Sparse-grid polynomial interpolation approximation and integration for parametric and stochastic elliptic PDEs with lognormal inputs

Dinh Dũng

Information Technology Institute, Vietnam National University
144 Xuan Thuy, Cau Giay, Hanoi, Vietnam
dinhzung@gmail.com

June 10, 2019

Abstract

By combining a certain approximation property in the spatial domain, and weighted ℓ_2-summability of the Hermite polynomial expansion coefficients in the parametric domain obtained in [M. Bachmayr, A. Cohen, R. DeVore and G. Migliorati, ESAIM Math. Model. Numer. Anal. **51**(2017), 341-363] and [M. Bachmayr, A. Cohen, D. Dũng and C. Schwab, SIAM J. Numer. Anal. **55**(2017), 2151-2186], we investigate linear non-adaptive methods of fully discrete polynomial interpolation approximation as well as fully discrete weighted quadrature methods of integration for parametric and stochastic elliptic PDEs with lognormal inputs. We explicitly construct such methods and prove corresponding convergence rates of the approximations by them. The linear non-adaptive methods of fully discrete polynomial interpolation approximation are sparse-grid collocation methods which are certain sums taken over finite nested Smolyak-type indices sets $G(\xi)$ parametrized by positive number ξ, of mixed tensor products of dyadic scale successive differences of spatial approximations of particular solvers, and of successive differences of their parametric Lagrange intepolating polynomials. The Smolyak sparse grids in the parametric domain are constructed from the roots of Hermite polynomials or their improved modifications. Moreover, they generate fully discrete weighted quadrature formulas in a natural way for integration of the solution to parametric and stochastic elliptic PDEs and its linear functionals, and the error of the corrsponding integration can be estimated via the error in the Bochner space $L_1(\mathbb{R}^\infty, V, \gamma)$ norm of the generating methods where γ is the Gaussian probability measure on $\mathbb{R}^\infty$ and V is the energy space. Our analysis leads to auxiliary convergence rates in parameter ξ of these approximations when ξ going to ∞. For a given $n \in \mathbb{N}$, we choose ξ_n so that the cardinality of $G(\xi_n)$ which in some sense characterizes computation complexity, does not exceed n, and hence obtain the convergence rates in increasing n, of the fully discrete polynomial approximation and integration.

Keywords and Phrases: High-dimensional approximation; Parametric and stochastic elliptic PDEs; Lognormal inputs; Collocation approximation; Fully discrete non-adaptive polynomial interpolation approximation; Fully discrete non-adaptive integration.

Mathematics Subject Classifications (2010): 65C30, 65D05, 65D32, 65N15, 65N30, 65N35.

1 Introduction

One of basic problems in Uncertainty Quantification are approximation and numerical methods for parametric and stochastic PDEs. Since the number of parametric variables may be very large or even infinite, they are treated as high-dimensional or infinite-dimensional approximation problems. Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Consider the diffusion elliptic equation

$$- \operatorname{div}(a\nabla u) \;=\; f \quad \text{in} \quad D, \quad u|_{\partial D} \;=\; 0, \tag{1.1}$$

for a given fixed right-hand side f and spatially variable scalar diffusion coefficient a. Denote $V := H_0^1(D)$ - the energy space and $V' := H^{-1}(D)$. If a satisfies the ellipticity assumption

$$0 < a_{\min} \le a \le a_{\max} < \infty,$$

by the well-known Lax-Milgram lemma, for any $f \in V'$, there exists a unique solution $u \in V$ in weak form which satisfies the variational equation

$$\int_D a\nabla u \cdot \nabla v \, \mathrm{d}\boldsymbol{x} \;=\; \langle f, v\rangle, \quad \forall v \in V.$$

We consider diffusion coefficients having a parametrized form $a = a(\boldsymbol{y})$, where $\boldsymbol{y} = (y_1, y_2, ...)$ is a sequence of real-valued parameters ranging in the set $\mathbb{U}^\infty$ which is either $\mathbb{R}^\infty$ or $\mathbb{I}^\infty := [-1,1]^\infty$. The resulting solution to parametric and stochastic elliptic PDEs map $\boldsymbol{y} \mapsto u(\boldsymbol{y})$ acts from $\mathbb{U}^\infty$ to the solution to parametric and stochastic elliptic PDEs space V. The objective is to achieve numerical approximation of this complex map by a small number of parameters with some guaranteed error in a given norm. Depending on the nature of the modeled object, the parameter $\boldsymbol{y}$ may be either deterministic or random. In the present paper, we consider the so-called lognormal case when $\mathbb{U}^\infty = \mathbb{R}^\infty$ and the diffusion coefficient a is of the form

$$a(\boldsymbol{y}) = \exp(b(\boldsymbol{y})), \quad b(\boldsymbol{y}) = \sum_{j=1}^{\infty} y_j \psi_j, \tag{1.2}$$

where the y_j are i.i.d. standard Gaussian random variables and $\psi_j \in L_\infty(D)$. Note that the above countably infinite-dimensional setting comprises its finite-dimensional counterpart by setting $\psi_j = 0$ for j large enough. We also birefly consider the affine case when $\mathbb{U}^\infty = \mathbb{I}^\infty$ and the diffusion coefficient a is of the form

$$a(\boldsymbol{y}) = \bar{a} + \sum_{j=1}^{\infty} y_j \psi_j. \tag{1.3}$$

In order to study fully discrete approximations of the solution $u(\boldsymbol{y})$ to the parametrized elliptic PDEs (1.1), we assume that $f \in L_2(D)$ and $a(\boldsymbol{y}) \in W^1_\infty(D)$, and hence we obtain that $u(\boldsymbol{y})$ has the second higher regularity, i. e., $u(\boldsymbol{y}) \in W$ where W is the space

$$W := \{v \in V \;:\; \Delta v \in L^2(D)\}$$

equipped with the norm

$$\|v\|_W := \|\Delta v\|_{L^2(D)},$$

which coincides with the Sobolev space $V \cap H^2(D)$ with equivalent norms if the domain D has $C^{1,1}$ smoothness, see [19, Theorem 2.5.1.1]. Moreover, we assume that there holds the following *approximation property* for the spaces V and W.

Assumption I There are a sequence $(V_n)_{n\in\mathbb{N}_0}$ of subspaces $V_n \subset V$ of dimension $\le n$, and sequence $(P_n)_{n\in\mathbb{N}_0}$ of linear operators from V into V_n, and a number $\alpha > 0$ such that

$$\|P_n(v)\|_V \le C, \quad \|v - P_n(v)\|_V \le C n^{-\alpha} \|v\|_W, \quad \forall n \in \mathbb{N}_0, \quad \forall v \in W. \tag{1.4}$$

A basic role in the approximation and integration for parametric and stochastic PDEs are generalized polynomial chaos expansions for the dependence on the parametric variables. We refer the reader to [9, 13, 20, 29, 28] and references there for different aspects in approximation for parametric and stochastic PDEs. In [6]–[11], based on the conditions $\big(\|\psi_j\|_{W^1_\infty(D)}\big)_{j\in\mathbb{N}} \in \ell_p(\mathbb{N})$ on the affine expansion (1.3) for some $0 < p < 1$, the authors have proven the ℓ_p-summability of the coefficients in a Taylor or Legendre polynomials expansion and hence proposed best adaptive n-term approximation methods in energy norm by choosing the set of the n most useful terms in these expansions. The obtained n-term approximands are then approximated by finite element methods. Simlilar results have been received in [24] for the lognormal case based on the conditions $\big(j\|\psi_j\|_{W^1_\infty(D)}\big)_{j\in\mathbb{N}} \in \ell_p(\mathbb{N})$ some $0 < p < 1$. In these papers, they did not take into account support properties of the functions ψ_j.

A different approach to studying summability that takes into account the support properties has been recently proposed in [1] for the affine case and [2] for the lognormal case. This approach leads to significant improvements on the results on ℓ_p-summability when the functions ψ_j have limited overlap, such as splines, finite elements or wavelet bases. These results by themselve do not imply practical applications, because they do not cover the approximation of the expansion coefficients which are functions of the spatial variable.

In the recent paper [3], the rates of fully discrete adaptive best n-term Taylor and Galerkin approximations for elliptic PDEs with affine or lognornal parametrizations of the diffusion coefficients have been obtained based on combining a certain approximation property on the spatial domain, and extensions of the results on ℓ_p-summability of [1, 2] to higher-order Sobolev norms of corresponding Taylor, Jacobi and Hermite expansion coefficients. These results providing a benchmark for convergence rates, are not constructive. In the case when ℓ_p-summable sequences of Sobolev norms of expansion coefficients have an ℓ_p-summable majorant sequence, these convergence rates can be achieved by non-adaptive linear methods of Galerkin and collocation approximations in the affine case [9, 14, 15, 16, 32, 33]. However, this non-adaptive approach is not applicable for the improvement of ℓ_p-summability in [1, 2, 3] since the weakened ℓ_p-assumption leads only to the ℓ_p-summability of expansion coefficients, but not to an ℓ_p-summable majorant sequence. Non-adaptive (not fully discrete) approach has been considered in [1, Remark 3.2] and [2, Remark 5.1] for Galerkin approximation, in [18] for polynomial collocation approximation, and in [4] for weighted integration.

In the present paper, by combining spatial and parametric aproximability, namely, the approximation property in Assumptions I in the spatial domain and weighted ℓ_2-summability of the V and W norms of Hermite polynomial expansion coefficients obtained in [2, 3], we investigate linear non-adaptive methods of fully discrete Galerkin approximation and polynomial interpolation approximation as well as fully discrete weighted quadrature methods of integration for parametric and stochastic elliptic PDEs with lognormal inputs (1.2). We explicitly construct such methods and prove corresponding convergence rates of the approximations by them. We show that the rate of best adaptive fully discrete n-term Galerkin approximation obtained in [3], is achieved by linear non-adaptive methods of fully discrete Galerkin approximation. The linear non-adaptive methods of fully discrete polynomial interpolation approximation are sparse-grid collocation methods which are certain sums taken over finite nested Smolyak-type indices sets $G(\xi)$ parametrized by positive number ξ, of tensor products of dyadic scale successive differences of spatial approximations of particular solvers, and of successive differences of their parametric Lagrange intepolating polynomials. The Smolyak sparse grids in the parametric domain are constructed from the roots of Hermite polynomials or their improved modifications. Moreover, these methods generate fully discrete weighted quadrature formulas in a natural way for integration of the solution $u(\boldsymbol{y})$ and its linear functionals, and the error of the corrsponding integration can be estimated via the error in the space $L_1(\mathbb{R}^\infty, V, \gamma)$ norm of the generating methods where γ is the Gaussian probability measure on $\mathbb{R}^\infty$. Our analysis leads to auxiliary convergence rates in parameter ξ of these approximations when ξ going to ∞. For a given $n \in \mathbb{N}$, we choose ξ_n so that the cardinality of $G(\xi_n)$ which in some sense measures computation complexity, does not exceed n, and hence obtain the convergence rates in increasing n, of the fully discrete polynomial approximation and integration. The convergence rate of fully discrete integration better than the convergence rate of the generating fully discrete polynomial interpolation approximation due to the simple but useful observation that the integral $\int_{\mathbb{R}} v(y)\, \mathrm{d}\gamma(y)$ is zero if $v(y)$ is an odd function and γ is the Gaussian probability measure on $\mathbb{R}$. (This property has been used in [32, 33, 34] for improving convergence rate of integration in the affine case.) This is the main contribution of our paper. We also briefly consider similar problems for parametric and stochastic elliptic PDEs with affine inputs (1.3) by using counterparts-results in [1, 3], and by-product problems of non-fully discrete polynomial interpolation approximation and integration similar to those treated in [18, 4]. In particular, the convergence rate of intergration obtained in this paper is better than that in [4].

The paper is organized as follows. In Sections 2–4, we explicitly construct general fully discrete linear methods of Galerkin and polynomial interpolation approximations in the Bochner space $L_p(\mathbb{R}^\infty, X^1, \gamma)$ and quadrature of functions taking values in X^2 and having a weighted ℓ_2-summability of Hermite expansion coefficients for Hilbert spaces X^1 and X^2 satisfying a certain "spatial" approximation property (see (2.3)). In particular, in Section 2, we prove convergence rates of general fully discrete linear Galerkin methods of approximation; in Section 3, we prove convergence rates of general fully discrete linear polynomial interpolation methods of approximation; in Section 4, we prove convergence rates of general fully discrete linear quadrature for integration. In Section 5, we apply the results of Sections 2–4 to obtain the main results of this paper on convergence rates of linear non-adaptive methods of fully discrete Galerkin and polynomial interpolation approximation as well as fully discrete weighted quadrature methods of integration for parametric and stochastic elliptic PDEs with lognormal inputs. In Section 6, by extending the theory in Sections 2–4, we briefly consider similar problems for parametric and stochastic elliptic PDEs with affine inputs. In all Sections 2–6, we also briefly investigate related non-fully approximation problems.

2 Galerkin approximation

In this section, we construct general fully discrete linear Galerkin methods of approximation in the Bochner space $L_p(\mathbb{R}^\infty, X^1, \gamma)$ with $0 < p \le 2$ and the infinite tensor product Gausian probability measure γ of functions taking values in X^2 and having a weighted ℓ_2-summability of Hermite expansion coefficients for Hilbert spaces X^1 and X^2 satisfying a certain "spatial" approximation property. In particular, we prove convergence rates for these methods of approximation.

We first recall a concept of infinite tensor product of probability measures. (For details see, e.g., [23, pp. 429–435].) Let $\mu(y)$ be a probability measrure on $\mathbb{U}$, where $\mathbb{U}$ is either $\mathbb{R}$ or $\mathbb{I} := [-1, 1]$. We introduce the probability measure $\mu(\boldsymbol{y})$ on $\mathbb{U}^\infty$ as the infinite tensor product of probability measures $\mu(y_i)$:

$$\mu(\boldsymbol{y}) := \bigotimes_{j \in \mathbb{N}} \mu(y_j), \quad \boldsymbol{y} = (y_j)_{j \in \mathbb{N}} \in \mathbb{U}^\infty.$$

The sigma algebra for $\mu(\boldsymbol{y})$ is generated by the set of cylinders $A := \prod_{j \in \mathbb{N}} A_j$, where $A_j \subset \mathbb{U}$ are univariate μ-measurable sets and only a finite number of A_i are different from $\mathbb{U}$. For such a set A, we have $\mu(A) := \prod_{j \in \mathbb{N}} \mu(A_j)$. If $\varrho(y)$ is the density of $\mu(y)$, i.e., $\mathrm{d}\mu(y) = \varrho(y)\mathrm{d}y$, then we write

$$\mathrm{d}\mu(\boldsymbol{y}) := \bigotimes_{j \in \mathbb{N}} \varrho(y_j)\mathrm{d}(y_j), \quad \boldsymbol{y} = (y_j)_{j \in \mathbb{N}} \in \mathbb{U}^\infty.$$

Let X be a Hilbert space and $0 < p < \infty$. The probability measure $\mu(\boldsymbol{y})$ induces the Bochner space $L_p(\mathbb{U}^\infty, X, \mu)$ of strongly μ-measurable mappings v from $\mathbb{U}^\infty$ to X which are p-summable. The (quasi-)norm in $L_p(\mathbb{U}^\infty, X, \mu)$ is defined by

$$\|v\|_{L_p(\mathbb{U}^\infty,X,\mu)} := \left(\int_{\mathbb{U}^\infty} \|v(\cdot,\boldsymbol{y})\|_X^p \, \mathrm{d}\mu(\boldsymbol{y})\right)^{1/p}.$$

In the present paper, we focus our attention mainly to the lognormal case with $\mathbb{U}^\infty = \mathbb{R}^\infty$ and $\mu(\boldsymbol{y}) = \gamma(\boldsymbol{y})$, the infinite tensor product Gausian probability measure. Let $\gamma(y)$ be the probability measrure on $\mathbb{R}$ with the standard Gaussian density:

$$\mathrm{d}\gamma(y) := g(y)\,\mathrm{d}y, \quad g(y) := \frac{1}{\sqrt{2\pi}} e^{-y^2/2}.$$

Then the infinite tensor product Gausian probability measure $\gamma(\boldsymbol{y})$ on $\mathbb{R}^\infty$ can be defined by

$$\mathrm{d}\gamma(\boldsymbol{y}) := \bigotimes_{j\in\mathbb{N}} g(y_j)\mathrm{d}(y_j), \quad \boldsymbol{y} = (y_j)_{j\in\mathbb{N}} \in \mathbb{R}^\infty.$$

A powerful strategy for the approximation of functions v in $L_2(\mathbb{U}^\infty, X, \gamma)$ is based on the truncation of polynomial expansions by the Hermite series

$$v(\boldsymbol{y}) = \sum_{\boldsymbol{s}\in\mathbb{F}} v_{\boldsymbol{s}}\, H_{\boldsymbol{s}}(\boldsymbol{y}), \quad v_{\boldsymbol{s}} \in X. \tag{2.1}$$

Here $\mathbb{F}$ is the set of all sequences of non-negative integers $\boldsymbol{s} = (s_j)_{j\in\mathbb{N}}$ such that their support $\operatorname{supp}(\boldsymbol{s}) := \{j \in \mathbb{N} : s_j > 0\}$ is a finite set, and

$$H_{\boldsymbol{s}}(\boldsymbol{y}) = \bigotimes_{j\in\mathbb{N}} H_{s_j}(y_j), \quad v_{\boldsymbol{s}} := \int_{\mathbb{R}^\infty} v(\boldsymbol{y})\, H_{\boldsymbol{s}}(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}), \quad \boldsymbol{s} \in \mathbb{F}, \tag{2.2}$$

with $(H_k)_{k\geq 0}$ being the Hermite polynomials normalized according to $\int_{\mathbb{R}} |H_k(y)|^2\, g(y)\,\mathrm{d}y = 1$. It is well-known that $(H_{\boldsymbol{s}})_{\boldsymbol{s}\in\mathbb{F}}$ is an orthonormal basis of $L_2(\mathbb{R}^\infty, \mathbb{R}, \gamma)$. Moreover, $L_2(\mathbb{U}^\infty, X, \gamma) = L_2(\mathbb{U}^\infty, \mathbb{R}, \gamma) \otimes X$, and for every $v \in L_2(\mathbb{U}^\infty, X, \gamma)$ represented by the series (2.1) there holds Parseval's identity

$$\|v\|^2_{L_2(\mathbb{U}^\infty,X,\gamma)} = \sum_{\boldsymbol{s}\in\mathbb{F}} \|v_{\boldsymbol{s}}\|_X^2.$$

We make use the obbreviations: $L_p(\mathbb{R}^\infty, \mu) := L_p(\mathbb{R}^\infty, \mathbb{R}, \mu)$; $\mathcal{V}_p(X) := L_p(\mathbb{R}^\infty, X, \gamma)$ for $0 < p < \infty$. We use letter C to denote a general positive constant which may take different values, and $C_{p,q,\alpha,D,\ldots}$ a constant depending on $p, q, \alpha, D, \ldots$.

Let X^1 and X^2 be Hilbert spaces. To construct general fully discrete linear Galerkin methods of approximation in the Bochner space $L_p(\mathbb{R}^\infty, X^1, \gamma)$ of functions taking values in X^2 , we need the following assumption on approximation property for X^1 and X^2.

Assumption II The Hilbert space X^2 a linear subspace of the Hilbert space X^1 and $\|\cdot\|_{X^1} \leq C\,\|\cdot\|_{X^2}$. There are a sequence $(V_n)_{n\in\mathbb{N}_0}$ of subspaces $V_n \subset X^1$ of dimension $\leq n$, and sequence $(P_n)_{n\in\mathbb{N}_0}$ of linear operators from X^1 into V_n, and a number $\alpha > 0$ such that

$$\|P_n(v)\|_{X^1} \leq C, \quad \|v - P_n(v)\|_{X^1} \leq Cn^{-\alpha}\|v\|_{X^2}, \quad \forall n \in \mathbb{N}_0, \quad \forall v \in X^2. \tag{2.3}$$

For $k \in \mathbb{N}_0$, we define

$$\delta_k(v) := P_{2^k}(v) - P_{2^{k-1}}(v), \; k \in \mathbb{N}, \quad \delta_0(v) = P_0(v).$$

We can represent every $v \in X^2$ by the series

$$v = \sum_{k=0}^{\infty} \delta_k(v)$$

converging in X^1 and satisfying the estimate

$$\|\delta_k(v)\|_{X^1} \leq C\, 2^{-\alpha k}\, \|v\|_{X^2}, \quad k \in \mathbb{N}_0. \tag{2.4}$$

For a subset G in $\mathbb{N}_0 \times \mathbb{F}$, denote by $\mathcal{V}(G)$ the subspace in $\mathcal{V}_2(X^1)$ of all functions v of the form

$$v = \sum_{(k,\boldsymbol{s}) \in G} v_k\, H_{\boldsymbol{s}}, \quad v_k \in V_{2^k}.$$

Let Assumption II hold for Hilbert spaces X^1 and X^2. We define the linear operator $\mathcal{S}_G : \mathcal{V}_2(X^2) \to \mathcal{V}(G)$ by

$$\mathcal{S}_G v := \sum_{(k,\boldsymbol{s}) \in G} \delta_k(v_{\boldsymbol{s}})\, H_{\boldsymbol{s}}$$

for $v \in \mathcal{V}_2(X^2)$ represented by the series

$$v = \sum_{\boldsymbol{s} \in \mathbb{F}} v_{\boldsymbol{s}} H_{\boldsymbol{s}}, \quad v_{\boldsymbol{s}} \in X^2. \tag{2.5}$$

Lemma 2.1 *Let Assumption II hold for Hilbert spaces X^1 and X^2. Then for every $v \in \mathcal{V}_2(X^2)$,*

$$\lim_{K \to \infty} \|v - \mathcal{S}_{G_K} v\|_{\mathcal{V}_2(X^1)} = 0, \tag{2.6}$$

where $G_K := \{(k, \boldsymbol{s}) \in \mathbb{N}_0 \times \mathbb{F} : 0 \leq k \leq K\}$.

Proof. Obviously, by the definition,

$$\mathcal{S}_{G_K} v = \sum_{\boldsymbol{s} \in \mathbb{F}} \sum_{k=0}^{K} \delta_k(v_{\boldsymbol{s}})\, H_{\boldsymbol{s}} = \sum_{\boldsymbol{s} \in \mathbb{F}} P_{2^K}(v_{\boldsymbol{s}})\, H_{\boldsymbol{s}}.$$

From Parseval's identity and (2.3) it follows that

$$\begin{aligned} \|\mathcal{S}_{G_K} v\|^2_{\mathcal{V}_2(X^1)} &= \sum_{\boldsymbol{s} \in \mathbb{F}} \|P_{2^K}(v_{\boldsymbol{s}})\|^2_{X^1} \leq 2 \sum_{\boldsymbol{s} \in \mathbb{F}} \|v_{\boldsymbol{s}}\|^2_{X^1} + 2 \sum_{\boldsymbol{s} \in \mathbb{F}} \|v_{\boldsymbol{s}} - P_{2^K}(v_{\boldsymbol{s}})\|^2_{X^1} \\ &\leq 2 \sum_{\boldsymbol{s} \in \mathbb{F}} \|v_{\boldsymbol{s}}\|^2_{X^1} + 2C^2 2^{-\alpha K} \sum_{\boldsymbol{s} \in \mathbb{F}} \|v_{\boldsymbol{s}}\|^2_{X^2} < \infty. \end{aligned} \tag{2.7}$$

This means that $\mathcal{S}_{G_K} v \in \mathcal{V}_2(X^1)$. Hence, by Parseval's identity and (2.3) we deduce that

$$\|v - \mathcal{S}_{G_K} v\|^2_{\mathcal{V}_2(X^1)} = \sum_{\boldsymbol{s} \in \mathbb{F}} \|v_{\boldsymbol{s}} - P_{2^K}(v_{\boldsymbol{s}})\|^2_{X^1} \leq C^2\, 2^{-2\alpha K} \sum_{\boldsymbol{s} \in \mathbb{F}} \|v_{\boldsymbol{s}}\|^2_{X^2} = C^2\, 2^{-2\alpha K} \|v\|^2_{\mathcal{V}(X^2)}$$

which prove the lemma. ∎

Theorem 2.1 *Let $0 < p \le 2$. Let Assumption II hold for Hilbert spaces X^1 and X^2. Let $v \in \mathcal{V}_2(X^2)$ be represented by the series (2.5). Assume that for $r = 1, 2$ there exist sequences $(\sigma_{r;\boldsymbol{s}})_{\boldsymbol{s}\in\mathbb{F}}$ of numbers strictly larger than 1 such that*

$$\sum_{\boldsymbol{s}\in\mathbb{F}} (\sigma_{r;\boldsymbol{s}} \|v_{\boldsymbol{s}}\|_{X^r})^2 < \infty$$

and $(\sigma_{r;\boldsymbol{s}}^{-1})_{\boldsymbol{s}\in\mathbb{F}} \in \ell_{q_r}(\mathbb{F})$ for some $0 < q_1 \le q_2 < \infty$. Define for $\xi > 0$

$$G(\xi) := \begin{cases} \left\{(k, \boldsymbol{s}) \in \mathbb{N}_0 \times \mathbb{F} : \, 2^k \sigma_{2;\boldsymbol{s}}^{q_2} \le \xi \right\} & \text{if } \alpha \le 1/q_2; \\ \left\{(k, \boldsymbol{s}) \in \mathbb{N}_0 \times \mathbb{F} : \, \sigma_{1;\boldsymbol{s}}^{q_1} \le \xi, \; 2^{\alpha q_1 k} \sigma_{2;\boldsymbol{s}}^{q_1} \le \xi \right\} & \text{if } \alpha > 1/q_2. \end{cases} \tag{2.8}$$

Then for each $n \in \mathbb{N}$ there exists a number ξ_n such that $\dim(\mathcal{V}(G(\xi_n)) \le n$ and

$$\|v - \mathcal{S}_{G(\xi_n)} v\|_{\mathcal{V}_p(X^1)} \le C n^{-\min(\alpha, \beta)}. \tag{2.9}$$

The rate α corresponds to the approximation of a single function in X^2 as given by (2.3), and the rate β is given by

$$\beta := \frac{1}{q_1} \frac{\alpha}{\alpha + \delta}, \quad \delta := \frac{1}{q_1} - \frac{1}{q_2}. \tag{2.10}$$

The constant C in (2.9) is independent of v and n.

Proof. Due to the inequality $\|\cdot\|_{\mathcal{V}_p(X^1)} \le \|\cdot\|_{\mathcal{V}_2(X^1)}$, it is sufficient to prove the theorem for $p = 2$.

We first consider the case $\alpha \le 1/q_2$. Let $\xi > 0$ be given and take arbitrary positive number ε. Since $G(\xi)$ is finite, from the definition of G_K and Lemma 2.1 it follows that there exists $K = K(\xi, \varepsilon)$ such that $G(\xi) \subset G_K$ and

$$\|v - \mathcal{S}_{G_K} v\|_{\mathcal{V}_2(X^1)} \; \le \; \varepsilon. \tag{2.11}$$

By the triangle inequality,

$$\|v - \mathcal{S}_{G(\xi)} v\|_{\mathcal{V}_2(X^1)} \; \le \; \|v - \mathcal{S}_{G_K} v\|_{\mathcal{V}_2(X^1)} \; + \; \|\mathcal{S}_{G_K} v - \mathcal{S}_{G(\xi)} v\|_{\mathcal{V}_2(X^1)}. \tag{2.12}$$

We have by Parseval's identity and (2.4) that

$$\begin{aligned} \|\mathcal{S}_{G_K} v - \mathcal{S}_{G(\xi)} v\|_{\mathcal{V}_2(X^1)}^2 \; &= \; \Big\| \sum_{\boldsymbol{s}\in\mathbb{F}} \sum_{k=0}^{K} \delta_k(v_{\boldsymbol{s}}) \, H_{\boldsymbol{s}} - \sum_{\boldsymbol{s}\in\mathbb{F}} \sum_{2^k \sigma_{2;\boldsymbol{s}}^{q_2} \le \xi} \delta_k(v_{\boldsymbol{s}}) \, H_{\boldsymbol{s}} \Big\|_{\mathcal{V}_2(X^1)}^2 \\ &= \; \Big\| \sum_{\boldsymbol{s}\in\mathbb{F}} \sum_{\xi \sigma_{2;\boldsymbol{s}}^{-q_2} < 2^k \le 2^K} \delta_k(v_{\boldsymbol{s}}) \, H_{\boldsymbol{s}} \Big\|_{\mathcal{V}_2(X^1)}^2 \; = \; \sum_{\boldsymbol{s}\in\mathbb{F}} \Big\| \sum_{\xi \sigma_{2;\boldsymbol{s}}^{-q_2} < 2^k \le 2^K} \delta_k(v_{\boldsymbol{s}}) \Big\|_{X^1}^2 \\ &\le \; \sum_{\boldsymbol{s}\in\mathbb{F}} \Big(\sum_{\xi \sigma_{2;\boldsymbol{s}}^{-q_2} < 2^k \le 2^K} \|\delta_k(v_{\boldsymbol{s}})\|_{X^1} \Big)^2 \; \le \; \sum_{\boldsymbol{s}\in\mathbb{F}} \Big(\sum_{\xi \sigma_{2;\boldsymbol{s}}^{-q_2} < 2^k \le 2^K} C \, 2^{-\alpha k} \|v_{\boldsymbol{s}}\|_{X^2} \Big)^2 \\ &\le \; C \sum_{\boldsymbol{s}\in\mathbb{F}} \|v_{\boldsymbol{s}}\|_{X^2}^2 \Big(\sum_{2^k > \xi \sigma_{2;\boldsymbol{s}}^{-q_2}} 2^{-\alpha k} \Big)^2 \; \le \; C \sum_{\boldsymbol{s}\in\mathbb{F}} \|v_{\boldsymbol{s}}\|_{X^2}^2 \; (\xi \sigma_{2;\boldsymbol{s}}^{-q_2})^{-2\alpha}. \end{aligned}$$

Hence, by the inequalities $q_2 \alpha \le 1$ and $\sigma_{2;\boldsymbol{s}} > 1$ we derive that

$$\|\mathcal{S}_{G_K} v - \mathcal{S}_{G(\xi)} v\|_{\mathcal{V}_2(X^1)}^2 \; \le \; C \sum_{\boldsymbol{s}\in\mathbb{F}} (\sigma_{2;\boldsymbol{s}} \|v_{\boldsymbol{s}}\|_{X^2})^2 \; = \; C \, \xi^{-2\alpha}.$$

Since $\varepsilon > 0$ is arbitrary, from the last estimates and (2.11) and (2.12) we derive that

$$\|v - \mathcal{S}_{G(\xi)} v\|_{\mathcal{V}_2(X^1)} \ \leq \ C\,\xi^{-\alpha}. \tag{2.13}$$

For the dimension of the space $\mathcal{V}(G(\xi))$ we have that

$$\begin{aligned}
\dim \mathcal{V}(G(\xi)) \ &\leq \ \sum_{(k,\boldsymbol{s})\in G(\xi)} \dim V_{2^k} \ \leq \ \sum_{(k,\boldsymbol{s})\in G(\xi)} 2^k \\
&\leq \ \sum_{\boldsymbol{s}\in\mathbb{F}} \ \sum_{2^k \ \leq \xi \sigma_{2;\boldsymbol{s}}^{-q_2}} 2^k \ \leq \ 2 \sum_{\boldsymbol{s}\in\mathbb{F}} \xi \sigma_{2;\boldsymbol{s}}^{-q_2} \ = \ M\,\xi,
\end{aligned}$$

where $M := 2\big\|\big(\sigma_{2;\boldsymbol{s}}^{-1}\big)\big\|_{\ell_{q_2}(\mathbb{F})}^{q_2}$. For any $n \in \mathbb{N}$, letting ξ_n be a number satisfying the inequalities

$$M\,\xi_n \ \leq \ n \ < \ 2M\,\xi_n, \tag{2.14}$$

we derive that $\dim \mathcal{V}(G(\xi_n)) \leq n$. On the other hand, from (2.14) it follows that $\xi_n^{-\alpha} \ \leq \ (2M)^\alpha\, n^{-\alpha}$. This together with (2.13) proves that

$$\|v - \mathcal{S}_{G(\xi_n)} v\|_{\mathcal{V}_2(X^1)} \ \leq \ C\,n^{-\alpha}, \quad \alpha \leq 1/q_2. \tag{2.15}$$

We now consider the case $\alpha > 1/q_2$. Putting

$$v_\xi := \sum_{\sigma_{1;\boldsymbol{s}}^{q_1} \leq \xi} v_{\boldsymbol{s}} H_{\boldsymbol{s}},$$

we get

$$\|v - \mathcal{S}_{G(\xi)} v\|_{\mathcal{V}_2(X^1)} \ \leq \ \|v - v_\xi\|_{\mathcal{V}_2(X^1)} + \|v_\xi - \mathcal{S}_{G(\xi)} v\|_{\mathcal{V}_2(X^1)}.$$

The norms in the right hand side can be estimated using Parseval's identity and the hypothesis of the theorem. Thus, or the norm $\|v - v_\xi\|_{\mathcal{V}_2(X^1)}$ we have that

$$\begin{aligned}
\|v - v_\xi\|_{\mathcal{V}_2(X^1)}^2 \ &= \ \sum_{\sigma_{1;\boldsymbol{s}} > \xi^{1/q_1}} \|v_{\boldsymbol{s}}\|_{X^1}^2 \ = \ \sum_{\sigma_{1;\boldsymbol{s}} > \xi^{1/q_1}} (\sigma_{1;\boldsymbol{s}} \|v_{\boldsymbol{s}}\|_{X^1})^2 \sigma_{1;\boldsymbol{s}}^{-2} \\
&\leq \ \xi^{-2/q_1} \sum_{\sigma_{1;\boldsymbol{s}} > \xi^{1/q_1}} (\sigma_{1;\boldsymbol{s}} \|v_{\boldsymbol{s}}\|_{X^1})^2 \leq C\xi^{-2/q_1}.
\end{aligned} \tag{2.16}$$

Anh for the norm $\|v_\xi - \mathcal{S}_{G(\xi)} v\|_{\mathcal{V}_2(X^1)}$, with $N = N(\xi, \boldsymbol{s}) := 2^{\lfloor \log_2 (\xi^{1/q_1 \alpha} \sigma_{2;\boldsymbol{s}}^{-1/\alpha}) \rfloor}$ we have that

$$\begin{aligned}
\|v_\xi - \mathcal{S}_{G(\xi)} v\|_{\mathcal{V}_2(X^1)}^2 \ &= \ \sum_{\sigma_{1;\boldsymbol{s}} > \xi^{1/q_1}} \Big\| v_{\boldsymbol{s}} - \sum_{2^{\alpha q_1 k} \sigma_{2;\boldsymbol{s}}^{q_1} \leq \xi} \delta_k(v_{\boldsymbol{s}}) \Big\|_{X^1}^2 \ \leq \ \sum_{\boldsymbol{s}\in\mathbb{F}} \Big\| v_{\boldsymbol{s}} - P_N(v_{\boldsymbol{s}}) \Big\|_{X^1}^2 \\
&\leq \ \sum_{\boldsymbol{s}\in\mathbb{F}} C N^{-2\alpha} \|v_{\boldsymbol{s}}\|_{X^2}^2 \leq C\xi^{-2/q_1} \sum_{\boldsymbol{s}\in\mathbb{F}} (\sigma_{2;\boldsymbol{s}} \|v_{\boldsymbol{s}}\|_{X^2})^2 \leq C\xi^{-2/q_1}.
\end{aligned}$$

These estimates yield that

$$\|v - \mathcal{S}_{G(\xi)} v\|_{\mathcal{V}_2(X^1)} \ \leq \ C\xi^{-1/q_1}. \tag{2.17}$$

For the dimension of the space $\mathcal{V}(G(\xi))$, with $q := q_2\alpha > 1$ and $1/q' + 1/q = 1$ we have that

$$\begin{aligned}
\dim \mathcal{V}(G(\xi)) \ &\le \ \sum_{(k,\boldsymbol{s})\in G(\xi)} \dim V_{2^k} \ \le \ \sum_{\sigma_{1;\boldsymbol{s}}^{q_1}\le \xi} \ \sum_{2^{\alpha q_1 k}\sigma_{2;\boldsymbol{s}}^{q_1}\le\xi} 2^k \\
&\le \ 2 \sum_{\sigma_{1;\boldsymbol{s}}^{q_1}\le \xi} \xi^{1/(q_1\alpha)} \sigma_{2;\boldsymbol{s}}^{-1/\alpha} \ \le \ 2\xi^{1/(q_1\alpha)} \left(\sum_{\sigma_{1;\boldsymbol{s}}^{q_1}\le \xi} \sigma_{2;\boldsymbol{s}}^{-q_2} \right)^{1/q} \left(\sum_{\sigma_{1;\boldsymbol{s}}^{q_1}\le \xi} 1 \right)^{1/q'} \\
&\le 2\xi^{1/(q_1\alpha)} \left(\sum_{\boldsymbol{s}\in\mathbb{F}} \sigma_{2;\boldsymbol{s}}^{-q_2} \right)^{1/q} \left(\sum_{\boldsymbol{s}\in\mathbb{F}} \xi \sigma_{1;\boldsymbol{s}}^{-q_1} \right)^{1/q'} \ = \ M\xi^{1+\delta/\alpha},
\end{aligned}$$

where $M := 2\left\|\left(\sigma_{2;\boldsymbol{s}}^{-1}\right)\right\|_{\ell_{q_2}(\mathbb{F})}^{q_2/q} \left\|\left(\sigma_{1;\boldsymbol{s}}^{-1}\right)\right\|_{\ell_{q_1}(\mathbb{F})}^{q_1/q'}$. For any $n \in \mathbb{N}$, letting ξ_n be a number satisfying the inequalities

$$M\,\xi_n^{1+\delta/\alpha} \ \le \ n \ < \ 2M\,\xi_n^{1+\delta/\alpha}, \tag{2.18}$$

we derive that $\dim \mathcal{V}(G(\xi_n)) \le n$. On the other hand, by (2.18),

$$\xi_n^{-1/q_1} \ \le \ (2M)^{\frac{\alpha}{\alpha+\delta}}\, n^{-\frac{1}{q_1}\frac{\alpha}{\alpha+\delta}}.$$

This together with (2.17) proves that

$$\|v - \mathcal{S}_{G(\xi_n)} v\|_{\mathcal{V}_2(X^1)} \ \le \ Cn^{-\beta}, \quad \alpha > 1/q_2. \tag{2.19}$$

By combining the last esimate and (2.15) we obtain (2.9). ◻

3 Polynomial interpolation approximation

In this section, we construct general fully discrete linear polynomial interpolation methods of approximation in the Bochner space $\mathcal{V}_p(X^1)$ of functions taking values in X^2 and having a weighted ℓ_2-summability of Hermite expansion coefficients for Hilbert spaces X^1 and X^2 satisfying a certain "spatial" approximation property. In particular, we prove convergence rates for these methods of approximation. We also briefly consider non-fully discrete linear polynomial interpolation methods of approximation.

3.1 Auxiliary results

Let $w = \exp(-Q)$, where Q is an even function on $\mathbb{R}$ and $yQ'(y)$ is positive and increasing in $(0,\infty)$, with limits 0 and ∞ at 0 and ∞. For $n \in \mathbb{N}$, the nth Mhaskar-Rakhmanov-Saff number $a_n = a_n(w)$ is defined as the positive root of the equation

$$n \ = \ \frac{2}{\pi}\int_0^1 \frac{a_n y Q'(a_n y)}{\sqrt{1-y^2}}\, \mathrm{d}y.$$

From [25, Page 11] we have for the weight $\sqrt{g}$,

$$a_n(\sqrt{g}) \ = \ \sqrt{n}. \tag{3.1}$$

For $0 < p, q \le \infty$, we introduce the quantity

$$\delta(p,q) := \ \frac{1}{2}\left|\frac{1}{p} - \frac{1}{q}\right|.$$

Lemma 3.1 *Let $0 < p,q \le \infty$. Then exists a positive constant $C_{p,q}$ such that for every polynomial φ of degree $\le n$, there holds the Nikol'skii-type inequality*

$$\|\varphi\sqrt{g}\|_{L_p(\mathbb{R})} \ \le \ C_{p,q} n^{\delta(p,q)} \, \|\varphi\sqrt{g}\|_{L_q(\mathbb{R})}.$$

Proof. This lemma is an immediate consequence of (3.1) and the inequality

$$\|\varphi\sqrt{g}\|_{L_p(\mathbb{R})} \ \le \ C_{p,q} N_n(p,q) \, \|\varphi\sqrt{g}\|_{L_q(\mathbb{R})}$$

which follows from [25, Theorem 9.1, p. 61], where

$$N_n(p,q) := \begin{cases} a_n^{1/p-1/q}, & p < q, \\ \left(\frac{n}{a_n}\right)^{1/q-1/p}, & p > q. \end{cases}$$

□

Lemma 3.2 *We have*

$$\|H_n\sqrt{g}\|_{L_\infty(\mathbb{R})} \ \le \ 1, \quad n \in \mathbb{N}_0. \tag{3.2}$$

Proof. From Cramér's bound (see, e.g., [17, Page 208, (19)]) we have for every $n \in \mathbb{N}_0$ and every $x \in \mathbb{R}$, $|H_n(x)\sqrt{g(x)}| \le K(2\pi)^{-1/4}$, where $K := 1.086435$. This implies (3.2). □

For our application the estimate (3.2) is sufficient, see [12] for an improvement.

For $\theta, \lambda \ge 0$, we define the sequence

$$p_{\boldsymbol{s}}(\theta,\lambda) := \prod_{j\in\mathbb{N}} (1+\lambda s_j)^\theta, \quad \boldsymbol{s} \in \mathbb{F}. \tag{3.3}$$

Lemma 3.3 *Let $0 < p \le 2$ and X be a Hilbert space. Let $v \in L_2(\mathbb{R}^\infty, X, \gamma)$ be represented by the series (2.1). Assume that there exists a sequence $\boldsymbol{\sigma} = (\sigma_{\boldsymbol{s}})_{\boldsymbol{s}\in\mathbb{F}}$ of positive numbers such that*

$$\sum_{\boldsymbol{s}\in\mathbb{F}} (\sigma_{\boldsymbol{s}}\|v_{\boldsymbol{s}}\|_X)^2 < \infty.$$

We have the following.

(i) *If $\left(p_{\boldsymbol{s}}(\theta,\lambda)\sigma_{\boldsymbol{s}}^{-1}\right)_{\boldsymbol{s}\in\mathbb{F}} \in \ell_q(\mathbb{F})$ for some $0 < q \le 2$ and $\theta, \lambda \ge 0$, then $(p_{\boldsymbol{s}}(\theta,\lambda)\|v_{\boldsymbol{s}}\|_X)_{\boldsymbol{s}\in\mathbb{F}} \in \ell_{\bar q}(\mathbb{F})$ for $\bar q$ such that $1/\bar q = 1/2 + 1/q$.*

(ii) *If $\left(\sigma_{\boldsymbol{s}}^{-1}\right)_{\boldsymbol{s}\in\mathbb{F}} \in \ell_q(\mathbb{F})$ for some $0 < q \le 2$, then the series (2.1) converges absulutely in $\mathcal{V}_p(X)$ to v.*

Proof. Since $\tau := 2/\bar q \ge 1$, with $1/\tau + 1/\tau' = 1$ and $p_{\boldsymbol{s}} = p_{\boldsymbol{s}}(\theta,\lambda)$ by the Hölder inequality we have that

$$\begin{aligned}\sum_{\boldsymbol{s}\in\mathbb{F}} (p_{\boldsymbol{s}}\|v_{\boldsymbol{s}}\|_X)^{\bar q} &\le \left(\sum_{\boldsymbol{s}\in\mathbb{F}} (\sigma_{\boldsymbol{s}}^{\bar q}\|v_{\boldsymbol{s}}\|_X^{\bar q})^\tau\right)^{1/\tau} \left(\sum_{\boldsymbol{s}\in\mathbb{F}} \left(p_{\boldsymbol{s}}^{\bar q}\sigma_{\boldsymbol{s}}^{-\bar q}\right)^{\tau'}\right)^{1/\tau'} \\ &= \left(\sum_{\boldsymbol{s}\in\mathbb{F}} (\sigma_{\boldsymbol{s}}\|v_{\boldsymbol{s}}\|_X)^2\right)^{\bar q/2} \left(\sum_{\boldsymbol{s}\in\mathbb{F}} \left(p_{\boldsymbol{s}}\sigma_{\boldsymbol{s}}^{-1}\right)^q\right)^{1-\bar q/2} < \infty\,.\end{aligned}$$

This proves the assersion (i).

We have by the inequality $\bar{q} \le 1$ and (i) for $\theta = \lambda = 0$,

$$\sum_{\boldsymbol{s}\in\mathbb{F}} \|v_{\boldsymbol{s}} H_{\boldsymbol{s}}\|_{\mathcal{V}_2(X)} \ = \ \sum_{\boldsymbol{s}\in\mathbb{F}} \|v_{\boldsymbol{s}}\|_X \|H_{\boldsymbol{s}}\|_{L_2(\mathbb{R}^\infty,\gamma)}$$

$$\le \ \sum_{\boldsymbol{s}\in\mathbb{F}} \|v_{\boldsymbol{s}}\|_X \ \le \ \left(\sum_{\boldsymbol{s}\in\mathbb{F}} (\|v_{\boldsymbol{s}}\|_X)^{\bar{q}}\right)^{1/\bar{q}} \ < \ \infty.$$

This yields that the series (2.1) absolutely converges in $\mathcal{V}_2(X)$ to v, since by the assumption this series converges in $\mathcal{V}_2(X)$ to v. The assertion (ii) is proven for the case $p = 2$. The case $0 < p < 2$ is derived from the case $p = 2$ and the inequality $\|\cdot\|_{\mathcal{V}_p(X)} \le \|\cdot\|_{\mathcal{V}_2(X)}$. ◻

Lemma 3.4 *Let $0 < p \le 2$. Let Assumption II hold for Hilbert spaces X^1 and X^2, and let the assumptions of Lemma 3.3(ii) hold for the space X^2. Then every $v \in \mathcal{V}_2(X^2)$ can be represented as the series*

$$v \ = \sum_{(k,\boldsymbol{s})\in\mathbb{N}_0\times\mathbb{F}} \delta_k(v_{\boldsymbol{s}})\, H_{\boldsymbol{s}} \tag{3.4}$$

converging absolutely in $\mathcal{V}_p(X^1)$.

Proof. Similarly to the proof of Lemma 3.3, it is sufficient to prove the lemma for the case $p = 2$. With $\bar{q}$ as in Lemma 3.3, we have by (2.3), Lemma 3.3(ii) and the inequality $\bar{q} \le 1$,

$$\sum_{(k,\boldsymbol{s})\in\mathbb{N}_0\times\mathbb{F}} \|\delta_k(v_{\boldsymbol{s}}) H_{\boldsymbol{s}}\|_{\mathcal{V}_2(X^1)} \ = \sum_{(k,\boldsymbol{s})\in\mathbb{N}_0\times\mathbb{F}} \|\delta_k(v_{\boldsymbol{s}})\|_{X^1} \ \le \ C \sum_{(k,\boldsymbol{s})\in\mathbb{N}_0\times\mathbb{F}} 2^{-\alpha k} \|v_{\boldsymbol{s}}\|_{X^2}$$

$$\le \ C \sum_{\boldsymbol{s}\in\mathbb{F}} \|v_{\boldsymbol{s}}\|_{X^2} \sum_{k\in\mathbb{N}_0} 2^{-\alpha k} \ \le \ C \sum_{\boldsymbol{s}\in\mathbb{F}} \|v_{\boldsymbol{s}}\|_{X^2}$$

$$\le \ C \left(\sum_{\boldsymbol{s}\in\mathbb{F}} \|v_{\boldsymbol{s}}\|_{X^2}^{\bar{q}}\right)^{1/\bar{q}} \ < \ \infty.$$

This means that the series in (3.4) converges absolutely to v, since by Lemma 2.6 the sum S_{G_K} converges in $\mathcal{V}_2(X)$ to v when $K \to \infty$. ◻

We will need the following two lemmata for application in estimating the convergence rate of the fully discrete polynomial interpolation approximation in this section and of integration in Section 4.

Lemma 3.5 *Under the hypothesis of Theorem 2.1, assume in addition that $q_1 < 2$. Define for $\xi > 0$*

$$G(\xi) := \begin{cases} \left\{(k,\boldsymbol{s}) \in \mathbb{N}_0 \times \mathbb{F} : \, 2^k \sigma_{2;\boldsymbol{s}}^{q_2} \le \xi\right\} & \text{if } \alpha \le 1/q_2 - 1/2; \\ \left\{(k,\boldsymbol{s}) \in \mathbb{N}_0 \times \mathbb{F} : \, \sigma_{1;\boldsymbol{s}}^{q_1} \le \xi, \ 2^{\alpha q_1 k} \sigma_{2;\boldsymbol{s}}^{q_1} \le \xi\right\} & \text{if } \alpha > 1/q_2 - 1/2. \end{cases} \tag{3.5}$$

Then for each $\xi > 0$,

$$\|v - \mathcal{S}_{G(\xi_n)} v\|_{\mathcal{V}_p(X^1)} \ \le \ C \times \begin{cases} \xi^{-\alpha} & \text{if } \alpha \le 1/q_2 - 1/2; \\ \xi^{-(1/q_1 - 1/2)} & \text{if } \alpha > 1/q_2 - 1/2. \end{cases} \tag{3.6}$$

The rate α is given by (2.3). The constant C in (3.6) is independent of v and ξ.

Proof. Similarly to the proof of Lemma 3.3, it is sufficient to prove the lemma for the case $p = 2$. Since in the case $\alpha \le 1/q_2 - 1/2$, the formulas (2.8) and (3.5) define the same set $G(\xi)$ for $\xi > 0$, from (2.13) follows the lemma for this case. Let us consider the case $\alpha > 1/q_2 - 1/2$. Putting

$$v_\xi := \sum_{\sigma_{1;\boldsymbol{s}}^{q_1} \le \xi} v_{\boldsymbol{s}} H_{\boldsymbol{s}},$$

we get

$$\|v - \mathcal{S}_{G(\xi)} v\|_{\mathcal{V}_2(X^1)} \le \|v - v_\xi\|_{\mathcal{V}_2(X^1)} + \|v_\xi - \mathcal{S}_{G(\xi)} v\|_{\mathcal{V}_2(X^1)}. \tag{3.7}$$

It is well kown that the unconditional convergence in a Banach space follows from the absolute convergence. Thus, by Lemma 3.4 the series (3.4) converges unconditionally to v. Hence the norm $\|v - v_\xi\|_{\mathcal{V}_2(X^1)}$ can be esitimated as

$$\begin{aligned}
\|v - v_\xi\|_{\mathcal{V}_2(X^1)} &\le \sum_{\sigma_{1;\boldsymbol{s}} > \xi^{1/q_1}} \|v_{\boldsymbol{s}}\|_{X^1} \, \|H_{\boldsymbol{s}}\|_{L_2(\mathbb{R}^\infty,\gamma)} = \sum_{\sigma_{1;\boldsymbol{s}} > \xi^{1/q_1}} \|v_{\boldsymbol{s}}\|_{X^1} \\
&\le \left(\sum_{\sigma_{1;\boldsymbol{s}} > \xi^{1/q_1}} (\sigma_{1;\boldsymbol{s}} \|v_{\boldsymbol{s}}\|_{X^1})^2 \right)^{1/2} \left(\sum_{\sigma_{1;\boldsymbol{s}} > \xi^{1/q_1}} \sigma_{1;\boldsymbol{s}}^{-2} \right)^{1/2} \\
&\le C \left(\sum_{\sigma_{1;\boldsymbol{s}} > \xi^{1/q_1}} \sigma_{1;\boldsymbol{s}}^{-q_1} \sigma_{1;\boldsymbol{s}}^{-(2-q_1)} \right)^{1/2} \\
&\le C \xi^{-(1/q_1 - 1/2)} \left(\sum_{\boldsymbol{s} \in \mathbb{F}} \sigma_{1;\boldsymbol{s}}^{-q_1} \right)^{1/2} \le C \xi^{-(1/q_1 - 1/2)}.
\end{aligned} \tag{3.8}$$

For the norm $\|v_\xi - \mathcal{S}_{G(\xi)} v\|_{\mathcal{V}_2(X^1)}$, with $N = N(\xi, \boldsymbol{s}) := 2^{\lfloor \log_2 (\xi^{1/(q_1\alpha)} \sigma_{2;\boldsymbol{s}}^{-1/\alpha}) \rfloor}$ we have

$$\begin{aligned}
\|v_\xi - \mathcal{S}_{G(\xi)} v\|_{\mathcal{V}_2(X^1)} &\le \sum_{\sigma_{1;\boldsymbol{s}}^{q_1} \le \xi} \Big\| v_{\boldsymbol{s}} - \sum_{2^{\alpha q_1 k} \sigma_{2;\boldsymbol{s}}^{q_1} \le \xi} \delta_k(v_{\boldsymbol{s}}) \Big\|_{X^1} \|H_{\boldsymbol{s}}\|_{L_2(\mathbb{R}^\infty,\gamma)} \\
&= C \sum_{\sigma_{1;\boldsymbol{s}}^{q_1} \le \xi} \Big\| v_{\boldsymbol{s}} - P_N(v_{\boldsymbol{s}}) \Big\|_{X^1} \le C \sum_{\sigma_{1;\boldsymbol{s}}^{q_1} \le \xi} N^{-\alpha} \|v_{\boldsymbol{s}}\|_{X^2} \\
&\le C \sum_{\sigma_{1;\boldsymbol{s}}^{q_1} \le \xi} (\xi^{1/(q_1\alpha)} \sigma_{2;\boldsymbol{s}}^{-1/\alpha})^{-\alpha} \|v_{\boldsymbol{s}}\|_{X^2} \le C \, \xi^{-1/q_1} \sum_{\sigma_{1;\boldsymbol{s}}^{q_1} \le \xi} \sigma_{2;\boldsymbol{s}} \|v_{\boldsymbol{s}}\|_{X^2} \\
&\le C \, \xi^{-1/q_1} \left(\sum_{\sigma_{1;\boldsymbol{s}}^{q_1} \le \xi} (\sigma_{2;\boldsymbol{s}} \|v_{\boldsymbol{s}}\|_{X^2})^2 \right)^{1/2} \left(\sum_{\sigma_{1;\boldsymbol{s}}^{q_1} \le \xi} 1 \right)^{1/2} \\
&\le C \, \xi^{-1/q_1} \left(\sum_{\sigma_{1;\boldsymbol{s}}^{q_1} \le \xi} \sigma_{1;\boldsymbol{s}}^{-q_1} \xi \right)^{1/2} \le C \, \xi^{-(1/q_1 - 1/2)} \left(\sum_{\boldsymbol{s} \in \mathbb{F}} \sigma_{1;\boldsymbol{s}}^{-q_1} \right)^{1/2} \le C \, \xi^{-(1/q_1 - 1/2)}.
\end{aligned}$$

This, (3.7) and (3.8) prove the lemma for the case $\alpha > 1/q_2 - 1/2$. ∎

We make use the notation: $\mathbb{F}_{\rm ev} := \{\boldsymbol{s} \in \mathbb{F} : s_j \text{ even}, \ j \in \mathbb{N}\}$. The following lemma can be proven in a similar way.

Lemma 3.6 *Let $0 < p \le 2$. Let Assumption II hold for Hilbert spaces X^1 and X^2. Let $v \in \mathcal{V}_2(X^2)$ be represented by the series*

$$v = \sum_{\boldsymbol{s} \in \mathbb{F}_{\rm ev}} v_{\boldsymbol{s}} H_{\boldsymbol{s}}, \quad v_{\boldsymbol{s}} \in X^2. \tag{3.9}$$

Assume that for $r = 1, 2$ there exist sequences $(\sigma_{r;\boldsymbol{s}})_{\boldsymbol{s} \in \mathbb{F}_{\rm ev}}$ of numbers strictly larger than 1 such that

$$\sum_{\boldsymbol{s} \in \mathbb{F}_{\rm ev}} (\sigma_{r;\boldsymbol{s}} \|v_{\boldsymbol{s}}\|_{X^r})^2 < \infty$$

and $(\sigma_{r;\boldsymbol{s}}^{-1})_{\boldsymbol{s} \in \mathbb{F}_{\rm ev}} \in \ell_{q_r}(\mathbb{F}_{\rm ev})$ for some $0 < q_1 \le q_2 < \infty$ with $q_1 < 2$. Define for $\xi > 0$,

$$G_{\rm ev}(\xi) := \begin{cases} \{(k, \boldsymbol{s}) \in \mathbb{N}_0 \times \mathbb{F}_{\rm ev} : 2^k \sigma_{2;\boldsymbol{s}}^{q_2} \le \xi\} & \text{if } \alpha \le 1/q_2 - 1/2; \\ \{(k, \boldsymbol{s}) \in \mathbb{N}_0 \times \mathbb{F}_{\rm ev} : \sigma_{1;\boldsymbol{s}}^{q_1} \le \xi, \ 2^{\alpha q_1 k} \sigma_{2;\boldsymbol{s}}^{q_1} \le \xi\} & \text{if } \alpha > 1/q_2 - 1/2. \end{cases} \tag{3.10}$$

Then for each $\xi > 0$,

$$\|v - \mathcal{S}_{G_{\rm ev}(\xi_n)} v\|_{\mathcal{V}_p(X^1)} \ \le \ C \times \begin{cases} \xi^{-\alpha} & \text{if } \alpha \le 1/q_2 - 1/2; \\ \xi^{-(1/q_1 - 1/2)} & \text{if } \alpha > 1/q_2 - 1/2. \end{cases} \tag{3.11}$$

The rate α is given by (2.3). The constant C in (3.11) is independent of v and ξ.

3.2 Interpolation approximation

For every $n \in \mathbb{N}_0$, let $Y_n = (y_{n;k})_{k=0}^n$ be a sequence of points in $\mathbb{R}$ such that

$$-\infty < y_{n;0} < \cdots < y_{n;n-1} < y_{n;n} < +\infty; \quad y_{0;0} = 0. \tag{3.12}$$

If v is a function defined on $\mathbb{R}$ taking values in a Hilbert space X, we define the interpolation formula $I_n(v)$ for $n \in \mathbb{N}_0$ by

$$I_n(v) := \sum_{k=0}^n v(y_{n;k}) \ell_{n;k}, \quad \ell_{n;k}(y) := \prod_{j=0, \ j \ne k}^n \frac{y - y_{n;j}}{y_{n;k} - y_{n;j}}, \tag{3.13}$$

as the unique Lagrange polynomial interpolating v at $y_{n;k}$. Notice that for a function v on $\mathbb{R}$, the function $I_n(v)$ is polynomial of degree $\le n$ and $I_n(\varphi) = \varphi$ for every polynomial φ of degree $\le n$.

For univariate functions on $\mathbb{R}$, let

$$\lambda_n(Y_n) := \sup_{\|v\sqrt{g}\|_{L_\infty(\mathbb{R})} \le 1} \|I_n(v)\sqrt{g}\|_{L_\infty(\mathbb{R})}$$

be the Lebesgue constant. We want to choose sequences Y_n so that for some positive numbers τ and C, there holds the inequlity

$$\lambda_n(Y_n) \ \le \ (Cn + 1)^\tau, \quad \forall n \in \mathbb{N}_0. \tag{3.14}$$

We present two examples of Y_n satisfying (3.14). The first example is the strictly increasing sequence $Y_n^* = (y_{n;k}^*)_{k=0}^n$ of the roots of H_{n+1}. Indeed, it was proven by Matjila and Szabados [26, 27, 30] that

$$\lambda_n(Y_n^*) \ \le \ C(n+1)^{1/6}, \quad n \in \mathbb{N},$$

for some positive constant C independent of n (with the obvious inequality $\lambda_0(Y_0^*) \le 1$). Hence, for every $\varepsilon > 0$, there exists a positive constant C_ε independent of n such that

$$\lambda_n(Y_n^*) \ \le \ (C_\varepsilon n + 1)^{1/6+\varepsilon}, \quad \forall n \in \mathbb{N}_0. \tag{3.15}$$

The minimum distance between consecutive roots d_{n+1} satisfies the inequalities $\frac{\pi\sqrt{2}}{\sqrt{2n+3}} < d_{n+1} < \frac{\sqrt{21}}{\sqrt{2n+3}}$, see [31, pp. 130–131]. The sequences Y_n^* have been used in [4] for sparse quadrature for high-dimensional integration with the measure γ, and in [18] polynomial interpolation approximation with the measure γ.

The inequality (3.15) can be improved if Y_{n-2}^* is slightly modified by the "method of adding points" suggested by Szabados [30] (for details, see also [25, Section 11]). More precisely, for $n > 2$, he added to Y_{n-2}^* two points $\pm\zeta_{n-1}$, near $\pm a_{n-1}(g)$, which are defined by the condition $|H_{n-1}\sqrt{g}|(\zeta_{n-1}) = \|H_{n-1}\sqrt{g}\|_{L_\infty(\mathbb{R})}$. By this way, he obtained the strictly increasing sequence

$$\bar{Y}_n^* := \{-\zeta_n, y_{n-2;0}^*, \cdots, y_{n-2,n-2}^*, \zeta_n\}$$

satisfying the inequality

$$\lambda_n(\bar{Y}_n^*) \ \le \ C\log(n-1) \quad (n > 2)$$

which yields that for every $\varepsilon > 0$, there exists a positive constant C_ε independent of n such that

$$\lambda_n(\bar{Y}_n^*) \ \le \ (C_\varepsilon n + 1)^{\varepsilon}, \quad \forall n \in \mathbb{N}_0.$$

For a given sequence $(Y_n)_{n=0}^\infty$, we define the univariate operator $\Delta_n^{\rm I}$ for $n \in \mathbb{N}_0$ by

$$\Delta_n^{\rm I} := \ I_n - I_{n-1},$$

with the convention $I_{-1} = 0$.

Lemma 3.7 *Assume that $(Y_n)_{n=0}^\infty$ is a sequence such that every Y_n satisfies the condition* (3.14) *for some positive numbers τ and C. Then for every $\varepsilon > 0$, there exists a positive constant C_ε independent of n such that for every function v on $\mathbb{R}$,*

$$\|\Delta_n^{\rm I}(v)\sqrt{g}\|_{L_\infty(\mathbb{R})} \ \le \ (C_\varepsilon n + 1)^{\tau+\varepsilon}\|v\sqrt{g}\|_{L_\infty(\mathbb{R})}, \quad \forall n \in \mathbb{N}_0, \tag{3.16}$$

whenever the norm in the right-hand side is finite.

Proof. From the assumptions we have that

$$\|\Delta_n^{\rm I}(v)\sqrt{g}\|_{L_\infty(\mathbb{R})} \ \le \ 2(Cn + 1)^{\tau}\|v\sqrt{g}\|_{L_\infty(\mathbb{R})}, \quad \forall n \in \mathbb{N}_0,$$

which simlilarly to (3.15) gives (3.16). □

We introduce the operator $\Delta_{\boldsymbol{s}}^{\rm I}$ for $\boldsymbol{s} \in \mathbb{F}$ by

$$\Delta_{\boldsymbol{s}}^{\rm I}(v) := \bigotimes_{j\in\mathbb{N}} \Delta_{s_j}^{\rm I}(v)$$

for functions v defined on $\mathbb{R}^\infty$ taking values in a Hilbert space X, where the univariate operator $\Delta_{s_j}^{\rm I}$ is applied to the univariate function v by considering v as a function of variable y_i with the other variables held fixed. Next, we introduce the operator I_Λ for a given finite set $\Lambda \subset \mathbb{F}$ by

$$I_\Lambda := \sum_{\boldsymbol{s}\in\Lambda} \Delta_{\boldsymbol{s}}^{\rm I}.$$

Let Assumption II hold for Hilbert spaces X^1 and X^2. We introduce the operator $\mathcal{I}_G$ for a given finite set $G \subset \mathbb{N}_0 \times \mathbb{F}$ by

$$\mathcal{I}_G v := \sum_{(k,\boldsymbol{s}) \in G} \delta_k \Delta^{\mathrm{I}}_{\boldsymbol{s}}(v)$$

for functions v defined on $\mathbb{R}^\infty$ taking values in a Hilbert space X^2.

Notice that $\mathcal{I}_G v$ is a linear non-adaptive method of fully discrete polynomial interpolation approximation which is the sum taken over the indices set G, of mixed tensor products of dyadic scale successive differences of "spatial" approximations to v, and of successive differences of their parametric Lagrange intepolating polynomials. It has been introduced in [14] (see also [16]). A similar construction for the multi-index stochastic collocation method for computing the expected value of a functional of the solution to elliptic PDEs with random data has been introduced in [21, 22] by using Clenshaw-Curtis points for quadrature.

A set $\Lambda \subset \mathbb{F}$ is called downward closed if the inclusion $\boldsymbol{s} \in \Lambda$ yields the inclusion $\boldsymbol{s}' \in \Lambda$ for every $\boldsymbol{s}' \in \mathbb{F}$ such that $\boldsymbol{s}' \le \boldsymbol{s}$. The inequality $\boldsymbol{s}' \le \boldsymbol{s}$ means that $s'_j \le s_j$, $j \in \mathbb{N}$. A sequence $(\sigma_{\boldsymbol{s}})_{\boldsymbol{s} \in \mathbb{F}}$ is called increasing if $\sigma_{\boldsymbol{s}'} \le \sigma_{\boldsymbol{s}}$ for $\boldsymbol{s}' \le \boldsymbol{s}$. Put $R_{\boldsymbol{s}} := \{\boldsymbol{s}' \in \mathbb{F} : \boldsymbol{s}' \le \boldsymbol{s}\}$.

Theorem 3.1 *Let $0 < p \le 2$. Let Assumption II hold for Hilbert spaces X^1 and X^2. Assume that $(Y_n)_{n \in \mathbb{N}_0}$ is a sequence such that every Y_n satisfies the condition* (3.14) *for some positive numbers τ and C. Let $v \in \mathcal{V}_2(X^2)$ be represented by the series* (2.5)*. Assume that for $r = 1, 2$ there exist increasing sequences $(\sigma_{r;\boldsymbol{s}})_{\boldsymbol{s} \in \mathbb{F}}$ of numbers strictly larger than 1 such that*

$$\sum_{\boldsymbol{s} \in \mathbb{F}} (\sigma_{r;\boldsymbol{s}} \|v_{\boldsymbol{s}}\|_{X^r})^2 < \infty$$

and $\left(p_{\boldsymbol{s}}(2\theta, \lambda)\sigma_{r;\boldsymbol{s}}^{-1}\right)_{\boldsymbol{s} \in \mathbb{F}} \in \ell_{q_r}(\mathbb{F})$ for some $0 < q_1 \le q_2 < \infty$ with $q_1 < 2$, where

$$\theta := \tau + \varepsilon + 5/4, \quad \lambda := \max\left(C_{\infty,2}, C_{2,\infty}, C_\varepsilon, 1\right), \tag{3.17}$$

$C_{\infty,2}$, $C_{2,\infty}$ are as in Lemma 3.1, ε is arbitrary positive number and C_ε is as in Lemma 3.7. For $\xi > 0$ let $G(\xi)$ be the set defined as in (3.5)*.*

Then for each $n \in \mathbb{N}$ there exists a number ξ_n such that $|G(\xi_n)| \le n$ and

$$\|v - \mathcal{I}_{G(\xi_n)} v\|_{\mathcal{V}_p(X^1)} \le C n^{-\min(\alpha,\beta)}. \tag{3.18}$$

The rate α corresponds to the approximation of a single function in X^2 as given by (2.3)*. The rate β is given by*

$$\beta := \left(\frac{1}{q_1} - \frac{1}{2}\right) \frac{\alpha}{\alpha + \delta}, \quad \delta := \frac{1}{q_1} - \frac{1}{q_2}. \tag{3.19}$$

The constant C in (3.18) *is independent of v and n.*

Proof. Clearly, by the inequality $\|\cdot\|_{\mathcal{V}_p(X^1)} \le \|\cdot\|_{\mathcal{V}_2(X^1)}$ it is sufficient to prove the theorem for $p = 2$. By Lemmata 3.3 and 3.4 the series (2.5) and (3.4) converge absolutely, and therefore, unconditionally in the Hilbert space $\mathcal{V}_2(X^1)$ to v. We have that $\Delta^{\mathrm{I}}_{\boldsymbol{s}} H_{\boldsymbol{s}'} = 0$ for every $\boldsymbol{s} \not\le \boldsymbol{s}'$. Moreover, if $\Lambda \subset \mathbb{F}$ is downward closed set, $I_\Lambda H_{\boldsymbol{s}} = H_{\boldsymbol{s}}$ for every $\boldsymbol{s} \in \Lambda$, and hence we can write

$$I_\Lambda v \;=\; I_\Lambda\Big(\sum_{\boldsymbol{s} \in \mathbb{F}} v_{\boldsymbol{s}}\, H_{\boldsymbol{s}}\Big) \;=\; \sum_{\boldsymbol{s} \in \mathbb{F}} v_{\boldsymbol{s}}\, I_\Lambda H_{\boldsymbol{s}} \;=\; \sum_{\boldsymbol{s} \in \Lambda} v_{\boldsymbol{s}}\, H_{\boldsymbol{s}} \;+\; \sum_{\boldsymbol{s} \notin \Lambda} v_{\boldsymbol{s}}\, I_{\Lambda \cap R_{\boldsymbol{s}}}\, H_{\boldsymbol{s}}. \tag{3.20}$$

Let us first consider the case $\alpha \le 1/q_2 - 1/2$. Let $\xi > 0$ be given. For $k \in \mathbb{N}_0$, put

$$\Lambda_k := \{\boldsymbol{s} \in \mathbb{F} : \; (k, \boldsymbol{s}) \in G(\xi)\} \;=\; \{\boldsymbol{s} \in \mathbb{F} : \sigma_{2,\boldsymbol{s}}^{q_2} \le 2^{-k}\xi\}.$$

Observe that $\Lambda_k = \emptyset$ for all $k > k^* := \lfloor \log_2 \xi \rfloor$, and consequently, we have that

$$\mathcal{I}_{G(\xi)} v \;=\; \sum_{k=0}^{k^*} \delta_k \Big(\sum_{\boldsymbol{s} \in \Lambda_k} \Delta^{\rm I}_{\boldsymbol{s}} \Big) v \;=\; \sum_{k=0}^{k^*} \delta_k I_{\Lambda_k} v. \tag{3.21}$$

Since the sequence $(\sigma_{2;\boldsymbol{s}})_{\boldsymbol{s} \in \mathbb{F}}$ is increasing, Λ_k are downward closed sets, and consequently, the sequence $\{\Lambda_k\}_{k=0}^{k^*}$ is nested in the inverse order, i.e., $\Lambda_{k'} \subset \Lambda_k$ if $k' > k$, and Λ_0 is the largest and $\Lambda_{k^*} = \{0_{\mathbb{F}}\}$. Therefore, from the unconditional convergence of the series (3.4) to v, (3.21) and (3.20) we derive that

$$\begin{aligned} \mathcal{I}_{G(\xi)} v \;&=\; \sum_{k=0}^{k^*} \sum_{\boldsymbol{s} \in \Lambda_k} \delta_k(v_{\boldsymbol{s}})\, H_{\boldsymbol{s}} \;+\; \sum_{k=0}^{k^*} \sum_{\boldsymbol{s} \notin \Lambda_k} \delta_k(v_{\boldsymbol{s}})\, I_{\Lambda_k \cap R_{\boldsymbol{s}}}\, H_{\boldsymbol{s}} \\ &=\; \mathcal{S}_{G(\xi)} v \;+\; \sum_{k=0}^{k^*} \sum_{\boldsymbol{s} \notin \Lambda_k} \delta_k(v_{\boldsymbol{s}})\, I_{\Lambda_k \cap R_{\boldsymbol{s}}}\, H_{\boldsymbol{s}}. \end{aligned}$$

This implies that

$$v \;-\; \mathcal{I}_{G(\xi)} v \;=\; v \;-\; \mathcal{S}_{G(\xi)} v \;-\; \sum_{k=0}^{k^*} \sum_{\boldsymbol{s} \notin \Lambda_k} \delta_k(v_{\boldsymbol{s}})\, I_{\Lambda_k \cap R_{\boldsymbol{s}}}\, H_{\boldsymbol{s}}.$$

Hence,

$$\big\| v - \mathcal{I}_{G(\xi)} v \big\|_{\mathcal{V}_2(X^1)} \;\le\; \big\| v - \mathcal{S}_{G(\xi)} v \big\|_{\mathcal{V}_2(X^1)} + \sum_{(k,\boldsymbol{s}) \notin G(\xi)} \|\delta_k(v_{\boldsymbol{s}})\|_{X^1} \, \big\| I_{\Lambda_k \cap R_{\boldsymbol{s}}}\, H_{\boldsymbol{s}} \big\|_{L_2(\mathbb{R}^\infty,\gamma)}. \tag{3.22}$$

Lemma 3.5 gives

$$\Big\| v - \mathcal{S}_{G(\xi)} u \Big\|_{\mathcal{V}_2(X^1)} \;\le\; C\, \xi^{-\alpha}. \tag{3.23}$$

Let us estimate the sum in the right-hand side of (3.22). We have that

$$\big\| I_{\Lambda_k \cap R_{\boldsymbol{s}}}\, H_{\boldsymbol{s}} \big\|_{L_2(\mathbb{R}^\infty,\gamma)} \;\le\; \sum_{\boldsymbol{s}' \in \Lambda_k \cap R_{\boldsymbol{s}}} \big\| \Delta^{\rm I}_{\boldsymbol{s}'}(H_{\boldsymbol{s}}) \big\|_{L_2(\mathbb{R}^\infty,\gamma)}. \tag{3.24}$$

We estimate the norm in the sum in the right-hand side. Assuming $\boldsymbol{s} \in \mathbb{F}$ to be such that $\operatorname{supp}(\boldsymbol{s}) \subset \{1, ..., J\}$, we have $\Delta^{\rm I}_{\boldsymbol{s}'}(H_{\boldsymbol{s}}) = \prod_{j=1}^J \Delta^{\rm I}_{s'_j}(H_{s_j})$. Since $\Delta^{\rm I}_{s'_j}(H_{s_j})$ is a polynomial of degree $\le s'_j$ in variable y_j, from Lemma 3.1 we obtain that

$$\begin{aligned} \big\| \Delta^{\rm I}_{\boldsymbol{s}'}(H_{\boldsymbol{s}}) \big\|_{L_2(\mathbb{R}^\infty,\gamma)} \;&=\; \prod_{j=1}^J \|\Delta^{\rm I}_{s'_j}(H_{s_j})\|_{L_2(\mathbb{R},\gamma)} \;=\; \prod_{j=1}^J \|\Delta^{\rm I}_{s'_j}(H_{s_j})\sqrt{g}\|_{L_2(\mathbb{R})} \\ &\le\; p_{\boldsymbol{s}'}(\theta,\lambda) \prod_{j=1}^J \|\Delta^{\rm I}_{s'_j}(H_{s_j})\sqrt{g}\|_{L_\infty(\mathbb{R})} \end{aligned}$$

where $\theta = 1/4$, $\lambda := C_{2,\infty}$ and $C_{2,\infty}$ is the constant in Lemma 3.1. Due to the assumption (3.14), we have by Lemmata 3.7 and 3.2 that

$$\begin{aligned} \|\Delta^{\rm I}_{s'_j}(H_{s_j})\sqrt{g}\|_{L_\infty(\mathbb{R})} \;&\le\; (1 + C_\varepsilon s'_j)^{\tau+\varepsilon}\, \|H_{s_j}\sqrt{g}\|_{L_\infty(\mathbb{R})} \\ &\le\; (1 + C_\varepsilon s'_j)^{\tau+\varepsilon} (1 + C_{\infty,2,2} s_j)^{1/4}\, \|H_{s_j}\sqrt{g}\|_{L_2(\mathbb{R})} \\ &=\; (1 + C_\varepsilon s'_j)^{\tau+\varepsilon} (1 + C_{\infty,2,2} s_j)^{1/4} \end{aligned}$$

and consequently,

$$\|\Delta^{\mathrm{I}}_{\boldsymbol{s}'}(H_{\boldsymbol{s}})\|_{L_2(\mathbb{R}^\infty,\gamma)} \ \le\ p_{\boldsymbol{s}'}(\theta,\lambda) \ \le\ p_{\boldsymbol{s}}(\theta,\lambda), \tag{3.25}$$

where

$$\theta := \tau+\varepsilon+1/4, \quad \lambda := \max\left(C_{\infty,2}, C_{2,\infty}, C_\varepsilon\right). \tag{3.26}$$

Substituting $\|\Delta^{\mathrm{I}}_{\boldsymbol{s}'}(H_{\boldsymbol{s}})\|_{L_2(\mathbb{R}^\infty,\gamma)}$ in (3.24) with the right-hand side of (3.25) gives that

$$\left\|I_{\Lambda_k\cap R_{\boldsymbol{s}}} H_{\boldsymbol{s}}\right\|_{L_2(\mathbb{R}^\infty,\gamma)} \ \le \sum_{\boldsymbol{s}'\in\Lambda_k\cap R_{\boldsymbol{s}}} p_{\boldsymbol{s}}(\theta,\lambda) \ \le\ |R_{\boldsymbol{s}}|\, p_{\boldsymbol{s}}(\theta,\lambda) \ \le\ p_{\boldsymbol{s}}(1,1)\, p_{\boldsymbol{s}}(\theta,\lambda).$$

where θ and λ are as in (3.26). Hence,

$$\left\|I_{\Lambda_k\cap R_{\boldsymbol{s}}} H_{\boldsymbol{s}}\right\|_{L_2(\mathbb{R}^\infty,\gamma)} \ \le\ p_{\boldsymbol{s}}(\theta,\lambda), \tag{3.27}$$

where θ and λ are as in (3.17). Denote by $\Sigma(\xi)$ the sum in the right-hand side of (3.22). By using (3.27) and (2.4) we estimate $\Sigma(\xi)$ as

$$\begin{aligned}\Sigma(\xi) \ &\le\ C \sum_{(k,\boldsymbol{s})\notin G(\xi)} 2^{-\alpha k} p_{\boldsymbol{s}}(\theta,\lambda)\|v_{\boldsymbol{s}}\|_{X^2} \ =\ C\sum_{\boldsymbol{s}\in\mathbb{F}} p_{\boldsymbol{s}}(\theta,\lambda)\|v_{\boldsymbol{s}}\|_{X^2} \sum_{2^k>\xi\sigma_{2;\boldsymbol{s}}^{-q_2}} 2^{-\alpha k} \\ &\le\ C\sum_{\boldsymbol{s}\in\mathbb{F}} p_{\boldsymbol{s}}(\theta,\lambda)\|v_{\boldsymbol{s}}\|_{X^2}\ (\xi\sigma_{2;\boldsymbol{s}}^{-q_2})^{-\alpha} \ \le\ C\xi^{-\alpha}\sum_{\boldsymbol{s}\in\mathbb{F}} p_{\boldsymbol{s}}(\theta,\lambda)\sigma_{2;\boldsymbol{s}}^{q_2\alpha}\|v_{\boldsymbol{s}}\|_{X^2}.\end{aligned} \tag{3.28}$$

By the inequalities $2(1-q_2\alpha)\ge q_2$ and $\sigma_{2;\boldsymbol{s}}>1$ and the assumptions we have that

$$\begin{aligned}\sum_{\boldsymbol{s}\in\mathbb{F}} p_{\boldsymbol{s}}(\theta,\lambda)\sigma_{2;\boldsymbol{s}}^{q_2\alpha}\|v_{\boldsymbol{s}}\|_{X^2} \ &\le\ \left(\sum_{\boldsymbol{s}\in\mathbb{F}}(\sigma_{2;\boldsymbol{s}}\|v_{\boldsymbol{s}}\|_{X^2})^2\right)^{1/2}\left(\sum_{\boldsymbol{s}\in\mathbb{F}} p_{\boldsymbol{s}}^2(\theta,\lambda)\sigma_{2;\boldsymbol{s}}^{-2(1-q_2\alpha)}\right)^{1/2} \\ &\le\ \left(\sum_{\boldsymbol{s}\in\mathbb{F}}(\sigma_{2;\boldsymbol{s}}\|v_{\boldsymbol{s}}\|_{X^2})^2\right)^{1/2}\left(\sum_{\boldsymbol{s}\in\mathbb{F}} p_{\boldsymbol{s}}(2\theta,\lambda)\sigma_{2;\boldsymbol{s}}^{-q_2}\right)^{1/2} \ =\ C\ <\infty.\end{aligned}$$

Thus, we obtain the estimate

$$\Sigma(\xi) := \sum_{(k,\boldsymbol{s})\notin G(\xi)} \|\delta_k(v_{\boldsymbol{s}})\|_{X^1}\left\|I_{\Lambda_k\cap R_{\boldsymbol{s}}} H_{\boldsymbol{s}}\right\|_{L_2(\mathbb{R}^\infty,\gamma)} \ \le\ C\xi^{-\alpha}.$$

This together with (3.22) and (3.23) implies that

$$\|v-\mathcal{I}_{G(\xi)}u\|_{\mathcal{V}_2(X^1)} \ \le\ C\xi^{-\alpha}. \tag{3.29}$$

Hence, similarly to (2.14)–(2.15), for each $n\in\mathbb{N}$ we can find a number ξ_n such that $|G(\xi_n)|\le n$ and

$$\|v-\mathcal{I}_{G(\xi_n)}v\|_{\mathcal{V}_p(X^1)} \le Cn^{-\alpha}, \quad \alpha\le 1/q_2-1/2. \tag{3.30}$$

We now consider the case $\alpha>1/q_2-1/2$. Observe that the unconditional convergence of the series (2.5) and the uniform boundedness of the operators P_n in X^1 imply that

$$\delta_k\Delta^{\mathrm{I}}_{\boldsymbol{s}}v \ =\ \Delta^{\mathrm{I}}_{\boldsymbol{s}}\delta_k v \ =\ \sum_{\boldsymbol{s}'\in\mathbb{F}}\delta_k(v_{\boldsymbol{s}'})\Delta^{\mathrm{I}}_{\boldsymbol{s}}(H_{\boldsymbol{s}'})$$

and

$$P_n v \;=\; \sum_{\boldsymbol{s}\in\mathbb{F}} P_n(v_{\boldsymbol{s}}) H_{\boldsymbol{s}}$$

with convergence of the series in $\mathcal{V}_2(X^1)$. Put $\Lambda(\xi) := \{\boldsymbol{s}\in\mathbb{F}: \sigma_{1;\boldsymbol{s}}^{q_1} \le \xi\}$ and $B(\xi,\boldsymbol{s}) := \{k\in\mathbb{N}_0 : 2^{\alpha q_1 k} \le \xi\sigma_{2;\boldsymbol{s}}^{-q_1}\}$ for $\xi > 0$. By using of these equalities and the uniconditional convergence of the series (2.5) and (3.4), with $N(\xi,\boldsymbol{s}) := 2^{\lfloor \log_2(\xi^{1/(q_1\alpha)}\sigma_{2;\boldsymbol{s}}^{-1/\alpha})\rfloor}$ we derive the estimates

$$\begin{aligned}
\mathcal{I}_{G(\xi)} v \;&=\; \sum_{(k,\boldsymbol{s})\in G(\xi)} \delta_k \Delta_{\boldsymbol{s}}^{\mathrm{I}} v \;=\; \sum_{\boldsymbol{s}\in\Lambda(\xi)} \Delta_{\boldsymbol{s}}^{\mathrm{I}} \left(\sum_{k\in B(\xi,\boldsymbol{s})} \delta_k v \right) \\
&=\; \sum_{\boldsymbol{s}\in\Lambda(\xi)} P_{N(\xi,\boldsymbol{s})}(v_{\boldsymbol{s}}) H_{\boldsymbol{s}} \;+\; \sum_{\boldsymbol{s}\notin\Lambda(\xi)} P_{N(\xi,\boldsymbol{s})}(v_{\boldsymbol{s}}) I_{\Lambda(\xi)\cap R_{\boldsymbol{s}}}(H_{\boldsymbol{s}}) \\
&=\; \sum_{\boldsymbol{s}\in\Lambda(\xi)} \left(\sum_{k\in B(\xi,\boldsymbol{s})} \delta_k v \right) H_{\boldsymbol{s}} \;+\; \sum_{\boldsymbol{s}\notin\Lambda(\xi)} P_{N(\xi,\boldsymbol{s})}(v_{\boldsymbol{s}}) I_{\Lambda(\xi)\cap R_{\boldsymbol{s}}}(H_{\boldsymbol{s}}) \\
&=\; \mathcal{S}_{G(\xi)} v \;+\; \sum_{\boldsymbol{s}\notin\Lambda(\xi)} P_{N(\xi,\boldsymbol{s})}(v_{\boldsymbol{s}}) I_{\Lambda(\xi)\cap R_{\boldsymbol{s}}}(H_{\boldsymbol{s}}).
\end{aligned} \tag{3.31}$$

Hence,

$$\left\| v - \mathcal{I}_{G(\xi)} v \right\|_{\mathcal{V}_2(X^1)} \;\le\; \left\| v - \mathcal{S}_{G(\xi)} v \right\|_{\mathcal{V}_2(X^1)} + \sum_{\boldsymbol{s}\notin\Lambda(\xi)} \|P_{N(\xi,\boldsymbol{s})}(v_{\boldsymbol{s}})\|_{X^1} \left\| I_{\Lambda(\xi)\cap R_{\boldsymbol{s}}} H_{\boldsymbol{s}} \right\|_{L_2(\mathbb{R}^\infty,\gamma)}. \tag{3.32}$$

Lemma 3.5 gives

$$\left\| v - \mathcal{S}_{G(\xi)} v \right\|_{\mathcal{V}_2(X^1)} \;\le\; C\xi^{-(1/q_1 - 1/2)}. \tag{3.33}$$

The sum in the right-hand side of (3.32) can be estimated by

$$\sum_{\boldsymbol{s}\notin\Lambda(\xi)} \|P_{N(\xi,\boldsymbol{s})}(v_{\boldsymbol{s}})\|_{X^1} \left\| I_{\Lambda(\xi)\cap R_{\boldsymbol{s}}} H_{\boldsymbol{s}} \right\|_{L_2(\mathbb{R}^\infty,\gamma)} \;\le\; C \sum_{\boldsymbol{s}\notin\Lambda(\xi)} \|v_{\boldsymbol{s}}\|_{X^1} \left\| I_{\Lambda(\xi)\cap R_{\boldsymbol{s}}} H_{\boldsymbol{s}} \right\|_{L_2(\mathbb{R}^\infty,\gamma)}.$$

Similarly to (3.27), we have

$$\left\| I_{\Lambda(\xi)\cap R_{\boldsymbol{s}}} H_{\boldsymbol{s}} \right\|_{L_2(\mathbb{R}^\infty,\gamma)} \;\le\; p_{\boldsymbol{s}}(\theta,\lambda)$$

with the same θ and λ as in (3.17). This gives the estimate

$$\sum_{\boldsymbol{s}\notin\Lambda(\xi)} \|P_{N(\xi,\boldsymbol{s})}(v_{\boldsymbol{s}})\|_{X^1} \left\| I_{\Lambda(\xi)\cap R_{\boldsymbol{s}}} H_{\boldsymbol{s}} \right\|_{L_2(\mathbb{R}^\infty,\gamma)} \;\le\; C \sum_{\boldsymbol{s}\notin\Lambda(\xi)} \|v_{\boldsymbol{s}}\|_{X^1}\, p_{\boldsymbol{s}}(\theta,\lambda). \tag{3.34}$$

We have by the Hölder inequality and the hypothesis of the theorem,

$$\begin{aligned}
\sum_{\boldsymbol{s}\notin\Lambda(\xi)} \|v_{\boldsymbol{s}}\|_{X^1}\, p_{\boldsymbol{s}}(\theta,\lambda) \;&\le\; \left(\sum_{\sigma_{1;\boldsymbol{s}} > \xi^{1/q_1}} (\sigma_{1;\boldsymbol{s}} \|v_{\boldsymbol{s}}\|_{X^1})^2 \right)^{1/2} \left(\sum_{\sigma_{1;\boldsymbol{s}} > \xi^{1/q_1}} p_{\boldsymbol{s}}^2(\theta,\lambda) \sigma_{1;\boldsymbol{s}}^{-2} \right)^{1/2} \\
&\le\; C \left(\sum_{\sigma_{1;\boldsymbol{s}} > \xi^{1/q_1}} p_{\boldsymbol{s}}^2(\theta,\lambda) \sigma_{1;\boldsymbol{s}}^{-q_1} \sigma_{1;\boldsymbol{s}}^{-(2-q_1)} \right)^{1/2} \\
&\le\; C\xi^{-(1/q_1 - 1/2)} \left(\sum_{\boldsymbol{s}\in\mathbb{F}} p_{\boldsymbol{s}}(2\theta,\lambda) \sigma_{1;\boldsymbol{s}}^{-q_1} \right)^{1/2} \;\le C\xi^{-(1/q_1 - 1/2)}.
\end{aligned} \tag{3.35}$$

Combining (3.32)–(3.35) leads to the estimate

$$\|v - \mathcal{I}_{G(\xi)} v\|_{\mathcal{V}_2(X^1)} \ \leq \ C\xi^{-(1/q_1 - 1/2)}. \tag{3.36}$$

Hence, similarly to (2.18)–(2.19), for each $n \in \mathbb{N}$ we can find a number ξ_n such that $|G(\xi_n)| \leq n$ and

$$\|v - \mathcal{I}_{G(\xi_n)} v\|_{\mathcal{V}_p(X^1)} \leq Cn^{-\beta}, \quad \alpha > 1/q_2 - 1/2. \tag{3.37}$$

By combining the last esimate and (3.30) we derive (3.18). ∎

Denote by $\Gamma_{\boldsymbol{s}}$ and $\Gamma(\Lambda)$ the set of interpolation points in the operators $\Delta^{\rm I}_{\boldsymbol{s}}$ and I_Λ, respectively. We have that $\Gamma(\Lambda) = \cup_{\boldsymbol{s} \in \Lambda} \Gamma_{\boldsymbol{s}}$. From the definitions we can see that $\Gamma_{\boldsymbol{s}} = \{\boldsymbol{y}_{\boldsymbol{s}-\boldsymbol{e};\boldsymbol{m}} : e \in \mathbb{E}_{\boldsymbol{s}};\ m_j = 0, ..., s_j,\ j \in \mathbb{N}\}$, where $\mathbb{E}_{\boldsymbol{s}}$ is the subset in $\mathbb{F}$ of all $\boldsymbol{e}$ such that e_j is 1 or 0 if $s_j > 0$, and e_j is 0 if $s_j = 0$, and $\boldsymbol{y}_{\boldsymbol{s};\boldsymbol{m}} := (y_{s_j;m_j})_{j \in \mathbb{N}}$.

Corollary 3.1 *Let $0 < p \leq 2$. Let $v \in \mathcal{V}_2(X)$ be represented by the series (2.1) for a Hilbert space X. Assume that $(Y_n)_{n \in \mathbb{N}_0}$ is a sequence such that every Y_n satisfies the condition (3.14) for some positive numbers τ and C. Assume that there exists an increasing sequence $(\sigma_{\boldsymbol{s}})_{\boldsymbol{s} \in \mathbb{F}}$ of numbers strictly larger than 1 such that*

$$\sum_{\boldsymbol{s} \in \mathbb{F}} (\sigma_{\boldsymbol{s}} \|v_{\boldsymbol{s}}\|_X)^2 < \infty$$

and $(p_{\boldsymbol{s}}(2\theta, \max(2, \lambda))\sigma_{\boldsymbol{s}}^{-1})_{\boldsymbol{s} \in \mathbb{F}} \in \ell_q(\mathbb{F})$ for some $0 < q < 2$, where θ and λ are as in (3.17). For $\xi > 0$ define

$$\Lambda(\xi) := \{\boldsymbol{s} \in \mathbb{F} :\ \sigma_{\boldsymbol{s}}^q \leq \xi\}. \tag{3.38}$$

Then for each $n \in \mathbb{N}$ there exists a number ξ_n such that $|\Gamma(\Lambda(\xi_n))| \leq n$ and

$$\|v - I_{\Lambda(\xi_n)} v\|_{\mathcal{V}_p(X)} \leq Cn^{-(1/q - 1/2)}. \tag{3.39}$$

The constant C in (3.39) is independent of v and n.

Proof. Similarly to the proof of Theorem 3.1 it is sufficient to prove (3.39) for $p = 2$. In the same way as in proving (3.32), we can show that

$$\left\|v - I_{\Lambda(\xi)} v\right\|_{\mathcal{V}_2(X^1)} \ \leq \ \left\|v - S_{\Lambda(\xi)} v\right\|_{\mathcal{V}_2(X^1)} + \sum_{\boldsymbol{s} \notin \Lambda(\xi)} \|v_{\boldsymbol{s}}\|_{X^1} \left\|I_{\Lambda(\xi) \cap R_{\boldsymbol{s}}} H_{\boldsymbol{s}}\right\|_{L_2(\mathbb{R}^\infty, \gamma)},$$

where

$$S_{\Lambda(\xi)} v := \sum_{\boldsymbol{s} \in \Lambda(\xi)} v_{\boldsymbol{s}} H_{\boldsymbol{s}}.$$

By estimating $\left\|v - S_{\Lambda(\xi)} v\right\|_{\mathcal{V}_2(X^1)}$ and $\sum_{\boldsymbol{s} \notin \Lambda(\xi)} \|v_{\boldsymbol{s}}\|_{X^1} \left\|I_{\Lambda(\xi) \cap R_{\boldsymbol{s}}} H_{\boldsymbol{s}}\right\|_{L_2(\mathbb{R}^\infty, \gamma)}$ similarly to (2.16) and (3.34)–(3.35), respectively, we derive

$$\left\|v - I_{\Lambda(\xi)} v\right\|_{\mathcal{V}_2(X^1)} \ \leq \ C\xi^{-(1/q - 1/2)}.$$

Since $|\Gamma_{\boldsymbol{s}}| \leq \prod_{j \in \mathbb{N}} (2s_j + 1) = p_{\boldsymbol{s}}(1, 2)$, we have from the definition

$$|\Gamma(\Lambda(\xi))| \ \leq \ \sum_{\boldsymbol{s} \in \Lambda(\xi)} |\Gamma_{\boldsymbol{s}}| \ \leq \ \sum_{\xi \sigma_{\boldsymbol{s}}^{-q} \geq 1} p_{\boldsymbol{s}}(1, 2) \ \leq \ M\xi,$$

where $M := \sum_{\boldsymbol{s} \in \mathbb{F}} p_{\boldsymbol{s}}(1, 2) \sigma_{\boldsymbol{s}}^{-q} < \infty$ by the assumption. For any $n \in \mathbb{N}$, by choosing a number ξ_n satisfying the inequalities $M\xi_n \leq n < 2M\xi_n$, we derive (3.39). ∎

The bound $\|v - I_{\Lambda_n} v\|_{L_2(\mathbb{R}^\infty,\mathcal{H},\gamma)} \le C n^{-(1/q-1/2)}$ has been obtained in [18] for a Hilbert space $\mathcal{H}$, where Λ_n is the set of $\boldsymbol{s}$ corresponding to the n largest elements of an ℓ_q-summable majorant of the sequence $(\sigma_{\boldsymbol{s}}^{-1}\, p_{\boldsymbol{s}}(\theta,\lambda))_{\boldsymbol{s}\in\mathbb{F}}$.

Theorem 3.2 *Let $0 < p \le 2$. Let Assumption II hold for Hilbert spaces X^1 and X^2. Let $v \in \mathcal{V}_2(X^2)$ be represented by the series (3.9). Assume that $(Y_n)_{n\in\mathbb{N}_0}$ is a sequence such that every Y_n satisfies the condition (3.14) for some positive numbers τ and C. Assume that for $r = 1, 2$ there exist increasing sequences $(\sigma_{r;\boldsymbol{s}})_{\boldsymbol{s}\in\mathbb{F}_{\rm ev}}$ of numbers strictly larger than 1 such that*

$$\sum_{\boldsymbol{s}\in\mathbb{F}_{\rm ev}} (\sigma_{r;\boldsymbol{s}}\|v_{\boldsymbol{s}}\|_{X^r})^2 < \infty$$

and $(p_{\boldsymbol{s}}(2\theta,\lambda)\sigma_{r;\boldsymbol{s}}^{-1})_{\boldsymbol{s}\in\mathbb{F}_{\rm ev}} \in \ell_{q_r}(\mathbb{F}_{\rm ev})$ for some $0 < q_1 \le q_2 < \infty$ with $q_1 < 2$, where θ and λ are as in (3.17). For $\xi > 0$ let $G_{\rm ev}(\xi)$ be the set defined as in (3.10).

Then for each $n \in \mathbb{N}$ there exists a number ξ_n such that $|G(\xi_n)| \le n$ and

$$\|v - \mathcal{I}_{G_{\rm ev}(\xi_n)} v\|_{\mathcal{V}_p(X^1)} \le C n^{-\min(\alpha,\beta)}. \tag{3.40}$$

The rate α corresponds to the approximation of a single function in X^2 as given by (2.3). The rate β is given by (3.19). The constant C in (3.40) is independent of v and n.

Proof. The proof of this theorem is similar to the proof of Theorem 3.1 with some modification. For example, all the indices sets are taken from the sets $\mathbb{F}_{\rm ev}$ and $\mathbb{N}_0 \times \mathbb{F}_{\rm ev}$ instead $\mathbb{F}$ and $\mathbb{N}_0 \times \mathbb{F}$; estimates similar to (3.23) and (3.33) are given by Lemma 3.6 instead Lemma 3.5. ∎

Similarly to Corollary 3.1 we have the following

Corollary 3.2 *Let $v \in \mathcal{V}_2(X)$ be represented by the series (3.9) for a Hilbert space X. Assume that $(Y_n)_{n\in\mathbb{N}_0}$ is a sequence such that every Y_n satisfies the condition (3.14) for some positive numbers τ and C. Assume that there exists an increasing sequence $(\sigma_{\boldsymbol{s}})_{\boldsymbol{s}\in\mathbb{F}_{\rm ev}}$ of numbers strictly larger than 1 such that*

$$\sum_{\boldsymbol{s}\in\mathbb{F}_{\rm ev}} (\sigma_{\boldsymbol{s}}\|v_{\boldsymbol{s}}\|_{X})^2 < \infty$$

and $(p_{\boldsymbol{s}}(2\theta,\max(2,\lambda))\sigma_{\boldsymbol{s}}^{-1})_{\boldsymbol{s}\in\mathbb{F}_{\rm ev}} \in \ell_q(\mathbb{F}_{\rm ev})$ for some $0 < q < 2$, where θ and λ are as in (3.17). For $\xi > 0$, define

$$\Lambda_{\rm ev}(\xi) := \{\boldsymbol{s} \in \mathbb{F}_{\rm ev} : \sigma_{\boldsymbol{s}}^q \le \xi\}. \tag{3.41}$$

Then for each $n \in \mathbb{N}$ there exists a number ξ_n such that $|\Gamma(\Lambda_{\rm ev}(\xi_n))| \le n$ and

$$\|v - I_{\Lambda_{\rm ev}(\xi_n)} v\|_{\mathcal{V}_p(X)} \le C n^{-(1/q-1/2)}. \tag{3.42}$$

The constant C in (3.42) is independent of v and n.

Theorem 3.2 and Corollary 3.2 will be applied in proving the convergence rates of fully and non-fully discrete integration in the next section.

4 Integration

In this section, we construct general fully discrete linear methods for integration of functions taking values in X^2 and having a weighted ℓ_2-summability of Hermite expansion coefficients for Hilbert spaces X^1 and X^2 satisfying a certain "spatial" approximation property, and their bounded linear functionals. In particular, we give convergence rates for these methods of integration which are derived from results on convergence rate of polynomial interpolation approximation in $\mathcal{V}_1(X^1)$ in Theorem 3.2. We also briefly consider non-fully discrete methods for integration.

If v is a function defined on $\mathbb{R}$ taking values in a Hilbert space X, an interpolation formula $I_n(v)$ of the form (3.13) generates the quadrature formula defined as

$$Q_n(v) := \int_{\mathbb{R}} I_n(v)(y)\,\mathrm{d}\gamma(y) \;=\; \sum_{k=0}^{n} \omega_{n;k}\, v(y_{n;k}),$$

where

$$\omega_{n;k} := \int_{\mathbb{R}} \ell_{n;k}(y)\,\mathrm{d}\gamma(y).$$

Notice that

$$Q_n(\varphi) \;=\; \int_{\mathbb{R}} \varphi(y)\,\mathrm{d}\gamma(y)$$

for every polynomial φ of degree $\le n$, due to the identity $I_n(\varphi) = \varphi$.

For integration purpose, we additionally assume that the sequence Y_n as in (3.12) is symmetric for every $n \in \mathbb{N}_0$, i. e., $y_{n;n-k} = y_{n;k}$ for $k = 0, ..., n$. The sequences Y_n^* of the of the roots of the Hermite polynomilas H_{n+1} and their modifications $\bar{Y}_n^*$ are symmetric. Also, for the sequence Y_n^*, it is well-known that

$$\omega_{n;k} \;=\; \frac{1}{(n+1)H_n^2(y^*_{n;k})}.$$

For a given sequence $(Y_n)_{n=0}^{\infty}$, we define the univariate operator Δ_n^{Q} for $n \in \mathbb{N}_0$ by

$$\Delta_n^{\mathrm{Q}} := \; Q_n - Q_{n-1},$$

with the convention $Q_{-1} := 0$. We introduce the operator $\Delta_{\boldsymbol{s}}^{\mathrm{I}}$ for $\boldsymbol{s} \in \mathbb{F}$ by

$$\Delta_{\boldsymbol{s}}^{\mathrm{Q}}(v) := \bigotimes_{j\in\mathbb{N}} \Delta_{s_j}^{\mathrm{Q}}(v)$$

for γ-measurable functions v on $\mathbb{R}^\infty$ taking values in a Hilbert space X, where the univariate operator $\Delta_{s_j}^{\mathrm{Q}}$ is applied to the univariate function v by considering v as a function of variable y_i with the other variables held fixed.

We introduce the operator I_Λ for a finite set $\Lambda \subset \mathbb{F}$ by

$$Q_\Lambda := \sum_{\boldsymbol{s}\in\Lambda} \Delta_{\boldsymbol{s}}^{\mathrm{I}}.$$

Let Assumption II hold for Hilbert spaces X^1 and X^2. We introduce the operator $\mathcal{Q}_G$ for a finite set $G \subset \mathbb{N}_0 \times \mathbb{F}$ by

$$\mathcal{Q}_G v := \sum_{(k,\boldsymbol{s})\in G} \delta_k \Delta_{\boldsymbol{s}}^{\mathrm{Q}}(v)$$

for γ-measurable functions v on $\mathbb{R}^\infty$ taking values in a Hilbert space X^2. Notice that

$$Q_\Lambda v \;=\; \int_{\mathbb{R}^\infty} I_\Lambda v(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}),$$

and

$$\mathcal{Q}_G v \;=\; \int_{\mathbb{R}^\infty} \mathcal{I}_G v(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}). \tag{4.1}$$

For a function v defined on $\mathbb{R}^\infty$ taking values in a Hilbert space X such that $v \in \mathcal{V}_2(X)$ and is represented by the series (2.1), consider the function $v_{\mathrm{ev}} \in \mathcal{V}_2(X)$ defined by

$$v_{\mathrm{ev}} := \sum_{\boldsymbol{s}\in\mathbb{F}_{\mathrm{ev}}} v_{\boldsymbol{s}} H_{\boldsymbol{s}}.$$

From the obvious equality $\int_{\mathbb{R}} v(y)\,\mathrm{d}\gamma(y) = 0$ for every odd function v, we have that

$$\int_{\mathbb{R}^\infty} H_{\boldsymbol{s}}(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}) \;=\; 0,\;\; \boldsymbol{s} \notin \mathbb{F}_{\mathrm{ev}},$$

and hence,

$$\int_{\mathbb{R}^\infty} v(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}) \;=\; \int_{\mathbb{R}^\infty} v_{\mathrm{ev}}(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}). \tag{4.2}$$

Moreover, if Y_n is symmetric for every $n \in \mathbb{N}_0$,

$$\Delta^{\mathrm{Q}}_{\boldsymbol{s}'} H_{\boldsymbol{s}}(\boldsymbol{y}) \;=\; 0, \quad \boldsymbol{s} \notin \mathbb{F}_{\mathrm{ev}},\;\; \boldsymbol{s}' \in \mathbb{F}. \tag{4.3}$$

Let Assumption II hold for Hilbert spaces X^1 and X^2. If v is a function defined on $\mathbb{R}^\infty$ taking values in X^2 and $\phi \in (X^1)'$ a bounded linear functional on X^1. For a finite set $G \subset \mathbb{N}_0 \times \mathbb{F}$, the quadrature formula $Q_\Lambda v$ generates the quadrature formula $Q_G\langle \phi, v\rangle$ for integration of $\langle \phi, v\rangle$ by

$$Q_G\langle \phi, v\rangle := \sum_{(k,\boldsymbol{s})\in G} \langle \phi, \delta_k \Delta^{\mathrm{Q}}_{\boldsymbol{s}} v\rangle \;=\; \int_{\mathbb{R}^\infty} \langle \phi, Q_G v(\boldsymbol{y})\rangle\,\mathrm{d}\gamma(\boldsymbol{y}).$$

Theorem 4.1 *Under the hypothesis of Theorem 3.1, assume additionally that the sequences Y_n, $n \in \mathbb{N}_0$, are symmetric. For $\xi > 0$ let $G_{\mathrm{ev}}(\xi)$ be the set defined as in* (3.10)*. Then we have the following.*

(i) *For each $n \in \mathbb{N}$ there exists a number ξ_n such that $|G_{\mathrm{ev}}(\xi_n)| \le n$ and*

$$\left\| \int_{\mathbb{R}^\infty} v(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}) - \mathcal{Q}_{G_{\mathrm{ev}}(\xi_n)} v \right\|_{X^1} \le C n^{-\min(\alpha,\beta)}. \tag{4.4}$$

(ii) *Let $\phi \in (X^1)'$ be a bounded linear functional on X^1. Then for each $n \in \mathbb{N}$ there exists a number ξ_n such that $|G_{\mathrm{ev}}(\xi_n)| \le n$ and*

$$\left| \int_{\mathbb{R}^\infty} \langle \phi, v(\boldsymbol{y})\rangle\,\mathrm{d}\gamma(\boldsymbol{y}) - \mathcal{Q}_{G_{\mathrm{ev}}(\xi_n)} \langle \phi, v\rangle \right| \le C n^{-\min(\alpha,\beta)}. \tag{4.5}$$

The rate α corresponds to the approximation of a single function in X^2 as given by (2.3)*. The rate β is given by* (3.19)*. The constants C in* (4.4) *and* (4.5) *are independent of v and n.*

Proof. For a given $n \in \mathbb{N}$, we approximate the integral in the right-hand side of (4.2) by $\mathcal{Q}_{G_{\rm ev}(\xi_n)}$ where ξ_n is as in Theorem 3.2. By Lemmata 3.3 and 3.4 the series (2.5) and (3.4) converge absolutely, and therefore, unconditionally in the Hilbert space $\mathcal{V}_1(X^1)$ to v. Hence, by (4.3) we derive that $\mathcal{Q}_{G_{\rm ev}(\xi_n)} v = \mathcal{Q}_{G_{\rm ev}(\xi_n)} v_{\rm ev}$. Due to (4.1) and (4.2) there holds the equality

$$\int_{\mathbb{R}^\infty} v(\boldsymbol{y})\,{\rm d}\gamma(\boldsymbol{y}) - \mathcal{Q}_{G_{\rm ev}(\xi_n)} v \ = \ \int_{\mathbb{R}^\infty} \big(v_{\rm ev}(\boldsymbol{y}) - \mathcal{I}_{G_{\rm ev}(\xi_n)} v_{\rm ev}(\boldsymbol{y})\big)\,{\rm d}\gamma(\boldsymbol{y}). \tag{4.6}$$

Hence, applying (3.40) in Theorem 3.2 for $p=1$, we obtain (i):

$$\left\| \int_{\mathbb{R}^\infty} v(\boldsymbol{y})\,{\rm d}\gamma(\boldsymbol{y}) - \mathcal{Q}_{G_{\rm ev}(\xi_n)} v \right\|_{X^1} \ \le \ \left\| v_{\rm ev} - \mathcal{I}_{G_{\rm ev}(\xi_n)} v_{\rm ev}\right\|_{\mathcal{V}_1(X^1)} \ \le \ C n^{-\min(\alpha,\beta)}.$$

For a given $n \in \mathbb{N}$, we approximate the integral $\int_{\mathbb{R}^\infty} \langle \phi, v(\boldsymbol{y})\rangle\,{\rm d}\gamma(\boldsymbol{y})$ by $\mathcal{Q}_{\Lambda_{\rm ev}(\xi_n)}\langle \phi, v\rangle$ where ξ_n is as in Corollary 3.2. Similarly to (4.6), there holds the equality

$$\int_{\mathbb{R}^\infty} \langle \phi, v_{\rm ev}(\boldsymbol{y})\rangle\,{\rm d}\gamma(\boldsymbol{y}) - \mathcal{Q}_{G_{\rm ev}(\xi_n)}\langle \phi, v_{\rm ev}(\boldsymbol{y})\rangle \ = \ \int_{\mathbb{R}^\infty} \langle \phi, v_{\rm ev}(\boldsymbol{y}) - \mathcal{I}_{G_{\rm ev}(\xi_n)} v_{\rm ev}(\boldsymbol{y})\rangle\,{\rm d}\gamma(\boldsymbol{y}).$$

Hence, applying (3.40) in Theorem 3.2 for $p=1$, we prove (ii):

$$\begin{aligned}
\left| \int_{\mathbb{R}^\infty} \langle \phi, v_{\rm ev}(\boldsymbol{y})\rangle\,{\rm d}\gamma(\boldsymbol{y}) - \mathcal{Q}_{G_{\rm ev}(\xi_n)}\langle \phi, v\rangle \right| \ &\le \ \int_{\mathbb{R}^\infty} \big|\langle \phi, v_{\rm ev}(\boldsymbol{y}) - \mathcal{I}_{G_{\rm ev}(\xi_n)} v_{\rm ev}(\boldsymbol{y})\rangle\big|\,{\rm d}\gamma(\boldsymbol{y}) \\
&\le \ \int_{\mathbb{R}^\infty} \|\phi\|_{(X^1)'} \|v_{\rm ev}(\boldsymbol{y}) - \mathcal{I}_{G_{\rm ev}(\xi_n)} v_{\rm ev}(\boldsymbol{y})\|_{X^1}\,{\rm d}\gamma(\boldsymbol{y}) \\
&\le \ C \int_{\mathbb{R}^\infty} \|v_{\rm ev}(\boldsymbol{y}) - \mathcal{I}_{G_{\rm ev}(\xi_n)} v_{\rm ev}(\boldsymbol{y})\|_{X^1}\,{\rm d}\gamma(\boldsymbol{y}) \\
&\le \ C \|v_{\rm ev} - \mathcal{I}_{G_{\rm ev}(\xi_n)} v_{\rm ev}\|_{\mathcal{V}_1(X^1)} \ \le \ C n^{-\min(\alpha,\beta)}.
\end{aligned}$$

□

Let v be a γ-measurable on $\mathbb{R}^\infty$ taking values in a Hilbert space X and $\phi \in X'$ a bounded linear functional on X. Denote by $\langle \phi, v\rangle$ the value of ϕ in v. For a finite set $\Lambda \subset \mathbb{F}$, the quadrature formula $Q_\Lambda v$ generates the quadrature formula $Q_\Lambda \langle \phi, v\rangle$ for integration of $\langle \phi, v\rangle$ by

$$Q_\Lambda \langle \phi, v\rangle := \sum_{\boldsymbol{s}\in\Lambda} \langle \phi, \Delta^{\rm Q}_{\boldsymbol{s}} v\rangle \ = \ \int_{\mathbb{R}^\infty} \langle \phi, I_\Lambda v(\boldsymbol{y})\rangle\,{\rm d}\gamma(\boldsymbol{y}).$$

Similarly to the proof of Theorem 4.1, applying (3.42) in Corollary 3.2 for $p=1$, we can derive the following

Corollary 4.1 *Under the hypothesis of Corollary 3.1, assume additionally that the sequences Y_n, $n \in \mathbb{N}_0$, are symmetric. For $\xi > 0$ let $\Lambda_{\rm ev}(\xi)$ be the set defined as in* (3.41)*.Then we have the following.*

(i) *For each $n \in \mathbb{N}$ there exists a number ξ_n such that $|\Gamma(\Lambda_{\rm ev}(\xi_n))| \le n$ and*

$$\left\| \int_{\mathbb{R}^\infty} v(\boldsymbol{y})\,{\rm d}\gamma(\boldsymbol{y}) - Q_{\Lambda_{\rm ev}(\xi_n)} v \right\|_X \ \le \ C n^{-(1/q - 1/2)}. \tag{4.7}$$

(ii) *Let $\phi \in X'$ be a bounded linear functional on X. Then for each $n \in \mathbb{N}$ there exists a number ξ_n such that $|\Gamma(\Lambda_{\rm ev}(\xi_n))| \le n$ and*

$$\left| \int_{\mathbb{R}^\infty} \langle \phi, v(\boldsymbol{y}) \rangle \, \mathrm{d}\gamma(\boldsymbol{y}) - Q_{\Lambda_{\rm ev}(\xi_n)} \langle \phi, v \rangle \right| \ \le \ C n^{-(1/q-1/2)}. \tag{4.8}$$

The constants C in (4.7) and (4.8) are independent of v and n.

5 Elliptic PDEs with lognormal inputs

In this section, we apply the results in Sections 2–4 to fully discrete linear Galerkin and polynomial interpolation approximations as well as integration for parametric and stochastic elliptic PDEs with lognormal inputs (1.2).

We approximate the solusion $u(\boldsymbol{y})$ to the parametrized elliptic PDEs (1.1) by truncation of the Hermite series

$$u(\boldsymbol{y}) = \sum_{\boldsymbol{s} \in \mathbb{F}} u_{\boldsymbol{s}} \, H_{\boldsymbol{s}}(\boldsymbol{y}), \quad u_{\boldsymbol{s}} \in V. \tag{5.1}$$

For convenience, we introduce the convention $W^1 := V$ and $W^2 := W$. Constructions of fully discrete approximations and integration are based on the approximation property (1.4) in Assumption I and the weighted ℓ_2-summability of the series $(\|u_{\boldsymbol{s}}\|_{W^r})_{\boldsymbol{s} \in \mathbb{F}}$ in following lemma which has been proven in [2, 3].

Lemma 5.1 *Let $r = 1, 2$. Assume that the right side f in (1.1) belongs to $H^{r-2}(D)$, that the domain D has $C^{r-2,1}$ smoothness, that all functions ψ_j belong to $W^{r-1,\infty}(D)$. Assume that there exist a number $0 < q_r < \infty$ and a sequence $\boldsymbol{\rho}_r = (\rho_{r;j})_{j \in \mathbb{N}}$ of positive numbers such that $(\rho_{r;j}^{-1})_{j \in \mathbb{N}} \in \ell_{q_r}(\mathbb{N})$ and*

$$\left\| \sum_{j \in \mathbb{N}} \rho_{r;j} |\psi_j| \right\|_{L^\infty(D)} + \sup_{|\alpha| \le r-1} \left\| \sum_{j \in \mathbb{N}} \rho_{r;j} |D^\alpha \psi_j| \right\|_{L^\infty(D)} < \infty \, .$$

Then we have that for any $\eta \in \mathbb{N}$,

$$\sum_{\boldsymbol{s} \in \mathbb{F}} (\sigma_{r;\boldsymbol{s}} \|u_{\boldsymbol{s}}\|_{W^r})^2 < \infty \quad \textit{with} \quad \sigma_{r;\boldsymbol{s}}^2 := \sum_{\|\boldsymbol{s}'\|_{\ell_\infty(\mathbb{F})} \le \eta} \binom{\boldsymbol{s}}{\boldsymbol{s}'} \boldsymbol{\rho}_r^{2\boldsymbol{s}'}.$$

We make use the following notation: for $\nu \in \mathbb{N}$,

$$\mathbb{F}_\nu := \{ \boldsymbol{s} \in \mathbb{F} : s_j \in \mathbb{N}_{0,\nu}, \ j \in \mathbb{N} \}; \quad \mathbb{N}_{0,\nu} := \{ n \in \mathbb{N}_0 : n = 0, \nu, \nu+1, ... \}.$$

The set $\mathbb{F}_\nu$ has been introduced in [34]. The set $\mathbb{F}_2$ plays an important role in establishing improved convergence rates for sparse-grid Smolyak quadrature in [33, 34].

The following lemma is a generalization of [2, Lemma 5.1].

Lemma 5.2 *Let $0 < q < \infty$, $\eta \in \mathbb{N}$, $\boldsymbol{\rho} = (\rho_j)_{j \in \mathbb{N}}$ of positive numbers such the sequence $(\rho_j^{-1})_{j \in \mathbb{N}}$ belongs to $\ell_q(\mathbb{N})$. Let θ, λ be arbitrary positive numbers and $(p_{\boldsymbol{s}}(\theta, \lambda))_{\boldsymbol{s} \in \mathbb{F}}$ the sequence given in (3.3). Let for numbers $\eta \in \mathbb{N}$ the sequence $(\sigma_{\boldsymbol{s}})_{\boldsymbol{s} \in \mathbb{F}}$ be defined by*

$$\sigma_{\boldsymbol{s}}^2 := \sum_{\|\boldsymbol{s}'\|_{\ell_\infty(\mathbb{F})} \le \eta} \binom{\boldsymbol{s}}{\boldsymbol{s}'} \rho^{2\boldsymbol{s}'}.$$

Then for any $\eta > \frac{2\nu(\theta+1)}{q}$*, we have*

$$\sum_{\boldsymbol{s}\in\mathbb{F}_\nu} p_{\boldsymbol{s}}(\theta,\lambda)\sigma_{\boldsymbol{s}}^{-q/\nu} < \infty.$$

Proof. With $\theta' := 2\theta\nu/q$, we have that

$$\sum_{\boldsymbol{s}\in\mathbb{F}_\nu} p_{\boldsymbol{s}}(\theta,\lambda)\sigma_{\boldsymbol{s}}^{-q/\nu} = \sum_{\boldsymbol{s}\in\mathbb{F}_\nu}\prod_{j\in\mathbb{N}}\left(\sum_{k=0}^{\eta}\binom{s_j}{k}(1+\lambda s_j)^{\theta'}\rho_j^{2k}\right)^{-q/2\nu}$$
$$= \prod_{j\in\mathbb{N}}\sum_{n\in\mathbb{N}_{0,\nu}}\left(\sum_{k=0}^{\eta}\binom{n}{k}(1+\lambda n)^{\theta'}\rho_j^{2k}\right)^{-q/2\nu} =: \prod_{j\in\mathbb{N}} B_j,$$

and

$$B_j \le \sum_{n\in\mathbb{N}_{0,\nu}}\left(\binom{n}{\min(n,\eta)}(1+\lambda n)^{\theta'}\rho_j^{2\min(n,\eta)}\right)^{-q/2\nu}$$
$$\le \sum_{n\in\mathbb{N}_{0,\nu},\ n<\eta}\left(\binom{n}{n}(1+\lambda n)^{\theta'}\rho_j^{2n}\right)^{-q/2\nu} + \sum_{n\ge\eta}\left((1+\lambda n)^{\theta'}\rho_j^{2\eta}\right)^{-q/2\nu}$$
$$\le \sum_{n\in\mathbb{N}_{0,\nu},\ n<\eta}(1+\lambda n)^{\theta}\rho_j^{-nq/\nu} + \rho_j^{-\eta q/\nu}\sum_{n\ge\eta}\binom{n}{\eta}^{-q/2\nu}(1+\lambda n)^{\theta} =: B_{j,1}+B_{j,2}.$$

We estimate $B_{j,1}$ and $B_{j,2}$. We have

$$B_{j,1} \le 1 + \sum_{n=\nu}^{\eta-1}(1+\lambda n)^{\theta}\rho_j^{-nq/\nu} \le 1 + (1+(\eta-1)\lambda)^{\theta}\sum_{n=\nu}^{\eta-1}\rho_j^{-nq/\nu}.$$

From the inequalities $\left(\frac{n}{\eta}\right)^{\eta} \le \binom{n}{\eta}$ and $\eta q/2\nu - \theta > 1$ we derive that

$$B_{j,2} \le \rho_j^{-\eta q/\nu}\sum_{n\ge\eta}\left(\frac{n}{\eta}\right)^{-\eta q/2\nu}(1+\lambda n)^{\theta} \le C\rho_j^{-\eta q/\nu}\sum_{n\ge\eta} n^{-(\eta q/2\nu-\theta)} \le C\rho_j^{-\eta q/\nu}.$$

Summing up we obtain that

$$B_j \le B_{j,1}+B_{j,2} \le 1 + C\sum_{n=\nu}^{\eta}\rho_j^{-nq/\nu}.$$

Since the sequence $(\rho_j^{-1})_{j\in\mathbb{N}}$ belongs to $\ell_q(\mathbb{N})$, there exists j^* large enough such that $\rho_j > 1$ for all $j \ge j^*$. Hence, there exists a constant C independent of j such that $B_j \le 1 + C\rho_j^{-q}$ for all $j \in \mathbb{N}$, and consequently,

$$\sum_{\boldsymbol{s}\in\mathbb{F}_\nu} p_{\boldsymbol{s}}(\theta,\lambda)\sigma_{\boldsymbol{s}}^{-q/2\nu} \le \prod_{j\in\mathbb{N}} B_j \le \prod_{j\in\mathbb{N}}(1+C\rho_j^{-q}) \le \exp\left(\sum_{j\in\mathbb{N}} C\rho_j^{-q}\right) < \infty.$$

∎

To treat fully discrete approximations we assume that $f \in L_2(D)$ and there holds the approximation property (1.4) in Assumption I for the spaces V and W. Notice here that classical error estimates [5] yield the convergence rate $\alpha = 1/d$ by using Lagrange finite elements of order at least 1 on quasi-uniform partitions. Also, the spaces W do not always coincide with $H^2(D)$. For example, for $d = 2$, we know that W is strictly larger than $H^2(D)$ when D is a polygon with re-entrant corner. In this case, it is well known that the optimal rate $\alpha = 1/2$ is yet attained, when using spaces V_n associated to meshes $(\mathcal{T}_n)_{n>0}$ with proper refinement near the re-entrant corners where the functions $v \in W$ might have singularities.

Theorem 5.1 *Let $0 < p \le 2$. Let $f \in L_2(D)$ and Assumption I hold. Let the assumptions of Lemma 5.1 hold for the spaces $W^1 = V$ and $W^2 = W$ with some $0 < q_1 \le q_2 < \infty$. For $\xi > 0$ let $G(\xi)$ be the set defined as in* (2.8).

Then for each $n \in \mathbb{N}$ there exists a number ξ_n such that $\dim(\mathcal{V}(G(\xi_n)) \le n$ *and*

$$\|u - \mathcal{S}_{G(\xi_n)}u\|_{\mathcal{V}_p(V)} \le Cn^{-\min(\alpha,\beta)}. \tag{5.2}$$

The rate α corresponds to the spatial approximation of a single function in W as given by (1.4)*, and the rate β is given by* (2.10)*. The constant C in* (5.2) *is independent of u and n.*

Proof. To prove the theorem it is sufficient to notice that the assumptions of Theorem 2.1 are satisfied for $X^1 = V$ and $X^2 = W$. This can be done by using Lemmata 5.1 and 5.2. (By multiplying the sequences $\boldsymbol{\rho}_r$ in Lemma 5.1 with a positive constant we can get $\sigma_{r;\boldsymbol{s}} > 1$ for $\boldsymbol{s} \in \mathbb{F}$.) ∎

The rate $\min(\alpha, \beta)$ in (5.2) is the rate of best adaptive n-term Galerkin approximation in $\mathcal{V}_2(V)$ based on ℓ_{p_1}-summability of $(\|u_{\boldsymbol{s}}\|_V)_{\boldsymbol{s}\in\mathbb{F}}$ and ℓ_{p_2}-summability of $(\|u_{\boldsymbol{s}}\|_W)_{\boldsymbol{s}\in\mathbb{F}}$ proven in [3], where $1/p_r = 1/q_r + 1/2$ for $r = 1, 2$.

In the same way, from Theorems 2.1 and 3.1 and Corollary 3.1 we derive the following two theorems and corollary.

Theorem 5.2 *Let $0 < p \le 2$. Let $f \in L_2(D)$ and Assumption I hold. Let the assumptions of Lemma 5.1 hold for the spaces $W^1 = V$ and $W^2 = W$ with some $0 < q_1 \le q_2 < \infty$ with $q_1 < 2$. Assume that $(Y_n)_{n\in\mathbb{N}_0}$ is a sequence such that every Y_n satisfies the condition* (3.14) *for some positive numbers τ and C. For $\xi > 0$ let $G(\xi)$ be the set defined as in* (3.5).

Then for each $n \in \mathbb{N}$ there exists a number ξ_n such that $|G(\xi_n)| \le n$ and

$$\|u - \mathcal{I}_{G(\xi_n)}u\|_{\mathcal{V}_p(V)} \le Cn^{-\min(\alpha,\beta)}. \tag{5.3}$$

The rate α corresponds to the spatial approximation of a single function in W as given by (1.4)*. The rate β is given by* (3.19)*. The constant C in* (5.3) *is independent of u and n.*

Observe that the approximands $\mathcal{I}_{G(\xi_n)}u$ in this theorem can be represented in the form

$$\mathcal{I}_{G(\xi_n)}u \;=\; \sum_{k=0}^{k_n} \delta_k \Big(\sum_{\boldsymbol{s}\in\Lambda_k(\xi_n)} \Delta^{\mathrm{I}}_{\boldsymbol{s}} \Big) u \;=\; \sum_{k=0}^{k_n} \delta_k I_{\Lambda_k(\xi_n)} u,$$

where $k_n := \lfloor \log_2 \xi_n \rfloor$ and for $k \in \mathbb{N}_0$ and $\xi > 0$,

$$\Lambda_k(\xi) := \begin{cases} \big\{\boldsymbol{s} \in \mathbb{F} : \sigma_{2;\boldsymbol{s}}^{q_2} \le 2^{-k}\xi\big\} & \text{if } \alpha \le 1/q_2; \\ \big\{\boldsymbol{s} \in \mathbb{F} : \sigma_{1;\boldsymbol{s}}^{q_1} \le \xi, \ \sigma_{2;\boldsymbol{s}}^{q_1} \le 2^{-k}\xi\big\} & \text{if } \alpha > 1/q_2. \end{cases}$$

Moreover, $\Lambda_k(\xi_n)$ are downward closed sets, and consequently, the sequence $\big\{\Lambda_k(\xi_n)\big\}_{k=0}^{k_n}$ is nested in the inverse order, i.e., $\Lambda_{k'}(\xi_n) \subset \Lambda_k(\xi_n)$ if $k' > k$, and $\Lambda_0(\xi_n)$ is the largest and $\Lambda_{k_n}(\xi_n) = \{0_{\mathbb{F}}\}$. Further, the fully discrete polynomial interpolation approximation by operators $\mathcal{I}_{G(\xi_n)}$ is a collocation approximation based on the finite number $|\Gamma(\Lambda_0(\xi_n))| \le \sum_{\boldsymbol{s}\in\Lambda_0(\xi_n)} p_{\boldsymbol{s}}(1,2)$ of the particular solvers $u(\boldsymbol{y})$, $\boldsymbol{y} \in \Gamma(\Lambda_0(\xi_n))$, where, we recall, $\Gamma(\Lambda_0(\xi_n)) = \cup_{\boldsymbol{s}\in\Lambda_0(\xi_n)}\Gamma_{\boldsymbol{s}}$ and $\Gamma_{\boldsymbol{s}} = \{\boldsymbol{y}_{\boldsymbol{s}-\boldsymbol{e};\boldsymbol{m}} : e \in \mathbb{E}_{\boldsymbol{s}};\ m_j = 0, ..., s_j,\ j \in \mathbb{N}\}$ ($\mathbb{E}_{\boldsymbol{s}}$ denotes the subset in $\mathbb{F}$ of all $\boldsymbol{e}$ such that e_j is 1 or 0 if $s_j > 0$, and e_j is 0 if $s_j = 0$, and $\boldsymbol{y}_{\boldsymbol{s};\boldsymbol{m}} := (y_{s_j;m_j})_{j\in\mathbb{N}}$.)

Corollary 5.1 *Let $0 < p \le 2$. Under the hypothesis of Lemma 5.1 for the spaces $W^1 = V$ with some $0 < q < 2$, for $\xi > 0$ let $\Lambda(\xi)$ be the set defined as in* (3.38)*. Then for each $n \in \mathbb{N}$ there exists a number ξ_n such that $|\Gamma(\Lambda(\xi_n))| \le n$ and*

$$\|u - I_{\Lambda(\xi_n)}u\|_{\mathcal{V}_p(V)} \le Cn^{-(1/q-1/2)}. \tag{5.4}$$

The constant C in (5.4) *is independent of v and n.*

The rate in Corollary 5.1 hase been obtained in [18] for a similar approximation in $\mathcal{V}_2(V)$.

Theorem 5.3 *Let $f \in L_2(D)$ and Assumption I hold. Let the assumptions of Lemma 5.1 hold for the spaces $W^1 = V$ and $W^2 = W$ with some $0 < q_1 \le q_2 < \infty$ with $q_1 < 4$. Assume that $(Y_n)_{n\in\mathbb{N}_0}$ is a symmetric sequence such that every Y_n satisfies the condition* (3.14) *for some positive numbers τ and C. For $\xi > 0$ let $G_{\mathrm{ev}}(\xi)$ be the set defined as in* (3.10)*. Then we have the following.*

(i) *For each $n \in \mathbb{N}$ there exists a number ξ_n such that $|G_{\mathrm{ev}}(\xi_n)| \le n$ and*

$$\left\| \int_{\mathbb{R}^\infty} u(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}) - \mathcal{Q}_{G_{\mathrm{ev}}(\xi_n)}u \right\|_V \le Cn^{-\min(\alpha,\beta)}. \tag{5.5}$$

(ii) *Let $\phi \in V'$ be a bounded linear functional on V. For each $n \in \mathbb{N}$ there exists a number ξ_n such that $|G_{\mathrm{ev}}(\xi_n)| \le n$ and*

$$\left| \int_{\mathbb{R}^\infty} \langle\phi, u(\boldsymbol{y})\rangle\,\mathrm{d}\gamma(\boldsymbol{y}) - \mathcal{Q}_{G_{\mathrm{ev}}(\xi_n)}\langle\phi, u\rangle \right| \le Cn^{-\min(\alpha,\beta)}. \tag{5.6}$$

The rate α corresponds to the spatial approximation of a single function in W as given by (1.4)*. The rate β is given by*

$$\beta := \left(\frac{2}{q_1} - \frac{1}{2}\right)\frac{\alpha}{\alpha+\delta}, \quad \delta := \frac{2}{q_1} - \frac{2}{q_2}. \tag{5.7}$$

The constants C in (5.5) *and* (5.6) *are independent of u and n.*

Proof. Observe that $\mathbb{F}_{\mathrm{ev}} \subset \mathbb{F}_2$. From Lemma 5.1 and Lemma 5.2 we can see that the assumptions of Theorem 3.1 hold for $X^1 = V$ and $X^2 = W$ with $0 < q_1/2 \le q_2/2 < \infty$ and $q_1/2 < 2$. Hence, by applying Theorem 4.1 we prove the theorem. ∎

Observe that the rate in (5.5) and (5.6) can be improved as $\min(\alpha, \frac{2}{q_1}\frac{\alpha}{\alpha+\delta})$ if the sequences $(\|u_{\boldsymbol{s}}\|_V)_{\boldsymbol{s}\in\mathbb{F}}$ and $(\|u_{\boldsymbol{s}}\|_{W^r})_{\boldsymbol{s}\in\mathbb{F}}$ have ℓ_{p_1}- and ℓ_{p_r}-summable majorant sequences, respectively, where $1/p_1 = 1/q_1 + 1/2$ and $1/p_r = 1/q_r + 1/2$.

In the same way, from Corollary 4.1 we derive the following

Corollary 5.2 *Let the assumptions of Lemma 5.1 hold for the spaces $W^1 = V$ with some $0 < q < 4$. Assume that $(Y_n)_{n\in\mathbb{N}_0}$ is a symmetric sequence such that every Y_n satisfies the condition* (3.14) *for some positive numbers τ and C. For $\xi > 0$ let $\Lambda_{\mathrm{ev}}(\xi)$ be the set defined as in* (3.41)*. Then we have the following.*

(i) *For each $n \in \mathbb{N}$ there exists a number ξ_n such that $|\Gamma(\Lambda_{\rm ev}(\xi_n))| \le n$ and*

$$\left\| \int_{\mathbb{R}^\infty} u(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}) - \mathcal{Q}_{\Lambda_{\rm ev}(\xi_n)} u \right\|_V \;\le\; C n^{-(2/q-1/2)}. \tag{5.8}$$

(ii) *Let $\phi \in V'$ a bounded linear functional on V. For each $n \in \mathbb{N}$ there exists a number ξ_n such that $|\Gamma(\Lambda_{\rm ev}(\xi_n))| \le n$ and*

$$\left| \int_{\mathbb{R}^\infty} \langle \phi, u(\boldsymbol{y})\rangle \,\mathrm{d}\gamma(\boldsymbol{y}) - \mathcal{Q}_{\Lambda_{\rm ev}(\xi_n)} \langle \phi, u\rangle \right| \;\le\; C n^{-(2/q-1/2)}. \tag{5.9}$$

The constants C in (5.8) *and* (5.9) *are independent of u and n.*

The rate $2/q - 1/2$ in Corollary 5.2 is an improvement of the rate $1/q - 1/2$ which has been recently obtained in [4]. Observe that this rate can be improved as $2/q$ if the sequence $(\|u_{\boldsymbol{s}}\|_V)_{\boldsymbol{s}\in\mathbb{F}}$ has an ℓ_p-summable majorant sequence, where $1/p = 1/q + 1/2$.

6 Elliptic PDEs with affine inputs

The theory of non-addaptive fully discrete approximations in Bochner spaces with infinite tensor product Gausian measure in Sections 2–4 can be extended to other situations. In this section, we present some results on similar problems for parametric and stochastic PDEs with the affine inputs (1.3).

In the affine case, for given $a, b > -1$, we consider the orthogonal Jacobi expansion of the solution $u(\boldsymbol{y})$ of the form

$$\sum_{\boldsymbol{s}\in\mathbb{F}} u_{\boldsymbol{s}} J_{\boldsymbol{s}}(\boldsymbol{y}), \quad J_{\boldsymbol{s}}(\boldsymbol{y}) = \bigotimes_{j\in\mathbb{N}} J_{s_j}(y_j), \quad u_{\boldsymbol{s}} := \int_{\mathbb{I}} u(\boldsymbol{y}) J_{\boldsymbol{s}}(\boldsymbol{y}) \mathrm{d}\sigma_{a,b}(\boldsymbol{y}),$$

where

$$\mathrm{d}\sigma_{a,b}(\boldsymbol{y}) := \bigotimes_{j\in\mathbb{N}} \delta_{a,b}(y_j)\,\mathrm{d}y_j, \tag{6.1}$$

$$\delta_{a,b}(y) := c_{a,b}(1-y)^a(1+y)^b, \quad c_{a,b} := \frac{\Gamma(a+b+2)}{2^{a+b+1}\Gamma(a+1)\Gamma(b+1)},$$

and $(J_k)_{k\ge 0}$ is the sequence of Jacobi polynomials on $\mathbb{I} := [-1,1]$ normalized with respect to the Jacobi probability measure $\int_{\mathbb{I}} |J_k(y)|^2 \delta_{a,b}(y)\mathrm{d}y = 1$. One has the Rodrigues' formula

$$J_k(y) \;=\; \frac{c_k^{a,b}}{k!2^k}(1-y)^{-a}(1+y)^{-b}\frac{\mathrm{d}^k}{\mathrm{d}y^k}\left((y^2-1)^k(1-y)^a(1+y)^b\right),$$

where $c_0^{a,b} := 1$ and

$$c_k^{a,b} := \sqrt{\frac{(2k+a+b+1)k!\Gamma(k+a+b+1)\Gamma(a+1)\Gamma(b+1)}{\Gamma(k+a+1)\Gamma(k+b+1)\Gamma(a+b+2)}}, \; k \in \mathbb{N}.$$

Examples corresponding to the values $a = b = 0$ is the family of the Legendre polynomials, and to the values $a = b = -1/2$ the family of the Chebyshev polynomials.

We introduce the space $W^r := \{v \in V \;:\; \Delta v \in H^{r-2}(D)\}$ for $r \ge 2$ with the convention $W^1 := V$. This space is equipped with the norm $\|v\|_{W^r} := \|\Delta v\|_{H^{r-2}(D)}$, and coincides with the Sobolev space $V \cap H^{r-2}(D)$ with equivalent norms if the domain D has $C^{r-1,1}$ smoothness, see [19, Theorem 2.5.1.1]. The following lemma has been proven in [1, 3].

Lemma 6.1 *For a given $r \in \mathbb{N}$, assume that $\bar{a} \in L^\infty(D)$ is such that* $\operatorname{ess\,inf} \bar{a} > 0$, *and that there exists a sequence $\boldsymbol{\rho}_r = (\rho_{r;j})_{j\in\mathbb{N}}$ of positive numbers such that*

$$\left\| \frac{\sum_{j\in\mathbb{N}} \rho_{1;j} |\psi_j|}{\bar{a}} \right\|_{L^\infty(D)} < 1 \, .$$

Assume that the right side f in (1.1) belongs to $H^{r-2}(D)$, that the domain D has $C^{r-2,1}$ smoothness, that $\bar{a}$ and all functions ψ_j belong to $W^{r-1,\infty}(D)$ and that

$$\sup_{|\alpha|\le r-1} \left\| \sum_{j\in\mathbb{N}} \rho_{r;j} |D^\alpha \psi_j| \right\|_{L^\infty(D)} < \infty \, .$$

Then

$$\sum_{\boldsymbol{s}\in\mathbb{F}} (\sigma_{r;\boldsymbol{s}} \|u_{\boldsymbol{s}}\|_{W^r})^2 < \infty, \quad \sigma_{r;\boldsymbol{s}} := \boldsymbol{\rho}_r^{\boldsymbol{s}} \prod_{j\in\mathbb{N}} c_{s_j}^{a,b}.$$

Lemma 6.2 *Let $0 < q < \infty$, $\boldsymbol{\rho} = (\rho_j)_{j\in\mathbb{N}}$ of numbers larger than one such the sequence $(\rho_j^{-1})_{j\in\mathbb{N}}$ belongs to $\ell_q(\mathbb{N})$, $(p_{\boldsymbol{s}}(\theta,\lambda))_{\boldsymbol{s}\in\mathbb{F}}$ is a sequence of the form (3.3) with arbitrary nonnegative θ, λ. Then for every $\nu \in \mathbb{N}_0$, we have*

$$\sum_{\boldsymbol{s}\in\mathbb{F}_\nu} p_{\boldsymbol{s}}(\theta,\lambda)(\rho^{-\boldsymbol{s}})^{q/\nu} < \infty.$$

Proof. We have

$$\sum_{\boldsymbol{s}\in\mathbb{F}_\nu} p_{\boldsymbol{s}}(\theta,\lambda)(\rho^{-\boldsymbol{s}})^{q/\nu} = \prod_{j\in\mathbb{N}} \sum_{s_j\in\mathbb{N}_{0,\nu}} \rho_j^{-s_j q/\nu}(1+\lambda s_j)^\theta =: \prod_{j\in\mathbb{N}} A_j.$$

Since $\boldsymbol{\rho} = (\rho_j)_{j\in\mathbb{N}}$ of numbers larger than one, and such the sequens $(\rho_j^{-1})_{j\in\mathbb{N}}$ belongs to $\ell_q(\mathbb{N})$, we have $\min_{j\in\mathbb{N}} \rho_j > 1$. Hence, there exists a constant C independent of j such that

$$A_j = 1 + \sum_{k=\nu}^{\infty} \rho_j^{-kq/\nu}(1+\lambda k)^\theta \le 1 + C\rho_j^{-q},$$

and consequently,

$$\sum_{\boldsymbol{s}\in\mathbb{F}_\nu} p_{\boldsymbol{s}}(\theta,\lambda)(\rho^{-\boldsymbol{s}})^{q/\nu} \le \prod_{j\in\mathbb{N}} (1 + C\rho_j^{-q}) \le \exp\Big(\sum_{j\in\mathbb{N}} C\rho_j^{-q}\Big) < \infty.$$

□

We assume that there holds the following *approximation property* for V and W^r with $r > 1$.

Assumption III There are a sequence $(V_n)_{n\in\mathbb{N}_0}$ of subspaces $V_n \subset V$ of dimension $\le n$, and sequence $(P_n)_{n\in\mathbb{N}_0}$ of linear operators from V into V_n, and a number $\alpha > 0$ such that

$$\|P_n(v)\|_V \le C, \quad \|v - P_n(v)\|_V \le Cn^{-\alpha}\|v\|_{W^r}, \quad \forall n \in \mathbb{N}_0, \quad \forall v \in W^r. \tag{6.2}$$

In what follows, we make use the abbreviation: $\mathcal{V}_p(V) := L_p(\mathbb{I}^\infty, V, \sigma_{a,b})$ and assume that $r > 1$. From Lemmata 6.1 and 6.2 we can prove the following results on non-adaptive fully Galerkin and polynomial interpolation approximations and integration for the affine case.

Theorem 6.1 *Let $0 < p \leq 2$. Let $f \in H^{r-2}(D)$ and Assumption III hold. Let the assumptions of Lemma 6.1 hold for the spaces $W^1 = V$ and W^r with some $0 < q_1 \leq q_r < \infty$. For $\xi > 0$ let $G(\xi)$ be the set defined as in* (2.8)*. Then for each $n \in \mathbb{N}$ there exists a number ξ_n such that* $\dim(\mathcal{V}(G(\xi_n)) \leq n$ *and*

$$\|u - \mathcal{S}_{G(\xi_n)} u\|_{\mathcal{V}_p(V)} \leq C n^{-\min(\alpha,\beta)}. \tag{6.3}$$

The rate α corresponds to the spatial approximation of a single function in W^r as given by (6.2)*, and the rate β is given by* (2.10)*. The constant C in* (6.3) *is independent of u and n.*

The rate $\min(\alpha, \beta)$ in (6.3) is the same rate of fully discrete best adaptive n-term approximation in $\mathcal{V}_2(V)$ based on ℓ_{p_1}-summability of $(\|u_{\boldsymbol{s}}\|_V)_{\boldsymbol{s}\in\mathbb{F}}$ and ℓ_{p_r}-summability of $(\|u_{\boldsymbol{s}}\|_{W^r})_{\boldsymbol{s}\in\mathbb{F}}$ proven in [3], where $1/p_1 = 1/q_1 + 1/2$ and $1/p_r = 1/q_r + 1/2$. This rate can be achieved by fully discrete linear non-adaptive approximation when $(\|u_{\boldsymbol{s}}\|_V)_{\boldsymbol{s}\in\mathbb{F}}$ and $(\|u_{\boldsymbol{s}}\|_{W^r})_{\boldsymbol{s}\in\mathbb{F}}$ have ℓ_{p_1}-summable and ℓ_{p_r}-summable majorant sequences, respectively [33].

Theorem 6.2 *Let $1 \leq p \leq \infty$. Let $f \in H^{r-2}(D)$ and Assumption III hold. Let the assumptions of Lemma 6.1 hold for the spaces $W^1 = V$ and W^r with some $0 < q_1 \leq q_r < \infty$ with $q_1 < 2$. For $\xi > 0$ let $G(\xi)$ be the set defined as in* (3.5)*. Then for each $n \in \mathbb{N}$ there exists a number ξ_n such that* $\dim(\mathcal{V}(G(\xi_n)) \leq n$ *and*

$$\|u - \mathcal{S}_{G(\xi_n)} u\|_{\mathcal{V}_p(V)} \leq C n^{-\min(\alpha,\beta)}. \tag{6.4}$$

The rate α corresponds to the spatial approximation of a single function in W^r as given by (6.2)*. The rate β is given by* (3.19) *The constant C in* (6.4) *is independent of u and n.*

For polynomial interpolation approximation and integration, we keep all definitions and notations in Section 3 with a proper modification for the affine case. For example, for univariate interpolation and integration we take a sequence of points $Y_n = (y_{n;k})_{k=0}^n$ in $\mathbb{I}$ such that

$$-\infty < y_{n;0} < \cdots < y_{n;n-1} < y_{n;n} < +\infty; \quad y_{0;0} = 0.$$

Sequences of points $Y_n = (y_{n;k})_{k=0}^n$ satisfying the inequality (3.14), are the symmetric sequences of the Chebyshev points, the symmetric sequences of the Gauss-Lobatto (Clenshaw-Curtis) points and the nested sequence of the $\Re$-Leja points, see [9] for details.

Theorem 6.3 *Let $1 \leq p \leq \infty$. Let $f \in H^{r-2}(D)$ and Assumption III hold. Let the assumptions of Lemma 6.1 hold for the spaces $W^1 = V$ and W^r with some $0 < q_1 \leq q_r < \infty$ with $q_1 < 2$. Assume that $(Y_n)_{n\in\mathbb{N}_0}$ is a sequence such that every Y_n satisfies the condition* (3.14) *for some positive numbers τ and C. For $\xi > 0$ let $G(\xi)$ be the set defined as in* (3.5)*. Then for each $n \in \mathbb{N}$ there exists a number ξ_n such that $|G(\xi_n)| \leq n$ and*

$$\|u - \mathcal{I}_{G(\xi_n)} u\|_{\mathcal{V}_p(V)} \leq C n^{-\min(\alpha,\beta)}. \tag{6.5}$$

The rate α corresponds to the spatial approximation of a single function in W^r as given by (6.2)*. The rate β is given by* (3.19)*. The constant C in* (6.5) *is independent of u and n.*

The rates in (6.3)–(6.5) for some non-adaptive approximations have been proven in the case when $(\|u_{\boldsymbol{s}}\|_V)_{\boldsymbol{s}\in\mathbb{F}}$ and $(\|u_{\boldsymbol{s}}\|_{W^r})_{\boldsymbol{s}\in\mathbb{F}}$ have ℓ_{p_1}-summable and ℓ_{p_r}-summable majorant sequences, respectively, which are derived from the analyticity of the solution u, where $1/p_1 = 1/q_1 + 1/2$ and $1/p_r = 1/q_r + 1/2$, see [33].

Theorem 6.4 *Let $f \in H^{r-2}(D)$ and Assumption III hold. Let $a = b$ in the definition of the Jacobi probability measure $\sigma_{a,b}(\boldsymbol{y})$ in* (6.1)*, and the assumptions of Lemma 6.1 hold for the spaces $W^1 = V$ and W^r with some $0 < q_1 \leq q_r < \infty$ with $q_1 < 4$. Assume that $(Y_n)_{n\in\mathbb{N}_0}$ is a sequence such that every Y_n is symmetric and satisfies the condition* (3.14) *for some positive numbers τ and C. For $\xi > 0$ let $G_{\rm ev}(\xi)$ be the set defined as in* (3.10)*. Then we have the following.*

(i) *For each $n \in \mathbb{N}$ there exists a number ξ_n such that $|G_{\rm ev}(\xi_n)| \le n$ and*

$$\left\| \int_{\mathbb{I}^\infty} u(\boldsymbol{y}) \, \mathrm{d}\gamma(\boldsymbol{y}) - \mathcal{Q}_{G_{\rm ev}(\xi_n)} u \right\|_V \le C n^{-\min(\alpha,\beta)}. \tag{6.6}$$

(ii) *Let $\phi \in V'$ be a bounded linear functional on V. For each $n \in \mathbb{N}$ there exists a number ξ_n such that $|G_{\rm ev}(\xi_n)| \le n$ and*

$$\left| \int_{\mathbb{I}^\infty} \langle \phi, u(\boldsymbol{y}) \rangle \, \mathrm{d}\gamma(\boldsymbol{y}) - \mathcal{Q}_{G_{\rm ev}(\xi_n)} \langle \phi, u \rangle \right| \le C n^{-\min(\alpha,\beta)}. \tag{6.7}$$

The rate α corresponds to the spatial approximation of a single function in W^r as given by (6.2)*. The rate β is given by*

$$\beta := \left(\frac{2}{q_1} - \frac{1}{2} \right) \frac{\alpha}{\alpha + \delta}, \quad \delta := \frac{2}{q_1} - \frac{2}{q_r}. \tag{6.8}$$

The constants C in (6.6) *and* (6.7) *are independent of u and n.*

The rate in (6.6)–(6.7) can be improved as $\min(\alpha, \frac{2}{q_1} \frac{\alpha}{\alpha+\delta})$ if $(\|u_{\boldsymbol{s}}\|_V)_{\boldsymbol{s} \in \mathbb{F}}$ and $(\|u_{\boldsymbol{s}}\|_{W^r})_{\boldsymbol{s} \in \mathbb{F}}$ have ℓ_{p_1}- and ℓ_{p_r}-summable majorant sequences, respectively, where $1/p_1 = 1/q_1 + 1/2$ and $1/p_r = 1/q_r + 1/2$, see [33].

Corollary 6.1 *Let $a = b$ in the definition of the Jacobi probability measure $\sigma_{a,b}(\boldsymbol{y})$ in* (6.1)*, and the assumptions of Lemma 6.1 hold for the spaces $W^1 = V$ with some $0 < q < 4$. Assume that $(Y_n)_{n \in \mathbb{N}_0}$ is a sequence such that every Y_n is symmetric and satisfies the condition* (3.14) *for some positive numbers τ and C. For $\xi > 0$ let $\Lambda_{\rm ev}(\xi)$ be the set defined as in* (3.41)*. Then we have the following.*

(i) *For each $n \in \mathbb{N}$ there exists a number ξ_n such that $|\Gamma(\Lambda_{\rm ev}(\xi_n))| \le n$ and*

$$\left\| \int_{\mathbb{I}^\infty} u(\boldsymbol{y}) \, \mathrm{d}\gamma(\boldsymbol{y}) - \mathcal{Q}_{\Lambda_{\rm ev}(\xi_n)} u \right\|_V \le C n^{-(2/q - 1/2)}. \tag{6.9}$$

(ii) *Let $\phi \in V'$ be a bounded linear functional on V. For each $n \in \mathbb{N}$ there exists a number ξ_n such that $|\Gamma(\Lambda_{\rm ev}(\xi_n))| \le n$ and*

$$\left| \int_{\mathbb{I}^\infty} \langle \phi, u(\boldsymbol{y}) \rangle \, \mathrm{d}\gamma(\boldsymbol{y}) - \mathcal{Q}_{\Lambda_{\rm ev}(\xi_n)} \langle \phi, u \rangle \right| \le C n^{-(2/q - 1/2)}. \tag{6.10}$$

The constants C in (6.9) *and* (6.10) *are independent of u and n.*

The rate in (6.9) and (6.10) can be improved as $2/q$ if the sequence $(\|u_{\boldsymbol{s}}\|_V)_{\boldsymbol{s} \in \mathbb{F}}$ has an ℓ_p-summable majorant sequence, where $1/p = 1/q + 1/2$, see [34].

Acknowledgments. This work is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under Grant No. 102.01-2017.05. It was partially supported by a grant from the Simon Foundation and EPSRC Grant Number EP/R014604/1. The author would like to thank the Isaac Newton Institute for Mathematical Sciences for partial support and hospitality during the programme Approximation, sampling and compression in data science when work on this paper was partially undertaken. He expresses special thanks to Dr. Nguyen Van Kien for valuable remarks and comments.

References

[1] M. Bachmayr, A. Cohen, and G. Migliorati, Sparse polynomial approximation of parametric elliptic PDEs. Part I: affine coefficients, ESAIM Math. Model. Numer. Anal. **51**(2017), 321-339.

[2] M. Bachmayr, A. Cohen, R. DeVore and G. Migliorati, Sparse polynomial approximation of parametric elliptic PDEs. Part II: lognormal coefficients, ESAIM Math. Model. Numer. Anal. **51**(2017), 341-363.

[3] M. Bachmayr, A. Cohen, D. Dũng and C. Schwab, Fully discrete approximation of parametric and stochastic elliptic PDEs, SIAM J. Numer. Anal. **55**(2017), 2151-2186.

[4] P. Chen, Sparse quadrature for high-dimensional integration with Gaussian measure. ESAIM Math. Model. Numer. Anal. **52**(2018), 631-657.

[5] P. Ciarlet, *The Finite Element Method for Elliptic Problems*, North Holland Publ., 1978.

[6] A. Chkifa, A. Cohen and C. Schwab, Breaking the curse of dimensionality in sparse polynomial approximation of parametric PDEs, Journal de Mathmatiques Pures et Appliques **103**(2)(2015), 400-428.

[7] A. Chkifa, A. Cohen, R. DeVore, Ch. Schwab, Sparse adaptive Taylor approximation algorithms for parametric and stochastic elliptic PDEs, Modl. Math. Anal. Numr. **47**(1)(2013), 253-280.

[8] A. Chkifa, A. Cohen, and C. Schwab. High-dimensional adaptive sparse polynomial interpolation and applications to parametric PDEs. Found. Comp. Math **14**(2014), 601-633.

[9] A. Cohen and R. DeVore, Approximation of high-dimensional parametric PDEs, Acta Numerica **24**(2015), 1-159.

[10] A. Cohen, R. DeVore, Ch. Schwab, Convergence rates of best N-term Galerkin approximations for a class of elliptic sPDEs, Found. Comput. Math. **10**(6)(2010), 615-646.

[11] A. Cohen, R. DeVore, Ch. Schwab, Analytic regularity and polynomial approximation of parametric and stochastic PDEs, Anal. Appl. **9**(2011), 11-47.

[12] J. Dick, Ch. Irrgeher, G. Leobacher and F. Pillichshammer, On the optimal order of integration in Hermite spaces with finite smoothness SIAM Journal on Numerical Analysis, **56**(2)(2018), 684-707.

[13] J. Dick, F. Kuo, and I. Sloan. High-dimensional integration: the quasi-Monte Carlo way. Acta Numerica, **22**(2013),133-288.

[14] D. Dũng, Linear collective collocation and Galerkin approximations for parametric and stochastic elliptic PDEs (2015), arXiv:1511.03377v5 [math.NA].

[15] D. Dũng, Galerkin approximation for parametric and stochastic elliptic PDEs, Bulletin of L.N. Gumilyov Euroasian National University, Mathematics-Computer Science-Mechanics Series, No 1(122)(2018), 76-89.

[16] D. Dũng, Linear collocation approximation for parametric and stochastic elliptic PDEs, Mat. Sb. **210**(2019), 103-227.

[17] A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, Higher transcendental functions, Vol. II, McGraw-Hill, 1955.

[18] O. Ernst, B. Sprungk, and L. Tamellini. Convergence of sparse collocation for functions of countably many Gaussian random variables - with application to lognormal elliptic diffusion problems, SIAM J. Numer. Anal. **56**(2018), 887-905.

[19] P. Grisvard, Elliptic Problems in Nonsmooth Domains, Monogr. Stud. Math. 24, Pitman (Advanced Publishing Program), Boston, 1985.

[20] M. Gunzburger, C. Webster and G. Zang, Stochastic finite element methods for partial diferential equations with random input data, Acta Numerica **23**(2014), 521-650.

[21] A.-L. Haji-Ali, F. Nobile, L. Tamellini, and R. Tempone. Multi-index stochastic collocation convergence rates for random PDEs with parametric regularity. Found. Com- put. Math., **16**(6)(2016),1555-1605.

[22] A.-L. Haji-Ali, F. Nobile, L. Tamellini, and R. Tempone. Multi-Index Stochastic Collocation for random PDEs. Comput. Methods Appl. Mech. Engrg., **306**(2016), 95-122.

[23] E. Hewitt and K. Stromberg, Real and abstract analysis, Springer, 1965.

[24] V.H. Hoang, Ch. Schwab, N-term Wiener chaos approximation rate for elliptic PDEs with lognormal Gausian random inputs, Math. Models Methods Applied Sciences, **24**(2014), 796-826.

[25] D. S. Lubinsky, A survey of weighted polynomial approximation with exponential weights, Surveys in Approximation Theory **3**(2007), 1-105.

[26] D. M. Matjila, Bounds for the weighted Lebesgue functions for Freud weights, J. Approx. Theory, **79**(1994), 385–406.

[27] D. M. Matjila, Convergence of Lagrange interpolation for Freud weights in weighted L_p, $0 < p \leq 1$, in: Nonlinear numerical methods and rational approximation, Kluwer, Dordrecht, 1994, pp. 25–35.

[28] F. Nobile, R. Tempone, C.G. Webster, An anisotropic sparse grid stochastic collocation method for elliptic partial differential equations with random input data, SIAM J. Numer. Anal. **46**(2008), 2411-2442.

[29] Ch. Schwab and C. Gittelson, Sparse tensor discretizations high-dimensional parametric and stochastic PDEs, Acta Numerica **20**(2011), 291-467.

[30] J. Szabados, Weighted Lagrange and Hermite-Fejer interpolation on the real line, J. Inequal. and Applns., **1**(1997), 99-123.

[31] G. Szegö, Orthogonal Polynomials, Amer. Math. Soc. Colloq. Publ., Vol. 23, American Mathematical Society, Providence, Rhode Island, 1939.

[32] J. Zech, Sparse-Grid Approximation of High-Dimensional Parametric PDEs, Dissertation 25683, ETH Zurich, 2018.

[33] J. Zech, D. Dũng and Ch. Schwab, Multilevel approximation of parametric and stochastic PDEs, Research Report No. 2018-05, Seminar for Appied Mathematics, ETH Zurich, 2018.

[34] J. Zech and C. Schwab, Convergence rates of high dimensional Smolyak quadrature, Research Report No. 2017-27, Seminar for Appied Mathematics, ETH Zurich, 2017.