_{n\in\mathbb{N}} \frac{|e_n^*(x)|-\delta}{4K_b}.$$

Therefore, for $m > n_0$

$$\mathscr{D}_m^*(x) \geq \sup_{n\in\mathbb{N}} \frac{|e_n^*(x)|-\delta}{4K_b}.$$

□

3. Main result: equivalence with the norm

The issue of when $\liminf_m \mathscr{D}_m^*(x)$ (resp. $\liminf_m \mathscr{D}_m(x)$) is equivalent to $\|x\|$ is going to be determined by the behaviour of the superdemocracy functions (resp. democracy functions), see Section 1 for the definitions. Along the present section we are going to focus on proving the results for superdemocracy case, namely for $h_l^*(m)$, $h_r^*(m)$ and the error $\mathscr{D}_m^*(x)$. The arguments for the case $h_l(m)$, h_r and the error $\mathscr{D}_m(x)$ are completely analogous. First of all, we recall a trivial estimates of the superdemocray functions for any basis:

$$h_l^*(k) \leq K_b\, h_l^*(m)\,, \quad h_r^*(k) \leq K_b\, h_r^*(m) \quad \text{ for every } k \leq m\,.$$

These relations together with the trivial inequality $h_l^*(m) \leq h_r^*(m)$ $(m\in\mathbb{N})$ yield that there are three possible cases:

- ▷ $h_l^*(m)$ and $h_r^*(m)$ are bounded.
- ▷ $h_l^*(m)$ is bounded and $h_r^*(m) \to +\infty$ as $m \to +\infty$.
- ▷ $h_l^*(m), h_r^*(m) \to +\infty$ as $m \to +\infty$.

Definition 3.1. The functions $h_l^*(m)$ and $h_r^*(m)$ (resp. $h_l(m)$ and $h_r(m)$) are said to be *comparable* if they are both bounded or divergent to infinity.

The main result of the section is the following theorem.

Theorem 3.2. *Let $\mathscr{B}$ be a basis of a Banach space $\mathbb{X}$. The following assertions are equivalent:*

(i) *There is a positive constant $c > 0$ such that*

$$c\,\|x\| \leq \liminf_{m\to+\infty} \mathscr{D}_m^*(x) \leq \limsup_{m\to+\infty} \mathscr{D}_m^*(x) \leq \|x\| \quad \textit{for every } x\in\mathbb{X}.$$

(ii) *$h_l^*(m)$ and $h_r^*(m)$ are comparable.*

Moreover, if $\mathscr{B}$ is monotone and $h_l^*(m) \to +\infty$ as $m \to +\infty$, then

$$\lim_{m\to+\infty} \mathscr{D}_m^*(x) = \|x\|\,. \tag{4}$$

(*The theorem also holds if we replace $\mathscr{D}_m^*(x)$, $h_l^*(m)$, $h_r^*(m)$ respectively by $\mathscr{D}_m(x)$, $h_l(m)$, $h_r(m)$.*)

Before going into the proof let us make a few observations:

- ▷ From Theorem 3.2 we recover (2) and (3). Indeed, if $\mathbb{H}$ is a (separable) Hilbert space and $\mathscr{B}$ is an orthonormal basis of $\mathbb{H}$ then $h_l(m) = h_l^*(m) = m^{1/2}$. On the other hand, for $\mathbb{X} = \ell_p$ with $1 \leq p < \infty$ and $\mathscr{B}$ is the canonical basis, it holds that $h_l(m) = h_l^*(m) = m^{1/p}$.
- ▷ For $\mathbb{X} = L_p[0,1]$ we have that the Haar basis $\mathscr{B}$ is monotone (see [7, Theorem 5.18]) and satisfies $h_l^*(m) = h_l(m) \approx m^{1-1/p}$ for $1 \leq p < \infty$. Hence, it satisfies that $\lim_m \mathscr{D}_m^*(x) = \lim_m \mathscr{D}_m(x) = \|x\|$ for every $x\in\mathbb{X}$.

▷ If $\mathscr{B}$ is superdemocratic (resp. democratic), then it satisfies Theorem 3.2.(ii) (resp. Theorem 3.2.(ii) for $h_r(m)$ and $h_l(m)$). However, there are easy examples showing that converse is not true. For instance, the canonical basis of $\ell^2 \oplus_1 \ell^4$ satisfies that $h_l(m) = h_l^*(m) \approx m^{1/4}$ and $h_r(m) = h_r^*(m) \approx m^{1/2}$.

▷ *Example of basis not satisfying Theorem 3.2.(ii)*: Let us consider $\mathbb{X} = \ell_1$ and let $\mathscr{B} = (\mathbf{x}_n)_{n=1}^\infty$ be the *difference basis*, which in terms of the canonical basis $(e_n)_{n=1}^\infty$ is given by

$$\mathbf{x}_1 = e_1\,, \quad \mathbf{x}_n = e_n - e_{n-1}\,, \quad n = 2,3,...$$

By [4, Lemma 8.1], it holds that $h_l^*(m) = h_l(m) = 1$ and $h_r^*(m) = h_r(m) = 2m$.

▷ *Example of basis satisfying* $\lim_m \mathscr{D}_m(x) = \|x\|$ *for every* $x \in \mathbb{X}$*, but* $\liminf_m \mathscr{D}^*(x)$ *is not even equivalent to* $\|x\|$: Let $\mathbb{X} = \mathbf{c}$ be the space of convergent sequences and let $\mathscr{B} = (\mathbf{s}_n)_{n=1}^\infty$ be the *summing basis*, defined as

$$\mathbf{s}_n := (\underbrace{0,\dots,0}_{n-1},1,1,\dots)\,, \quad n \in \mathbb{N}\,.$$

By [4, Lemma 8.1] we know that $h_l^*(m) \approx 1$ and $h_r^*(m) \approx m$, so Theorem 3.2.(ii) does not hold. On the other hand, $\mathscr{B}$ is monotone and $h_l(m) \approx h_r(m) \approx m$ by the same reference. Thus, $\lim_m \mathscr{D}_m(x) = \|x\|$ for every $x \in \mathbb{X}$.

▷ *Condition Theorem 3.2.(ii) is not preserved for dual bases*: If $(e_n)_{n=1}^\infty$ is the canonical basis of ℓ_1, let us consider the sequence $\mathbf{x}_n = e_n - (e_{2n+1} + e_{2n+2})/2$, $n \in \mathbb{N}$ and the space

$$\mathbb{X} := \overline{\operatorname{span}\{\mathbf{x}_n : n \in \mathbb{N}\}}^{\ell^1}\,.$$

This is known as the *Lindenstrauss space* [8] and the sequence $\mathscr{B} = (\mathbf{x}_n)_{n=1}^\infty$ is actually a monotone basis for $\mathbb{X}$ (see [11, pg 457]). In [4, Section 8.2] it is shown that $h_l^*(m) \approx m$. On the other hand, in the same reference it is proved that the dual space $\mathbb{X}^*$ with the corresponding dual basis $\mathscr{B}^*$ satisfies $h_l^*(m) \approx 1$ and $h_r^*(m) \approx \ln(m)$.

3.1. **Proof of the main result.**

Proposition 3.3. *Let $\mathscr{B}$ be a basis of a Banach space $\mathbb{X}$. Then,*

$$\sup_{\substack{A\subset\mathbb{N} \\ finite, |\eta|=1}} \liminf_{m\to+\infty} \mathscr{D}_m^*(\mathbf{1}_{\eta A}) \;\leq\; (1+K_b) \liminf_{m\to+\infty} h_l^*(m) \;\leq\; \infty\,, \tag{5}$$

$$\sup_{\substack{A\subset\mathbb{N} \\ finite}} \liminf_{m\to+\infty} \mathscr{D}_m(\mathbf{1}_{A}) \;\leq\; (1+K_b) \liminf_{m\to+\infty} h_l(m) \;\leq\; \infty\,. \tag{6}$$

Proof. We explain the argument for (5), as the proof of (6) is completely analogous with the obvious replacements. Let us fix a finite set $A \subset \mathbb{N}$ and $\eta \in \{\pm 1\}^A$, and let us take $\lambda \in \mathbb{R}$ satisfying

$$\lambda < \liminf_{m\to+\infty} \mathscr{D}_m^*(\mathbf{1}_{\eta A}). \tag{7}$$

We can then find $m_0, n_0 \in \mathbb{N}$ with the following properties:

▷ $\lambda \leq \|\mathbf{1}_{\eta A} - \alpha \mathbf{1}_{\varepsilon B}\|$ for every $\alpha \in \mathbb{R}$, $|\varepsilon| = 1$ and $B \subset \mathbb{N}$ with $|B| \geq m_0$,

▷ $A \subset \{1,\dots,n_0\}$.

Let $C \subset \mathbb{N}$ be a finite set with $|C| \geq m_0 + n_0$. Then,

$$\mathbf{1}_{\varepsilon C} - P_{n_0}(\mathbf{1}_{\varepsilon C}) = \mathbf{1}_{\varepsilon C'}$$

where $C' := C \setminus \{1, \ldots, n_0\}$. Notice that $|C'| \geq m_0$, so in particular

$$\lambda \leq \|\mathbf{1}_{\eta A} - \mathbf{1}_{(\eta A)\cup(\varepsilon C')}\| = \|\mathbf{1}_{\varepsilon C'}\| \leq \|\operatorname{Id} - P_{n_0}\| \, \|\mathbf{1}_{\varepsilon C}\| \leq (1+K_b)\, \|\mathbf{1}_{\varepsilon C}\|.$$

Thus, we have the relation

$$\lambda \leq (1+K_b) \liminf_{m\to+\infty} h_l^{\varepsilon}(m).$$

Taking supremums on λ according to (7) we conclude that

$$\liminf_{m\to+\infty} \mathscr{D}_m^*(\mathbf{1}_{\eta A}) \;\leq\; (1+K_b) \liminf_{m\to+\infty} h_l^{\varepsilon}(m).$$

□

Theorem 3.4. *Let $\mathscr{B}$ be a basis of a Banach space $\mathbb{X}$. Assume that there is a constant $C > 0$ satisfying*

$$\sup_{n\in\mathbb{N}} h_r^*(n) \leq C \sup_{n\in\mathbb{N}} h_l^*(n) \leq \infty.$$

Then, for every $x \in \mathbb{X}$

$$\frac{1}{C+K_b(1+C)}\, \|x\| \leq \liminf_m \mathscr{D}_m^*(x) \leq \limsup_m \mathscr{D}_m^*(x) \leq \|x\|. \tag{8}$$

Proof. Let us fix $x \in \mathbb{X}$. We just have to show that the left hand-side of (8) holds. For, let $0 < \delta < 1$ and $m_0, n_0 \in \mathbb{N}$ such that

$$\begin{aligned} \|P_n(x) - x\| &\leq \delta\, \|x\| \quad \text{for every } n \geq n_0, \\ h_r^*(n_0) &\leq C\,(1-\delta)\, h_l^*(m_0). \end{aligned}$$

Given $\alpha \in \mathbb{R}$, $A \subset \mathbb{N}$ with $|A| \geq m_0 + n_0$ and $\varepsilon \in \{\pm 1\}^A$, we are going to establish two lower bounds for $\|x - \alpha \mathbf{1}_{\varepsilon A}\|$.

▷ Since $|A \cap (n_0, +\infty)| \geq m_0$ we can find $n \geq n_0$ such that $|A \cap (n, +\infty)| = m_0$. Thus, applying the operator $\operatorname{Id} - P_n$ to $x - \alpha \mathbf{1}_{\varepsilon A}$ we have that

$$\|x - \alpha \mathbf{1}_{\varepsilon A}\| \geq \frac{1}{K_b+1} \|(\operatorname{Id} - P_n)(x) - \alpha \mathbf{1}_{\varepsilon(A\cap(n,+\infty))}\| \geq \frac{1}{K_b+1}\left(|\alpha|\, h_l^*(m_0) - \delta\, \|x\|\right). \tag{9}$$

▷ As $|A| \geq n_0$ we can find $n \geq n_0$ with $|A \cap [1,n]| = n_0$, so that

$$\begin{aligned} \|x - \alpha \mathbf{1}_{\varepsilon A}\| \geq \frac{1}{K_b}\left(\|P_n(x) - \alpha \mathbf{1}_{\varepsilon(A\cap[1,n])}\|\right) &\geq \frac{1}{K_b}\left(\|x\|(1-\delta) - |\alpha|\, h_r^*(n_0)\right) && (10) \\ &\geq \frac{1-\delta}{K_b}\left(\|x\| - C\,|\alpha|\, h_l^*(m_0)\right) && (11) \end{aligned}$$

Note that the lower estimations (9) and (11) are respectively increasing and decreasing linear functions $f(t)$ and $g(t)$ on $t = |\alpha|$. Moreover these functions have a unique point of intersection $t_0 > 0$ which can be easily checked to satisfy

$$t_0 = \frac{\|x\|}{h_l^*(m_0)} \cdot \frac{(1-\delta)\,(1+K_b) + \delta\, K_b}{C(1-\delta)(1+K_b) + K_b}. \tag{12}$$

Thus

$$\|x - \alpha \mathbf{1}_{\varepsilon A}\| \geq \max\{f(|\alpha|), g(|\alpha|)\} \geq f(t_0) = g(t_0) = \frac{\|x\|}{1+K_b}\left[\frac{(1-\delta)\,(1+K_b) + \delta\, K_b}{C(1-\delta)(1+K_b) + K_b} - \delta\right].$$

Taking the infimum of $\|x-\alpha \mathbf{1}_{\varepsilon A}\|$ on $\alpha \in \mathbb{R}$ and A satisfying the conditions above, we deduce that

$$\liminf_{k\to+\infty} \mathscr{D}_k^*(x) \geq \inf_{k\geq m_0+n_0} \mathscr{D}_k^*(x) \geq \frac{\|x\|}{1+K_b}\left[\frac{(1-\delta)(1+K_b)+\delta K_b}{C(1-\delta)(1+K_b)+K_b} - \delta\right].$$

Finally, making $\delta \to 0^+$ we get the desired conclusion. □

Proof of Theorem 3.2. To check (i) ⇒ (ii), note that using Proposition 3.3 we then deduce that

$$\sup_{m\in\mathbb{N}} h_r^*(m) = \sup_{\substack{A\subset\mathbb{N} \\ finite, |\eta|=1}} \|\mathbf{1}_{\eta A}\| \leq \sup_{\substack{A\subset\mathbb{N} \\ finite, |\eta|=1}} \liminf_{m\to+\infty} \mathscr{D}_m^*(\mathbf{1}_{\eta A}) \leq (1+K_b) \liminf_{m\to+\infty} h_l^*(m) \leq \infty.$$

It is clear from this inequality that $h_l^*(m)$ and $h_r^*(m)$ are then comparable. To see the converse (ii) ⇒ (i), note first that if $h_l^*(m)$ and $h_r^*(m)$ are comparable, then there exists $C>0$ such that

$$\sup_{m\in\mathbb{N}} h_r^*(m) \leq \sup_{m\in\mathbb{N}} C\, h_l^*(m) \tag{13}$$

and so Theorem 3.4 applies. The second statement of the theorem follows also from Theorem 3.4 since $\mathscr{B}$ being monotone means that $K_b = 1$, and condition $\lim_m h_l^*(m) = +\infty$ means that (13) holds for every $C>0$. □

4. Almost-greediness and polynomials with constant coefficients

Definition 4.1. Let $\mathscr{B} = (e_n)_{n=1}^\infty$ be a basis of a Banach space $\mathbb{X}$. We say that $\mathscr{B}$ is *almost-greedy* if there exists a constant $C>0$ such that

$$\|x - \mathscr{G}_m(x)\| \leq C\,\widetilde{\sigma}_m(x)$$

where

$$\widetilde{\sigma}_m(x,\mathscr{B})_{\mathbb{X}} = \widetilde{\sigma}_m(x) := \inf\{\|x - \sum_{n\in A} e_n^*(x)\, e_n\| : A\subset\mathbb{N}, |A| = m\}.$$

This notion was introduced by S. J. Dilworth, N. J. Kalton, D. Kutzarova and V. N. Temlyakov in [5], together with two characterizations. First, that a basis is almost-greedy if and only if it is quasi-greedy and democratic. The second characterization is given in the next theorem.

Theorem 4.2 ([5, Theorem 3.3])**.** *Let $\mathscr{B}$ be a basis of a Banach space $\mathbb{X}$. Then, $\mathscr{B}$ is almost-greedy if and only if for some (resp. every) $\lambda>1$, there exists a positive constant C_λ such that*

$$\|x - \mathscr{G}_{[\lambda m]}(x)\| \leq C_\lambda\, \sigma_m(x), \quad \textit{for every } x\in\mathbb{X}, m\in\mathbb{N}.$$

Indeed, $C_\lambda \approx \frac{1}{\lambda-1}$.

As in the case of greedy basis, we can replace the error $\sigma_m(x)$ by the m-th error of approximation by polynomials with constant (resp. modulus-constant) coefficients.

Theorem 4.3. *Let $\mathscr{B}$ be a basis of a Banach space $\mathbb{X}$ and let $\lambda>1$. The following assertions are equivalent:*

(i) *$\mathscr{B}$ is almost-greedy.*

(ii) *There is $C>0$ such that $\|x-\mathscr{G}_{[\lambda m]}(x)\| \leq C_\lambda\, \mathscr{D}_m(x)$ for every $x\in\mathbb{X}$ and every $m\in\mathbb{N}$.*

(iii) *There is $C>0$ such that $\|x-\mathscr{G}_{[\lambda m]}(x)\| \leq C_\lambda\, \mathscr{D}_m^*(x)$ for every $x\in\mathbb{X}$ and every $m\in\mathbb{N}$.*

Proof. Implication (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) are clear using Theorem 4.2 and the inequalities $\sigma_m(x) \leq \mathscr{D}_m^*(x) \leq \mathscr{D}_m(x)$. To show that (ii) $\Rightarrow$ (i) we follow the ideas from the proof of Theorem 4.2: using the hypothesis, we argue that $\mathscr{B}$ is democratic and quasi-greedy.

To see that it is democratic, let $m \in \mathbb{N}$ and $A, B \subset \mathbb{N}$ with $|A| = m$ and $|B| = [\lambda m]$. Let us consider a set $E \supset A, B$ with $|E| = m + [\lambda m]$, let $\delta > 0$ and consider the element $x = \mathbf{1}_A + (1+\delta)\mathbf{1}_{E\setminus A}$. Then,

$$\|\mathbf{1}_A\| = \|x - \mathscr{G}_{[\lambda m]}(x)\| \leq C_\lambda\, \mathscr{D}_m(x) \leq C_\lambda \|\mathbf{1}_{B\setminus A} + (1+\delta)\mathbf{1}_{B\cap A}\|\,.$$

As $\delta > 0$ is arbitrary, taking supremum over A and infimum over B we deduce that

$$h_r(m) \leq C_\lambda\, h_l(\lambda m) \leq C_\lambda\, K_b\, h_l(m)\,,$$

where in the last inequality we have used the estimations mentioned at the beginning of Section 2.

Let show now that the basis $\mathscr{B}$ is quasi-greedy. For, take $m \in \mathbb{N}$ and $r \in \mathbb{N} \cup \{0\}$ such that $[\lambda r] \leq m < [\lambda(r+1)]$. Then,

$$\|x - \mathscr{G}_m(x)\| \leq \|x - \mathscr{G}_{[\lambda r]}(x)\| + \|\mathscr{G}_{[\lambda r]}(x) - \mathscr{G}_m(x)\|\,.$$

Note that $\mathscr{G}_{[\lambda r]}(x) - \mathscr{G}_m(x)$ contains at most $m - [\lambda r] < \lambda$ summands of the form $e_n^*(x)\, e_n$, so that

$$\|\mathscr{G}_{[\lambda r]}(x) - \mathscr{G}_m(x)\| \leq \big(\lambda \sup_{n\in\mathbb{N}} \|e_n\| \sup_{n\in\mathbb{N}} \|e_n^*\|\big)\, \|x\|\,.$$

On the other hand, using the hypothesis

$$\|x - \mathscr{G}_{[\lambda r]}(x)\| \leq C_\lambda\, \mathscr{D}_m(x) \leq C_\lambda\, \|x\|\,.$$

Thus, the basis is quasi-greedy. □

Recently, S. J. Dilworth and D. Khurana provided the following characterization of almost-greedy bases in the same spirit of Theorem 1.4. In order to present it we have to introduce some notation: if $A, B \subset \mathbb{N}$ are finite sets, we will write $A < B$ if $\max A < \min B$.

$$\mathscr{H}_m(x) \quad := \quad \inf\{\|x - \alpha \mathbf{1}_A\| \colon \alpha \in \mathbb{R}, |A| = m \text{ and either } A < \Lambda_m(x) \text{ or } A > \Lambda_m(x)\}$$

where recall that $\Lambda_m(x)$ is the m-th greedy set associated to x introduced in Section 1.

Theorem 4.4. [6] *Let $\mathscr{B}$ be a basis of a Banach space $\mathbb{X}$. Then, $\mathscr{B}$ is almost-greedy if and only if there exists $C > 0$ such that*

$$\|x - \mathscr{G}_m(x)\| \leq C \inf_{1\leq n\leq m} \mathscr{H}_n(x) \quad \textit{for every } x \in \mathbb{X} \textit{ and } m \in \mathbb{N}.$$

Inspiring on the previous theorem , we can prove the following result which is again strinking as $\mathscr{D}_m(x) \leq \mathscr{H}_m(x)$ and so $\liminf \mathscr{H}_m(x) \approx \|x\|$ when $h_l(m)$ and $h_r(m)$ are comparable by Theorem 3.2.

Corollary 4.5. *Let $\mathscr{B}$ be a basis of a Banach space $\mathbb{X}$. Then, $\mathscr{B}$ is almost-greedy if and only if there exists $C > 0$ such that*

$$\|x - \mathscr{G}_m(x)\| \leq C\, \mathscr{H}_m(x) \quad \textit{for every } x \in \mathbb{X} \textit{ and } m \in \mathbb{N}. \tag{14}$$

Proof. If $\mathscr{B}$ is quasi-greedy then 14 holds by Theorem 4.4. To see the converse we use the aforementioned characterization of almost-greedy bases as those being quasi-greedy and democratic. The fact that $\mathscr{B}$ is quasi-greedy follows from the hypothesis and the trivial inequality $\mathscr{H}_m(x) \leq \|x\|$. Let us show that $\mathscr{B}$ is democratic. Let $A, B \subset \mathbb{N}$ be finite subsets of cardinality m, and take $E \subset \mathbb{N}$ also with $|E| = m$ and moreover $A < E$ and $B < E$. Fixed $\delta > 0$ consider the elements $x = \mathbf{1}_A + (1+\delta)\mathbf{1}_E$ and $y = \mathbf{1}_E + (1+\delta)\mathbf{1}_B$. Then,

$$\|\mathbf{1}_A\| = \|x - \mathbf{1}_E\| = \|x - \mathscr{G}_m(x)\| \leq C\, \mathscr{H}_m(x) \leq C\, \|x - \mathbf{1}_A\| = C\,(1+\delta)\, \|\mathbf{1}_E\|\,.$$

Analogously,

$$\|\mathbf{1}_E\| = \|y - \mathbf{1}_B\| = \|y - \mathscr{G}_m(y)\| \leq C\,\mathscr{H}_m(y) \leq C\|y - \mathbf{1}_E\| = C\,(1+\delta)\,\|\mathbf{1}_B\|\,.$$

Since $\delta > 0$ was arbitraty, we conclude that $h_r(m) \leq C^2\, h_l(m)$ for every $m \in \mathbb{N}$, and so the basis is democratic. □

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