EXTRA INVARIANCE OF PRINCIPAL SHIFT INVARIANT SPACES AND THE ZAK TRANSFORM.

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Abstract. We prove a necessary and sufficient condition for a principal shift invariant space of $L^2(\mathbb{R})$ to be invariant under translations by the subgroup $\frac{1}{N}\mathbb{Z}\,, N > 1$. This condition is given in terms of the Zak transform of the group $\frac{1}{N}\mathbb{Z}$. This result is extended to principal shift invariant spaces generated by a lattice in a general locally compact abelian (LCA) group.

1. Introduction and main result

Let H be an additive subgroup of $\mathbb{R}$. A closed subspace V of $L^2(\mathbb{R})$ is called **H-invariant** if it is invariant under translations by elements of H. That is, when $f \in V$, then $T_h(f) \in V$ for all $h \in H$, where $T_h(x) = f(x-h)$. A $\mathbb{Z}$ invariant subspace of $L^2(\mathbb{R})$ is called **shift invariant**.

Shift invariant spaces are the core spaces of Multiresolution Analysis ([Mey90, Dau92, HW96, Mal99]), and as such they are used to study signals and images. They are also used as models to approximate functional data ([dBDVR94, ACHM07]).

Shift invariant spaces are also the natural spaces for sampling. For a measurable set $A \subset \mathbb{R}$, the Paley-Wiener space $PW(A)$ is defined by

$$PW(A) := \{f \in L^2(\mathbb{R}) : \operatorname{supp} \widehat{f} \subset A\}\,,$$

where $\widehat{f}(\xi) = \displaystyle\int_{\mathbb{R}} f(x) e^{-2\pi i x\xi}\, dx$ denotes the Fourier transform of f. The Whittaker-Shannon-Kotel'nikov sampling theorem establishes that any signal f in the space $PW([-M/2\,, M/2]), M > 0$, can be recovered with the samples $\{f(k/M)\}_{k\in\mathbb{Z}}$ by the formula

$$f(x) = \sum_{k\in\mathbb{Z}} f\left(\frac{k}{M}\right) \frac{\sin \pi(Mx-k)}{\pi(Mx-k)}\,, \tag{1.1}$$

with convergence in $L^2(\mathbb{R})$ and pointwise uniformly. The space $PW(A)$ is shift invariant, that is invariant under translations by the group $\mathbb{Z}$. It has extra invariance, since it si also invariant under the elements of the group $\mathbb{R}$, a bigger group than $\mathbb{Z}$.

There are other closed additive subgroups of $\mathbb{R}$ that contain $\mathbb{Z}$. All of them are of the form $\frac{1}{N}\mathbb{Z}$ for some natural number $N > 1$. In [ACHKM10] several equivalent conditions are given to determine if a shift invariant space is also $\frac{1}{N}\mathbb{Z}$ invariant, $N \in$

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$\mathbb{N}, N > 1$. Their results are given in terms of cut-off spaces in the Fourier transform side, gramians and range functions.

A particular important class of H invariant spaces is the one whose elements are generated by the H-translations of a single function $\psi \in L^2(\mathbb{R})$. They are called **principal** and define as

$$\langle \psi \rangle_H := \overline{span}\{T_h \psi : h \in H\},$$

where the closure is taken in $L^2(\mathbb{R})$. It can be seen using (1.1) that the Paley-Wiener space $PW([1/2\,,1/2])$ is principal and generated by the function $\psi \in L^2(\mathbb{R})$ given by $\widehat{\psi} = \chi_{[-1/2\,,1/2]}$.

For principal shift invariant spaces $\langle \psi \rangle_{\mathbb{Z}} \subset L^2(\mathbb{R})$ it is shown in [SW11] that $\langle \psi \rangle_{\mathbb{Z}}$ is also $\frac{1}{N}\mathbb{Z}$ invariant, $N \in \mathbb{N}, N > 1$, if and only if for all $p = 1, 2, \ldots, N-1$,

$$P_{\psi,N}(\xi) P_{\psi,N}(\xi + p) = 0\,,\ a.e.\ \xi \in \mathbb{R}\,,$$

where $P_{\psi,N}$ is the periodization function of ψ for the group $\frac{1}{N}\mathbb{Z}$, that is

$$P_{\psi,N}(\xi) := \sum_{k\in\mathbb{Z}} |\widehat{\psi}(\xi + Nk)|^2\,.$$

The Zak transform of a function $f \in L^1(\mathbb{R})$ for the group $\frac{1}{N}\mathbb{Z}, N \in \mathbb{N}$, is given by

$$Z_N(f)(x,\xi) := \frac{1}{N} \sum_{k\in\mathbb{Z}} f(x + \frac{k}{N}) e^{-2\pi i \frac{k}{N}\xi}\,,\ x, \xi \in \mathbb{R}\,. \tag{1.2}$$

It can be extended to be an isometric isomorphism from $L^2(\mathbb{R})$ onto $L^2([0,1/N) \times [0,N))$. For the proof of this result and other properties of the Zak transform, together with historical background and references, see [Jan88]. It is a very useful tool in time-frequency analysis ([Gro01]) and in situations where the Fourier transform is not available, such as in Harmonic Analysis in non-commutative discrete groups ([BHP14, BHP15]). And it is also of great value in abelian Fourier Analysis: as an example see the simple proof of the Plancherel Theorem given in [HSWW10] using the Zak transform.

The main purpose of this article is to give a characterization of $\frac{1}{N}\mathbb{Z}$ extra invariant of principal shift invariant spaces of $L^2(\mathbb{R})$ using the Zak transform of the group $\frac{1}{N}\mathbb{Z}$. The statement is the following:

Theorem 1.1. *Let $\psi \neq 0, \psi \in L^2(\mathbb{R})$ and $N \in \mathbb{N}, N > 1$. The following are equivalent:*
(a) $\langle \psi \rangle_{\mathbb{Z}}$ is $\frac{1}{N}\mathbb{Z}$ invariant.
(b) $Z_N(\psi)(x, \xi + p)\, Z_N(\psi)(y, \xi + q) = 0$ a.e. $x, y \in [0\,,1/N)$, a.e. $\xi \in [0,1)$, for all $p, q = 0, 1, \ldots N-1, p \neq q$.

Let $I_1 = [0,1/2) \cup [1,3/2)$ and $I_2 = [0,1/2) \cup [3/2,2)$. The functions ψ_1 and ψ_2 given by $\widehat{\psi_1} = \chi_{I_1}$ and $\widehat{\psi_2} = \chi_{I_2}$ are exhibited in [SW11] to show that although both allow sampling formulas with the lattice $\frac{1}{2}\mathbb{Z}$, the second one is better with respect to sampling since it also allows sampling with the coarser lattice $\mathbb{Z}$, while the first one does not.

We can witness, using Theorem 1.1, that they are also different for $\frac{1}{2}\mathbb{Z}$ extra invariance. To see this, we borrow from Proposition 2.3 the following formula for the Zak transform of a function $f \in L^1(\mathbb{R})$ for the group $\frac{1}{N}\mathbb{Z}, N \in \mathbb{N}$:

$$Z_N(f)(x,\xi) = \sum_{k\in\mathbb{Z}} \widehat{f}(\xi + Nk)e^{2\pi i x\cdot(\xi+Nk)}\,, \quad x,\ \xi \in \mathbb{R}\,. \tag{1.3}$$

Using (1.3) it is easy to see that for $\xi \in [0,1)$, and $x, y \in \mathbb{R}$,

$$Z_2(\psi_1)(x,\xi) = e^{2\pi i x\cdot\xi}\chi_{[0,1/2)}(\xi)\,, \qquad \text{and} \qquad Z_2(\psi_1)(y,\xi+1) = e^{2\pi i x\cdot(\xi+1)}\chi_{[0,1/2)}(\xi)\,.$$

Since $Z_2(\psi_1)(x,\xi)Z_2(\psi_1)(y,\xi+1) \neq 0$ for all $x, y \in \mathbb{R}$ and all $\xi \in [0,1/2)$, we deduce from Theorem 1.1 that $\langle\psi_1\rangle_{\mathbb{Z}}$ is not $\frac{1}{2}\mathbb{Z}$ invariant. On the other hand, using again (1.3), for all $\xi \in [0,1)$, and $x, y \in \mathbb{R}$, we have

$$Z_2(\psi_2)(x,\xi) = e^{2\pi i x\cdot\xi}\chi_{[0,1/2)}(\xi)\,, \qquad \text{and} \qquad Z_2(\psi_2)(y,\xi+1) = e^{2\pi i x\cdot(\xi+1)}\chi_{[1/2,1)}(\xi)$$

Hence, $Z_2(\psi_2)(x,\xi)Z_2(\psi_2)(y,\xi+1) = 0$ for all $\xi \in [0,1)$, and $x, y \in \mathbb{R}$. This shows, by Theorem 1.1, that $\langle\psi_2\rangle_{\mathbb{Z}}$ is $\frac{1}{2}\mathbb{Z}$ invariant.

Theorem 1.1 will be proved in Section 3. Section 2 contains the tools needed for the proof. In Section 5 we generalize Theorem 1.1 to the case of locally compact abelian (LCA) groups. We need an LCA group G and two lattices $\mathcal{K} \subset \mathcal{L}$ of G. The dual group of G will be denoted by $\widehat{G}$ and $\mathcal{L}^\perp \subset \mathcal{K}^\perp$ denote the dual lattices of $\mathcal{L}$ and $\mathcal{K}$ respectively. We denote by $C_{\mathcal{L}}$ a measurable tiling set of G by $\mathcal{L}$, and similarly by $C_{\mathcal{K}^\perp}$ a measurable tiling set of $\widehat{G}$ by $\mathcal{K}^\perp$.

We also need the notion of Zak transform with respect to a lattice that the reader can find in (4.5).

Theorem 1.2. *Let G be an LCA group and let $\mathcal{K} \subset \mathcal{L}$ be two lattices in G. Let $\psi \neq 0, \psi \in L^2(G)$. The following are equivalent:*

(a) $\langle\psi\rangle_{\mathcal{K}}$ is $\mathcal{L}$ invariant.

(b) $Z_{\mathcal{L}}(\psi)(\alpha+\beta_1,x)\,Z_{\mathcal{L}}(\psi)(\alpha+\beta_2,y) = 0$ for all β_1,β_2 such that $[\beta_1] \neq [\beta_2]$ in $\mathcal{K}^\perp/\mathcal{L}^\perp$, and a.e. $x, y \in C_{\mathcal{L}}$, $\alpha \in C_{\mathcal{K}^\perp}$.

The proof of Theorem 1.2 will be given in Section 5. Subsections 4.1, 4.2, and 4.3 contain the tools needed for the proof.

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2. Preliminaries

2.1. **Properties of the Zak transform.** Recall from (1.2) that the Zak transform of a function $f \in L^1(\mathbb{R})$ for the group $\frac{1}{N}\mathbb{Z}, N \in \mathbb{N}$, is given by

$$Z_N(f)(x,\xi) := \frac{1}{N}\sum_{k\in\mathbb{Z}} f(x+\frac{k}{N})e^{-2\pi i\frac{k}{N}\xi}\,,\ x,\xi \in \mathbb{R}\,. \tag{2.1}$$

It can be extended to be an isometric isomorphism from $L^2(\mathbb{R})$ onto $L^2([0,1/N)\times[0,N))$. It follows from the definition that if $\ell \in \mathbb{Z}$ and $x,\xi \in \mathbb{R}$

$$Z_N(f)(x,\xi+\ell N) = Z_N(f)(x,\xi)\,, \tag{2.2}$$

and

$$Z_N(f)(x+\frac{\ell}{N}\,,\xi) = e^{2\pi i\frac{\ell}{N}\xi}Z_N(f)(x,\xi)\,. \tag{2.3}$$

Therefore, $Z_N(f)(x,\xi)$ is determined as soon as we know its values in the rectangle $[0,1/N)\times[0,N)$.

The following result relates the usual Zak transform Z_1 with the Zak transform defined by (2.1).

Proposition 2.1. *For $f \in L^2(\mathbb{R})$, $x\,,\xi \in \mathbb{R}$, and $N \in \mathbb{N}$,*

$$Z_1(f)(x,\xi) = \sum_{q=0}^{N-1} Z_N(f)(x,\xi+q)\,.$$

Proof. By density, it is enough to prove the result for $f \in L^1(\mathbb{R})\cap L^2(\mathbb{R})$. Using definition (2.1) and collecting terms we obtain

$$\begin{aligned}\sum_{q=0}^{N-1} Z_N(f)(x,\xi+q) &= \sum_{q=0}^{N-1}\frac{1}{N}\sum_{k\in\mathbb{Z}} f(x+\frac{k}{N})e^{-2\pi i\frac{k}{N}(\xi+q)}\\ &= \sum_{k\in\mathbb{Z}}\frac{1}{N}\left(\sum_{q=0}^{N-1} e^{-2\pi i\frac{k}{N}q}\right) f(x+\frac{k}{N})e^{-2\pi i\frac{k}{N}\xi}\,.\end{aligned}$$

Let $\Phi_N(k) := \frac{1}{N}\left(\sum_{q=0}^{N-1} e^{-2\pi i\frac{k}{N}q}\right)$. If $K=\ell N$, $\Phi_N(k)=1$. On the other hand if k is not an integer multiple of N, using the sum of a geometric progression, $\Phi_N(k)=0$. Therefore,

$$\sum_{q=0}^{N-1} Z_N(f)(x,\xi+q) = \sum_{\ell\in\mathbb{Z}} f(x+\ell)e^{-2\pi i\ell\xi} = Z_1(f)(x,\xi).$$

□

Recall that $T_x(f)(y) = f(y-x)$ denotes de translation by $x \in \mathbb{R}$.

Proposition 2.2. *For $f \in L^2(\mathbb{R})$, $x\,,\xi \in \mathbb{R}$, and $N \in \mathbb{N}$,*

$$Z_1(T_{1/N}(f))(x,\xi) = \sum_{q=0}^{N-1} e^{-\frac{2\pi i(\xi+q)}{N}} Z_N(f)(x,\xi+q)\,.$$

Proof. By density, it is enough to prove the result for $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Now, the result follows by using Proposition 2.1 and equation (2.3):

$$\begin{aligned} Z_1(T_{1/N}(f))(x,\xi) &= Z_1(f)(x-\frac{1}{N},\xi) = \sum_{q=0}^{N-1} Z_N(f)(x-\frac{1}{N},\xi+q) \\ &= \sum_{q=0}^{N-1} e^{-2\pi i \frac{1}{N}(\xi+q)} Z_N(f)(x,\xi+q)\,. \end{aligned}$$

□

The following result will be needed in the sequel. It gives a way to compute the Zak transform of a function using its Fourier transform.

Proposition 2.3. *For $f \in L^2(\mathbb{R})$, $x\,,\xi \in \mathbb{R}$, and $N \in \mathbb{N}$,*

$$Z_N(f)(x,\xi) = \sum_{k\in\mathbb{Z}} \widehat{f}(\xi+Nk) e^{2\pi i x\cdot(\xi+Nk)}\,.$$

Proof. It is enough to show the result for $f \in C_c(\mathbb{R})$, the continuous functions with compact support in $\mathbb{R}$. For each $x,\xi \in \mathbb{R}$ define $F_{x,\xi}(t) := f(x+\dfrac{t}{N})e^{-2\pi i \frac{t}{N}\xi}$, $t \in \mathbb{R}$. By the Poisson Summation Formula,

$$Z_N(f)(x,\xi) = \frac{1}{N}\sum_{k\in\mathbb{Z}} F_{x,\xi}(k) = \frac{1}{N}\sum_{k\in\mathbb{Z}} \widehat{F_{x,\xi}}(k)\,. \tag{2.4}$$

With the change of variables $x + \frac{t}{N} = z$ we obtain

$$\begin{aligned} \widehat{F_{x,\xi}}(k) &= \int_{\mathbb{R}} F_{x,\xi}(t) e^{-2\pi i k t}\,dt = \int_{\mathbb{R}} f(x+\frac{t}{N})e^{-2\pi i \frac{t}{N}\xi}\, e^{-2\pi i k t}\,dt \\ &= N\int_{\mathbb{R}} f(z)\, e^{-2\pi i (z-x)\xi} e^{-2\pi i k (z-x)N}\,dz \\ &= N e^{2\pi i x\dot{(}\xi+Nk)} \int_{\mathbb{R}} f(z)\, e^{-2\pi i z\cdot(\xi+Nk)}\,dz \\ &= N e^{2\pi i x\dot{(}\xi+Nk)} \widehat{f}(\xi+Nk)\,. \end{aligned}$$

The result follows by replacing this equality in (2.4). □

Remark 2.4. *Propositions 2.1 and 2.2 can also be proved using Proposition 2.3. We leave the details for the reader.*

2.2. **Principal invariant spaces.** We start by giving a condition to determine if a principal shift invariant subspace is also $\frac{1}{N}\mathbb{Z}$ invariant.

Proposition 2.5. *Let $\psi \neq 0, \psi \in L^2(\mathbb{R})$, and $N \in \mathbb{N}, N > 1$. The following are equivalent:*
(a) $\langle\psi\rangle_{\mathbb{Z}}$ is $\frac{1}{N}\mathbb{Z}$ invariant.
(b) $T_{1/N}(\psi) \in \langle\psi\rangle_{\mathbb{Z}}$.

Proof. $(a) \Rightarrow (b)$ is clear by definition. To prove $(b) \Rightarrow (a)$ let $f \in \langle\psi\rangle_{\mathbb{Z}}$. We have to show that $T_{k/N}(f) \in \langle\psi\rangle_{\mathbb{Z}}$ for all $k \in \mathbb{Z}$. Write $k = \ell N + q, \ell \in \mathbb{Z}, q \in \mathbb{Z}, 0 \le q \le N-1$. Then, $T_{k/N}(f) = T_{q/N}T_\ell(f) \in T_{q/N}(\langle\psi\rangle_{\mathbb{Z}})$. Thus, it is enough to prove that $T_{q/N}(\langle\psi\rangle_{\mathbb{Z}}) \subset \langle\psi\rangle_{\mathbb{Z}}$ for all $q \in \mathbb{Z}, 0 \le q \le N-1$. The result is clear for $q = 0$. If $q = 1$,

$$T_{1/N}(\langle\psi\rangle_{\mathbb{Z}}) \subset \langle T_{1/N}(\psi)\rangle_{\mathbb{Z}} \subset \langle\psi\rangle_{\mathbb{Z}}\,,$$

since $T_{1/N}\psi \in \langle\psi\rangle_{\mathbb{Z}}$ by (b). Proceed now by induction on q. If $T_{q/N}(\langle\psi\rangle_{\mathbb{Z}}) \subset \langle\psi\rangle_{\mathbb{Z}}$, then

$$T_{\frac{q+1}{N}}(\langle\psi\rangle_{\mathbb{Z}}) = T_{\frac{1}{N}}T_{\frac{q}{N}}(\langle\psi\rangle_{\mathbb{Z}}) \subset T_{\frac{1}{N}}(\langle\psi\rangle_{\mathbb{Z}}) \subset \langle T_{1/N}(\psi)\rangle_{\mathbb{Z}} \subset \langle\psi\rangle_{\mathbb{Z}}\,,$$

since $T_{1/N}\psi \in \langle\psi\rangle_{\mathbb{Z}}$ by (b). □

The following result characterizes the elements of $\langle\psi\rangle_{\mathbb{Z}}$ in terms of a multiplier. It was first proved in [dBDVR94], Theorem 2.14 (see also Theorem 2.1 in [HSWW10]).

Proposition 2.6. *Let $\psi \neq 0, \psi \in L^2(\mathbb{R})$.*

(a) If $f \in \langle\psi\rangle_{\mathbb{Z}}$, there exists a $\mathbb{Z}$-periodic function m_f on $\mathbb{R}$ such that $\widehat{f} = m_f\widehat{\psi}$.

(b) If m is a $\mathbb{Z}$-periodic function on $\mathbb{R}$ such that $m\widehat{\psi} \in L^2(\mathbb{R})$ then, the function f defined by $\widehat{f} = m\widehat{\psi}$ belongs to $\langle\psi\rangle_{\mathbb{Z}}$.

We will need a similar result to the one stated in the above Proposition, but in terms of multipliers of the Zak transform. It is a corollary of Proposition 2.6.

Corollary 2.7. *Let $\psi \neq 0, \psi \in L^2(\mathbb{R})$.*

(a) If $f \in \langle\psi\rangle_{\mathbb{Z}}$, there exists a $\mathbb{Z}$-periodic function m_f on $\mathbb{R}$ with $m_f\widehat{\psi} \in L^2(\mathbb{R})$ such that $Z_1(f)(x,\xi) = m_f(\xi)Z_1(\psi)(x,\xi)$, a. e. $x, \xi \in \mathbb{R}$.

(b) If m is a $\mathbb{Z}$-periodic function on $\mathbb{R}$ such that $m\widehat{\psi} \in L^2(\mathbb{R})$ then, the function f defined by $Z_1(f)(x,\xi) = m(\xi)Z_1(\psi)(x,\xi)$, a. e. $x, \xi \in \mathbb{R}$, belongs to $\langle\psi\rangle_{\mathbb{Z}}$.

Proof. (a) By (a) of Proposition 2.6 there exists a $\mathbb{Z}$-periodic function m_f on $\mathbb{R}$ such that $\widehat{f} = m_f\widehat{\psi}$. Hence, $m_f\widehat{\psi} \in L^2(\mathbb{R})$ and by Proposition 2.3 for $N = 1$ we deduce,

$$\begin{aligned} Z_1(f)(x,\xi) &= \sum_{k\in\mathbb{Z}} \widehat{f}(\xi+k)e^{2\pi i x\cdot(\xi+k)} = \sum_{k\in\mathbb{Z}} m_f(\xi+k)\widehat{\psi}(\xi+k)e^{2\pi i x\cdot(\xi+k)} \\ &= m_f(\xi)\sum_{k\in\mathbb{Z}} \widehat{\psi}(\xi+k)e^{2\pi i x\cdot(\xi+k)} = m_f(\xi)Z_1\psi(x,\xi)\,. \end{aligned}$$

(b) First notice that since $m\widehat{\psi} \in L^2(\mathbb{R})$, by Proposition 2.3 for $N = 1$,

$$\begin{aligned} m(\xi)Z_1(\psi)(x,\xi) &= m(\xi)\sum_{k\in\mathbb{Z}} \widehat{\psi}(\xi+k)e^{2\pi i x\cdot(\xi+k)} = \sum_{k\in\mathbb{Z}} m(\xi+k)\widehat{\psi}(\xi+k)e^{2\pi i x\cdot(\xi+k)} \\ &= \sum_{k\in\mathbb{Z}} m\widehat{\psi}(\xi+k)e^{2\pi i x\cdot(\xi+k)} = Z_1(\mathcal{F}^{-1}(m\widehat{\psi}))(x,\xi)\,, \end{aligned}$$

where $\mathcal{F}^{-1}$ is our notation for the inverse Fourier transform of an $L^2(\mathbb{R})$ function. This shows that $mZ_1(\psi)$ coincides with the Zak transform of $\mathcal{F}^{-1}(m\widehat{\psi})) \in L^2(\mathbb{R})$. Since f satisfies $Z_1(f)(x,\xi) = m(\xi)Z_1(\psi)(x,\xi) = Z_1(\mathcal{F}^{-1}(m\widehat{\psi}))(x,\xi)$ and Z_1 is an isometry, we conclude $\widehat{f} = m\widehat{\psi}$. By (b) of Proposition 2.6, $f \in \langle\psi\rangle_{\mathbb{Z}}$. □

3. Proof of Theorem 1.1

3.1. **Proof of (a) implies (b) of Theorem 1.1.** Assume that $\langle\psi\rangle_{\mathbb{Z}}$ is $\frac{1}{N}\mathbb{Z}$ invariant, $N \in \mathbb{N}, N > 1$. Then 2.5, $T_{1/N}(\psi) \in \langle\psi\rangle_{\mathbb{Z}}$. By Corollary 2.7, there exists a $\mathbb{Z}$-periodic function m on $\mathbb{R}$, with $m\widehat{\psi} \in L^2(\mathbb{R})$, such that

$$Z_1(T_{1/N}(\psi))(x,\xi) = m(\xi)Z_1(\psi)(x,\xi)\,,\ a.\,e.\ x\,,\xi \in \mathbb{R}.$$

Equivalently,

$$Z_1(\psi)(x - \frac{1}{N},\xi) = m(\xi)Z_1(\psi)(x,\xi)\,,\ a.\,e.\ x\,,\xi \in \mathbb{R}.$$

Iterating, for $p = 0, 1, 2, \ldots$

$$Z_1(\psi)(x - \frac{p}{N},\xi) = m(\xi)^p Z_1(\psi)(x,\xi)\,,\ a.\,e.\ x\,,\xi \in \mathbb{R}. \tag{3.1}$$

On the other hand, by Proposition 2.1 and equation (2.3), for $p \in \mathbb{Z}$ we obtain,

$$\begin{aligned} Z_1(\psi)(x-\frac{p}{N},\xi) &= \sum_{q=0}^{N-1} Z_N(\psi)(x-\frac{p}{N},\xi+q) \\ &= \sum_{q=0}^{N-1} e^{-2\pi i\frac{p(\xi+q)}{N}} Z_N(\psi)(x,\xi+q) \\ &= e^{-2\pi i\frac{p\xi}{N}} \sum_{q=0}^{N-1} e^{-2\pi i\frac{pq}{N}} Z_N(\psi)(x,\xi+q)\,. \end{aligned} \tag{3.2}$$

For $q,p \in \mathbb{Z}$ and $x,\xi \in \mathbb{R}$, let $\alpha_q(x,\xi) := Z_N(\psi)(x,\xi{+}q)$ and $A_p(x,\xi) := \sum_{q=0}^{N-1} e^{-2\pi i\frac{pq}{N}}\alpha_q(x,\xi)$. Observe that for x,ξ fixed, $\alpha_q(x,\xi)$ is $N\mathbb{Z}$-periodic in q (see 2.2). Also $A_p(x,\xi)$ are the discrete Fourier coefficients of the sequence $\{\alpha_q(x,\xi)\}_{q=0}^{N-1}$. Thus, by inversion,

$$\alpha_q(x,\xi) = \frac{1}{N}\sum_{p=0}^{N-1} e^{2\pi i\frac{pq}{N}} A_p(x,\xi)\,,\ \ x\,,\xi \in \mathbb{R}\,. \tag{3.3}$$

For these coefficients $A_p(x,\xi)$ the following crucial relation can be proved:

Lemma 3.1. *Let $x,y,\xi \in \mathbb{R}$. If $p,q,p_1.q_1 \in \mathbb{N}$ and $p+q = p_1+q_1(modN)$, then*

$$A_p(x,\xi)A_q(y,\xi) = A_{p_1}(x,\xi)A_{q_1}(y,\xi).$$

Proof. By equation (3.2),

$$e^{-\frac{2\pi i(p+q)\xi}{N}} A_p(x,\xi)A_q(y,\xi) = Z_1(\psi)(x-\frac{p}{N},\xi)Z_1(\psi)(y-\frac{q}{N},\xi)\,.$$

By equation (3.1),

$$e^{-\frac{2\pi i(p+q)\xi}{N}} A_p(x,\xi)A_q(y,\xi) = m(\xi)^{p+q} Z_1(\psi)(x,\xi)Z_1(\psi)(y,\xi)\,. \tag{3.4}$$

Similarly,

$$e^{-\frac{2\pi i(p_1+q_1)\xi}{N}} A_{p_1}(x,\xi)A_{q_1}(y,\xi) = m(\xi)^{p_1+q_1} Z_1(\psi)(x,\xi)Z_1(\psi)(y,\xi)\,. \tag{3.5}$$

For $k = 0,1,2,\dots$, use (3.1) with $p = kN$ and then (2.3) with $k = \ell$ and $N = 1$ to obtain

$$m(\xi)^{kN} Z_1(\psi)(x,\xi) = Z_1(\psi)(x-k,\xi) = e^{-2\pi ik\xi} Z_1(\psi)(x,\xi)\,. \tag{3.6}$$

Assume $p_1+q_1 = p+q+kN$ for some $k = 0,1,2,\dots$. Then, by (3.5), (3.6) and (3.4),

$$\begin{aligned} A_{p_1}(x,\xi)A_{q_1}(y,\xi) &= e^{\frac{2\pi i(p_1+q_1)\xi}{N}} m(\xi)^{p_1+q_1} Z_1(\psi)(x,\xi)Z_1(\psi)(y,\xi) \\ &= e^{\frac{2\pi i(p+q)\xi}{N}} e^{2\pi ik\xi} m(\xi)^{p+q} m(\xi)^{kN} Z_1(\psi)(x,\xi)Z_1(\psi)(y,\xi) \\ &= e^{\frac{2\pi i(p+q)\xi}{N}} m(\xi)^{p+q} Z_1(\psi)(x,\xi)Z_1(\psi)(y,\xi) \\ &= A_p(x,\xi)A_q(y,\xi)\,. \end{aligned}$$

□

We continue now with the proof. With the notation introduced above, we need to show that $\alpha_p(x,\xi)\,\alpha_q(y,\xi) = 0\ \ a.e.\, x,y \in [0,1/N)$, $a.e.\, \xi \in [0,1)$, for all $p,q = 0,1,\dots N-1, p \neq$

q . By equation (3.3)

$$\begin{aligned}\alpha_p(x,\xi)\,\alpha_q(y,\xi) &= \frac{1}{N^2}\left(\sum_{j=0}^{N-1} e^{2\pi i\frac{jp}{N}} A_j(x,\xi)\right)\left(\sum_{\ell=0}^{N-1} e^{2\pi i\frac{\ell q}{N}} A_\ell(y,\xi)\right)\\ &= \frac{1}{N^2}\sum_{j=0}^{N-1}\sum_{\ell=0}^{N-1} e^{2\pi i\frac{(jp+\ell q)}{N}} A_j(x,\xi)A_\ell(y,\xi)\,.\end{aligned}$$

Let $\ell = N-1-j-k$. Then,

$$\alpha_p(x,\xi)\,\alpha_q(y,\xi) = \frac{1}{N^2}\sum_{j=0}^{N-1}\sum_{k=-j}^{N-1-j} e^{\frac{2\pi i j(p-q)}{N}} e^{-\frac{2\pi i(k+1)q}{N}} A_j(x,\xi)A_{N-1-j-k}(y,\xi)\,.$$

By Lemma 3.1, $A_j(x,\xi)A_{N-1-j-k}(y,\xi) = A_0(x,\xi)A_{N-1-k}(y,\xi)$. Thus,

$$\alpha_p(x,\xi)\,\alpha_q(y,\xi) = \frac{1}{N^2}\sum_{j=0}^{N-1}\sum_{k=-j}^{N-1-j} e^{\frac{2\pi i j(p-q)}{N}} e^{-\frac{2\pi i(k+1)q}{N}} A_0(x,\xi)A_{N-1-k}(y,\xi)\,.$$

Interchanging, carefully, the above summations, and using that $A_0(x,\xi)A_{2N-1-\ell}(y,\xi) = A_0(x,\xi)A_{N-1-\ell}(y,\xi)$ by Lemma 3.1, we obtain,

$$\begin{aligned}\alpha_p(x,\xi)\,\alpha_q(y,\xi) &= \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1-k} e^{\frac{2\pi i j(p-q)}{N}} e^{-\frac{2\pi i(k+1)q}{N}} A_0(x,\xi)A_{N-1-k}(y,\xi)\\ &\quad+\frac{1}{N^2}\sum_{k=-N+1}^{-1}\sum_{j=-k}^{N-1} e^{\frac{2\pi i j(p-q)}{N}} e^{-\frac{2\pi i(k+1)q}{N}} A_0(x,\xi)A_{N-1-k}(y,\xi)\\ &= \frac{1}{N^2}\sum_{\ell=0}^{N-1}\sum_{j=0}^{N-1-\ell} e^{\frac{2\pi i j(p-q)}{N}} e^{-\frac{2\pi i(\ell+1)q}{N}} A_0(x,\xi)A_{N-1-\ell}(y,\xi)\\ &\quad+\frac{1}{N^2}\sum_{\ell=1}^{N-1}\sum_{j=N-\ell}^{N-1} e^{\frac{2\pi i j(p-q)}{N}} e^{-\frac{2\pi i(\ell+1)q}{N}} A_0(x,\xi)A_{2N-1-\ell}(y,\xi)\\ &= \frac{1}{N^2}\sum_{\ell=0}^{N-1}\left(\sum_{j=0}^{N-1} e^{\frac{2\pi i j(p-q)}{N}}\right) e^{-\frac{2\pi i(\ell+1)q}{N}} A_0(x,\xi)A_{N-1-\ell}(y,\xi)\,.\end{aligned}$$

Since, when $p\neq q$, $\displaystyle\sum_{j=0}^{N-1} e^{\frac{2\pi i j(p-q)}{N}} = 0$ the result is established.

3.2. **Proof of (b) implies (a) of Theorem 1.1.** Suppose that

$$Z_N(\psi)(x,\xi+p)\,Z_N(\psi)(y,\xi+q) = 0 \tag{3.7}$$

a.e. $x,y\in[0\,,1/N)$, a.e. $\xi\in[0,1)$, for all $p,q=0,1,\dots N-1, p\neq q$. By (2.3), equation (3.7) holds for a. e. $x,y\in\mathbb{R}$. By Proposition 2.5 and Corollary 2.7 it is enough to find a $\mathbb{Z}$-periodic function m defined on $\mathbb{R}$ such that $m\widehat{\psi}\in L^2(\mathbb{R})$ and

$$Z_1(T_{1/N}(\psi))(x,\xi) = m(\xi)Z_1(\psi)(x,\xi) \tag{3.8}$$

a. e. $x,\xi\in\mathbb{R}$. By the quasi-periodicity properties of Z_1 (see (2.2) and (2.3)) it is enough to prove (3.8) for a. e. $x,\xi\in[0,1)$.

For $0\le q\le N-1$ and $0\le x<1$, let

$$S_\psi^{(q)}(x) := \{\xi\in[0,1) : Z_N(\psi)(x,\xi+q)\neq 0\}\,,$$

and

$$S_\psi^{(q)} := \bigcup_{x\in[0,1)} S_\psi^{(q)}(x)\,.$$

Note that $S_\psi^{(q)}$ is a measurable subset of $[0,1)\times[0,1)$. From (3.7) we conclude $|S_\psi^{(q)}\cap S_\psi^{(p)}|=0$ when $p,q=0,1,2,\dots,N-1, p\neq q$. Finally, define

$$S_\psi = [0,1)\setminus \bigcup_{q=0}^{N-1} S_\psi^{(q)}\,.$$

For $0\le \xi < 1$, define

$$m(\xi)=\begin{cases} e^{\frac{-2\pi i(\xi+q)}{N}} & \text{if } \xi\in S^{(q)}, 0\le q\le N-1\\ 1 & \text{if } \xi\in S_\psi\end{cases},$$

and extend m to $\mathbb{R}$ to be $\mathbb{Z}$-periodic. Since $|m(\xi)|=1$ and $\psi\in L^2(\mathbb{R})$, we conclude $m\widehat{\psi}\in L^2(\mathbb{R})$.

We need to show that (3.8) holds for a. e. $x,\xi\in[0,1)$. For almost every $x,\xi\in[0,1)$ either $Z_N(\psi)(x,\xi+q)=0$ for all $q=0,1,2,\dots,N-1$ or there exists only one value of $q\in\{0,1,2,\dots,N-1\}$ such that $Z_N(\psi)(x,\xi+q)\neq 0$. In the first case, by Propositions 2.1 and 2.2 we have

$$Z_1(\psi)(x,\xi)=0 \qquad\text{and}\qquad Z_1(T_{1/N})(x,\xi)=0\,,$$

so that (3.8) holds trivially. In the second case, again by Propositions 2.1 and 2.2 we have

$$Z_1(\psi)(x,\xi)=Z_N(\psi)(x,\xi+q) \qquad\text{and}\qquad Z_1(T_{1/N})(x,\xi)=e^{-\frac{2\pi i(\xi+q)}{N}}Z_N(\psi)(x,\xi+q)\,.$$

Since, in this case, $\xi\in S_\psi^{(q)}(x)$, we have $m(\xi)=e^{-\frac{2\pi i(\xi+q)}{N}}$ and the equality (3.8) also holds in this case.

4. Tools and results for LCA groups

A natural question is to ask if Theorem 1.1 can be extended to locally compact abelian (LCA) groups. In [ACP10] the authors characterize the extra invariance of shift invariant spaces on LCA groups in terms of cut–off spaces in the Fourier transform side, and also in terms of range functions. Here, we give a characterization using the Zak transform relative to a given lattice.

We start by describing the results we need for our extension. For a detailed introduction to LCA groups see [Rud92].

4.1. **Background on LCA groups.** A group $(G,+)$ is an LCA (locally compact abelian) group if it is endowed with a separable, locally compact, Hausdorff topology, the map $x\longrightarrow -x$ is continuous from G into G, and the map $(x,y)\longrightarrow x+y$ is continuous from $G\times G$ into G. Every LCA group G has a non-zero Borel measure which is translation invariant and unique, up to a possible scalar multiple, called Haar measure, and denoted by μ_G.

A character of an LCA group G is a continuous homomorphism $\alpha: G\longrightarrow S^1=\{z\in\mathbb{C}: |z|=1\}$. The set of all characters of G, with the compact open topology, is an LCA group, denoted by $\widehat{G}$, the **dual group** of G. We write $(x,\alpha)=\alpha(x)$ when $x\in G$ and $\alpha\in\widehat{G}$. Notice that for $x,y\in G$ and $\alpha\in\widehat{G}$, $(x+y,\alpha)=(x,\alpha)(y,\alpha)$ since α is a homomorphism. Thus, $(0,\alpha)=1$, for any $\alpha\in\widehat{G}$. Similarly, for $x\in G$ and $\alpha,\beta\in\widehat{G}$, $(x,\alpha+\beta)=(x,\alpha)(x,\beta)$ and $(x,0)=1$.

A subgroup $\mathcal{L}$ fo G is called a **lattice** it is discrete with respect to the topology of G and $T_{\mathcal{L}} = G/\mathcal{L}$ is compact in the quotient topology. In particular $\mathcal{L}$ is countable. Associated to a lattice $\mathcal{L}$ of G there is a **dual lattice** given by

$$\mathcal{L}^{\perp} = \{\alpha \in \widehat{G} : (\ell, \alpha) = 0 \text{ for all } \ell \in \mathcal{L}\}.$$

It is well known (see [Rud92], Theorem 2.1.2) that

$$\widehat{(G/\mathcal{L})} \approx \mathcal{L}^{\perp} \qquad \text{and} \qquad \widehat{G}/\widehat{\mathcal{L}} \approx \mathcal{L}^{\perp}. \tag{4.1}$$

Given two lattices $\mathcal{K} \subset \mathcal{L}$ of G, the quotient group $\mathcal{L}/\mathcal{K} \approx (G/\mathcal{L})/(G/\mathcal{K}) = T_{\mathcal{L}}/T_{\mathcal{K}}$, is a finite abelian group since $T_{\mathcal{L}}$ and $T_{\mathcal{K}}$ are compact.

We have $\mathcal{L}^{\perp} \subset \mathcal{K}^{\perp}$, and therefore $\mathcal{K}^{\perp}/\mathcal{L}^{\perp}$ is also a finite abelian group. In fact, $\mathcal{K}^{\perp}/\mathcal{L}^{\perp}$ and $\mathcal{L}/\mathcal{K}$ have the same number of elements. To see this, use (4.1) with $G = \mathcal{L}$ and $\mathcal{L} = \mathcal{K}$ to deduce $\widehat{(\mathcal{L}/\mathcal{K})} \approx \widehat{\mathcal{L}}/\widehat{\mathcal{K}}$. Again by (4.1),

$$\widehat{(\mathcal{L}/\mathcal{K})} \approx \widehat{\mathcal{L}}/\widehat{\mathcal{K}} \approx \left(\widehat{G}/\widehat{\mathcal{K}}\right) / \left(\widehat{G}/\widehat{\mathcal{L}}\right) \approx \mathcal{K}^{\perp}/\mathcal{L}^{\perp}.$$

Since $\mathcal{L}/\mathcal{K}$ is a finite abelian group, $\widehat{(\mathcal{L}/\mathcal{K})} \approx \mathcal{L}/\mathcal{K}$ and the result follows.

If $[\ell] \in \mathcal{L}/\mathcal{K}$ and $[\alpha] \in \mathcal{K}^{\perp}/\mathcal{L}^{\perp}$, the number $([\ell], [\alpha]) := (\ell, \alpha)$ is well defined. Since $\mathcal{L}/\mathcal{K}$ and $\mathcal{K}^{\perp}/\mathcal{L}^{\perp}$ are finite abelian groups, by Theorem 1.2.5 in [Rud92],

$$\sum_{[\ell] \in \mathcal{L}/\mathcal{K}} ([\ell], [\alpha]) = \begin{cases} |\mathcal{L}/\mathcal{K}| & \text{if } [\alpha] = [0] \in \mathcal{K}^{\perp}/\mathcal{L}^{\perp} \\ 0 & \text{if } [\alpha] \neq [0] \in \mathcal{K}^{\perp}/\mathcal{L}^{\perp} \end{cases}, \tag{4.2}$$

By duality we also have,

$$\sum_{[\alpha] \in \mathcal{K}^{\perp}/\mathcal{L}^{\perp}} ([\ell], [\alpha]) = \begin{cases} |\mathcal{K}^{\perp}/\mathcal{L}^{\perp}| = |\mathcal{L}/\mathcal{K}| & \text{if } [\ell] = [0] \in \mathcal{L}/\mathcal{K} \\ 0 & \text{if } [\ell] \neq [0] \in \mathcal{L}/\mathcal{K} \end{cases}, \tag{4.3}$$

The **Fourier transform** of $f \in L^1(G, \mu_G)$ is defined by

$$\widehat{f}(\alpha) = \int_G f(x)(-x, \alpha) d\mu_G(x), \quad \alpha \in \widehat{G},$$

and extends to an unique isometry $\mathcal{F}(f) = \widehat{f}$ from $L^2(G, \mu_G)$ into $L^2(\widehat{G}, \mu_{\widehat{G}})$, where $\mu_{\widehat{G}}$ is the Plancherel measure in $\widehat{G}$.

In the sequel we will use the **Poisson Summation Formula** in this situation (see Theorem 5.5.2 in [Rei68]). Let $\mathcal{L}$ be a lattice in an LCA group G and $F \in C_c(G)$ (the set of continuous functions with compact support on G), then

$$|T_{\mathcal{L}}| \sum_{\ell \in \mathcal{L}} F(\ell) = \sum_{\gamma \in \mathcal{L}^{\perp}} \widehat{F}(\gamma). \tag{4.4}$$

4.2. **The Zak transform on LCA groups.** Let $\mathcal{L}$ be a lattice in an LCA group. For $f \in L^1(G)$ the **Zak transform** of f with respect to the lattice $\mathcal{L}$ is given by

$$Z_{\mathcal{L}}(f)(\alpha, x) = |T_{\mathcal{L}}| \sum_{\ell \in \mathcal{L}} f(x + \ell)(-\ell, \alpha), \quad \alpha \in \widehat{G},\ x \in G,. \tag{4.5}$$

It can be extended to an isometric isomorphism from $L^2(G)$ onto $L^2(\widehat{\mathcal{L}}, L^2(C_{\mathcal{L}}))$, where $C_{\mathcal{L}}$ is a measurable set of representatives of $G/\mathcal{L}$. (For a proof see Proposition 3.3 in [BHP15].)

We list now some properties of the Zak transform just defined. The first one is the following: if $[\alpha_1] = [\alpha_2]$ in $\widehat{G}/\mathcal{L}^\perp$, then

$$Z_{\mathcal{L}}(f)(\alpha_1, x) = Z_{\mathcal{L}}(f)(\alpha_2, x), \quad x \in G. \tag{4.6}$$

Indeed, since $[\alpha_1] = [\alpha_2]$ in $\widehat{G}/\mathcal{L}^\perp$, there exists $\gamma \in \mathcal{L}^\perp$ such that $\alpha_1 - \alpha_2 = \gamma$. Then

$$\begin{aligned} Z_{\mathcal{L}}(f)(\alpha_1, x) &= |T_{\mathcal{L}}| \sum_{\ell \in \mathcal{L}} f(x+\ell)(-\ell, \alpha_1) \\ &= |T_{\mathcal{L}}| \sum_{\ell \in \mathcal{L}} f(x+\ell)(-\ell, \alpha_2 + \gamma) \\ &= |T_{\mathcal{L}}| \left(\sum_{\ell \in \mathcal{L}} f(x+\ell)(-\ell, \alpha_2) \right) (-\ell, \gamma) \\ &= Z_{\mathcal{L}}(f)(\alpha_2, x), \end{aligned}$$

since $(-\ell, \gamma) = 1$ by definition of $\mathcal{L}^\perp$. The second one is related to translations in G: if $\ell \in \mathcal{L}$, then

$$Z_{\mathcal{L}}(f)(\alpha, x - \ell) = (-\ell, \alpha) Z_{\mathcal{L}}(f)(\alpha, x), \quad x \in G, \ \alpha \in \widehat{G}. \tag{4.7}$$

In fact,

$$\begin{aligned} Z_{\mathcal{L}}(f)(\alpha, x - \ell) &= |T_{\mathcal{L}}| \sum_{\ell' \in \mathcal{L}} f(x - \ell + \ell')(-\ell', \alpha) \\ &= |T_{\mathcal{L}}| \sum_{\ell'' \in \mathcal{L}} f(x + \ell'')(-\ell'' - \ell, \alpha) \\ &= |T_{\mathcal{L}}| \left(\sum_{\ell'' \in \mathcal{L}} f(x + \ell'')(-\ell'', \alpha) \right) (-\ell, \alpha) \\ &= (-\ell, \alpha) Z_{\mathcal{L}}(f)(\alpha, x). \end{aligned}$$

Remark 4.1. *It follows from* (4.7) *that if* $[x_1] = [x_2]$ *in* $G/\mathcal{L}$ *and* $\alpha \in \mathcal{L}^\perp$, *then* $Z_{\mathcal{L}}(f)(\alpha, x_1) = Z_{\mathcal{L}}(f)(\alpha, x_2)$.

Proposition 4.2. *Let* $\mathcal{K} \subset \mathcal{L}$ *be two lattices in an LCA group* G. *For* $f \in L^2(G)$, $\alpha \in \widehat{G}$, $x \in G$,

$$Z_{\mathcal{K}}(f)(\alpha, x) = \sum_{[\beta] \in \mathcal{K}^\perp/\mathcal{L}^\perp} Z_{\mathcal{L}}(f)(\alpha + \beta, x).$$

Proof. Observe that for $[\beta] \in \mathcal{K}^\perp/\mathcal{L}^\perp$, $Z_{\mathcal{L}}(f)(\alpha + \beta, x)$ is well defined by (4.6), that is the formula is independent of the representative chosen in $[\beta]$. By density, it is enough to prove the result for $f \in C_c(G)$. Using definition (4.5),

$$\begin{aligned} \sum_{[\beta] \in \mathcal{K}^\perp/\mathcal{L}^\perp} Z_{\mathcal{L}}(f)(\alpha + \beta, x) &= \sum_{[\beta] \in \mathcal{K}^\perp/\mathcal{L}^\perp} |T_{\mathcal{L}}| \sum_{\ell \in \mathcal{L}} f(x+\ell)(-\ell, \alpha + \beta) \\ &= \sum_{\ell \in \mathcal{L}} |T_{\mathcal{L}}| \left(\sum_{[\beta] \in \mathcal{K}^\perp/\mathcal{L}^\perp} (-\ell, \beta) \right) f(x+\ell)(-\ell, \alpha). \end{aligned}$$

By (4.3), $\sum_{[\beta] \in \mathcal{K}^\perp/\mathcal{L}^\perp} (-\ell, \beta) = |\mathcal{L}/\mathcal{K}|$ if $[\ell] = [0]$ in $\mathcal{L}/\mathcal{K}$ and equals 0 ir $[\ell] \neq [0]$ in $\mathcal{L}/\mathcal{K}$. Since $|\mathcal{L}/\mathcal{K}| = |T_{\mathcal{K}}|/|T_{\mathcal{L}}|$ we obtain

$$\sum_{[\beta] \in \mathcal{K}^\perp/\mathcal{L}^\perp} Z_{\mathcal{L}}(f)(\alpha + \beta, x) = \sum_{k \in \mathcal{K}} |T_{\mathcal{K}}| f(x+k)(-k, \alpha) = Z_{\mathcal{K}}(f)(\alpha, x).$$

□

Recall that $T_x(f)(y) = f(y-x)$ denotes the translation by $x \in G$ of the function f defined in G.

Proposition 4.3. *Let $\mathcal{K} \subset \mathcal{L}$ be two lattices in an LCA group G. For $\ell \in \mathcal{L}, f \in L^2(G),\ \alpha \in \widehat{G},\ x \in G$,*

$$Z_{\mathcal{K}}(T_\ell f)(\alpha, x) = \sum_{[\beta] \in \mathcal{K}^\perp/\mathcal{L}^\perp} (-\ell, \alpha+\beta) Z_{\mathcal{L}}(f)(\alpha+\beta, x).$$

Proof. By density, it is enough to prove the result for $f \in C_c(G)$. Use Proposition 4.2 and (4.7) to obtain

$$\begin{aligned} Z_{\mathcal{K}}(T_\ell f)(\alpha, x) &= Z_{\mathcal{K}}(f)(\alpha, x-\ell) \\ &= \sum_{[\beta] \in \mathcal{K}^\perp/\mathcal{L}^\perp} Z_{\mathcal{L}}(f)(\alpha+\beta, x-\ell) \\ &= \sum_{[\beta] \in \mathcal{K}^\perp/\mathcal{L}^\perp} (-\ell, \alpha+\beta) Z_{\mathcal{L}}(f)(\alpha+\beta, x). \end{aligned}$$

□

As in te case of $G = \mathbb{R}$ we are going to need an expression for the Zak transform of $f \in L^2(G)$ in terms of the Fourier transform of f in G. This is possible due to the Poisson Summation Formula (4.4).

Proposition 4.4. *For $f \in L^2(G),\ x \in G,\ \alpha \in \widehat{G}$, and $\mathcal{L}$ a lattice in G,*

$$Z_{\mathcal{L}}(f)(\alpha, x) = \sum_{\gamma \in \mathcal{L}^\perp} \widehat{f}(\alpha+\gamma)(x, \alpha+\gamma).$$

Proof. As before, it is enough to prove the result for $f \in C_c(G)$. Consider the function $F_{\alpha,x}(y) = f(x+y)(-y,\alpha)\,,\ y \in G$. By the Poisson Summation Formula,

$$Z_{\mathcal{L}}(\alpha, x) = |T_{\mathcal{L}}| \sum_{\ell \in \mathcal{L}} F_{\alpha,x}(\ell) = \sum_{\gamma \in \mathcal{L}^\perp} \widehat{F_{\alpha,x}}(\gamma)\,. \tag{4.8}$$

We now compute $\widehat{F_{\alpha,x}}(\gamma)$:

$$\begin{aligned} \widehat{F_{\alpha,x}}(\gamma) &= \int_G F_{\alpha,x}(y)(-y,\gamma)\, d\mu_G(y) \\ &= \int_G f(x+y)(-y,\alpha)(-y,\gamma)\, d\mu_G(y) \\ &= \int_G f(x+y)(-y,\alpha+\gamma)\, d\mu_G(y) \\ &= \int_G f(z)(x-z,\alpha+\gamma)\, d\mu_G(z) \\ &= (x,\alpha+\gamma)\ \widehat{f}(\alpha+\gamma)\,. \end{aligned} \tag{4.9}$$

The result now follows replacing (4.9) in (4.8). □

4.3. **Principal invariant spaces in LCA groups.** Let $\mathcal{L}$ be a lattice in an LCA group G. A closed subspace V of $L^2(G)$ is $\mathcal{L}$ **invariant** if when $f \in V$, $T_\ell(f) \in V$ for all $\ell \in \mathcal{L}$. If $\psi \in L^2(G)$, the subspace

$$\langle \psi \rangle_{\mathcal{L}} := \overline{span}\{T_\ell(\psi) : \ell \in \mathcal{L}\}$$

is an $\mathcal{L}$ invariant subspace of $L^2(G)$ that is called **principal**.

As in subsection 2.2, given two lattices $\mathcal{K} \subset \mathcal{L}$ in G, we are interested in finding necessary and sufficient conditions on $\psi \in L^2(G)$ for $\langle \psi \rangle_{\mathcal{L}}$ to be $\mathcal{L}$ invariant. A preliminary result is the following:

Proposition 4.5. *Let $\psi \neq 0$, $\psi \in L^2(G)$, and $\mathcal{K} \subset \mathcal{L}$ be two lattices in G. The following are equivalent:*
(a) $\langle \psi \rangle_{\mathcal{K}}$ is $\mathcal{L}$ invariant.
(b) $T_\ell(\psi) \in \langle \psi \rangle_{\mathcal{K}}$ for all $\ell \in \mathcal{L}$.

Proof. $(a) \Rightarrow (b)$ is clear by definition. To prove $(b) \Rightarrow (a)$ let $f \in \langle \psi \rangle_{\mathcal{K}}$. We have to show $T_\ell(f) \in \langle \psi \rangle_{\mathcal{K}}$ for all $\ell \in \mathcal{L}$. But

$$T_\ell(f) \in T_\ell(\langle \psi \rangle_{\mathcal{K}}) \subset \langle T_\ell(\psi) \rangle_{\mathcal{K}} \subset \langle \psi \rangle_{\mathcal{K}},$$

since $T_\ell(\psi) \in \langle \psi \rangle_{\mathcal{K}}$ by (b). □

We need now a characterization of $\langle \psi \rangle_{\mathcal{K}}$ in terms of a multiplier. In the case of $\mathbb{R}$ this was accomplished by means of the Fourier transform. For LCA groups, the right tool is the periodization mapping introduced by H. Helson (see [Hel92]) for the case $G = \mathbb{T}$ and extended to LCA groups in [CP10]. For $f \in L^2(G)$ the **periodization** mapping of f relative to the lattice $\mathcal{K}$ is given by

$$\mathcal{T}_{\mathcal{K}}(f)(\alpha) = \{\widehat{f}(\alpha + \gamma)\}_{\gamma \in \mathcal{K}^\perp}, \quad \alpha \in \widehat{G}.$$

It can be shown (see Proposition 3.3 in [CP10]) that $\mathcal{T}$ is an isometric isomorphism from $L^2(G)$ onto $L^2(C_{\mathcal{K}^\perp}, \ell^2(\mathcal{K}^\perp))$, where $C_{\mathcal{K}^\perp}$ is a measurable section of $\widehat{G}/\mathcal{K}^\perp$. For our purposes we need the following statement of Proposition 3.3 in [CP10] adapted to principal invariant subspaces.

Proposition 4.6. *Let $\psi \neq 0$, $\psi \in L^2(G)$, and $\mathcal{K}$ a lattice in G.*
(a) If $f \in \langle \psi \rangle_{\mathcal{K}}$, there exists a $\mathcal{K}^\perp$-periodic function m_f on $\widehat{G}$ such that $\mathcal{T}_{\mathcal{K}}(f)(\alpha) = m_f(\alpha)\, T_{\mathcal{K}}(\psi)(\alpha)$, $\alpha \in \widehat{G}$.
(b) If m is a $\mathcal{K}^\perp$-periodic function on $\widehat{G}$ such that $m\, T_{\mathcal{K}}(\psi) \in L^2(C_{\mathcal{K}^\perp}, \ell^2(\mathcal{K}^\perp))$, the function f defined by $\mathcal{T}_{\mathcal{K}}(f) = m\, T_{\mathcal{K}}(\psi)$ belongs to $\langle \psi \rangle_{\mathcal{K}}$.

We need a similar result in terms of multipliers of the Zak transform.

Corollary 4.7. *Let $\psi \neq 0$, $\psi \in L^2(G)$, and $\mathcal{K}$ a lattice in G.*
(a) If $f \in \langle \psi \rangle_{\mathcal{K}}$, there exists a $\mathcal{K}^\perp$-periodic function m_f on $\widehat{G}$ such that $Z_{\mathcal{K}}(f)(\alpha, x) = m_f(\alpha) Z_{\mathcal{K}}(\alpha, x)$, $\alpha \in \widehat{G}$, $x \in G$.
(b) If m is a $\mathcal{K}^\perp$-periodic function on $\widehat{G}$ such that $m\, \mathcal{T}_{\mathcal{K}}(\psi) \in L^2(C_{\mathcal{K}^\perp}, \ell^2(\mathcal{K}^\perp))$, the function f defined by $Z_{\mathcal{K}}(f)(\alpha, x) = m(\alpha) Z_{\mathcal{K}}(\psi)(\alpha, x)$ belongs to $\langle \psi \rangle_{\mathcal{K}}$.

Proof. (a) Choose m_f as in part (a) of Proposition 4.6. Then, by Proposition 4.4 for $\mathcal{L} = \mathcal{K}$, since m_f is $\mathcal{K}^\perp$-periodic, we have

$$\begin{aligned} Z_{\mathcal{K}}(f)(\alpha, x) &= \sum_{\gamma \in \mathcal{K}^\perp} \widehat{f}(\alpha + \gamma)(x, \alpha + \gamma) \\ &= \langle \mathcal{T}_{\mathcal{K}}(f)(\alpha), (x, \cdot)\rangle_{\ell^2(\mathcal{K}^\perp)}(x, \alpha) \\ &= \langle m_f(\alpha)\mathcal{T}_{\mathcal{K}}(\psi)(\alpha), (x, \cdot)\rangle_{\ell^2(\mathcal{K}^\perp)}(x, \alpha) \\ &= m_f(\alpha)\langle \mathcal{T}_{\mathcal{K}}(\psi)(\alpha), (x, \cdot)\rangle_{\ell^2(\mathcal{K}^\perp)}(x, \alpha) \\ &= m_f(\alpha) Z_{\mathcal{K}}(\psi)(\alpha, x)\,. \end{aligned}$$

(b) If $\alpha \in \widehat{G}$ and $x \in G$, by Proposition 4.4 and the $\mathcal{K}^\perp$-periodicity of m, we can write:

$$\begin{aligned} m(\alpha) Z_{\mathcal{K}}(\psi)(\alpha, x) &= m(\alpha)\langle \mathcal{T}_{\mathcal{K}}(\psi)(\alpha), (x, \cdot)\rangle_{\ell^2(\mathcal{K}^\perp)}(x, \alpha) \\ &= \langle m(\alpha)\mathcal{T}_{\mathcal{K}}(\psi)(\alpha), (x, \cdot)\rangle_{\ell^2(\mathcal{K}^\perp)}(x, \alpha) \\ &= Z_{\mathcal{K}}(\mathcal{T}_{\mathcal{K}}^{-1}(m\mathcal{T}_{\mathcal{K}}(\psi)))\,. \end{aligned}$$

By (b), $Z_{\mathcal{K}}(\mathcal{T}_{\mathcal{K}}^{-1}(m\mathcal{T}_{\mathcal{K}}(\psi))) = Z_{\mathcal{K}}(f)$, and since $Z_{\mathcal{K}}$ is an isometry, we conclude $m\mathcal{T}_{\mathcal{K}}(\psi) = \mathcal{T}_{\mathcal{K}}(f)$. The result now follows from (b) of Proposition 4.6. □

5. Proof of Theorem 1.2

5.1. **Proof of (a) implies (b) of Theorem 1.2.** Assume that $\langle \psi \rangle_{\mathcal{K}}$ is $\mathcal{L}$ invariant. By Proposition 4.5, for every $\ell \in \mathcal{L}$, we have $T_\ell(\psi) \in \langle \psi \rangle_{\mathcal{K}}$. By Corollary 4.7, there exists a $\mathcal{K}^\perp$-periodic function m_ℓ on $\widehat{G}$ such that

$$Z_{\mathcal{K}}(T_\ell(\psi))(\alpha, x) = m_\ell(\alpha) Z_{\mathcal{K}}(\psi)(\alpha, x), \quad \alpha \in \widehat{G},\ x \in G. \tag{5.1}$$

On the other hand, by Proposition 4.3, for $\ell \in \mathcal{L}$,

$$Z_{\mathcal{K}}(T_\ell \psi)(\alpha, x) = (-\ell, \alpha) \sum_{[\beta] \in \mathcal{K}^\perp/\mathcal{L}^\perp} (-\ell, \beta) Z_{\mathcal{L}}(\psi)(\alpha + \beta, x), \tag{5.2}$$

for $\ell \in \mathcal{L},\ \alpha \in \widehat{G},\ x \in G$. Define

$$A_\ell(\alpha, x) := \sum_{[\beta] \in \mathcal{K}^\perp/\mathcal{L}^\perp} (-\ell, \beta) Z_{\mathcal{L}}(\psi)(\alpha + \beta, x)\,.$$

We know that for $[\ell] \in \mathcal{L}/\mathcal{K}$ and $[\alpha] \in \mathcal{K}^\perp/\mathcal{L}^\perp$, $([\ell], [\alpha]) = (\ell, \alpha)$ is well defined. Also, if $[\ell_1] = [\ell_2]$ in $\mathcal{L}/\mathcal{K}$ it can be shown that $A_{\ell_1}(\alpha, x) = A_{\ell_2}(\alpha, x)$. Thus, for $[\ell] \in \mathcal{L}/\mathcal{K}$, $\alpha \in \widehat{G}$, $x \in G$, there is no ambiguity in defining

$$A_{[\ell]}(\alpha, x) := \sum_{[\beta] \in \mathcal{K}^\perp/\mathcal{L}^\perp} (-[\ell], [\beta]) Z_{\mathcal{L}}(\psi)(\alpha + \beta, x)\,. \tag{5.3}$$

Use the orthogonality relations (4.2) to obtain, for $\beta \in \mathcal{K}^\perp$, $\alpha \in \widehat{G},\ x \in G$,

$$Z_{\mathcal{L}}(\psi)(\alpha + \beta, x) := \frac{1}{|\mathcal{L}/\mathcal{K}|} \sum_{[\ell] \in \mathcal{L}/\mathcal{K}} (\ell, \beta) A_{[\ell]}(\alpha, x)\,. \tag{5.4}$$

Lemma 5.1. *If $[\ell_1] + [\ell_2] = [s_1] + [s_2]$ in $\mathcal{L}/\mathcal{K}$, $\alpha \in \widehat{G}$, $x \in G$, then*

$$A_{[\ell_1]}(\alpha, x) A_{[\ell_2]}(\alpha, y) = A_{[s_1]}(\alpha, x) A_{[s_2]}(\alpha, y).$$

Proof. By (5.3), (5.2), and (5.1),

$$A_{[\ell_1]}(\alpha,x)A_{[\ell_2]}(\alpha,y) = (\ell_1+\ell_2,\alpha)m_{\ell_1}(\alpha)m_{\ell_2}(\alpha)Z_{\mathcal{K}}(\psi)(\alpha,x)Z_{\mathcal{K}}(\psi)(\alpha,y)\,.$$

Similarly,

$$A_{[s_1]}(\alpha,x)A_{[s_2]}(\alpha,y) = (s_1+s_2,\alpha)m_{s_1}(\alpha)m_{s_2}(\alpha)Z_{\mathcal{K}}(\psi)(\alpha,x)Z_{\mathcal{K}}(\psi)(\alpha,y)\,.$$

Since $[\ell_1]+[\ell_2]=[s_1]+[s_2]$ in $\mathcal{L}/\mathcal{K}$, there exists $k\in\mathcal{K}$ such that $s_1+s_2=\ell_1+\ell_2+k$. Hence, by (4.7) with $\mathcal{L}=\mathcal{K}$,

$$\begin{aligned} m_{s_1}(\alpha)m_{s_2}(\alpha)Z_{\mathcal{K}}(\psi)(\alpha,x) &= Z_{\mathcal{K}}(T_{s_1+s_2}(\psi))(\alpha,x)\\ &= Z_{\mathcal{K}}(T_kT_{\ell_1+\ell_2}(\psi))(\alpha,x)\\ &= (-k,\alpha)Z_{\mathcal{K}}(T_{\ell_1+\ell_2}(\psi))(\alpha,x)\\ &= (-k,\alpha)m_{\ell_1}(\alpha)m_{\ell_2}(\alpha)Z_{\mathcal{K}}(\psi)(\alpha,x)\,. \end{aligned}$$

Thus,

$$\begin{aligned} A_{[s_1]}(\alpha,x)A_{[s_2]}(\alpha,y) &= (\ell_1+\ell_2+k,\alpha)(-k,\alpha)m_{\ell_1}(\alpha)m_{\ell_2}(\alpha)Z_{\mathcal{K}}(\psi)(\alpha,x)Z_{\mathcal{K}}(\psi)(\alpha,y)\\ &= A_{[\ell_1]}(\alpha,x)A_{[\ell_2]}(\alpha,y) \end{aligned}$$

since $(k,\alpha)(-k,\alpha)=|(k,\alpha)|^2=1$. □

We continue with the proof of (a) implies (b) of Theorem 1.2. Choose $[\beta_1]\neq[\beta_2]\in\mathcal{K}^\perp/\mathcal{L}^\perp$, $\alpha\in\widehat{G}$, $x\in G$. By (5.4),

$$\begin{aligned} &Z_{\mathcal{L}}(\psi)(\alpha+\beta_1,x)\,Z_{\mathcal{L}}(\psi)(\alpha+\beta_2,y)\\ =\;&\frac{1}{|\mathcal{L}/\mathcal{K}|^2}\sum_{[\ell]\in\mathcal{L}/\mathcal{K}}\sum_{[m]\in\mathcal{L}/\mathcal{K}}([\ell],[\beta_1])\,([m],[\beta_2])A_{[\ell]}(\alpha,x)\,A_{[m]}(\alpha,y)\\ =\;&\frac{1}{|\mathcal{L}/\mathcal{K}|^2}\sum_{[\ell]\in\mathcal{L}/\mathcal{K}}\sum_{[s]\in\mathcal{L}/\mathcal{K}}([\ell],[\beta_1])\,([s-\ell],[\beta_2])A_{[\ell]}(\alpha,x)\,A_{[s-\ell]}(\alpha,y)\,. \end{aligned}$$

Since $[\ell]+[s-\ell]=[s]=[0]+[s]$, by Lemma 5.1,

$$\begin{aligned} &Z_{\mathcal{L}}(\psi)(\alpha+\beta_1,x)\,Z_{\mathcal{L}}(\psi)(\alpha+\beta_2,y)\\ =\;&\frac{1}{|\mathcal{L}/\mathcal{K}|^2}\sum_{[\ell]\in\mathcal{L}/\mathcal{K}}\sum_{[s]\in\mathcal{L}/\mathcal{K}}([\ell],[\beta_1]-[\beta_2])\,([s],[\beta_2])A_{[0]}(\alpha,x)\,A_{[s]}(\alpha,y)\\ =\;&\frac{1}{|\mathcal{L}/\mathcal{K}|^2}\sum_{[s]\in\mathcal{L}/\mathcal{K}}\left(\sum_{[\ell]\in\mathcal{L}/\mathcal{K}}([\ell],[\beta_1]-[\beta_2])\right)([s],[\beta_2])A_{[0]}(\alpha,x)\,A_{[s]}(\alpha,y)\,. \end{aligned}$$

Since, when $[\beta_1]\neq[\beta_2]$, $\displaystyle\sum_{[\ell]\in\mathcal{L}/\mathcal{K}}([\ell],[\beta_1]-[\beta_2])=0$ by (4.2), the result is established.

5.2. **Proof of (b) implies (a) of Theorem 1.2.** Assume that

$$Z_{\mathcal{L}}(\psi)(\alpha+\beta_1,x)\,Z_{\mathcal{L}}(\psi)(\alpha+\beta_2,y)=0 \tag{5.5}$$

when $[\beta_1]\neq[\beta_2]$ in $\mathcal{K}^\perp/\mathcal{L}^\perp$, and a. e. $x,y\in C_{\mathcal{L}}$, $\alpha\in C_{\mathcal{K}^\perp}$. Recall that

$$\bigcup_{\ell\in\mathcal{L}}C_{\mathcal{L}}+\ell=G\,,\qquad\text{and}\qquad\bigcup_{\gamma\in\mathcal{K}^\perp}C_{\mathcal{K}^\perp}+\gamma=\widehat{G}\,, \tag{5.6}$$

with disjoint unions. By Proposition 4.5 and Corollary 4.7 we have to show that for $\ell\in\mathcal{L}$ there exists a $\mathcal{K}^\perp$-periodic function m_ℓ defined on $\widehat{G}$ such that $m_\ell\,Z_{\mathcal{K}}(\psi)\in L^2(C_{\mathcal{K}^\perp},\ell^2(\mathcal{K}^\perp))$ and

$$Z_{\mathcal{K}}(T_\ell(\psi))(\alpha,x)=m_\ell(\alpha)Z_{\mathcal{K}}(\psi)(\alpha,x)\,,\quad\alpha\in\widehat{G}\,,\;x\in G. \tag{5.7}$$

For $x \in G$ and $[\beta] \in \mathcal{K}^\perp/\mathcal{L}^\perp$ let

$$S_\psi^{[\beta]}(x) := \{\alpha \in C_{\mathcal{K}^\perp} : Z_\mathcal{L}(\psi)(\alpha+\beta, x) \neq 0\}\,.$$

Notice that the definition of $S_\psi^{[\beta]}(x)$ does not depend on the representation chosen for $[\beta]$. Indeed, If $\beta_1 \in [\beta]$, there exists $\gamma \in \mathcal{L}^\perp$ such that $\beta_1 - \beta = \gamma$, and since $(-\ell, \gamma) = 1$ when $\ell \in \mathcal{L}$ and $\gamma \in \mathcal{L}^\perp$,

$$\begin{aligned} Z_\mathcal{L}(\psi)(\alpha+\beta_1, x) &= \sum_{\ell\in\mathcal{L}} \psi(x+\ell)(-\ell, \alpha+\beta_1) \\ &= \left(\sum_{\ell\in\mathcal{L}} \psi(x+\ell)(-\ell,\alpha+\beta)\right)(-\ell,\gamma) \\ &= Z_\mathcal{L}(\psi)(\alpha+\beta, x)\,. \end{aligned}$$

Consider

$$S_\psi^{[\beta]} := \bigcup_{x\in C_\mathcal{K}} S_\psi^{[\beta]}(x)\,, \qquad \text{and} \qquad S_\psi := C_{\mathcal{K}^\perp} \setminus \bigcup_{[\beta]\in\mathcal{K}^\perp/\mathcal{L}^\perp} S_\psi^{[\beta]}\,. \tag{5.8}$$

Observe that the union in the left hand side of (5.8) is disjoint due to (5.5).

For $\ell \in \mathcal{L}$ define

$$m_\ell(\alpha) := \sum_{[\beta]\in\mathcal{K}^\perp/\mathcal{L}^\perp} (-\ell,\alpha)\chi_{S_\psi^{[\beta]}}(\alpha)(-\ell,\beta) + \chi_{S_\psi}(\alpha)\,, \quad \alpha \in C_{\mathcal{K}^\perp}\,, \tag{5.9}$$

and extend m_ℓ to be $\mathcal{K}^\perp$-periodic in $\widehat{G}$.

Notice that, by Proposition 4.2,

$$Z_\mathcal{K}(\psi)(\alpha, x) = \sum_{[\beta]\in\mathcal{K}^\perp/\mathcal{L}^\perp} Z_\mathcal{L}(\psi)(\alpha+\beta, x)\,, \tag{5.10}$$

and, by Proposition 4.3,

$$Z_\mathcal{K}(T_\ell(\psi))(\alpha, x) = \sum_{[\beta]\in\mathcal{K}^\perp/\mathcal{L}^\perp} (-\ell, \alpha+\beta) Z_\mathcal{L}(\psi)(\alpha+\beta, x)\,. \tag{5.11}$$

If given $\alpha \in C_{\mathcal{K}^\perp}$ and $x \in C_\mathcal{L}$, $Z_\mathcal{L}(\psi)(\alpha+\beta, x) = 0$ for all $[\beta] \in \mathcal{K}^\perp/\mathcal{L}^\perp$, then by (5.10), $Z_\mathcal{K}(\psi)(\alpha, x) = 0$, and by (5.11), $Z_\mathcal{K}(T_\ell(\psi))(\alpha, x) = 0$. Therfore, (5.7) holds trivially for any value given to m_ℓ and in particular for the value given by the definition of m_ℓ in (5.9).

If $Z_\mathcal{L}(\psi)(\alpha+\beta, x) \neq 0$ for some $[\beta] \in \mathcal{K}^\perp/\mathcal{L}^\perp$, by (5.5) and (5.10) we have $Z_\mathcal{K}(\psi)(\alpha, x) = Z_\mathcal{L}(\psi)(\alpha+\beta, x)$, and by (5.5) and (5.11), $Z_\mathcal{K}(T_\ell(\psi))(\alpha, x) = (-\ell, \alpha+\beta) Z_\mathcal{L}(\psi)(\alpha+\beta, x)$. In this case $\alpha \in S_\psi^{[\beta]}$ and, by (5.9), $m_\ell(\alpha) = (-\ell, \alpha+\beta)$, so that (5.7) also holds. Observe that $|m_\ell(\alpha)| = 1$ and since $\mathcal{T}_\mathcal{K}(\psi) \in L^2(C_{\mathcal{K}^\perp}, \ell^2(\mathcal{K}^\perp))$, also $m_\ell \mathcal{T}_\mathcal{K}(\psi) \in L^2(C_{\mathcal{K}^\perp}, \ell^2(\mathcal{K}^\perp))$.

Finally, although we have only proved (5.7) for $\alpha \in C_{\mathcal{K}^\perp}$ and $x \in C_\mathcal{L}$, the quasi-periodicity properties of $Z_\mathcal{K}$ and the periodicity properties of m_ℓ, together with (5.6), prove the result for all $\alpha \in \widehat{G}$ and all $x \in G$.

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