ON BASIS CONSTRUCTIONS IN FINITE ELEMENT EXTERIOR CALCULUS

MARTIN W. LICHT

Abstract. We give a systematic and self-contained account of the construction of geometrically decomposed bases and degrees of freedom in finite element exterior calculus. In particular, we elaborate upon a previously overlooked basis for one of the families of finite element spaces, which is of interest for implementations. Moreover, we give details for the construction of isomorphisms and duality pairings between finite element spaces. These structural results show, for example, how to transfer linear dependencies between canonical spanning sets, or give a new derivation of the degrees of freedom.

1. Introduction

Exterior calculus is a canonical approach towards mathematical electromagnetism. Utilizing exterior calculus in numerical analysis hence seems to be a natural choice, and a multitude of finite element methods have been formulated in what is now known as *Finite Element Exterior Calculus* (FEEC [5]). A particular achievement of FEEC has been the identification of spaces of polynomial differential forms invariant under affine transformations, and subsequently the construction of *finite element de Rham complexes*. Research in numerical analysis has elaborated upon bases, degrees of freedom, and their geometric decompositions for those finite element spaces (see [3, 4, 16, 17, 22, 23], for example).

In this exposition we address the construction of spanning sets, bases, and degrees of freedom in finite element exterior calculus. A new result is a simple basis for one of the families of spaces in FEEC that includes the Brezzi-Douglas-Marini space and the Nédélec spaces of second kind. To the author's best knowledge, that basis is not yet commonly known and has been used only implicitly in the seminal work of Arnold, Falk, and Winther [3]. Furthermore, it differs from other explicit bases in FEEC ([4]). Our new bases is easy to define and can serve as a default in finite element implementations. Another contribution of our exposition a complete theory of isomorphisms and duality pairings between finite element spaces in FEEC. These serve as general tools which allow us to relate bases of different finite element spaces and enable a straightforward construction of geometrically decomposed degrees of freedom. These isomorphisms and duality pairings have been used implicitly in the initial work of Arnold, Falk, and Winther and have been made explicit recently by Christiansen and Rapetti [11]. We complete the theory of these isomorphisms and show that they are well-defined in terms of spanning sets.

2010 *Mathematics Subject Classification.* 65N30.

Key words and phrases. barycentric differential form, canonical spanning sets, degrees of freedom, finite element exterior calculus, geometrically decomposed bases.

This research was supported by the European Research Council through the FP7-IDEAS-ERC Starting Grant scheme, project 278011 STUCCOFIELDS. This research was supported in part by NSF DMS/RTG Award 1345013 and DMS/CM Award 1262982. Parts of this article are based on the author's PhD Thesis.

The wider purpose of this exposition is giving a concise and self-contained presentation of the construction of spanning sets, bases, and degrees of freedom in FEEC. For that purpose our exposition also collects and rearranges results in a manner which until now can only be found distributed throughout the literature.

Our construction of geometrically decomposed bases for finite element spaces builds upon previous publications in the literature. For the purpose of comparison and as a justficiation for our manner of exposition, we recall the approaches in two of the major references, beginning with the exposition [3, Chapter 4] of Arnold, Falk, and Winther. Their presentation begins by devising a basis for $\mathcal{P}_r^-\Lambda^k(T)$. Then they determine a geometrically decomposed basis of the dual space $\mathcal{P}_r\Lambda^k(T)^*$, and subsequently a geometrically decomposed basis of the dual space $\mathcal{P}_r^-\Lambda^k(T)^*$. After outlining bases for spaces with vanishing trace, they find geometrically decomposed bases for $\mathcal{P}_r^-\Lambda^k(T)$ and, implicitly, for $\mathcal{P}_r\Lambda^k(T)$. In [4], we are given bases for the spaces with vanishing trace, $\mathring{\mathcal{P}}_r^-\Lambda^k(T)$ and $\mathring{\mathcal{P}}_r\Lambda^k(T)$. The latter publication studies extension operators and geometrically decomposed bases, in explicit form also for $\mathcal{P}_r\Lambda^k(T)$, though the basis in [4] is generally different from the one in [3].

We arrange this material in a different manner. Most importantly, we directly construct geometrically decomposed bases of the spaces $\mathcal{P}_r\Lambda^k(T)$ which contain bases of $\mathring{\mathcal{P}}_r\Lambda^k(T)$. It does not seem to be widely known that explicit geometrically decomposed bases for the $\mathcal{P}_r\Lambda^k$-family can already be derived with the methods in [3] and that these bases are different from the ones in [4]. By contrast, our geometrically decomposed bases for $\mathcal{P}_r^-\Lambda^k(T)$ and $\mathring{\mathcal{P}}_r^-\Lambda^k(T)$ coincide with those in [3] and [4]. We emphasize that we develop bases for the finite element spaces without referring to any degrees of freedom in the first place.

Quite remarkably, finding explicit bases for the Brezzi-Douglas-Marini space and the Nédélec spaces of second kind seems to have been an open problem for quite some time. The original articles by Brezzi, Douglas, and Marini [9] and Nédélec [21] describe the degrees of freedom but do not address explicit formulas for bases. It seems that explicit bases for general polynomial degree have appeared in the literature only twenty years later: we point out the contributions by Arnold, Falk, and Winther [4], Ervin [12], and Bentley [6]. Explicit bases for the Raviart-Thomas space and the Nédélec spaces of first kind have started to appear around the same time (e.g. [15]).

In this article, we describe a basis for the Brezzi-Douglas-Marini space,

$$\left\{ \lambda_T^\alpha \nabla \lambda_p^T \;\middle|\; \alpha \in A(r,n),\ p \in \{0,1,2\},\ \min[\alpha] \neq p \right\}, \tag{1}$$

for the curl-conforming Nédélec elements of second kind,

$$\left\{ \lambda_T^\alpha \nabla \lambda_p^T \;\middle|\; \alpha \in A(r,n),\ p \in \{0,1,2,3\},\ \min[\alpha] \neq p \right\}, \tag{2}$$

and for the divergence-conforming Nédélec elements of second kind

$$\left\{ \lambda_T^\alpha \nabla \lambda_p^T \times \nabla \lambda_q^T \;\middle|\; \alpha \in A(r,n),\ p,q \in \{0,1,2,3\},\ p<q,\ \min[\alpha] \notin \{p,q\} \right\}, \tag{3}$$

where r denotes the polynomial degree; see also Remark 4.4 later in this article for a description of the notation. An illustration of these bases for vector-valued finite elements for low polynomial degree is given by Tables 1 – 3.

Having constructed bases for the finite element spaces, we study isomorphisms between finite element spaces. Whenever T is an n-dimensional simplex and k and r are non-negative integers, we have isomorphisms

$$\mathcal{P}_r\Lambda^k(T) \simeq \mathring{\mathcal{P}}^-_{r+n-k+1}\Lambda^{n-k}(T), \quad \mathcal{P}^-_{r+1}\Lambda^{n-k}(T) \simeq \mathring{\mathcal{P}}_{r+n-k+1}\Lambda^k(T). \tag{4}$$

$r=1$	$\lambda_0\{\nabla\lambda_1,\nabla\lambda_2\},\ \lambda_1\{\nabla\lambda_0,\nabla\lambda_2\},\ \lambda_2\{\nabla\lambda_0,\nabla\lambda_1\}$
$r=2$	$\lambda_0\{\lambda_0,\lambda_1,\lambda_2\}\{\nabla\lambda_1,\nabla\lambda_2\},\ \lambda_1\{\lambda_1,\lambda_2\}\{\nabla\lambda_0,\nabla\lambda_2\},\ \lambda_2^2\{\nabla\lambda_0,\nabla\lambda_1\}$
$r=3$	$\lambda_0\{\lambda_0,\lambda_1,\lambda_2\}^2\{\nabla\lambda_1,\nabla\lambda_2\},\ \lambda_1\{\lambda_1,\lambda_2\}^2\{\nabla\lambda_0,\nabla\lambda_2\},\ \lambda_2^3\{\nabla\lambda_0,\nabla\lambda_1\}$

TABLE 1. Bases for the Brezzi-Douglas-Marini space on a 2-simplex T for low r in terms of barycentric coordinates.

$r=1$	$\lambda_0\{\nabla\lambda_1,\nabla\lambda_2,\nabla\lambda_3\},\ \lambda_1\{\nabla\lambda_0,\nabla\lambda_2,\nabla\lambda_3\},\ \lambda_2\{\nabla\lambda_0,\nabla\lambda_1,\nabla\lambda_3\},$ $\lambda_3\{\nabla\lambda_0,\nabla\lambda_1,\nabla\lambda_2\}$
$r=2$	$\lambda_0\{\lambda_0,\lambda_1,\lambda_2,\lambda_3\}\{\nabla\lambda_1,\nabla\lambda_2,\nabla\lambda_3\},\ \lambda_1\{\lambda_1,\lambda_2,\lambda_3\}\{\nabla\lambda_0,\nabla\lambda_2,\nabla\lambda_3\},$ $\lambda_2\{\lambda_2,\lambda_3\}\{\nabla\lambda_0,\nabla\lambda_1,\nabla\lambda_3\},\ \lambda_3^2\{\nabla\lambda_0,\nabla\lambda_1,\nabla\lambda_2\},$
$r=3$	$\lambda_0\{\lambda_0,\lambda_1,\lambda_2,\lambda_3\}^2\{\nabla\lambda_1,\nabla\lambda_2,\nabla\lambda_3\},\ \lambda_1\{\lambda_1,\lambda_2,\lambda_3\}^2\{\nabla\lambda_0,\nabla\lambda_2,\nabla\lambda_3\},$ $\lambda_2\{\lambda_2,\lambda_3\}^2\{\nabla\lambda_0,\nabla\lambda_1,\nabla\lambda_3\},\ \lambda_3^3\{\nabla\lambda_0,\nabla\lambda_1,\nabla\lambda_2\},$

TABLE 2. Bases for the curl-conforming Nédélec space of second kind on a 3-simplex T for low r in terms of barycentric coordinates.

$r=1$	$\lambda_0\{\nabla\lambda_1\times\nabla\lambda_2,\nabla\lambda_1\times\nabla\lambda_3,\nabla\lambda_2\times\nabla\lambda_3\},$ $\lambda_1\{\nabla\lambda_0\times\nabla\lambda_2,\nabla\lambda_0\times\nabla\lambda_3,\nabla\lambda_2\times\nabla\lambda_3\},$ $\lambda_2\{\nabla\lambda_0\times\nabla\lambda_1,\nabla\lambda_0\times\nabla\lambda_3,\nabla\lambda_1\times\nabla\lambda_3\},$ $\lambda_3\{\nabla\lambda_0\times\nabla\lambda_1,\nabla\lambda_0\times\nabla\lambda_2,\nabla\lambda_1\times\nabla\lambda_2\}$
$r=2$	$\lambda_0\{\lambda_0,\lambda_1,\lambda_2,\lambda_3\}\{\nabla\lambda_1\times\nabla\lambda_2,\nabla\lambda_1\times\nabla\lambda_3,\nabla\lambda_2\times\nabla\lambda_3\},$ $\lambda_1\{\lambda_1,\lambda_2,\lambda_3\}\{\nabla\lambda_0\times\nabla\lambda_2,\nabla\lambda_0\times\nabla\lambda_3,\nabla\lambda_2\times\nabla\lambda_3\},$ $\lambda_2\{\lambda_2,\lambda_3\}\{\nabla\lambda_0\times\nabla\lambda_1,\nabla\lambda_0\times\nabla\lambda_3,\nabla\lambda_1\times\nabla\lambda_3\},$ $\lambda_3^2\{\nabla\lambda_0\times\nabla\lambda_1,\nabla\lambda_0\times\nabla\lambda_2,\nabla\lambda_1\times\nabla\lambda_2\},$
$r=3$	$\lambda_0\{\lambda_0,\lambda_1,\lambda_2,\lambda_3\}^2\{\nabla\lambda_1\times\nabla\lambda_2,\nabla\lambda_1\times\nabla\lambda_3,\nabla\lambda_2\times\nabla\lambda_3\},$ $\lambda_1\{\lambda_1,\lambda_2,\lambda_3\}^2\{\nabla\lambda_0\times\nabla\lambda_2,\nabla\lambda_0\times\nabla\lambda_3,\nabla\lambda_2\times\nabla\lambda_3\},$ $\lambda_2\{\lambda_2,\lambda_3\}^2\{\nabla\lambda_0\times\nabla\lambda_1,\nabla\lambda_0\times\nabla\lambda_3,\nabla\lambda_1\times\nabla\lambda_3\},$ $\lambda_3^3\{\nabla\lambda_0\times\nabla\lambda_1,\nabla\lambda_0\times\nabla\lambda_2,\nabla\lambda_1\times\nabla\lambda_2\},$

TABLE 3. Bases for the divergence-conforming Nédélec space of second kind on a 3-simplex T for low r in terms of barycentric coordinates.

In fact, more is true: to each of these isomorphic pairs corresponds a duality pairing. These isomorphic relations and duality pairings are used in the seminal publication by Arnold, Falk, and Winther [3].

A novel result of this article is that both linear isomorphisms preserve the canonical spanning sets. Consequently, these isomorphisms translate linear dependencies and independencies between these spaces. We subsequently show how the values of the duality pairings can be expressed in terms of the spanning sets. One application of these results is of expositional nature: they enable a new way to derive the degrees of freedom for the finite element spaces. The aforementioned results draw major inspiration from recent work by Christiansen and Rapetti [11], who have derived similar results for the first isomorphism in (4) but not for the second isomorphism. The techniques in this article allow us to extend upon the results in [11] on the first isomorphic pair and to give analogous results for the second isomorphic pair.

Finding geometrically decomposed bases and degrees of freedom is much more difficult for vector-valued than for scalar-valued finite element spaces. The reason is that the latter's canonical spanning sets are linearly independent. Our aim in this exposition is simply to establish explicit formulas for finite element basis forms in barycentric coordinates that are simple and readily implementable. We adapt techniques that from prior research on finite element differential forms [3, 4, 11, 16, 22, 23]. Of course, much research has also been devoted to other aspects of vector-valued finite element methods such as condition numbers, sparsity properties, and hierarchical structures, or fast evaluation [2, 24, 1, 7, 8, 18, 19], and finite element bases are also studied over quadrilaterals and general polytopes [14, 10]. This article prepares further algebraic and combinatorial studies of finite element spaces.

The remainder of this work is structured as follows. In Section 2 we review combinatorial results, exterior calculus, and polynomial differential forms. Section 3 summarizes some auxiliary lemmas. In Section 4 we introduce spaces of polynomial differential forms and construct geometrically decomposed bases. Subsequently, we study the isomorphy relations in Section 5 and the duality pairings in Section 6. We supplement applications to finite element spaces over triangulations in Section 7.

2. Notation and Definitions

We introduce or review notions regarding combinatorics and differential forms over simplices. All vector spaces in this publication are over the complex numbers unless noted otherwise; we write $\overline{z} \in \mathbb{C}$ for the complex conjugate of $z \in \mathbb{C}$.

2.1. Combinatorics.

We let $[m : n] = \{m, \dots, n\}$ for $m, n \in \mathbb{Z}$ with $m \leq n$. For $m, n \in \mathbb{Z}$ with $m \neq n$ we let $\epsilon(m,n) = 1$ if $m < n$ and $\epsilon(m,n) = -1$ if $m > n$.

For any mapping $\alpha \colon [m : n] \to \mathbb{N}_0$ we write $|\alpha| := \sum_{i=m}^{n} \alpha(i)$. Given $r, m, n \in \mathbb{N}_0$, we let $A(r, m : n)$ be the set of all mappings $\alpha \colon [m : n] \to \mathbb{N}_0$ for which $|\alpha| = r$. So $A(r, m : n)$ is the set of *multiindices* over the set $[m : n]$. We abbreviate $A(r,n) := A(r, 0 : n)$. Whenever $\alpha \in A(r, m : n)$, we write

$$[\alpha] := \{ \, i \in [m : n] \mid \alpha(i) > 0 \, \}, \tag{5}$$

and we write $\lfloor \alpha \rfloor$ for the minimal element of $[\alpha]$ provided that $[\alpha]$ is not empty, and $\lfloor \alpha \rfloor = \infty$ otherwise. The sum $\alpha + \beta$ of $\alpha, \beta \in A(r, m : n)$ is defined in the obvious manner. When $\alpha \in A(r, m : n)$ and $p \in [m : n]$, then $\alpha + p$ denotes the unique member of $A(r+1, m : n)$ with $(\alpha + p)(p) = \alpha(p) + 1$ and coincides with α otherwise; similarly, when $p \in [\alpha]$, then $\alpha - p$ denotes the unique member of $A(r-1, m : n)$ with $(\alpha - p)(p) = \alpha(p) - 1$ and coincides with α otherwise.

For $a, b, m, n \in \mathbb{N}_0$, we let $\Sigma(a : b, m : n)$ be the set of strictly ascending mappings from $[a : b]$ to $[m : n]$. We call those mappings also *alternator indices*. We write $\Sigma(a : b, m : n) := \{\emptyset\}$ whenever $a > b$. For any $\sigma \in \Sigma(a : b, m : n)$ we let

$$[\sigma] := \{\sigma(i) \mid i \in [a : b]\}, \tag{6}$$

and we write $\lfloor \sigma \rfloor$ for the minimal element of $[\sigma]$ provided that $[\sigma]$ is not empty, and $\lfloor \sigma \rfloor = \infty$ otherwise. Furthermore, if $q \in [m : n] \setminus [\sigma]$, then we write $\sigma + q$ for the unique element of $\Sigma(a : b+1, m : n)$ with image $[\sigma] \cup \{q\}$. In that case, we also write $\epsilon(q, \sigma)$ for the signum of the permutation that orders the sequence $q, \sigma(a), \dots, \sigma(b)$ in ascending order, and we write $\epsilon(\sigma, q)$ for the signum of the permutation that orders the sequence $\sigma(a), \dots, \sigma(b), q$ in ascending order. Similarly, if $p \in [\sigma]$, then we write $\sigma - p$ for the unique element of $\Sigma(a : b-1, m : n)$ with image $[\sigma] \setminus \{p\}$. Note that $\epsilon(\sigma, q) = (-1)^{b-a+1} \epsilon(q, \sigma)$.

We use the abbreviations $\Sigma(k,n) = \Sigma(1:k, 0:n)$ and $\Sigma_0(k,n) = \Sigma(0:k, 0:n)$. If n is understood and $k,l \in [0:n]$, then for any $\sigma \in \Sigma(k,n)$ we define $\sigma^c \in \Sigma_0(n-k,n)$ by the condition $[\sigma] \cup [\sigma^c] = [0:n]$, and for any $\rho \in \Sigma_0(l,n)$ we define $\rho^c \in \Sigma(n-l,n)$ by the condition $[\rho] \cup [\rho^c] = [0:n]$. In particular, $\sigma^{cc} = \sigma$ and $\rho^{cc} = \rho$. Note that σ^c and ρ^c depend on n, which we suppress in the notation.

When $\sigma \in \Sigma(k,n)$ and $\rho \in \Sigma_0(l,n)$ with $[\sigma] \cap [\rho] = \emptyset$, then $\epsilon(\sigma,\rho)$ denotes the signum of the permutation ordering the sequence $\sigma(1), \dots, \sigma(k), \rho(0), \dots, \rho(l)$ in ascending order, and we let $\sigma + \rho \in \Sigma(0:k+l, 0:n)$ be the unique strictly ascending mapping from $[0:k+l]$ to $[0:n]$ whose image is the set $[\sigma] \cup [\rho]$.

2.2. **Simplices.** Let $n \in \mathbb{N}_0$. An n-dimensional simplex T is the convex closure of pairwise distinct points $v_0^T, \dots, v_n^T$ in Euclidean space, called the *vertices* of T, such that the vertices are an affinely independent set. Note that the dimension of the ambient Euclidean space must be at least n but otherwise does not matter.

An *ordered simplex* is a simplex with an ordering of its set of vertices (see [13]) We henceforth assume that all simplices in this article are ordered.

We call $F \subseteq T$ a *subsimplex* of T if the set of vertices of F is a subset of the set of vertices of T. We write $\imath(F,T)\colon F \to T$ for the set inclusion of F into T.

Suppose that F is an m-dimensional subsimplex of T with ordered vertices $v_0^F, \dots, v_m^F$. With a mild abuse of notation, we let $\imath(F,T) \in \Sigma_0(m,n)$ denote the unique mapping that satisfies $v_{\imath(F,T)(i)}^T = v_i^F$.

2.3. **Barycentric Coordinates and Differential Forms.** Let T be a simplex of dimension n. Following the notation of [3], we write $\Lambda^k(T)$ for the space of *differential k-forms* over T with smooth bounded coefficients of all orders, where $k \in \mathbb{Z}$. Recall that these mappings take values in the k-th exterior power of the dual of the tangential space of the simplex T. In the case $k = 0$, the space $\Lambda^0(T) = C^\infty(T)$ is just the space of smooth functions over T with uniformly bounded derivatives. Furthermore, $\Lambda^k(T)$ is the trivial vector space unless $0 \leq k \leq n$.

We recall the *exterior product* $\omega \wedge \eta \in \Lambda^{k+l}(T)$ for $\omega \in \Lambda^k(T)$ and $\eta \in \Lambda^l(T)$ and that it satisfies $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$. We let $\mathsf{d}\colon \Lambda^k(T) \to \Lambda^{k+1}(T)$ denote the *exterior derivative*. It satisfies $\mathsf{d}(\omega \wedge \eta) = \mathsf{d}\omega \wedge \eta + (-1)^k \omega \wedge \mathsf{d}\eta$ for $\omega \in \Lambda^k(T)$ and $\eta \in \Lambda^l(T)$. We also recall that the integral $\int_T \omega$ of a differential n-form over T is well-defined. We refer to [3] and [20] for more background. In this article, we focus on a special class of differential forms, namely the *barycentric differential forms.*

The *barycentric coordinates* $\lambda_0^T, \dots, \lambda_n^T \in \Lambda^0(T)$ are the unique affine functions over T that satisfy the *Lagrange property*

$$\lambda_i^T(v_j) = \delta_{ij}, \quad i,j \in [0:n]. \tag{7}$$

The barycentric coordinate functions of T are linearly independent and constitute a partition of unity:

$$1 = \lambda_0^T + \cdots + \lambda_n^T. \tag{8}$$

We write $\mathsf{d}\lambda_0^T, \mathsf{d}\lambda_1^T, \dots, \mathsf{d}\lambda_n^T \in \Lambda^1(T)$ for the exterior derivatives of the barycentric coordinates. The exterior derivatives are differential 1-forms and constitute a partition of zero:

$$0 = \mathsf{d}\lambda_0^T + \cdots + \mathsf{d}\lambda_n^T. \tag{9}$$

It can be shown that this is the only linear independence between the exterior derivatives of the barycentric coordinate functions.

We consider several classes of differential forms over T that are expressed in terms of the barycentric polynomials and their exterior derivatives. When $r \in \mathbb{N}_0$ and $\alpha \in A(r,n)$, then the corresponding *barycentric polynomial* over T is

$$\lambda^T_\alpha := \prod_{i=0}^{n} (\lambda^T_i)^{\alpha(i)}. \tag{10}$$

When $a, b \in \mathbb{N}_0$ and $\sigma \in \Sigma(a:b, 0:n)$, the corresponding *barycentric alternator* is

$$\mathsf{d}\lambda^T_\sigma := \mathsf{d}\lambda^T_{\sigma(a)} \wedge \cdots \wedge \mathsf{d}\lambda^T_{\sigma(b)}. \tag{11}$$

Here, we treat the special case $\sigma = \emptyset$ by defining $\mathsf{d}\lambda^T_\emptyset = 1$. Finally, whenever $a, b \in \mathbb{N}_0$ and $\rho \in \Sigma(a:b, 0:n)$, then the corresponding *Whitney form* is

$$\phi^T_\rho := \sum_{p \in [\rho]} \epsilon(p, \rho - p) \lambda^T_p \mathsf{d}\lambda^T_{\rho - p}. \tag{12}$$

In the special case that $\rho_T \colon [0:n] \to [0:n]$ is the single member of $\Sigma_0(n,n)$, then we write $\phi_T := \phi_{\rho_T}$ for the associated Whitney form.

In the sequel, we call the differential forms (10), (11), (12), and their sums and exterior products, *barycentric differential forms* over T.

Remark 2.1.
Whenever a fixed simplex T is understood and there is no danger of ambiguity, we may simplify the notation by writing

$$\lambda_i \equiv \lambda^T_i, \quad \lambda^\alpha \equiv \lambda^\alpha_T, \quad \mathsf{d}\lambda_\sigma \equiv \mathsf{d}\lambda^T_\sigma, \quad \phi_\rho \equiv \phi^T_\rho.$$

2.4. **Traces.** Let T be an n-dimensional simplex and let $F \subseteq T$ be a subsimplex of T of dimension m. The inclusion $\imath(F,T) \colon F \to T$ introduced above naturally induces a mapping $\operatorname{tr}_{T,F} \colon \Lambda^k(T) \to \Lambda^k(F)$ by taking the pullback. We call $\operatorname{tr}_{T,F}$ the *trace* from T onto F. It is well-known that $\mathsf{d} \operatorname{tr}_{T,F} \omega = \operatorname{tr}_{T,F} \mathsf{d}\omega$ for all $\omega \in \Lambda^k(T)$, that is, the exterior derivative commutes with taking traces. In the case of 0-forms, the trace is just the natural restriction operator of functions. The trace does not depend on the order of the simplices.

Taking into account the ordering of the simplices, however, we obtain explicit formulas for the traces of barycentric differential forms. Write $[\imath(F,T)]$ for the set of indices of those vertices of T that are also vertices of F, which is compatible with prior definition of $[\imath(F,T)]$. We write $\imath(F,T)^\dagger \colon [\imath(F,T)] \to [0:m]$ for the inverse of the mapping $\imath(F,T) \colon [0:m] \to [\imath(F,T)]$

Consider $i \in [0:n]$. If $i \notin [\imath(F,T)]$, then v^T_i is a vertex of T that is not a vertex of F, and in that case we have $\operatorname{tr}_{T,F} \lambda^T_i = 0$. If instead $i \in [\imath(F,T)]$, then there exists $j \in [0:m]$ such that $i = \imath(F,T)(j)$, and in that case we have $\operatorname{tr}_{T,F} \lambda^T_i = \lambda^F_j$ or, equivalently, $\operatorname{tr}_{T,F} \lambda^T_i = \lambda^F_{\imath(F,T)^\dagger i}$. Analogous observations follow for the exterior derivatives of the barycentric coordinates.

Let $\alpha \in A(r, 0:n)$ be a multiindex. If $[\alpha] \nsubseteq [\imath(F,T)]$, then we have $\operatorname{tr}_{T,F} \lambda^\alpha_T = 0$. If instead $[\alpha] \subseteq [\imath(F,T)]$, then there exists $\hat\alpha \in A(r, m:n)$ with $\hat\alpha = \alpha \circ \imath(F,T)$, and we then have

$$\operatorname{tr}_{T,F} \lambda^\alpha_T = \lambda^{\hat\alpha}_F \tag{13}$$

Let $\sigma \in \Sigma(a:b, 0:n)$ be a basic alternator. If $[\sigma] \nsubseteq [\imath(F,T)]$, then we have $\operatorname{tr}_{T,F} \mathsf{d}\lambda^T_\sigma = 0$. If instead $[\sigma] \subseteq [\imath(F,T)]$, then there exists $\hat\sigma \in \Sigma(a:b, 0:n)$ with $\imath(F,T) \circ \hat\sigma = \sigma$, or equivalently, $\hat\sigma = \imath(F,T)^\dagger \circ \sigma$, and we then have

$$\operatorname{tr}_{T,F} \mathsf{d}\lambda^\sigma_T = \mathsf{d}\lambda^{\hat\sigma}_F, \quad \operatorname{tr}_{T,F} \phi^T_\sigma = \phi^F_{\hat\sigma}. \tag{14}$$

3. Auxiliary Lemmas

In this section we provide some auxiliary lemmas on barycentric differential forms over an n-dimensional simplex T.

Suppose that $k \in [1:n]$ and $\sigma \in \Sigma(k,n)$. For any $p \in [\sigma]$ we have

$$\mathsf{d}\lambda_\sigma = \epsilon(p, \sigma - p)\mathsf{d}\lambda_p \wedge \mathsf{d}\lambda_{\sigma-p}. \tag{15}$$

This follows from definitions and properties of the alternating product. The analogous result for the Whitney forms is folklore and slightly more technical to derive.

Lemma 3.1.
Let $k \in [0:n]$. If $\rho \in \Sigma_0(k,n)$ and $q \in [0:n]$ with $q \notin [\rho]$, then

$$\epsilon(q,\rho)\phi_{\rho+q} = \lambda_q \mathsf{d}\lambda_\rho - \mathsf{d}\lambda_q \wedge \phi_\rho. \tag{16}$$

Proof. If ρ and q are as in the statement of the lemma, then we find

$$\begin{aligned}\lambda_q \mathsf{d}\lambda_\rho - \epsilon(q,\rho)\phi_{\rho+q} &= \lambda_q \mathsf{d}\lambda_\rho - \epsilon(q,\rho) \sum_{l \in [\rho+q]} \epsilon(l, \rho+q-l)\lambda_l \mathsf{d}\lambda_{\rho+q-l} \\ &= -\epsilon(q,\rho) \sum_{l \in [\rho]} \epsilon(l, \rho+q-l)\lambda_l \mathsf{d}\lambda_{\rho+q-l}.\end{aligned}$$

Using definitions and (15), we get

$$\sum_{l \in [\rho]} \epsilon(l, \rho+q-l)\lambda_l \mathsf{d}\lambda_{\rho+q-l} = \mathsf{d}\lambda_q \wedge \sum_{l \in [\rho]} \epsilon(l, \rho+q-l)\epsilon(q, \rho-l)\lambda_l \mathsf{d}\lambda_{\rho-l}.$$

For any $l \in [\rho]$ one finds that

$$\epsilon(l, \rho+q-l)\epsilon(q, \rho-l) = \epsilon(l,q)\epsilon(l,\rho-l)\epsilon(q,l)\epsilon(q,\rho) = -\epsilon(l,\rho-l)\epsilon(q,\rho).$$

The desired statement now follows from the definition of ϕ_ρ. □

Whenever $k \in [0:n]$ and $\rho \in \Sigma_0(k,n)$, then it follows from definitions that we can express the differential of the corresponding Whitney form by

$$\mathsf{d}\phi_\rho = (k+1)\mathsf{d}\lambda_\rho. \tag{17}$$

The following result, which has appeared as Proposition 3.4 in [11], and also as Equation (6.6) in [4], can be seen as a converse to that.

Lemma 3.2.
Let $k \in [0:n]$ and $\rho \in \Sigma_0(k,n)$. Then

$$\mathsf{d}\lambda_\rho = \sum_{q \in [\rho^c]} \epsilon(q,\rho)\phi_{\rho+q}. \tag{18}$$

Proof. We use Lemma 3.1 and see

$$\sum_{q \in [\rho^c]} \epsilon(q,\rho)\phi_{\rho+q} = \sum_{q \in [\rho^c]} \lambda_q \mathsf{d}\lambda_\rho - \sum_{q \in [\rho^c]} \mathsf{d}\lambda_q \wedge \phi_\rho.$$

Using (12), (9), (15), and (8), we see that the last expression equals

$$\begin{aligned}\sum_{q \in [\rho^c]} \lambda_q \mathsf{d}\lambda_\rho + \sum_{p \in [\rho]} \mathsf{d}\lambda_p \wedge \phi_\rho &= \sum_{q \in [\rho^c]} \lambda_q \mathsf{d}\lambda_\rho + \sum_{p \in [\rho]} \mathsf{d}\lambda_p \wedge \epsilon(p, \rho-p)\lambda_p \mathsf{d}\lambda_{\rho-p} \\ &= \sum_{q \in [\rho^c]} \lambda_q \mathsf{d}\lambda_\rho + \sum_{p \in [\rho]} \lambda_p \mathsf{d}\lambda_\rho = \sum_{i=0}^{n} \lambda_i \mathsf{d}\lambda_\rho = \mathsf{d}\lambda_\rho,\end{aligned}$$

which had to be shown. □

The following identity describes an elementary linear dependence between Whitney forms of higher order; see also [4, Equation (6.5)] and [11, Proposition 3.3].

Lemma 3.3.
Let $k \in [0:n]$ and $\rho \in \Sigma_0(k,n)$. Then

$$\sum_{p \in [\rho]} \epsilon(p, \rho - p) \lambda_p \phi_{\rho - p} = 0. \tag{19}$$

Proof. Using (12), we expand the left-hand side of (19) to see

$$\begin{aligned}\sum_{p \in [\rho]} \epsilon(p, \rho - p) \lambda_p \phi_{\rho - p} &= \sum_{p \in [\rho]} \epsilon(p, \rho - p) \lambda_p \sum_{s \in [\rho - p]} \lambda_s \epsilon(s, \rho - p - s) \mathsf{d}\lambda_{\rho - p - s} \\ &= \sum_{\substack{p, s \in [\rho] \\ p \neq s}} \epsilon(p, \rho - p) \epsilon(s, \rho - p - s) \lambda_p \lambda_s \mathsf{d}\lambda_{\rho - p - s}.\end{aligned}$$

We have $\epsilon(s, \rho - p - s) = \epsilon(s, \rho - s)\epsilon(s, p)$ for $s, p \in [\rho]$ with $s \neq p$. The desired statement now follows by reasoning with antisymmetry of the summands. □

4. Finite Element Spaces

In this section we introduce two families of barycentric differential forms over simplices and find geometrically decomposed bases. Throughout this section, we let T be a simplex of dimension n, let $r \in \mathbb{N}_0$, and let $k \in [0:n]$.

In the sequel, we are particularly interested in the following two spaces of barycentric differential forms:

$$\mathcal{P}_r\Lambda^k(T) := \operatorname{span}\left\{ \lambda_T^\alpha \mathsf{d}\lambda_\sigma^T \;\middle|\; \alpha \in A(r,n),\ \sigma \in \Sigma(k,n) \right\}, \tag{20}$$

$$\mathcal{P}_r^-\Lambda^k(T) := \operatorname{span}\left\{ \lambda_T^\alpha \phi_\rho^T \;\middle|\; \alpha \in A(r-1,n),\ \rho \in \Sigma_0(k,n) \right\}. \tag{21}$$

We also consider subspaces of differential forms with vanishing traces:

$$\mathring{\mathcal{P}}_r\Lambda^k(T) := \left\{ \omega \in \mathcal{P}_r\Lambda^k(T) \;\middle|\; \forall F \subsetneq T : \operatorname{tr}_{T,F} \omega = 0 \right\}, \tag{22}$$

$$\mathring{\mathcal{P}}_r^-\Lambda^k(T) := \left\{ \omega \in \mathcal{P}_r^-\Lambda^k(T) \;\middle|\; \forall F \subsetneq T : \operatorname{tr}_{T,F} \omega = 0 \right\}. \tag{23}$$

It is evident that these spaces are nested, as follows from definitions and Lemma 3.2,

$$\mathcal{P}_r\Lambda^k(T) \subseteq \mathcal{P}_{r+1}^-\Lambda^k(T) \subseteq \mathcal{P}_{r+1}\Lambda^k(T),$$
$$\mathring{\mathcal{P}}_r\Lambda^k(T) \subseteq \mathring{\mathcal{P}}_{r+1}^-\Lambda^k(T) \subseteq \mathring{\mathcal{P}}_{r+1}\Lambda^k(T),$$

and that they are closed under taking traces: if $F \subseteq T$ is a subsimplex, then

$$\operatorname{tr}_{T,F} \mathcal{P}_r\Lambda^k(T) = \mathcal{P}_r\Lambda^k(F), \quad \operatorname{tr}_{T,F} \mathcal{P}_r^-\Lambda^k(T) = \mathcal{P}_r^-\Lambda^k(F).$$

We remark that our definitions (20) and (21) are different from but equivalent to the corresponding definitions in [3], as is easily checked.

4.1. Basis construction for $\mathcal{P}_r\Lambda^k(T)$ and $\mathring{\mathcal{P}}_r\Lambda^k(T)$.

In this subsection we study spanning sets and bases for the spaces $\mathcal{P}_r\Lambda^k(T)$ and $\mathring{\mathcal{P}}_r\Lambda^k(T)$. We introduce the sets of barycentric differential forms

$$\mathcal{S}\mathcal{P}_r\Lambda^k(T) := \left\{ \lambda_T^\alpha \mathsf{d}\lambda_\sigma^T \;\middle|\; \alpha \in A(r,n),\ \sigma \in \Sigma(k,n) \right\}, \tag{24}$$

$$\mathcal{S}\mathring{\mathcal{P}}_r\Lambda^k(T) := \left\{ \lambda_T^\alpha \mathsf{d}\lambda_\sigma^T \;\middle|\; \begin{array}{l} \alpha \in A(r,n),\ \sigma \in \Sigma(k,n), \\ [\alpha] \cup [\sigma] = [0:n] \end{array} \right\}. \tag{25}$$

Furthermore, under the restriction that $r \geq 1$, we consider the sets of barycentric differential forms

$$\mathcal{BP}_r\Lambda^k(T) := \left\{ \lambda_T^\alpha \mathsf{d}\lambda_\sigma^T \;\middle|\; \alpha \in A(r,n),\ \sigma \in \Sigma(k,n),\ \lfloor\alpha\rfloor \notin [\sigma] \right\}, \tag{26}$$

$$\mathcal{B}\mathring{\mathcal{P}}_r\Lambda^k(T) := \left\{ \lambda_T^\alpha \mathsf{d}\lambda_\sigma^T \;\middle|\; \begin{array}{c} \alpha \in A(r,n),\ \sigma \in \Sigma(k,n), \\ \lfloor\alpha\rfloor \notin [\sigma],\ [\alpha] \cup [\sigma] = [0:n] \end{array} \right\}. \tag{27}$$

We call $\mathcal{SP}_r\Lambda^k(T)$ the *canonical spanning set* of $\mathcal{P}_r\Lambda^k(T)$, and we call $\mathcal{S}\mathring{\mathcal{P}}_r\Lambda^k(T)$ the *canonical spanning set* of $\mathring{\mathcal{P}}_r\Lambda^k(T)$; these names are justified below. Evidently,

$$\mathcal{B}\mathring{\mathcal{P}}_r\Lambda^k(T) \subseteq \mathcal{S}\mathring{\mathcal{P}}_r\Lambda^k(T), \quad \mathcal{S}\mathring{\mathcal{P}}_r\Lambda^k(T) \subseteq \mathcal{SP}_r\Lambda^k(T),$$
$$\mathcal{B}\mathring{\mathcal{P}}_r\Lambda^k(T) \subseteq \mathcal{BP}_r\Lambda^k(T), \quad \mathcal{BP}_r\Lambda^k(T) \subseteq \mathcal{SP}_r\Lambda^k(T).$$

Suppose that $F \subseteq T$ is a subsimplex. From definitions it is clear that

$$\operatorname{tr}_{T,F} \mathcal{SP}_r\Lambda^k(T) = \mathcal{SP}_r\Lambda^k(F), \quad \operatorname{tr}_{T,F} \mathcal{BP}_r\Lambda^k(T) = \mathcal{BP}_r\Lambda^k(F).$$

In fact, the trace of any member of $\mathcal{SP}_r\Lambda^k(T)$ onto F is either zero or a member of $\mathcal{SP}_r\Lambda^k(F)$, and any member of $\mathcal{SP}_r\Lambda^k(F)$ has exactly one preimage under the trace in $\mathcal{SP}_r\Lambda^k(T)$. If $\lambda_T^\alpha \mathsf{d}\lambda_\sigma^T \in \mathcal{BP}_r\Lambda^k(T)$ with $[\alpha] \cup [\sigma] \subseteq [\imath(F,T)]$, then

$$\operatorname{tr}_{T,F} \lambda_T^\alpha \mathsf{d}\lambda_\sigma^T = \lambda_F^{\hat\alpha} \mathsf{d}\lambda_{\hat\sigma}^F \in \mathcal{BP}_r\Lambda^k(F),$$

where $\hat\alpha = \alpha \circ \imath(F,T)$ and $\hat\sigma = \imath(F,T)^\dagger \circ \sigma$. In turn, if $\lambda_F^\alpha \mathsf{d}\lambda_\sigma^F \in \mathcal{BP}_r\Lambda^k(F)$, then

$$\lambda_T^{\tilde\alpha} \mathsf{d}\lambda_{\tilde\sigma}^T \in \mathcal{BP}_r\Lambda^k(T), \quad \operatorname{tr}_{T,F} \lambda_T^{\tilde\alpha} \mathsf{d}\lambda_{\tilde\sigma}^T = \lambda_F^\alpha \mathsf{d}\lambda_\sigma^F,$$

where $\tilde\alpha = \alpha \circ \imath(F,T)^\dagger$ over $[\imath(F,T)]$ and zero otherwise, and where $\tilde\sigma = \imath(F,T) \circ \sigma$.

We call $\mathcal{SP}_r\Lambda^k(T)$ the canonical spanning set of $\mathcal{P}_r\Lambda^k(T)$ because

$$\mathcal{P}_r\Lambda^k(T) = \operatorname{span} \mathcal{SP}_r\Lambda^k(T)$$

by definition. However, $\mathcal{SP}_r\Lambda^k(T)$ is generally not linearly independent, and thus does not form a basis. However, its subset $\mathcal{BP}_r\Lambda^k(T)$ does.

Lemma 4.1.
Let $r \geq 1$. The set $\mathcal{BP}_r\Lambda^k(T)$ is a basis of $\mathcal{P}_r\Lambda^k(T)$.

Proof. The claim holds in the case $k = 0$, so let us assume that $k > 0$. First we show that $\mathcal{BP}_r\Lambda^k(T)$ spans $\mathcal{P}_r\Lambda^k(T)$. For any $\alpha \in A(r,n)$ and $\sigma \in \Sigma(k,n)$ with $\lfloor\alpha\rfloor \in [\sigma]$ we find

$$\begin{aligned}
\lambda^\alpha \mathsf{d}\lambda_\sigma^T &= \epsilon(\lfloor\alpha\rfloor, \sigma - \lfloor\alpha\rfloor) \lambda^\alpha \mathsf{d}\lambda_{\lfloor\alpha\rfloor}^T \wedge \mathsf{d}\lambda_{\sigma-\lfloor\alpha\rfloor}^T \\
&= -\epsilon(\lfloor\alpha\rfloor, \sigma - \lfloor\alpha\rfloor) \sum_{q \in [\sigma^c]} \lambda^\alpha \mathsf{d}\lambda_q^T \wedge \mathsf{d}\lambda_{\sigma-\lfloor\alpha\rfloor}^T \\
&= -\epsilon(\lfloor\alpha\rfloor, \sigma - \lfloor\alpha\rfloor) \sum_{q \in [\sigma^c]} \epsilon(q, \sigma - \lfloor\alpha\rfloor) \lambda^\alpha \mathsf{d}\lambda_{\sigma-\lfloor\alpha\rfloor+q}^T.
\end{aligned}$$

Hence $\mathcal{BP}_r\Lambda^k(T)$ is a spanning set. It remains to show that $\mathcal{BP}_r\Lambda^k(T)$ is linearly independent. Let $\omega \in \mathcal{P}_r\Lambda^k(T)$. Then there exist coefficients $\omega_{\alpha\sigma} \in \mathbb{C}$ such that

$$\omega = \sum_{\alpha \in A(r,n)} \sum_{\substack{\sigma \in \Sigma(k,n) \\ \lfloor\alpha\rfloor \notin [\sigma]}} \omega_{\alpha\sigma} \lambda_T^\alpha \mathsf{d}\lambda_\sigma^T.$$

Suppose that $\omega = 0$ while not all coefficients vanish. Consider the constant k-forms

$$V_\alpha := \sum_{\substack{\sigma \in \Sigma(k,n) \\ \lfloor\alpha\rfloor \notin [\sigma]}} \omega_{\alpha\sigma} \mathsf{d}\lambda_\sigma^T, \quad \alpha \in A(r,n).$$

For each $\alpha \in A(r,n)$ we have $V_\alpha = 0$ if and only if for all $\sigma \in \Sigma(k,n)$ with $\lfloor\alpha\rfloor \notin [\sigma]$ we have $\omega_{\alpha\sigma} = 0$. Since we assume that not all coefficients vanish, there exists $\alpha \in A(r,n)$ with $V_\alpha \neq 0$. Letting $\mathcal{V}_\alpha$ be the constant k-vector field dual to V_α,

$$0 = \omega(\mathcal{V}_\alpha) = \sum_{\beta \in A(r,n)} \lambda_T^\beta V_\beta(\mathcal{V}_\alpha) = \lambda_T^\alpha + \sum_{\substack{\beta \in A(r,n) \\ \beta \neq \alpha}} \lambda_T^\beta V_\beta(\mathcal{V}_\alpha).$$

But this contradicts the linear independence of the λ_T^α. Hence all coefficients must vanish. This shows linear independence, and thus completes the proof. □

The following result shows that the subset $\mathcal{B}\mathring{\mathcal{P}}_r\Lambda^k(T) \subseteq \mathcal{B}\mathcal{P}_r\Lambda^k(T)$ is a basis of subspace $\mathring{\mathcal{P}}_r\Lambda^k(T) \subseteq \mathcal{P}_r\Lambda^k(T)$. Moreover, it justifies why we call $\mathcal{S}\mathring{\mathcal{P}}_r\Lambda^k(T) \subseteq \mathcal{S}\mathcal{P}_r\Lambda^k(T)$ a canonical spanning set.

Theorem 4.2.
Let $r \geq 1$. The set $\mathcal{B}\mathring{\mathcal{P}}_r\Lambda^k(T)$ is a basis for $\mathring{\mathcal{P}}_r\Lambda^k(T)$, and $\mathcal{S}\mathring{\mathcal{P}}_r\Lambda^k(T)$ is a spanning set for that space.

Proof. Let $\omega \in \mathring{\mathcal{P}}_r\Lambda^k(T)$. Then $\omega \in \mathcal{P}_r\Lambda^k(T)$, and thus there exist unique coefficients $\omega_{\alpha\sigma} \in \mathbb{C}$ such that

$$\omega = \sum_{\alpha \in A(r,n)} \sum_{\substack{\sigma \in \Sigma(k,n) \\ \lfloor\alpha\rfloor \notin [\sigma]}} \omega_{\alpha\sigma} \lambda_T^\alpha \mathsf{d}\lambda_\sigma^T.$$

When F is a lower-dimensional subsimplex of T, then $0 = \operatorname{tr}_{T,F} \omega$ leads to

$$0 = \sum_{\substack{\alpha \in A(r,n) \\ \sigma \in \Sigma(k,n) \\ \lfloor\alpha\rfloor \notin [\sigma]}} \omega_{\alpha\sigma} \operatorname{tr}_{T,F} \lambda_T^\alpha \mathsf{d}\lambda_\sigma^T = \sum_{\substack{\alpha \in A(r,n) \\ \sigma \in \Sigma(k,n) \\ \lfloor\alpha\rfloor \notin [\sigma] \\ [\alpha]\cup[\sigma] \subseteq [\imath(F,T)]}} \omega_{\alpha\sigma} \lambda_F^{\alpha \circ \imath(F,T)} \mathsf{d}\lambda^F_{\imath(F,T)^\dagger \circ \sigma}.$$

Since the last sum runs over linearly independent differential forms, we thus find that $\omega_{\alpha\sigma} = 0$ for all $[\alpha] \cup [\sigma] \subseteq [\imath(F,T)]$. Since F was assumed to be an arbitrary proper subsimplex of T, we get that $\omega_{\alpha\sigma} = 0$ when $[\alpha] \cup [\sigma] \neq [0:n]$. So $\mathcal{B}\mathring{\mathcal{P}}_r\Lambda^k(T)$ is a spanning set of $\mathring{\mathcal{P}}_r\Lambda^k(T)$. It is linearly independent, being a subset of $\mathcal{B}\mathcal{P}_r\Lambda^k(T)$. Hence $\mathcal{S}\mathring{\mathcal{P}}_r\Lambda^k(T)$ is a spanning set, as claimed. □

We define an extension operator that facilitates a geometric decomposition. Whenever F is a subsimplex of T, we consider the operator

$$\operatorname{ext}^{r,k}_{F,T} : \mathring{\mathcal{P}}_r\Lambda^k(F) \to \mathcal{P}_r\Lambda^k(T),$$

which is defined by setting

$$\operatorname{ext}^{r,k}_{F,T} \lambda_F^\alpha \mathsf{d}\lambda_\sigma^F = \lambda_T^{\tilde\alpha} \mathsf{d}\lambda_{\tilde\sigma}^T, \quad \lambda_F^\alpha \mathsf{d}\lambda_\sigma^F \in \mathcal{B}\mathcal{P}_r\Lambda^k(F),$$

where $\tilde\alpha = \alpha \circ \imath(F,T)^\dagger$ over $[\imath(F,T)]$ and zero otherwise, and where $\tilde\sigma = \imath(F,T) \circ \sigma$.

We see that whenever $f \subseteq F$ is a subsimplex of F, then

$$\operatorname{tr}_{T,F} \operatorname{ext}^{r,k}_{f,T} = \operatorname{ext}^{r,k,-}_{f,F},$$

and that whenever $G \subset T$ is a subsimplex of T with $F \cap G = \emptyset$, then

$$\operatorname{tr}_{T,G} \operatorname{ext}^{r,k}_{F,T} = 0.$$

Remark 4.3.
We give a brief overview of the literature. Our basis $\mathcal{B}\mathring{\mathcal{P}}_r\Lambda^k(T)$ of $\mathring{\mathcal{P}}_r\Lambda^k(T)$ appears in [3] together with the same extension operators. Our basis $\mathcal{B}\mathcal{P}_r\Lambda^k(T)$ of $\mathcal{P}_r\Lambda^k(T)$, however, it is not explicitly described there even though it emerges naturally with their tools.

We remark that $\mathcal{B}\mathring{\mathcal{P}}_r\Lambda^k(T)$ can also be written as

$$\begin{aligned}\mathcal{B}\mathring{\mathcal{P}}_r\Lambda^k(T) &= \left\{ \lambda_T^\alpha \mathsf{d}\lambda_\sigma^T \;\middle|\; \begin{array}{l}\alpha \in A(r,n),\ \sigma \in \Sigma(k,n),\\ \lfloor\alpha\rfloor = \min([0:n]\setminus[\sigma]),\ [\alpha]\cup[\sigma] = [0:n]\end{array} \right\}\\ &= \left\{ \lambda_T^\alpha \mathsf{d}\lambda_\sigma^T \;\middle|\; \begin{array}{l}\alpha \in A(r,n),\ \sigma \in \Sigma(k,n),\\ \lfloor\alpha\rfloor \geq \min([0:n]\setminus[\sigma]),\ [\alpha]\cup[\sigma] = [0:n]\end{array} \right\}.\end{aligned}$$

To see this, suppose that $\alpha \in A(r,n)$ and $\sigma \in \Sigma(k,n)$ with $[\alpha]\cup[\sigma] = [0:n]$. We have equivalence of $\lfloor\alpha\rfloor \notin [\sigma]$ and $\lfloor\alpha\rfloor \in [0:n]\setminus[\sigma]$. And by $[\alpha]\cup[\sigma] = [0:n]$, we have $\lfloor\alpha\rfloor \in [0:n]\setminus[\sigma]$ if and only if $\lfloor\alpha\rfloor = \min([0:n]\setminus[\sigma])$ if and only if $\lfloor\alpha\rfloor \geq \min([0:n]\setminus[\sigma])$. In particular, we recover the basis description in Theorem 6.1 of [4]. The same basis of $\mathring{\mathcal{P}}\Lambda^k(T)$ is used implicitly in Theorem 4.22 of [3]. Furthermore, our extension operator is used in the seminal publication by Arnold, Falk, and Winther [3, p.56]. It differs from the extension operator in [4].

Remark 4.4.
Our basis for $\mathcal{P}_r\Lambda^k(T)$ does not seem to have been made explicit in the literature previously. We give some examples in the language of vector analysis. In the two-dimensional case, a basis for the Brezzi-Douglas-Marini space of polynomial degree r is

$$\left\{ \lambda_T^\alpha \nabla\lambda_p^T \;\middle|\; \alpha \in A(r,n),\ p \in \{0,1,2\},\ \lfloor\alpha\rfloor \neq p \right\}. \tag{28}$$

In the three-dimensional case, we consider the curl-conforming and the divergence-conforming Nédélec elements of the second kind of polynomial degree r. A basis for the former is

$$\left\{ \lambda_T^\alpha \nabla\lambda_p^T \;\middle|\; \alpha \in A(r,n),\ p \in \{0,1,2,3\},\ \lfloor\alpha\rfloor \neq p \right\} \tag{29}$$

and a basis for the latter is

$$\left\{ \lambda_T^\alpha \nabla\lambda_p^T \times \nabla\lambda_q^T \;\middle|\; \alpha \in A(r,n),\ p,q \in \{0,1,2,3\},\ p<q,\ \lfloor\alpha\rfloor \notin \{p,q\} \right\}. \tag{30}$$

These basis are already geometrically decomposed: for each member of these basis sets, the indices in the parameters $[\alpha]$, p, and q completely determine which subsimplex the corresponding member is associated with. For example, $\lambda_0\nabla\lambda_1^T$ and $\lambda_1\nabla\lambda_0^T$ are associated with the edge of T that contains the 0-th and the 1-st vertices, and $\lambda_0\lambda_1\nabla\lambda_2^T \times \nabla\lambda_3^T$ is associated with the entire tetrahedron T.

The basis $\mathcal{B}_r\Lambda^k(T)$ above allows for a geometric decomposition and can be therefore be used in the construction of basis for an entire finite element space. We outline another basis, which will be merely of technical interest in the next subsection. Consider the set of barycentric differential forms

$$\mathcal{B}_0\mathcal{P}_r\Lambda^k(T) := \left\{ \lambda_T^\alpha \mathsf{d}\lambda_\sigma^T \;\middle|\; \alpha \in A(r,n),\ \sigma \in \Sigma(k,n),\ \lfloor\sigma\rfloor > 0 \right\}. \tag{31}$$

That this is indeed a basis of $\mathcal{B}_0\mathcal{P}_r\Lambda^k(T)$ can be seen by transformation to a reference simplex. While $\mathcal{B}_0\mathcal{P}_r\Lambda^0(T)$ is easy to describe, it is not amenable for a geometric decomposition of the space $\mathcal{P}_r\Lambda^k(T)$. For example, restricting the elements of $\mathcal{B}_0\mathcal{P}_r\Lambda^k(T)$ to faces F of T produces the basis $\mathcal{B}_0\mathcal{P}_r\Lambda^k(F)$ generally only when F contains the 0-th vertex.

Remark 4.5.
Our basis $\mathcal{B}\mathcal{P}_r\Lambda^k(T)$ is generally different from $\mathcal{B}_0\mathcal{P}_r\Lambda^k(T)$. However, they adhere to the following same idea. If for each multiindex $\alpha \in A(r,n)$ we pick an index $j_\alpha \in [0:n]$, then a basis of $\mathcal{P}_r\Lambda^k(T)$ is given by the set

$$\left\{ \lambda_T^\alpha \mathsf{d}\lambda_\sigma^T \;\middle|\; \alpha \in A(r,n),\ \sigma \in \Sigma(k,n),\ j_\alpha \notin [\sigma] \right\}.$$

In the case $\mathcal{B}_0\mathcal{P}_r\Lambda^k(T)$ one always picks $j_\alpha = 0$ for every $\alpha \in A(r,n)$. In the case $\mathcal{B}\mathcal{P}_r\Lambda^k(T)$ one always picks $j_\alpha = \lfloor\alpha\rfloor$ for every $\alpha \in A(r,n)$. The basis $\mathcal{B}_0\mathcal{P}_r\Lambda^k(T)$ coincides with the basis given in Theorem 6.1 of [4] and also used implicitly in Theorem 4.16 of [3].

4.2. **Basis construction for $\mathcal{P}_r^-\Lambda^k(T)$ and $\mathring{\mathcal{P}}_r^-\Lambda^k(T)$.** This subsection follows a similar paths as the previous one. We study spanning sets and bases for the spaces $\mathcal{P}_r^-\Lambda^k(T)$ and $\mathring{\mathcal{P}}_r^-\Lambda^k(T)$. We introduce the sets of barycentric differential forms

$$\mathcal{S}\mathcal{P}_r^-\Lambda^k(T) := \left\{ \lambda_T^\alpha \phi_\rho^T \;\middle|\; \alpha \in A(r-1,n),\ \rho \in \Sigma_0(k,n) \right\}, \tag{32}$$

$$\mathcal{S}\mathring{\mathcal{P}}_r^-\Lambda^k(T) := \left\{ \lambda_T^\alpha \phi_\rho^T \;\middle|\; \begin{array}{l} \alpha \in A(r-1,n),\ \rho \in \Sigma_0(k,n), \\ [\alpha]\cup[\rho] = [0:n] \end{array} \right\}. \tag{33}$$

Furthermore, under the restriction that $r \geq 1$, we consider the sets of barycentric differential forms

$$\mathcal{B}\mathcal{P}_r^-\Lambda^k(T) := \left\{ \lambda_T^\alpha \phi_\rho^T \;\middle|\; \begin{array}{l} \alpha \in A(r-1,n),\ \rho \in \Sigma_0(k,n), \\ \lfloor\alpha\rfloor \geq \lfloor\rho\rfloor \end{array} \right\}, \tag{34}$$

$$\mathcal{B}\mathring{\mathcal{P}}_r^-\Lambda^k(T) := \left\{ \lambda_T^\alpha \phi_\rho^T \;\middle|\; \begin{array}{l} \alpha \in A(r-1,n),\ \rho \in \Sigma_0(k,n), \\ \lfloor\rho\rfloor = 0,\ [\alpha]\cup[\rho] = [0:n] \end{array} \right\}. \tag{35}$$

We call $\mathcal{S}\mathcal{P}_r^-\Lambda^k(T)$ the *canonical spanning set* of $\mathcal{P}_r^-\Lambda^k(T)$, and we call $\mathcal{S}\mathring{\mathcal{P}}_r^-\Lambda^k(T)$ the *canonical spanning set* of $\mathring{\mathcal{P}}_r^-\Lambda^k(T)$; again these names will be justified shortly. It is evident that

$$\mathcal{B}\mathring{\mathcal{P}}_r^-\Lambda^k(T) \subseteq \mathcal{S}\mathring{\mathcal{P}}_r^-\Lambda^k(T), \quad \mathcal{S}\mathring{\mathcal{P}}_r^-\Lambda^k(T) \subseteq \mathcal{S}\mathcal{P}_r^-\Lambda^k(T),$$
$$\mathcal{B}\mathring{\mathcal{P}}_r^-\Lambda^k(T) \subseteq \mathcal{B}\mathcal{P}_r^-\Lambda^k(T), \quad \mathcal{B}\mathcal{P}_r^-\Lambda^k(T) \subseteq \mathcal{S}\mathcal{P}_r^-\Lambda^k(T).$$

Suppose that $F \subseteq T$ is a subsimplex. From definitions it is clear that

$$\operatorname{tr}_{T,F} \mathcal{S}\mathcal{P}_r^-\Lambda^k(T) = \mathcal{S}\mathcal{P}_r^-\Lambda^k(F), \quad \operatorname{tr}_{T,F} \mathcal{B}\mathcal{P}_r^-\Lambda^k(T) = \mathcal{B}\mathcal{P}_r^-\Lambda^k(F).$$

In fact, the trace of any member of $\mathcal{S}\mathcal{P}_r^-\Lambda^k(T)$ onto F is either zero or a member of $\mathcal{S}\mathcal{P}_r^-\Lambda^k(F)$, and any member of $\mathcal{S}\mathcal{P}_r^-\Lambda^k(F)$ has exactly one preimage under the trace in $\mathcal{S}\mathcal{P}_r^-\Lambda^k(T)$. If $\lambda_T^\alpha\phi_\rho^T \in \mathcal{B}\mathcal{P}_r^-\Lambda^k(T)$ with $[\alpha]\cup[\rho] \subseteq [\imath(F,T)]$, then

$$\operatorname{tr}_{T,F} \lambda_T^\alpha \phi_\rho^T = \lambda_F^{\hat\alpha}\phi_{\hat\rho}^F \in \mathcal{B}\mathcal{P}_r^-\Lambda^k(F),$$

where $\hat\alpha = \alpha \circ \imath(F,T)$ and $\hat\rho = \imath(F,T)^\dagger \circ \rho$. In turn, if $\lambda_F^\alpha\phi_\rho^F \in \mathcal{B}\mathcal{P}_r^-\Lambda^k(F)$, then

$$\lambda_T^{\tilde\alpha}\phi_{\tilde\rho}^T \in \mathcal{B}\mathcal{P}_r^-\Lambda^k(T), \quad \operatorname{tr}_{T,F} \lambda_T^{\tilde\alpha}\phi_{\tilde\rho}^T = \lambda_F^\alpha\phi_\rho^F,$$

where $\tilde\alpha = \alpha \circ \imath(F,T)^\dagger$ over $[\imath(F,T)]$ and zero otherwise, and where $\tilde\rho = \imath(F,T)\circ\rho$.

We call $\mathcal{S}\mathcal{P}_r^-\Lambda^k(T)$ the canonical spanning set of $\mathcal{P}_r^-\Lambda^k(T)$ because by definition it is a spanning set for the higher-order Whitney forms,

$$\mathcal{P}_r^-\Lambda^k(T) = \operatorname{span} \mathcal{S}\mathcal{P}_r^-\Lambda^k(T).$$

However, $\mathcal{S}\mathcal{P}_r^-\Lambda^k(T)$ is generally not linearly independent, and thus does not form a basis. Analogously to the previous subsection, we show that its subset $\mathcal{B}\mathcal{P}_r^-\Lambda^k(T)$ is a basis, and we also show that $\mathcal{S}\mathring{\mathcal{P}}_r^-\Lambda^k(T)$ is a spanning set and that $\mathcal{B}\mathring{\mathcal{P}}_r^-\Lambda^k(T)$ is a basis of $\mathring{\mathcal{P}}_r^-\Lambda^k(T)$

Lemma 4.6.
The set $\mathcal{B}\mathcal{P}_r^-\Lambda^k(T)$ is a basis of $\mathcal{P}_r^-\Lambda^k(T)$. The set $\mathcal{B}\mathring{\mathcal{P}}_r^-\Lambda^k(T)$ is a basis of $\mathring{\mathcal{P}}_r^-\Lambda^k(T)$, and the set $\mathcal{S}\mathring{\mathcal{P}}_r^-\Lambda^k(T)$ is a spanning set for that space.

Proof. We first show that $\mathcal{B}\mathcal{P}_r^-\Lambda^k(T)$ spans $\mathcal{P}_r^-\Lambda^k(T)$. If $r = 1$, then $\mathcal{B}\mathcal{P}_r^-\Lambda^k(T) = \mathcal{S}\mathcal{P}_r^-\Lambda^k(T)$, so it remains to consider the case $r \geq 2$. Let $\alpha \in A(r-1, n)$ and $\rho \in \Sigma_0(k, n)$, let $p := \lfloor\alpha\rfloor$, and assume $p < \lfloor\rho\rfloor$. There exists $\beta \in A(r-2, n)$ with $\lambda_T^\alpha = \lambda_T^\beta \lambda_p^T$. Using Lemma 3.3, we find that

$$\lambda_T^\alpha \phi_\rho^T = \lambda_T^\beta \lambda_p^T \phi_\rho^T = \lambda_T^\beta \sum_{j=0}^{k} (-1)^j \lambda_{\rho(j)}^T \phi_{\rho+p-\rho(j)}^T.$$

Hence all members of $\mathcal{S}\mathcal{P}_r^-\Lambda^k(T)$ are linear combinations of members of $\mathcal{B}\mathcal{P}_r^-\Lambda^k(T)$.

Next we show that $\mathcal{B}\mathring{\mathcal{P}}_r^-\Lambda^k(T)$ is linearly independent. Let $\omega \in \mathcal{P}_r^-\Lambda^k(T)$ be in the span of $\mathcal{B}\mathring{\mathcal{P}}_r^-\Lambda^k(T)$. Thus we can write

$$\omega = \sum_{\alpha \in A(r-1,n)} \sum_{\substack{\rho \in \Sigma_0(k,n) \\ \lfloor\rho\rfloor = 0}} \omega_{\alpha\rho} \lambda_T^\alpha \phi_\rho^T$$

where $\omega_{\alpha\rho} \in \mathbb{C}$ for each $(\alpha, \rho) \in A(r-1, n) \times \Sigma_0(k, n)$. Hence $\omega = \omega_0 + \omega_+$, where

$$\begin{aligned}
\omega_0 &:= \sum_{\alpha \in A(r-1,n)} \sum_{\substack{\rho \in \Sigma_0(k,n) \\ \lfloor\rho\rfloor = 0}} \omega_{\alpha\rho} \lambda_T^\alpha \lambda_0^T \mathsf{d}\lambda_{\rho-0}^T, \\
\omega_+ &:= \sum_{\alpha \in A(r-1,n)} \sum_{\substack{\rho \in \Sigma_0(k,n) \\ \lfloor\rho\rfloor = 0}} \sum_{\substack{p \in [\rho] \\ p \neq 0}} \omega_{\alpha\rho} \epsilon(p, \rho - p) \lambda_T^\alpha \lambda_p^T \mathsf{d}\lambda_{\rho-p}^T \\
&= -\sum_{\alpha \in A(r-1,n)} \sum_{\substack{\rho \in \Sigma_0(k,n) \\ \lfloor\rho\rfloor = 0}} \sum_{\substack{p \in [\rho] \\ p \neq 0}} \sum_{i=1}^{n} \omega_{\alpha\rho} \epsilon(p, \rho - p) \lambda_T^{\alpha+p} \mathsf{d}\lambda_i^T \wedge \mathsf{d}\lambda_{\rho-p-0}^T \\
&= \sum_{\alpha \in A(r-1,n)} \sum_{\substack{\rho \in \Sigma_0(k,n) \\ \lfloor\rho\rfloor = 0}} \sum_{\substack{p \in [\rho] \\ p \neq 0}} \sum_{\substack{i \in [1:n] \\ i \notin [\rho-p]}} \omega_{\alpha\rho} \epsilon(p, \rho - p) \epsilon(i, \rho - p) \lambda_T^{\alpha+p} \mathsf{d}\lambda_{\rho-p-0+i}^T.
\end{aligned}$$

These differential forms are expressed in terms of $\mathcal{B}_0\mathcal{P}_r\Lambda^k(T)$. Suppose $\omega = 0$.

We use induction to prove that all $\omega_{\alpha\rho}$ vanish. First, it is evident that $\omega_{\alpha\rho} = 0$ for $\alpha(0) = r - 1$. Now let us assume that $s \in [1 : r-1]$ such that $\omega_{\alpha\rho} = 0$ for all $\alpha(0) \in [s : r-1]$. Since the terms $\lambda_T^\alpha \lambda_0^T$ with $\alpha(s) = s - 1$ in the definition of ω_0 always have a higher exponent in index 0 than the terms $\lambda_T^\alpha \lambda_p^T$ in the definition of ω_+, we conclude that $\omega_{\alpha\rho} = 0$ for $\alpha(s) = s - 1$. Repeating this argument yields $\omega_{\alpha\rho} = 0$ for all coefficients. Thus $\mathcal{B}\mathring{\mathcal{P}}_r^-\Lambda^k(T)$ is linearly independent.

It remains to show that $\mathcal{B}\mathring{\mathcal{P}}_r^-\Lambda^k(T)$ spans $\mathring{\mathcal{P}}_r^-\Lambda^k(T)$ and that $\mathcal{B}\mathcal{P}_r^-\Lambda^k(T)$ is linearly independent. We use induction over the dimension of T for both claims.

First, the two claims hold if $\dim T = k$ because $\mathcal{P}_r^-\Lambda^k(T) = \mathcal{P}_r\Lambda^k(T)$ and $\mathcal{B}\mathring{\mathcal{P}}_r^-\Lambda^k(T) = \mathcal{B}\mathcal{P}_r^-\Lambda^k(T)$ in that case.

Suppose that the two claims hold for simplices of dimension at most $m \geq k$ and that $\dim T = m + 1$. Let $\omega \in \mathcal{P}_r^-\Lambda^k(T)$, so there exist coefficients $\omega_{\alpha\rho} \in \mathbb{C}$ with

$$\omega = \sum_{\alpha \in A(r-1,n)} \sum_{\substack{\rho \in \Sigma_0(k,n) \\ \lfloor\alpha\rfloor \geq \lfloor\rho\rfloor}} \omega_{\alpha\rho} \lambda_T^\alpha \phi_\rho^T.$$

We prove that if $\omega \in \mathring{\mathcal{P}}_r^-\Lambda^k(T)$, then $\omega_{\alpha\rho} = 0$ for all $\alpha \in A(r-1, n)$ and $\sigma \in \Sigma_0(k, n)$ with $[\alpha] \cup [\rho] = [0 : n]$. Let us assume that $\omega \in \mathring{\mathcal{P}}_r^-\Lambda^k(T)$, and let F be any proper face of T. Then $0 = \operatorname{tr}_{T,F} \omega$ leads to

$$0 = \sum_{\substack{\alpha \in A(r-1,n) \\ \rho \in \Sigma_0(k,n) \\ \lfloor \alpha \rfloor \geq \lfloor \rho \rfloor}} \omega_{\alpha\rho} \operatorname{tr}_{T,F} \lambda_T^\alpha \phi_\rho^T = \sum_{\substack{\alpha \in A(r-1,n) \\ \rho \in \Sigma_0(k,n) \\ \lfloor \alpha \rfloor \geq \lfloor \rho \rfloor \\ [\alpha] \cup [\rho] \subseteq [\imath(F,T)]}} \omega_{\alpha\rho} \lambda_F^{\alpha \circ \imath(F,T)} \phi^F_{\imath(F,T)^\dagger \circ \rho}.$$

By the induction assumption, this expresses $0 = \operatorname{tr}_{T,F} \omega$ in terms of basis of $\mathcal{P}_r^- \Lambda^k(F)$. Hence $\omega_{\alpha\rho} = 0$ when $[\alpha] \cup [\rho] \subseteq [\imath(F,T)]$. Since F was assumed to be an abitrary proper face of T, we get that $\omega_{\alpha\rho} = 0$ when $[\alpha] \cup [\rho] \neq [0:n]$. So $\mathcal{B}\mathring{\mathcal{P}}_r^- \Lambda^k(T)$ spans $\mathring{\mathcal{P}}_r^- \Lambda^k(T)$. Thus $\mathcal{B}\mathring{\mathcal{P}}_r^- \Lambda^k(T)$ is a basis of $\mathring{\mathcal{P}}_r^- \Lambda^k(T)$, and $\mathcal{S}\mathring{\mathcal{P}}_r^- \Lambda^k(T)$ is a spanning set. Since $\omega = 0$ implies $\omega \in \mathring{\mathcal{P}}_r^- \Lambda^k(T)$, we now also see that $\mathcal{B}\mathcal{P}_r^- \Lambda^k(T)$ is linearly independent and thus a basis of $\mathcal{P}_r^- \Lambda^k(T)$. This completes the induction step, and the desired claim follows. □

Similar as before, we can define an extension operator that facilitates a geometric decomposition. Whenever F is a subsimplex of T, we consider the operator

$$\operatorname{ext}_{F,T}^{r,k,-} : \mathring{\mathcal{P}}_r^- \Lambda^k(F) \to \mathcal{P}_r^- \Lambda^k(T),$$

which is defined by setting

$$\operatorname{ext}_{F,T}^{r,k,-} \lambda_F^\alpha \phi_\rho^F = \lambda_T^{\tilde\alpha} \phi_{\tilde\rho}^T, \quad \lambda_F^\alpha \phi_\rho^F \in \mathcal{B}\mathcal{P}_r^- \Lambda^k(F),$$

where $\tilde\alpha = \alpha \circ \imath(F,T)^\dagger$ over $[\imath(F,T)]$ and zero otherwise, and where $\tilde\rho = \imath(F,T) \circ \rho$.

Similar as before, we note that whenever $f \subseteq F$ is a subsimplex of F, then

$$\operatorname{tr}_{T,F} \operatorname{ext}_{f,T}^{r,k,-} = \operatorname{ext}_{f,F}^{r,k,-},$$

and that whenever $G \subset T$ is a subsimplex of T with $F \cap G = \emptyset$, then

$$\operatorname{tr}_{T,G} \operatorname{ext}_{F,T}^{r,k,-} = 0.$$

Remark 4.7.
The bases for $\mathcal{P}_r^- \Lambda^k(T)$ and $\mathring{\mathcal{P}}_r^- \Lambda^k(T)$ are identical to the bases presented or implied in Section 4 of [3] (see Theorems 4.4 and 4.16 there) or in [4], which are all the same. Our extension operator coincides with the extension operator for the higher-order Whitney forms in [3].

Remark 4.8.
We illustrate the basis for the space of higher order Whitney forms $\mathcal{P}_r^- \Lambda^k(T)$ in the language of vector analysis. In the two-dimensional case, a basis for the Raviart-Thomas space containing up to polynomial degree $r-1$ is

$$\left\{ \lambda_T^\alpha \left(\lambda_p^T \nabla \lambda_q^T - \lambda_q^T \nabla \lambda_p^T \right) \;\middle|\; \begin{array}{l} \alpha \in A(r-1,n),\ p,q \in \{0,1,2\}, \\ p < q,\ \lfloor \alpha \rfloor \geq p \end{array} \right\}. \tag{36}$$

In the three-dimensional case, we consider the curl-conforming and the divergence-conforming Nédélec elements of the first kind of polynomial degree $r-1$. A basis for the former is

$$\left\{ \lambda_T^\alpha \left(\lambda_p^T \nabla \lambda_q^T - \lambda_q^T \nabla \lambda_p^T \right) \;\middle|\; \begin{array}{l} \alpha \in A(r-1,n),\ p,q \in \{0,1,2\}, \\ p < q,\ \lfloor \alpha \rfloor \geq p \end{array} \right\}. \tag{37}$$

and a basis for the latter is

$$\begin{aligned} \Big\{ & \lambda_T^\alpha \left(\lambda_p^T \nabla \lambda_q^T \times \nabla \lambda_s^T - \lambda_q^T \nabla \lambda_p^T \times \nabla \lambda_s^T + \lambda_s^T \nabla \lambda_p^T \times \nabla \lambda_q^T \right) \\ & \qquad \left| \begin{array}{l} \alpha \in A(r-1,n),\ p,q,s \in \{0,1,2,3\}, \\ p < q < s,\ \lfloor \alpha \rfloor \geq p \end{array} \right. \Big\}. \end{aligned} \tag{38}$$

5. Linear dependencies

We have previously encountered canonical spanning sets for the spaces of polynomial differential forms over a simplex T. The goal of this section is to improve our understanding of the linear dependencies of those spanning sets. As a byproduct, we improve our understanding of the isomorphisms

$$\mathcal{P}_r\Lambda^k(T) \simeq \mathring{\mathcal{P}}^-_{r+k}\Lambda^{n-k}(T), \quad \mathcal{P}^-_{r+1}\Lambda^{n-k}(T) \simeq \mathring{\mathcal{P}}_{r+n-k+1}\Lambda^k(T),$$

between the finite element spaces, which have been used earlier in [3].

Lemma 5.1.
Let $r \in \mathbb{N}_0$ and $k \in [0:n]$. Let $\omega_{\alpha\sigma} \in \mathbb{C}$ for $\sigma \in \Sigma(k,n)$ and $\alpha \in A(r,n)$. Then

$$\sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(k,n)}} \omega_{\alpha\sigma}\lambda^\alpha \mathsf{d}\lambda_\sigma = 0 \quad \Longleftrightarrow \quad \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(k,n)}} \epsilon(\sigma,\sigma^c)\omega_{\alpha\sigma}\lambda^\alpha\lambda_\sigma\phi_{\sigma^c} = 0, \tag{39}$$

each of which is the case if and only if

$$\omega_{\alpha\sigma} - \sum_{p\in[\sigma]} \epsilon(p,\sigma-p)\omega_{\alpha,\sigma-p+0} = 0 \tag{40}$$

holds for $\alpha \in A(r,n)$ and $\sigma \in \Sigma(k,n)$ with $0 \notin [\sigma]$.

Proof. The statement is trivial if $k = 0$, so assume that $1 \leq k \leq n$. Define

$$S_L := \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(k,n)}} \omega_{\alpha\sigma}\lambda^\alpha\mathsf{d}\lambda_\sigma = \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(k,n)\\ 0\notin[\sigma]}} \omega_{\alpha\sigma}\lambda^\alpha\mathsf{d}\lambda_\sigma + \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(k,n)\\ 0\in[\sigma]}} \omega_{\alpha\sigma}\lambda^\alpha\mathsf{d}\lambda_\sigma.$$

For $\sigma \in \Sigma(k,n)$ with $0 \in [\sigma]$ we observe

$$\mathsf{d}\lambda_\sigma = \mathsf{d}\lambda_0 \wedge \mathsf{d}\lambda_{\sigma-0} = -\sum_{q\in[\sigma^c]} \mathsf{d}\lambda_q \wedge \mathsf{d}\lambda_{\sigma-0} = \sum_{q\in[\sigma^c]} \epsilon(q,\sigma)\mathsf{d}\lambda_{\sigma-0+q},$$

Direct application of this observation gives

$$\begin{aligned}
S_L &= \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(k,n)\\ 0\notin[\sigma]}} \omega_{\alpha\sigma}\lambda^\alpha\mathsf{d}\lambda_\sigma + \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(k,n)\\ 0\in[\sigma]}} \omega_{\alpha\sigma}\lambda^\alpha \sum_{q\in[\sigma^c]} \epsilon(q,\sigma)\mathsf{d}\lambda_{\sigma-0+q}\\
&= \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(k,n)\\ 0\notin[\sigma]}} \left(\omega_{\alpha\sigma} + \sum_{p\in[\sigma]} \epsilon(p,\sigma-p+0)\omega_{\alpha,\sigma-p+0} \right) \lambda^\alpha\mathsf{d}\lambda_\sigma\\
&= \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(k,n)\\ 0\notin[\sigma]}} \left(\omega_{\alpha\sigma} - \sum_{p\in[\sigma]} \epsilon(p,\sigma-p)\omega_{\alpha,\sigma-p+0} \right) \lambda^\alpha\mathsf{d}\lambda_\sigma.
\end{aligned}$$

This is an expression in a basis of $\mathcal{P}_r\Lambda^k(T)$. On the other hand, define S_R by

$$\begin{aligned}
S_R &:= \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(k,n)}} \epsilon(\sigma,\sigma^c)\omega_{\alpha\sigma}\lambda^\alpha\lambda_\sigma\phi_{\sigma^c}\\
&= \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(k,n)\\ 0\notin[\sigma]}} \epsilon(\sigma,\sigma^c)\omega_{\alpha\sigma}\lambda^\alpha\lambda_\sigma\phi_{\sigma^c} + \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(k,n)\\ 0\in[\sigma]}} \epsilon(\sigma,\sigma^c)\omega_{\alpha\sigma}\lambda^\alpha\lambda_\sigma\phi_{\sigma^c}.
\end{aligned}$$

Using Lemma 3.3, for $\sigma \in \Sigma(k,n)$ with $0 \in [\sigma]$ we observe

$$\lambda_\sigma \phi_{\sigma^c} = \lambda_{\sigma-0}\lambda_0\phi_{\sigma^c} = \lambda_{\sigma-0} \sum_{q\in[\sigma^c]} \epsilon(q,\sigma)\lambda_q\phi_{\sigma^c-q+0}.$$

Using previous observations, we calculate that S_R equals

$$\sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(k,n)\\ 0\notin[\sigma]}} \left(\epsilon(\sigma,\sigma^c)\omega_{\alpha\sigma} + \sum_{p\in[\sigma]} \epsilon(\sigma-p+0,\sigma^c+p-0)\epsilon(p,\sigma^c-0)\omega_{\alpha,\sigma-p+0} \right) \lambda^\alpha\lambda_\sigma\phi_{\sigma^c}.$$

This is an expression in terms of a basis of $\mathring{\mathcal{P}}^-_{r+k}\Lambda^k(T)$. Note that

$$\begin{aligned} \epsilon(\sigma-p+0,\sigma^c+p-0)\epsilon(p,\sigma^c-0) &= (-1)^{k+1}\epsilon(\sigma-p,\sigma^c+p)\epsilon(p,\sigma^c) \\ &= -\epsilon(\sigma,\sigma^c)\epsilon(p,\sigma-p) \end{aligned}$$

for $\sigma \in \Sigma(k,n)$, $p \in [\sigma]$ and $0 \notin [\sigma]$. Thus $S_L = 0$ if and only if $S_R = 0$, which is the case if and only if (40) holds. This completes the proof. □

Lemma 5.2.
Let $r \in \mathbb{N}_0$ and $k \in [0:n]$. Let $\omega_{\alpha\sigma} \in \mathbb{C}$ for $\sigma \in \Sigma(k,n)$ and $\alpha \in A(r,n)$. Then

$$\sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(k,n)}} \epsilon(\sigma,\sigma^c)\omega_{\alpha\sigma}\lambda^\alpha\phi_{\sigma^c} = 0 \iff \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(k,n)}} \omega_{\alpha\sigma}\lambda^\alpha\lambda_{\sigma^c}\mathsf{d}\lambda_\sigma = 0, \tag{41}$$

each of which is the case if and only if

$$\omega_{\alpha\sigma} - \sum_{q\in[\sigma]\cap[\alpha]} \epsilon(\lfloor\sigma^c\rfloor,\sigma-q)\epsilon(q,\sigma-q)\omega_{\alpha+\lfloor\sigma^c\rfloor-q,\sigma+\lfloor\sigma^c\rfloor-q} = 0 \tag{42}$$

holds for $\alpha \in A(r,n)$ and $\sigma \in \Sigma(k,n)$ with $\lfloor\alpha\rfloor \geq \lfloor\sigma^c\rfloor$.

Proof. If $r = 0$, then the two sums in (41) are already stated in terms of bases of $\mathcal{P}^-_{r+1}\Lambda^{n-k}(T)$ and $\mathring{\mathcal{P}}_{r+n-k+1}\Lambda^k(T)$, and (42) just reduces to all coefficients vanishing. So it remains to study the case $r \geq 1$. In the special case $k = 0$, the statement is trivial. So let us assume $k > 0$. We define S_L by setting

$$S_L := \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(k,n)}} \epsilon(\sigma,\sigma^c)\omega_{\alpha\sigma}\lambda^\alpha\phi_{\sigma^c}.$$

Using Lemma 3.3, for each $\sigma \in \Sigma(k,n)$ and $\alpha \in A(r,n)$ with $\lfloor\alpha\rfloor < \lfloor\sigma^c\rfloor$ we have

$$\lambda^\alpha\phi_{\sigma^c} = \lambda^{\alpha-\lfloor\alpha\rfloor}\lambda_{\lfloor\alpha\rfloor}\phi_{\sigma^c} = \sum_{q\in[\sigma^c]} \epsilon(q,\sigma^c-q)\lambda^{\alpha-\lfloor\alpha\rfloor+q}\phi_{\sigma^c+\lfloor\alpha\rfloor-q}.$$

Therefore we can rewrite S_L as

$$\begin{aligned} &\sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(k,n)\\ \lfloor\alpha\rfloor\geq\lfloor\sigma^c\rfloor}} \epsilon(\sigma,\sigma^c)\omega_{\alpha\sigma}\lambda^\alpha\phi_{\sigma^c} + \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(k,n)\\ \lfloor\alpha\rfloor<\lfloor\sigma^c\rfloor}} \epsilon(\sigma,\sigma^c)\omega_{\alpha\sigma}\lambda^\alpha\phi_{\sigma^c} \\ &= \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(k,n)\\ \lfloor\alpha\rfloor\geq\lfloor\sigma^c\rfloor}} \epsilon(\sigma,\sigma^c)\omega_{\alpha\sigma}\lambda^\alpha\phi_{\sigma^c} + \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(k,n)\\ \lfloor\alpha\rfloor<\lfloor\sigma^c\rfloor\\ q\in[\sigma^c]}} \epsilon(\sigma,\sigma^c)\epsilon(q,\sigma^c-q)\omega_{\alpha\sigma}\lambda^{\alpha-\lfloor\alpha\rfloor+q}\phi_{\sigma^c+\lfloor\alpha\rfloor-q}. \end{aligned}$$

Let $\sigma \in \Sigma(k,n)$, $\alpha \in A(r,n)$ and $q \in [\sigma^c]$ with $\lfloor\alpha\rfloor < \lfloor\sigma^c\rfloor$. We set $\beta = \alpha - \lfloor\alpha\rfloor + q$ and $\rho = \sigma - \lfloor\alpha\rfloor + q$. Then $\lfloor\beta\rfloor \geq \lfloor\alpha\rfloor = \lfloor\rho^c\rfloor$, thus $\beta + \lfloor\rho^c\rfloor - q = \alpha$ and

$\rho + \lfloor \rho^c \rfloor - q = \sigma$. Hence $q \in [\rho]$ and $q \in [\beta]$. Based on these observations,

$$\sum_{\substack{\alpha \in A(r,n) \\ \sigma \in \Sigma(k,n) \\ \lfloor \alpha \rfloor < \lfloor \sigma^c \rfloor}} \sum_{q \in [\sigma^c]} \epsilon(\sigma, \sigma^c) \epsilon(q, \sigma^c - q) \omega_{\alpha\sigma} \lambda^{\alpha - \lfloor \alpha \rfloor + q} \phi_{\sigma^c + \lfloor \alpha \rfloor - q}$$
$$= \sum_{\substack{\beta \in A(r,n) \\ \rho \in \Sigma(k,n) \\ \lfloor \beta \rfloor \geq \lfloor \rho^c \rfloor \\ q \in [\rho] \cap [\beta]}} \epsilon(\rho + \lfloor \rho^c \rfloor - q, \rho^c - \lfloor \rho^c \rfloor + q) \epsilon(q, \rho^c - \lfloor \rho^c \rfloor) \omega_{\beta + \lfloor \rho^c \rfloor - q, \rho + \lfloor \rho^c \rfloor - q} \lambda^{\beta} \phi_{\rho^c}.$$

For $\rho \in \Sigma(k,n)$, $\beta \in A(r,n)$ and $q \in [\rho] \cap [\beta]$ such that $\lfloor \beta \rfloor \geq \lfloor \rho^c \rfloor$, we make the combinatorial observation

$$\begin{aligned} &\epsilon(\rho + \lfloor \rho^c \rfloor - q, \rho^c - \lfloor \rho^c \rfloor + q) \epsilon(q, \rho^c - \lfloor \rho^c \rfloor) \\ &\quad = -\epsilon(\rho, \rho^c) \epsilon(\rho - q, q) \epsilon(\lfloor \rho^c \rfloor, \rho^c - \lfloor \rho^c \rfloor) \epsilon(\rho - q, \lfloor \rho^c \rfloor) \\ &\quad = -\epsilon(\rho, \rho^c) \epsilon(\rho - q, q) \epsilon(\rho - q, \lfloor \rho^c \rfloor). \end{aligned}$$

It can thus be seen that S_L equals

$$\sum_{\substack{\beta \in A(r,n) \\ \rho \in \Sigma(k,n) \\ \lfloor \beta \rfloor \geq \lfloor \rho^c \rfloor}} \epsilon(\rho, \rho^c) \left(\omega_{\beta\rho} - \sum_{q \in [\rho] \cap [\beta]} \epsilon(\rho - q, q) \epsilon(\rho - q, \lfloor \rho^c \rfloor) \omega_{\beta + \lfloor \rho^c \rfloor - q, \rho + \lfloor \rho^c \rfloor - q} \right) \lambda^{\beta} \phi_{\rho^c}.$$

This an expression in terms of a basis of $\mathcal{P}^-_{r+1}\Lambda^{n-k}(T)$. Now, define S_R by

$$S_R := \sum_{\substack{\alpha \in A(r,n) \\ \sigma \in \Sigma(k,n)}} \omega_{\alpha\sigma} \lambda^{\alpha} \lambda_{\sigma^c} \mathsf{d}\lambda_{\sigma}.$$

For any $\alpha \in A(r,n)$ and $\sigma \in \Sigma(k,n)$ with $\lfloor \alpha \rfloor < \lfloor \sigma^c \rfloor$ we see

$$\begin{aligned} \lambda^{\alpha} \lambda_{\sigma^c} \mathsf{d}\lambda_{\sigma} &= \lambda^{\alpha} \lambda_{\sigma^c} \epsilon(\lfloor \alpha \rfloor, \sigma - \lfloor \alpha \rfloor) \mathsf{d}\lambda_{\lfloor \alpha \rfloor} \wedge \mathsf{d}\lambda_{\sigma - \lfloor \alpha \rfloor} \\ &= \omega_{\alpha\sigma} \lambda^{\alpha} \lambda_{\sigma^c} \epsilon(\lfloor \alpha \rfloor, \sigma - \lfloor \alpha \rfloor) \epsilon(q, \sigma - \lfloor \alpha \rfloor) \mathsf{d}\lambda_{\sigma - \lfloor \alpha \rfloor + q}. \end{aligned}$$

Hence

$$- \sum_{\substack{\alpha \in A(r,n) \\ \sigma \in \Sigma(k,n) \\ \lfloor \alpha \rfloor < \lfloor \sigma^c \rfloor}} \omega_{\alpha\sigma} \lambda^{\alpha} \lambda_{\sigma^c} \mathsf{d}\lambda_{\sigma}$$
$$= \sum_{\substack{\alpha \in A(r,n) \\ \sigma \in \Sigma(k,n) \\ \lfloor \alpha \rfloor < \lfloor \sigma^c \rfloor}} \sum_{\substack{q \in [\sigma^c] \\ q \neq \lfloor \alpha \rfloor}} \omega_{\alpha\sigma} \lambda^{\alpha} \lambda_{\sigma^c} \epsilon(\lfloor \alpha \rfloor, \sigma - \lfloor \alpha \rfloor) \epsilon(q, \sigma - \lfloor \alpha \rfloor) \mathsf{d}\lambda_{\sigma - \lfloor \alpha \rfloor + q}.$$

Arguing similarly as above, we see that the last expression is identical to

$$\sum_{\substack{\beta \in A(r,n) \\ \rho \in \Sigma(k,n) \\ \lfloor \beta \rfloor \geq \lfloor \rho^c \rfloor \\ q \in [\rho] \cap [\beta]}} \omega_{\beta + \lfloor \rho^c \rfloor - q, \rho + \lfloor \rho^c \rfloor - q} \lambda^{\beta + \lfloor \rho^c \rfloor - q} \lambda_{\rho^c - \lfloor \rho^c \rfloor + q} \epsilon(\lfloor \rho^c \rfloor, \rho - q) \epsilon(q, \rho - q) \mathsf{d}\lambda_{\rho}.$$

Note that we can simplify $\lambda^{\beta + \lfloor \rho^c \rfloor - q} \lambda_{\rho^c - \lfloor \rho^c \rfloor + q} = \lambda^{\beta} \lambda_{\rho^c}$ for each $\beta \in A(r,n)$, $\rho \in \Sigma(k,n)$, and $q \in [\rho] \cap [\beta]$ with $\lfloor \beta \rfloor \geq \lfloor \rho^c \rfloor$ in this sum. Consequently, we see that S_R equals

$$\sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(k,n)\\ \lfloor\alpha\rfloor\geq\lfloor\sigma^c\rfloor}} \left(\omega_{\alpha\sigma} - \sum_{q\in[\sigma]\cap[\alpha]} \epsilon(\lfloor\sigma^c\rfloor, \sigma - q)\epsilon(q, \sigma - q)\omega_{\alpha+\lfloor\sigma^c\rfloor-q,\sigma+\lfloor\sigma^c\rfloor-q} \right) \lambda^\alpha \lambda_{\sigma^c} \mathsf{d}\lambda_\sigma.$$

This is an expression in terms of a basis of $\mathring{\mathcal{P}}_{r+n-k+1}\Lambda^k(T)$. Thus $S_L = 0$ if and only if $S_R = 0$, which is the case if and only if (42) holds. The proof is complete. □

The point of these results is that we have a correspondence between the linear dependencies of the canonical spanning sets of $\mathcal{P}_r\Lambda^k(T)$ and $\mathring{\mathcal{P}}^-_{r+k+1}\Lambda^{n-k}(T)$, and a correspondence between the linear dependencies of the canonical spanning sets of $\mathring{\mathcal{P}}_{r+n-k+1}\Lambda^k(T)$ and $\mathcal{P}^-_{r+1}\Lambda^{n-k}(T)$.

An immediate application is the well-definedness of the following isomorphisms. There exists a linear isomorphism from $\mathcal{P}_r\Lambda^k(T)$ to $\mathring{\mathcal{P}}^-_{r+k+1}\Lambda^{n-k}(T)$ that in terms of coefficients can be written as

$$\sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(k,n)}} \omega_{\alpha\sigma}\lambda^\alpha \mathsf{d}\lambda_\sigma \mapsto \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(k,n)}} \omega_{\alpha\sigma}\lambda^\alpha\lambda_\sigma\phi_{\sigma^c} \tag{43}$$

and we have a linear isomorphism from $\mathcal{P}^-_{r+1}\Lambda^{n-k}(T)$ to $\mathring{\mathcal{P}}_{r+n-k+1}\Lambda^k(T)$, that in terms of coefficients can be written as

$$\sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(k,n)}} \omega_{\alpha\sigma}\lambda^\alpha \phi_{\sigma^c} \mapsto \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(k,n)}} \omega_{\alpha\sigma}\lambda^\alpha\lambda_{\sigma^c}\mathsf{d}\lambda_\sigma \tag{44}$$

That these mappings are indeed well-defined follows immediately from Lemma 5.1 and Lemma 5.2. We refer to Remark 5.5 below for an example.

We give two more auxiliary results, Lemma 5.3 and Lemma 5.4, which are stated and proven below. They give conditions on the coefficients that are equivalent to the ones encountered in the previous two lemmas, but which seem more "natural" than the latter. This not only rounds up the theory, but will also be instrumental in the next section.

Lemma 5.3.
Let $r \in \mathbb{N}_0$ and $k \in [0 : n]$. Let $\omega_{\alpha\sigma}$ be a family of complex numbers indexed over $\alpha \in A(r,n)$ and $\sigma \in \Sigma(k,n)$. We have that

$$\omega_{\alpha\sigma} - \sum_{p\in[\sigma]} \epsilon(p, \sigma - p)\omega_{\alpha,\sigma-p+0} = 0 \tag{45}$$

holds for all $\alpha \in A(r,n)$ and $\sigma \in \Sigma(k,n)$ with $0 \notin [\sigma]$ if and only

$$\sum_{p\in[\theta]} \epsilon(p, \theta - p)\omega_{\alpha,\theta-p} = 0 \tag{46}$$

holds for all $\alpha \in A(r,n)$ and $\theta \in \Sigma(k+1,n)$.

Proof. The lemma is trivial in the special case $k = 0$. So let us assume that $1 \leq k \leq n$. Clearly, the second claim implies the first. So let us suppose the first claim holds. Then the second claim holds for all θ with $0 \in [\theta]$. If instead $0 \notin [\theta]$, then we find

$$\sum_{p\in[\theta]} \epsilon(p,\theta-p)\omega_{\alpha,\theta-p} = \sum_{p\in[\theta]}\sum_{s\in[\theta-p]} \epsilon(p,\theta-p)\epsilon(s,\theta-p-s)\omega_{\alpha,\theta-p-s+0}$$
$$= \sum_{p\in[\theta]}\sum_{s\in[\theta-p]} \epsilon(p,s)\epsilon(p,\theta-p)\epsilon(s,\theta-p-s)\omega_{\alpha,\theta-p-s+0}.$$

This sum vanishes as follows by antisymmetry. The lemma follows. □

Lemma 5.4.
Let $r\in\mathbb{N}_0$ and $k\in[0:n]$. Let $\omega_{\alpha\sigma}$ be a family of complex numbers indexed over $\alpha\in A(r,n)$ and $\sigma\in\Sigma(k,n)$. We have that

$$\omega_{\alpha\sigma} - \sum_{q\in[\sigma]\cap[\alpha]} \epsilon(\lfloor\sigma^c\rfloor,\sigma-q)\epsilon(q,\sigma-q)\omega_{\alpha+\lfloor\sigma^c\rfloor-q,\sigma+\lfloor\sigma^c\rfloor-q} = 0$$

holds for all $\alpha\in A(r,n)$ and $\sigma\in\Sigma(k,n)$ with $\lfloor\alpha\rfloor\geq\lfloor\sigma\rfloor$ if and only

$$\sum_{p\in[\theta]\cap[\beta]} \epsilon(\theta-p,p)\omega_{\beta-p,\theta-p} = 0$$

holds for all $\beta\in A(r+1,n)$ and $\theta\in\Sigma(k+1,n)$.

Proof. The lemma is trivial in the special case $k=0$. So let us assume that $1\leq k\leq n$. The first condition has several equivalent formulations:

$$\omega_{\alpha\sigma} - \sum_{q\in[\sigma]\cap[\alpha]} \epsilon(\lfloor\sigma^c\rfloor,\sigma-q)\epsilon(q,\sigma-q)\omega_{\alpha+\lfloor\sigma^c\rfloor-q,\sigma+\lfloor\sigma^c\rfloor-q} = 0$$
$$\iff \quad \omega_{\alpha\sigma} - \sum_{q\in[\sigma]\cap[\alpha]} \epsilon(\lfloor\sigma^c\rfloor,q)\epsilon(\lfloor\sigma^c\rfloor,\sigma)\epsilon(q,\sigma-q)\omega_{\alpha+\lfloor\sigma^c\rfloor-q,\sigma+\lfloor\sigma^c\rfloor-q} = 0$$
$$\iff \quad \epsilon(\lfloor\sigma^c\rfloor,\sigma)\omega_{\alpha\sigma} + \sum_{q\in[\sigma]\cap[\alpha]} \epsilon(q,\lfloor\sigma^c\rfloor)\epsilon(q,\sigma-q)\omega_{\alpha+\lfloor\sigma^c\rfloor-q,\sigma+\lfloor\sigma^c\rfloor-q} = 0$$
$$\iff \quad \epsilon(\lfloor\sigma^c\rfloor,\sigma)\omega_{\alpha\sigma} + \sum_{q\in[\sigma]\cap[\alpha]} \epsilon(q,\sigma+\lfloor\sigma^c\rfloor-q)\omega_{\alpha+\lfloor\sigma^c\rfloor-q,\sigma+\lfloor\sigma^c\rfloor-q} = 0$$
$$\iff \quad \sum_{q\in[\sigma+\lfloor\sigma^c\rfloor]\cap[\alpha+\lfloor\sigma^c\rfloor]} \epsilon(q,\sigma+\lfloor\sigma^c\rfloor-q)\omega_{\alpha+\lfloor\sigma^c\rfloor-q,\sigma+\lfloor\sigma^c\rfloor-q} = 0.$$

It is now obvious that the second condition implies the first condition.

Let us assume in turn that the first condition holds, and derive the second condition. From the first condition we conclude that the second condition already holds for $\beta\in A(r+1,n)$ and $\theta\in\Sigma(k+1,n)$ for which there exists $\sigma\in\Sigma(k,n)$ and $\alpha\in A(r,n)$ such that $\theta=\sigma+\lfloor\sigma^c\rfloor$ and $\beta=\alpha+\lfloor\sigma^c\rfloor$.

But since $0\in[\sigma]\cup[\sigma^c]$, we know that $\theta=\sigma+\lfloor\sigma^c\rfloor$ if and only if $0\in[\theta]$ and $\lfloor\sigma^c\rfloor=0$. So it remains to show the second condition for the case $0\notin[\theta]\cap[\beta]$. For such θ and β, we find

$$\sum_{p\in[\theta]\cap[\beta]} \epsilon(\theta-p,p)\omega_{\beta-p,\theta-p}$$
$$= -\sum_{p\in[\theta]\cap[\beta]}\sum_{s\in[\theta]\cap[\beta]\setminus\{p\}} \epsilon(\theta-p,p)\epsilon(s,\theta-p+0-s)\omega_{\beta-p+0-s,\theta-p+0-s},$$

using the first condition. But with the combinatorial observation

$$\epsilon(\theta-p,p)\epsilon(s,\theta-p+0-s) = \epsilon(\theta+0-p,p)\epsilon(s,\theta-p+0-s)$$
$$= -\epsilon(\theta+0-p,p)\epsilon(s,p)\epsilon(s,\theta+0-s)$$

we conclude that the sum vanishes if and only if

$$0 = \sum_{\substack{s,p \in [\theta] \cap [\beta] \\ p \neq s}} \epsilon(\theta + 0 - p, p) \epsilon(s, p) \epsilon(s, \theta + 0 - s) \omega_{\beta - p + 0 - s, \theta - p + 0 - s}.$$

This holds because the terms in the sum cancel. The statement is proven. □

Remark 5.5.
The results of this section show the correspondence of linear independencies between finite element spaces: a basis for one space is induced by one and only one basis for the other space.

Note that the first identity in Lemma 5.1 is already contained Proposition 3.7 of [11]. The latter reference, however, does not state further details about the conditions on the coefficients. Our analogous result in Lemma 5.2 is a natural analogue of their result and has not appeared previously in the literature.

The isomorphism (43) is identical to the isomorphism used in Theorem 4.16 of [3], and the isomorphism (44) is identical to the isomorphism used in Theorem 4.22 of [3]. In that reference, the isomorphisms are only stated in terms of basis forms. We emphasize that the isomorphisms can be stated naturally in terms of the canonical spanning sets.

6. Duality Pairings

We have seen in the last section that there exist isomorphisms

$$\mathcal{P}_r \Lambda^k(T) \simeq \mathring{\mathcal{P}}^-_{r+k+1} \Lambda^{n-k}(T), \quad \mathcal{P}^-_{r+1} \Lambda^{n-k}(T) \simeq \mathring{\mathcal{P}}_{r+n-k+1} \Lambda^k(T).$$

In this section, we extend those results and introduce a non-degenerate bilinear pairings between the spaces $\mathcal{P}_r \Lambda^k(T)$ and $\mathring{\mathcal{P}}^-_{r+k+1} \Lambda^{n-k}(T)$, and between the spaces $\mathring{\mathcal{P}}_{r+n-k+1} \Lambda^k(T)$ and $\mathcal{P}^-_{r+1} \Lambda^{n-k}(T)$. We begin with a technical auxiliary result.

Lemma 6.1.
Let $r \in \mathbb{N}_0$ and $k \in [0:n]$. Let $\sigma, \rho \in \Sigma(k,n)$. Then

(47)
$$\mathsf{d}\lambda_\sigma \wedge \phi_{\rho^c} = \begin{cases} 0 & \text{if} \quad |[\sigma] \cap [\rho^c]| > 1, \\ (-1)^k \epsilon(\sigma, \sigma^c) \sum_{q \in [\sigma^c]} \lambda_q \phi_T & \text{if} \quad [\sigma] \cap [\rho^c] = \emptyset, \\ (-1)^{k+1} \epsilon(\rho, \rho^c) \epsilon(p, \sigma - p) \epsilon(q, \sigma - p) \lambda_p \phi_T & \text{if} \quad |[\sigma] \cap [\rho^c]| = 1, \end{cases}$$

where in the last case $q \in [\sigma^c]$ and $p \in [\sigma]$ are the unique solutions of $\rho = \sigma - p + q$.

Proof. Let $\sigma, \rho \in \Sigma(k,n)$, so $\rho^c \in \Sigma_0(n-k,n)$. Exactly one of the cases on the right-hand side of (47) is true.

Firstly, suppose that $|[\sigma] \cap [\rho^c]| > 1$. Then it is easy to verify that

$$\mathsf{d}\lambda_\sigma \wedge \phi_{\rho^c} = 0.$$

This can be seen by expanding the Whitney form ϕ_{ρ^c} according to (12) and using the properties of the alternating product.

Secondly, suppose that $[\sigma] \cap [\rho^c] = \emptyset$. This is equivalent to $|[\sigma] \cap [\rho^c]| = 0$ and, in particular, to $\sigma = \rho$. We see, using (12), (15) and Lemma 3.2, that

$$\begin{aligned}\mathsf{d}\lambda_\sigma \wedge \phi_{\sigma^c} &= \mathsf{d}\lambda_\sigma \wedge \sum_{q \in [\sigma^c]} \lambda_q \epsilon(q, \sigma^c - q) \mathsf{d}\lambda_{\sigma^c - q} \\ &= \sum_{q \in [\sigma^c]} \lambda_q \epsilon(q, \sigma^c - q)\epsilon(\sigma, \sigma^c - q) \mathsf{d}\lambda_{\sigma + \sigma^c - q} \\ &= \sum_{q \in [\sigma^c]} \lambda_q \epsilon(q, \sigma^c - q)\epsilon(\sigma, \sigma^c - q)\epsilon(q, \sigma + \sigma^c - q)\phi_T.\end{aligned}$$

From the combinatorial observation that

$$\epsilon(q, \sigma^c - q)\epsilon(\sigma, \sigma^c - q)\epsilon(q, \sigma + \sigma^c - q) = (-1)^k \epsilon(\sigma, \sigma^c),$$

we conclude the desired expression for $\mathsf{d}\lambda_\sigma \wedge \phi_{\sigma^c}$ in the second case.

Lastly, suppose that $|[\sigma] \cap [\rho^c]| = 1$. There exists a unique $p \in [\sigma] \cap [\rho^c]$. Then there exists a unique $q \in [\sigma^c] \cap [\rho]$ such that $\rho = \sigma - p + q$ and $\rho^c = \sigma^c - q + p$. We see that the right-hand side of (47) is well-defined. We have $[\sigma] \cap [\rho^c] = \{p\}$ and $[\sigma^c] \cap [\rho] = \{q\}$. We find, similar as above, that

$$\begin{aligned}\mathsf{d}\lambda_\sigma \wedge \phi_{\rho^c} &= \mathsf{d}\lambda_\sigma \wedge \phi_{\sigma^c - q + p} \\ &= \epsilon(p, \sigma^c - q)\lambda_p \mathsf{d}\lambda_\sigma \wedge \mathsf{d}\lambda_{\sigma^c - q} \\ &= \epsilon(p, \sigma^c - q)\epsilon(\sigma, \sigma^c - q)\lambda_p \mathsf{d}\lambda_{\sigma + \sigma^c - q} \\ &= \epsilon(p, \sigma^c - q)\epsilon(\sigma, \sigma^c - q)\epsilon(q, \sigma + \sigma^c - q)\lambda_p \phi_T \\ &= (-1)^k \epsilon(p, \sigma^c - q)\epsilon(\sigma, \sigma^c)\epsilon(q, \sigma^c - q)\lambda_p \phi_T.\end{aligned}$$

With the combinatorial observation

$$\begin{aligned}&\epsilon(\sigma - p + q, \sigma^c - q + p) \\ &\quad = \epsilon(\sigma, \sigma^c)\epsilon(\sigma - p, p)\epsilon(q, \sigma^c - q)(-1)\epsilon(\sigma - p, q)\epsilon(p, \sigma^c - q),\end{aligned}$$

we derive

$$\begin{aligned}&(-1)^k \epsilon(p, \sigma^c - q)\epsilon(\sigma, \sigma^c)\epsilon(q, \sigma^c - q) \\ &\quad = (-1)^{k+1} \epsilon(\sigma - p + q, \sigma^c - q + p)\epsilon(\sigma - p, p)\epsilon(\sigma - p, q) \\ &\quad = (-1)^{k+1} \epsilon(\sigma - p + q, \sigma^c - q + p)\epsilon(p, \sigma - p)\epsilon(q, \sigma - p).\end{aligned}$$

This, together with $\rho = \sigma - p + q$, leads to the identity

$$\begin{aligned}\mathsf{d}\lambda_\sigma \wedge \epsilon(\rho, \rho^c)\phi_{\rho^c} &= (-1)^{k+1} \epsilon(\sigma - p + q, \sigma^c - q + p)\epsilon(p, \sigma - p)\epsilon(q, \sigma - p)\lambda_p \phi_T \\ &= (-1)^{k+1} \epsilon(\rho, \rho^c)\epsilon(p, \sigma - p)\epsilon(q, \sigma - p)\lambda_p \phi_T.\end{aligned}$$

The proof is complete. □

Lemma 6.2.
Let $r \in \mathbb{N}_0$ and $k \in [0 : n]$. Let $\sigma, \rho \in \Sigma(k, n)$. Then

$$\mathsf{d}\lambda_\sigma \wedge \epsilon(\rho, \rho^c)\lambda_\rho \phi_{\rho^c} = \mathsf{d}\lambda_\rho \wedge \epsilon(\sigma, \sigma^c)\lambda_\sigma \phi_{\sigma^c}. \tag{48}$$

Moreover, we have

$$\mathsf{d}\lambda_\sigma \wedge \epsilon(\sigma, \sigma^c)\lambda_\sigma \phi_{\sigma^c} = (-1)^k \lambda_\sigma \sum_{q \in [\sigma^c]} \lambda_q \phi_T. \tag{49}$$

and, if $\rho = \sigma - p + q$ for $p \in [\sigma]$ and $q \in [\sigma^c]$, then we have

$$\mathsf{d}\lambda_\sigma \wedge \epsilon(\rho, \rho^c)\lambda_\rho \phi_{\rho^c} = (-1)^{k+1} \epsilon(p, \sigma - p)\epsilon(q, \sigma - p)\lambda_\rho \lambda_p \phi_T. \tag{50}$$

Proof. We use Lemma 6.1 above. Firstly, if $[\sigma] \cap [\rho] = \emptyset$, then we obtain (48) by

$$\mathsf{d}\lambda_\sigma \wedge \lambda_\rho \phi_{\rho^c} = \mathsf{d}\lambda_\rho \wedge \lambda_\sigma \phi_{\sigma^c} = 0.$$

Secondly, if $\sigma = \rho$, then (48) holds trivially and (49) is an easy observation.

Lastly, consider the case $|[\sigma] \cap [\rho]| = 1$. In that case, there exist $p \in [\sigma]$ and $q \in [\sigma^c]$ such that $\rho = \sigma - p + q$. Then

$$\mathsf{d}\lambda_\sigma \wedge \epsilon(\rho, \rho^c)\lambda_\rho \phi_{\rho^c} = (-1)^{k+1}\epsilon(p, \sigma - p)\epsilon(q, \sigma - p)\lambda_\rho \lambda_p \phi_T \tag{51}$$

on the one hand, proving (50), while

$$\begin{aligned} \mathsf{d}\lambda_\rho \wedge \epsilon(\sigma, \sigma^c)\lambda_\sigma \phi_{\sigma^c} &= (-1)^{k+1}\epsilon(q, \rho - q)\epsilon(p, \rho - q)\lambda_\sigma \lambda_q \phi_T \\ &= (-1)^{k+1}\epsilon(p, \sigma - p)\epsilon(q, \sigma - p)\lambda_\rho \lambda_p \phi_T \end{aligned} \tag{52}$$

on the other hand. The identity (48) follows. The proof is complete. □

Without much further ado, we give our first main result in this section:

Theorem 6.3.
Let $r \in \mathbb{N}_0$ and $k \in [0 : n]$. Let $\omega_{\alpha\sigma}$ be a family of complex numbers indexed over $\alpha \in A(r, n)$ and $\sigma \in \Sigma(k, n)$. Then we have

$$\begin{aligned} &\sum_{\alpha,\beta \in A(r,n)} \sum_{\sigma,\rho \in \Sigma(k,n)} \int_T \omega_{\alpha\sigma} \lambda^\alpha \mathsf{d}\lambda_\sigma \wedge \epsilon(\rho, \rho^c)\overline{\omega_{\beta\rho}} \lambda^\beta \lambda_\rho \phi_{\rho^c} \\ &\quad = (-1)^k \sum_{\theta \in \Sigma(k+1,n)} \int_T \lambda_\theta \Big| \sum_{\alpha \in A(r,n)} \sum_{p \in [\theta]} \epsilon(p, \theta - p)\lambda^\alpha \omega_{\alpha,\theta-p} \Big|^2. \end{aligned} \tag{53}$$

In particular, this term is zero if and only one of the equivalent conditions of Lemma 5.1 and Lemma 5.3 is satisfied.

Proof. For the proof, we introduce some additional notation. Let us write

$$S(\theta, \alpha, \omega) := \sum_{p \in [\theta]} \epsilon(p, \theta - p)\omega_{\alpha,\theta-p}, \quad \theta \in \Sigma(k+1, n), \quad \alpha \in A(r, n).$$

We write $S(\omega)$ for the left-hand side of (53), and we moreover write

$$S_d(\omega) := \sum_{\substack{\alpha,\beta \in A(r,n) \\ \sigma \in \Sigma(k,n)}} \int_T \lambda^{\alpha+\beta} \omega_{\alpha\sigma} \overline{\omega_{\beta\sigma}} \lambda_\sigma \mathsf{d}\lambda_\sigma \wedge \epsilon(\sigma, \sigma^c)\phi_{\sigma^c}$$

$$S_o(\omega) := \sum_{\substack{\alpha,\beta \in A(r,n) \\ \sigma,\rho \in \Sigma(k,n) \\ \sigma \neq \rho}} \int_T \lambda^{\alpha+\beta} \omega_{\alpha\sigma} \overline{\omega_{\beta\rho}} \mathsf{d}\lambda_\sigma \wedge \epsilon(\rho, \rho^c)\lambda_\rho \phi_{\rho^c}.$$

So $S(\omega) = S_d(\omega) + S_o(\omega)$ splits into a *diagonal part* $S_d(\omega)$ and an *off-diagonal part* $S_o(\omega)$. We apply our previous observations and find that $S(\omega)$ equals

$$\sum_{\substack{\alpha,\beta \in A(r,n) \\ \sigma \in \Sigma(k,n) \\ q \in [\sigma^c]}} \int_T \lambda^{\alpha+\beta} (-1)^k \lambda_{\sigma+q} \omega_{\alpha\sigma} \left(\overline{\omega_{\beta\sigma}} - \sum_{p \in [\sigma]} \epsilon(p, \sigma - p)\epsilon(q, \sigma - p)\overline{\omega_{\beta,\sigma-p+q}} \right) \phi_T.$$

With the combinatorial observation

$$\begin{aligned} \epsilon(p, \sigma - p)\epsilon(q, \sigma - p) &= \epsilon(p, \sigma + q - p)\epsilon(p, q)\epsilon(q, \sigma)\epsilon(q, p) \\ &= -\epsilon(p, \sigma + q - p)\epsilon(\sigma, q), \end{aligned}$$

we simplify this sum further to

$$\sum_{\substack{\alpha,\beta\in A(r,n)\\ \sigma\in\Sigma(k,n)\\ q\in[\sigma^c]}} \int_T \lambda^{\alpha+\beta}(-1)^k \lambda_{\sigma+q}\omega_{\alpha\sigma}\epsilon(q,\sigma)\left(\sum_{p\in[\sigma+q]} \epsilon(p,\sigma-p+q)\overline{\omega_{\beta,\sigma+q-p}}\right)\phi_T$$
$$= \sum_{\substack{\alpha,\beta\in A(r,n)\\ \sigma\in\Sigma(k,n)\\ q\in[\sigma^c]}} \int_T \lambda^{\alpha+\beta}(-1)^k \lambda_{\sigma+q}\omega_{\alpha\sigma}\epsilon(q,\sigma)\overline{S(\sigma+q,\beta,\omega)}\phi_T.$$

This leads to

$$\begin{aligned} S(\omega) &= (-1)^k \sum_{\alpha,\beta\in A(r,n)} \int_T \lambda^{\alpha+\beta} \sum_{\theta\in\Sigma(k+1,n)} \lambda_\theta \sum_{p\in[\theta]} \omega_{\alpha,\theta-p}\epsilon(p,\theta-p)\overline{S(\theta,\beta,\omega)}\phi_T \\ &= (-1)^k \sum_{\theta\in\Sigma(k+1,n)} \int_T \lambda_\theta \sum_{\alpha,\beta\in A(r,n)} \lambda^{\alpha+\beta} S(\theta,\alpha,\omega)\overline{S(\theta,\beta,\omega)}\phi_T \\ &= (-1)^k \sum_{\theta\in\Sigma(k+1,n)} \int_T \lambda_\theta \Big| \sum_{\alpha\in A(r,n)} \lambda^\alpha S(\theta,\alpha,\omega)\Big|^2 \phi_T. \end{aligned}$$

The integrand is non-negative. Hence the integral vanishes if and only if for all $\theta \in \Sigma(k+1,n)$ we have

$$0 = \sum_{\alpha\in A(r,n)} \lambda^\alpha S(\theta,\alpha,\omega).$$

Since the λ^α are linearly independent for $\alpha \in A(r,n)$, this holds if and only if one of the equivalent conditions of Lemma 5.1 and Lemma 5.3 is satisfied. □

Theorem 6.4.
Let $r \in \mathbb{N}_0$ and $k \in [0:n]$. Let $\omega_{\alpha\sigma}$ be a family of complex numbers indexed over $\alpha \in A(r,n)$ and $\sigma \in \Sigma(k,n)$. Then we have

$$\begin{aligned} &\sum_{\alpha,\beta\in A(r,n)} \sum_{\sigma,\rho\in\Sigma(k,n)} \int_T \omega_{\alpha\sigma}\lambda^\alpha\lambda_{\sigma^c}\mathsf{d}\lambda_\sigma \wedge \epsilon(\rho,\rho^c)\omega_{\beta\rho}\lambda^\beta\phi_{\rho^c} \\ &\quad = (-1)^k \sum_{\theta\in\Sigma(k+1,n)} \int_T \lambda_{\theta^c}\Big| \sum_{\alpha\in A(r,n)} \sum_{p\in[\theta]} \epsilon(p,\theta-p)\lambda^\alpha\lambda_p\omega_{\alpha,\theta-p}\Big|^2. \end{aligned} \tag{54}$$

In particular, this term is zero if and only one of the equivalent conditions of Lemma 5.2 and Lemma 5.4 is satisfied.

Proof. Let us write $S(\omega)$ for the left-hand side in the equality (54). We can split that sum into two parts. On the one hand, for the *diagonal part*,

$$\begin{aligned} S_d(\omega) &:= \sum_{\substack{\alpha,\beta\in A(r,n)\\ \sigma\in\Sigma(k,n)}} \int_T \omega_{\alpha\sigma}\lambda^\alpha\lambda_{\sigma^c}\mathsf{d}\lambda_\sigma \wedge \epsilon(\sigma,\sigma^c)\overline{\omega_{\beta\sigma}}\lambda^\beta\phi_{\sigma^c} \\ &= \sum_{\substack{\alpha,\beta\in A(r,n)\\ \sigma\in\Sigma(k,n)}} \int_T \omega_{\alpha\sigma}\overline{\omega_{\beta\sigma}}\lambda^{\alpha+\beta}\lambda_{\sigma^c}(-1)^k \sum_{q\in[\sigma]} \lambda_q\phi_T, \end{aligned}$$

while on the other hand, for the *off-diagonal part*,

$$S_o(\omega) := \sum_{\substack{\alpha,\beta\in A(r,n)\\ \sigma,\rho\in\Sigma(k,n)\\ \sigma\neq\rho}} \int_T \omega_{\alpha\sigma}\lambda^\alpha \lambda_{\sigma^c}\mathsf{d}\lambda_\sigma \wedge \epsilon(\rho,\rho^c)\overline{\omega_{\beta\rho}}\lambda^\beta\phi_{\rho^c}$$

$$= \sum_{\substack{\sigma\in\Sigma(k,n)\\ \alpha,\beta\in A(r,n)\\ p\in[\sigma]\\ q\in[\sigma^c]}} \int_T \omega_{\alpha\sigma}\overline{\omega_{\beta,\sigma-p+q}}\lambda^{\alpha+\beta}\lambda_{\sigma^c}(-1)^{k+1}\epsilon(p,\sigma-p)\epsilon(q,\sigma-p)\lambda_p\phi_T.$$

Since $S(\omega) = S_d(\omega) + S_o(\omega)$, we combine that $(-1)^k S(\omega)$ equals

$$\sum_{\substack{\alpha,\beta\in A(r,n)\\ \sigma\in\Sigma(k,n)\\ q\in[\sigma^c]}} \int_T \lambda^{\alpha+\beta}\omega_{\alpha\sigma}\lambda_{\sigma^c}\left(\overline{\omega_{\beta\rho}}\lambda_q - \sum_{p\in[\sigma]}\epsilon(p,\sigma-p)\epsilon(q,\sigma-p)\overline{\omega_{\beta,\sigma-p+q}}\lambda_p\right)\phi_T$$

$$= \sum_{\substack{\alpha,\beta\in A(r,n)\\ \sigma\in\Sigma(k,n)\\ q\in[\sigma^c]}} \int_T \lambda^{\alpha+\beta}\omega_{\alpha\sigma}\lambda_{\sigma^c}\epsilon(q,\sigma)\left(\sum_{p\in[\sigma+q]}\epsilon(p,\sigma-p+q)\overline{\omega_{\beta,\sigma-p+q}}\lambda_p\right)\phi_T$$

$$= \sum_{\substack{\alpha,\beta\in A(r,n)\\ \theta\in\Sigma(k+1,n)\\ p\in[\theta^c]}} \int_T \lambda^{\alpha+\beta}\epsilon(p,\theta-p)\omega_{\alpha,\theta-p}\lambda_{\theta^c}\lambda_p\left(\sum_{p\in[\theta]}\epsilon(p,\theta-p)\overline{\omega_{\beta,\theta-p}}\lambda_p\right)\phi_T$$

$$= \sum_{\theta\in\Sigma(k+1,n)} \int_T \lambda_{\theta^c}\left|\sum_{\alpha\in A(r,n)}\sum_{p\in[\theta]}\epsilon(p,\theta-p)\omega_{\alpha,\theta-p}\lambda_\alpha\lambda_p\right|^2\phi_T$$

$$= \sum_{\theta\in\Sigma(k+1,n)} \int_T \lambda_{\theta^c}\left|\sum_{\beta\in A(r+1,n)}\sum_{p\in[\theta]}\epsilon(\theta-p,p)\omega_{\beta-p,\theta-p}\lambda_\beta\right|^2\phi_T.$$

The integrand is non-negative. Moreover, we see that it vanishes if and only if the conditions of Lemma 5.2 and Lemma 5.4. This completes the proof. □

Remark 6.5.
A careful inspection of the foregoing proofs shows that the statements of Theorems 6.3 and 6.4 remain true even with the integral sign removed.

We apply the former two theorems in our study of duality pairings between spaces of finite element differential forms. Let us write

$$\mathcal{P}(r,k,n) := \mathbb{C}^{A(r,n)\times\Sigma(k,n)}$$

for the abstract complex vector space generated by the set $A(r,n)\times\Sigma(k,n)$. The members of that vector space represent the coefficients in linear combinations of the canonical spanning sets.

We have a bilinear form over $\mathcal{P}(r,k,n)$ which for $\omega,\eta\in\mathcal{P}(r,k,n)$ is given by

$$(\omega,\eta)\mapsto \sum_{\alpha,\beta\in A(r,n)}\sum_{\sigma,\rho\in\Sigma(k,n)}\int_T \omega_{\alpha\sigma}\lambda^\alpha\mathsf{d}\lambda_\sigma\wedge\epsilon(\rho,\rho^c)\overline{\eta_{\beta\rho}}\lambda^\beta\lambda_\rho\phi_{\rho^c},$$

and another bilinear form over $\mathcal{P}(r,k,n)$ which for $\omega,\eta\in\mathcal{P}(r,k,n)$ is given by

$$(\omega,\eta)\mapsto \sum_{\alpha,\beta\in A(r,n)}\sum_{\sigma,\rho\in\Sigma(k,n)}\int_T \omega_{\alpha\sigma}\lambda^\alpha\lambda_{\sigma^c}\mathsf{d}\lambda_\sigma\wedge\epsilon(\rho,\rho^c)\overline{\eta_{\beta\rho}}\lambda^\beta\phi_{\rho^c}.$$

Theorems 6.3 and 6.4 have the following implications. We see that these bilinear forms are symmetric and semi-definite. In fact, they are positive semidefinite for k even and negative semidefinite for k odd.

The degeneracy space of the first bilinear form is exactly the linear subspace of $\mathcal{P}(r,k,n)$ spanned by those coefficient vectors that satisfy the conditions of Lemma 5.1 and Lemma 5.3. In particular, it follows that the bilinear form

$$(\omega,\eta) \mapsto \int_T \omega \wedge \eta, \quad \omega \in \mathcal{P}_r\Lambda^k(T), \quad \eta \in \mathring{\mathcal{P}}^-_{r+k+1}\Lambda^{n-k}(T), \tag{55}$$

is non-degenerate.

The degeneracy space of the first bilinear form is exactly the linear subspace of $\mathcal{P}(r,k,n)$ spanned by those coefficient vectors that satisfy the conditions of Lemma 5.2 and Lemma 5.4. In particular, it follows that the bilinear form

$$(\omega,\eta) \mapsto \int_T \omega \wedge \eta, \quad \omega \in \mathring{\mathcal{P}}_{r+n-k+1}\Lambda^k(T), \quad \eta \in \mathcal{P}^-_{r+1}\Lambda^{n-k}(T), \tag{56}$$

is non-degenerate.

Remark 6.6.
Theorem 6.3 refines Proposition 3.7 in [11], while Theorem 6.4 states the natural but hitherto unpublished analogue for the second isomorphism relation. Our first duality pairing is also used in Lemma 4.11 of [3], whereas our second duality pairing is utilized in Lemma 4.7 of [3].

7. Geometric Decompositions and Degrees of Freedom

In this section we describe geometric decompositions of finite element spaces and construct the degrees of freedom. Most concepts and results of this section are, at least in principle, already known in the literature; we include them for the completion of exposition and because the results of the previous section allow us to construct the degrees of freedom in a manner different from prior expositions.

Throughout this section, we let $\mathcal{T}$ be a collection of simplices satisfying the following conditions: (i) for every $T \in \mathcal{T}$ and every subsimplex $F \subseteq T$ we have $F \in \mathcal{T}$, (ii) for every two $T, T' \in \mathcal{T}$ we either have $T \cap T' = \emptyset$ or $T \cap T' \in \mathcal{T}$, (iii) we have $\dim T \leq n$ for every $T \in \mathcal{T}$.

We formulate our results within an abstract framework. We assume to be given $X^k(T) \subseteq \Lambda^k(T)$ for each cell $T \in \mathcal{T}$ such that for every $F, T \in \mathcal{T}$ with $F \subseteq T$ we have the surjectivity condition $\operatorname{tr}_{T,F} X^k(T) = X^k(F)$. We write $\mathring{X}^k(T)$ for the subspace of forms with vanishing boundary traces:

$$\mathring{X}^k(T) = \left\{\, \omega \in X^k(T) \;\middle|\; \forall F \in \mathcal{T}, F \subsetneq T : \operatorname{tr}_{T,F}\omega = 0 \,\right\}. \tag{57}$$

Let us abbreviate $X^k_{-1}(\mathcal{T}) := \bigoplus_{T \in \mathcal{T}, \dim T = n} X^k(T)$ for the direct sum of vector spaces associated to the n-simplices. We say that $\omega \in X^k_{-1}(\mathcal{T})$ is *single-valued* if for all n-dimensional simplices $T, T' \in \mathcal{T}$ with non-empty intersection $F = T \cap T'$ we have $\operatorname{tr}_{T,F}\omega_T = \operatorname{tr}_{T',F}\omega_{T'}$. The single-valued members of $X^k_{-1}(\mathcal{T})$ constitute a vector space on their own that we denote by $X^k(\mathcal{T})$.

The definition suggests a natural way to define the trace of any $\omega \in X^k(\mathcal{T})$ onto any simplex $F \in \mathcal{T}$. We introduce the *global trace operators*

$$\operatorname{Tr}_{\mathcal{T},F} : X^k(\mathcal{T}) \to X^k(F). \tag{58}$$

Remark 7.1.
Let $r \in \mathbb{N}$ and $k \in [0:n]$. We consider two prototypical instances of our abstract framework. On the one hand, we have the full spaces of barycentric polynomial

differential forms, where $X^k(T) = \mathcal{P}_r\Lambda^k(T)$ and $\mathring{X}^k(T) = \mathring{\mathcal{P}}_r\Lambda^k(T)$ for each $T \in \mathcal{T}$. Here, $\mathcal{P}_r\Lambda^k(\mathcal{T}) = X(\mathcal{T})$ is common notation. On the other hand, we have the spaces of higher order Whitney forms, where $X^k(T) = \mathcal{P}_r^-\Lambda^k(T)$ and $\mathring{X}^k(T) = \mathring{\mathcal{P}}_r^-\Lambda^k(T)$ for each $T \in \mathcal{T}$. In this case, $\mathcal{P}_r^-\Lambda^k(\mathcal{T}) = X(\mathcal{T})$ is common notation. From these examples we see that $X^k(\mathcal{T})$ captures the idea of a *conforming* finite element space.

Our abstract framework relies on extension operators. For all $F, T \in \mathcal{T}$ with $F \subseteq T$ we assume to have a linear mapping

$$\operatorname{ext}_{F,T} : \mathring{X}^k(F) \to X^k(T).$$

We assume that these are generalized inverses of the trace operators,

$$\operatorname{tr}_{T,F} \operatorname{ext}_{F,T} \omega = \omega, \quad \omega \in \mathring{X}^k(F), \tag{59}$$

and satisfy the two conditions

$$\operatorname{ext}_{F,G} \omega = \operatorname{tr}_{T,G} \operatorname{ext}_{F,T} \omega, \quad \omega \in \mathring{X}^k(F), \quad F \subseteq G \subseteq T, \quad F, G, T \in \mathcal{T}, \tag{60}$$

$$\operatorname{tr}_{T,G} \operatorname{ext}_{F,T} \omega = 0, \quad \omega \in \mathring{X}^k(F), \quad F, G \subseteq T, \quad F \nsubseteq G, \quad F, G, T \in \mathcal{T}. \tag{61}$$

The identity (60) formalizes that extensions to different simplices have the same trace on common subsimplices, while the identity (61) formalizes that the extension is local in the sense that the extension has zero trace on all simplices of $\mathcal{T}$ that do not contain the original simplex.

Under these assumptions, we easily verify that the *global extension operators*

$$\operatorname{Ext}_{F,\mathcal{T}} : \mathring{X}^k(F) \to X^k(\mathcal{T}), \quad \omega_F \mapsto \sum_{\substack{F,T \in \mathcal{T} \\ F \subseteq T,\, \dim T = n}} \operatorname{ext}_{F,T} \omega_F, \tag{62}$$

are well-defined. We can state this section's main result.

Theorem 7.2.
Suppose that $\omega \in X^k(\mathcal{T})$. Then there exist unique $\omega_F \in \mathring{X}^k(F)$ for every $F \in \mathcal{T}$ such that

$$\omega = \sum_{F \in \mathcal{T}} \operatorname{Ext}_{F,\mathcal{T}} \omega_F. \tag{63}$$

Proof. Let $\omega \in X^k(\mathcal{T})$. We prove the statement of the theorem by a recursion argument. We let $\omega_V := \operatorname{Tr}_{\mathcal{T},V} \omega \in \mathring{X}^k(V)$ for every vertex $V \in \mathcal{T}$ of the simplicial complex. Set

$$\omega^{(0)} := \sum_{V \in \mathcal{T},\, \dim T = 0} \operatorname{Ext}_{V,\mathcal{T}} \omega_V.$$

Then $\operatorname{Tr}_{\mathcal{T},V}(\omega - \omega^{(0)}) = 0$ for every 0-dimensional $V \in \mathcal{T}$.

Now assume that for some $m \in [0 : n-1]$ the following holds: for every $F \in \mathcal{T}$ of dimension at most m there exists $\omega_F \in \mathring{X}^k(F)$ such that, letting

$$\omega^{(m)} := \sum_{F \in \mathcal{T},\, \dim F \leq m} \operatorname{Ext}_{F,\mathcal{T}} \omega_F,$$

we have $\operatorname{Tr}_{\mathcal{T},F}(\omega - \omega^{(m)}) = 0$ for every $F \in \mathcal{T}$ of dimension at most m. Now, for every $F \in \mathcal{T}$ of dimension $m+1$ we set $\omega_F := \operatorname{Tr}_{\mathcal{T},F} \omega \in \mathring{X}^k(F)$ for every $F \in \mathcal{T}$ of dimension at most $m+1$. Letting

$$\omega^{(m+1)} := \sum_{F \in \mathcal{T},\, \dim F \leq m+1} \operatorname{Ext}_{F,\mathcal{T}} \omega_F,$$

it follows that $\operatorname{Tr}_{\mathcal{T},F}(\omega - \omega^{(m)}) = 0$ for every $F \in \mathcal{T}$ of dimension at most $m+1$.

Iterating this construction produces $\omega_F \in \mathring{X}^k(F)$ for every $F \in \mathcal{T}$ such that

$$\omega - \sum_{F \in \mathcal{T}} \operatorname{Ext}_{F,\mathcal{T}} \omega_F$$

has vanishing trace on every $F \in \mathcal{T}$. Thus (63) follows, completing the proof. □

Remark 7.3.
The two families of extension operators defined previously,

$$\operatorname{ext}^{k,r}_{F,T} : \mathring{\mathcal{P}}_r\Lambda^k(F) \to \mathcal{P}_r\Lambda^k(T), \quad \operatorname{ext}^{k,r,-}_{F,T} : \mathring{\mathcal{P}}^-_r\Lambda^k(F) \to \mathcal{P}^-_r\Lambda^k(T), \tag{64}$$

satisfy the required conditions of this section, and thus lead to geometric decompositions of $\mathcal{P}_r\Lambda^k(\mathcal{T})$ and $\mathcal{P}^-_r\Lambda^k(\mathcal{T})$, respectively.

Remark 7.4.
Any basis of $\mathring{X}^k(F)$ induces a basis of $\operatorname{Ext}_{F,\mathcal{T}} \mathring{X}^k(F)$. In the light of the geometric decomposition (63), we see that choosing a basis for the space of vanishing trace for each simplex leads to a basis for the entire finite element space.

The extension operators are defined on the spaces with vanishing traces but using the geometric decomposition, we can extend them to the full space on each cell. For each $T \in \mathcal{T}$ we define the operator

$$\operatorname{ext}_{F,T} \colon X^k(F) \to X^k(T), \qquad \sum_{f \in \mathcal{T},\, f \subseteq F} \operatorname{ext}_{f,F} \omega_f \mapsto \sum_{f \in \mathcal{T},\, f \subseteq F} \operatorname{ext}_{f,T} \omega_f,$$

where the argument in is expressed in terms of the geometric decomposition with $\omega_f \in \mathring{X}^k(f)$ for each subsimplex of F. The operator $\operatorname{ext}_{F,T} \colon X^k(F) \to X^k(T)$ extends the operator $\operatorname{ext}_{F,T} \colon \mathring{X}^k(F) \to \mathring{X}^k(T)$, as is easily seen.

Remark 7.5.
The identities (59), (60) and (61) are satisfied by this definition of extension operator in the general case $\omega \in X^k(F)$. Moreover, for $\omega \in X^k(F)$ and $F, G, T \in \mathcal{T}$ with $F, G \subseteq T$ and $F \cap G \neq \emptyset$ one can verify

$$\operatorname{ext}_{F \cap G, G} \operatorname{tr}_{F, F \cap G} \omega = \operatorname{tr}_{T,G} \operatorname{ext}_{F,T} \omega, \tag{65}$$

and in particular, these extension operators are *consistent* in the terminology of [4, Section 4]. In that manner, we obtain operators

$$\operatorname{ext}^{k,r}_{F,T} : \mathcal{P}_r\Lambda^k(F) \to \mathcal{P}_r\Lambda^k(T), \quad \operatorname{ext}^{k,r,-}_{F,T} : \mathcal{P}^-_r\Lambda^k(F) \to \mathcal{P}^-_r\Lambda^k(T). \tag{66}$$

The mapping $\operatorname{ext}^{k,r}_{F,T}$ has not appeared before in the literature, whereas $\operatorname{ext}^{k,r,-}_{F,T}$ appears in [4].

We finish this section with an outline of the degrees of freedom. For each $F \in \mathcal{T}$ of dimension $\dim F = m$ we define the spaces of functionals

$$W_{r,k}(F) := \left\{ \phi \in \mathcal{P}_r\Lambda^k(\mathcal{T})^* \;\middle|\; \exists \eta \in \mathcal{P}^-_{r+k-m}\Lambda^{m-k}(F) : \phi(\cdot) = \int_F \eta \wedge \operatorname{Tr} \cdot \right\},$$

$$W^-_{r,k}(F) := \left\{ \phi \in \mathcal{P}^-_r\Lambda^k(\mathcal{T})^* \;\middle|\; \exists \eta \in \mathcal{P}_{r+k-m-1}\Lambda^{m-k}(F) : \phi(\cdot) = \int_F \eta \wedge \operatorname{Tr} \cdot \right\}.$$

We have isomorphisms $W_{r,k}(F) \simeq \mathring{\mathcal{P}}_r\Lambda^k(F)^*$ and $W^-_{r,k}(F) \simeq \mathring{\mathcal{P}}^-_r\Lambda^k(F)^*$, as is evident considering the pairings (55) and (56). We define the spaces

$$W_{r,k}(\mathcal{T}) := \sum_{F \in \mathcal{T}} W_{r,k}(F), \quad W^-_{r,k}(\mathcal{T}) := \sum_{F \in \mathcal{T}} W^-_{r,k}(F).$$

These are not only spaces of functionals over conforming finite element spaces but in fact the entire dual spaces, as expressed in the following two theorems.

Theorem 7.6.
For $r \in \mathbb{N}_0$ and $k \in [0:n]$ we have

$$\mathcal{P}_r\Lambda^k(\mathcal{T})^* = W_{r,k}(\mathcal{T}), \quad \mathcal{P}_r^-\Lambda^k(\mathcal{T})^* = W_{r,k}^-(\mathcal{T}). \tag{67}$$

Proof. We state the proof for the first identity; the proof for the second identity is completely analogous. Recall that $W_{r,k}(\mathcal{T}) \subseteq \mathcal{P}_r\Lambda^k(\mathcal{T})^*$. Let $\omega \in \mathcal{P}_r\Lambda^k(\mathcal{T})$ such that $\eta(\omega) = 0$ for all $\eta \in W_{r,k}(\mathcal{T})$. We show that $\omega = 0$.

By Theorem 7.2 we have $\omega = \sum_{F\in\mathcal{T}} \operatorname{Ext}_{F,\mathcal{T}} \omega_F$ with unique $\omega_F \in \mathring{\mathcal{P}}_r\Lambda^k(F)$ for each $F \in \mathcal{T}$. Suppose that for some $m \in \mathbb{N}_0$ we have $\omega_F = 0$ for each $F \in \mathcal{T}$ with $\dim F < m$ and consider some $F \in \mathcal{T}$ with $\dim F = m$. By assumption, $\operatorname{Tr}_F \omega = \omega_F$, and since $\eta(\omega) = 0$ for all $\eta \in W_{r,k}(F)$, we have $\omega_F = 0$. By induction, we find $\omega = 0$. Hence $W_{r,k}(\mathcal{T})$ spans $\mathcal{P}_r\Lambda^k(\mathcal{T})^*$. □

Theorem 7.7.
For $r \in \mathbb{N}_0$ and $k \in [0:n]$ we have direct sums

$$\mathcal{P}_r\Lambda^k(\mathcal{T})^* = \sum_{F\in\mathcal{T}} W_{r,k}(F), \quad \mathcal{P}_r^-\Lambda^k(\mathcal{T})^* = \sum_{F\in\mathcal{T}} W_{r,k}^-(F). \tag{68}$$

Proof. Suppose that we have $\eta_F \in W_{r,k}(F)$ for each $F \in \mathcal{T}$, not all zero. Write $\mathcal{T}^m$ for the set of m-dimensional simplices of $\mathcal{T}$ and abbreviate $\eta_m := \sum_{l=k}^m \sum_{F\in\mathcal{T}^l} \eta_F$. We use induction to find $\omega \in \mathcal{P}_r\Lambda^k(\mathcal{T})$ such that $\eta(\omega) > 0$.

First, consider the smallest $m \in \mathbb{N}_0$ for which there exists an m-dimensional $F \in \mathcal{T}$ with η_F nonzero. For each m-dimensional $F \in \mathcal{T}$ we choose $\omega_F \in \mathring{\mathcal{P}}_r\Lambda^k(F)$ such that $\eta_F(\omega_F) > 0$ if η_F is nonzero and let $\omega_F = 0$ otherwise. It follows that $\omega_k := \sum_{F\in\mathcal{T}^m} \operatorname{Ext}_{F,\mathcal{T}} \omega_F$ satisfies $\eta_k(\omega_k) > 0$.

For the induction step, suppose that for some $m \in \mathbb{N}$ we have $\omega_{m-1} \in \mathcal{P}_r\Lambda^k(\mathcal{T})$ such that $\eta_{m-1}(\omega_{m-1}) > 0$. For every m-dimensional $F \in \mathcal{T}$ we then choose $\omega_F \in \mathring{\mathcal{P}}_r\Lambda^k(F)$ satisfying $\eta_F(\operatorname{Ext}_{F,\mathcal{T}} \omega_F) > \eta_F(\omega_{m-1})$ if η_F is nonzero and $\omega_F = 0$ otherwise. It follows that $\omega_m := \omega_{m-1} + \sum_{F\in\mathcal{T}^m} \operatorname{Ext}_{F,\mathcal{T}} \omega_F$ satisfies $\eta_m(\omega_m) > 0$. Repeating this, we get $\omega \in \mathcal{P}_r\Lambda^k(\mathcal{T})$ such that $\eta(\omega) > 0$, finishing the proof. □

Acknowledgements. The author acknowledges helpful discussions with Douglas N. Arnold and Snorre H. Christiansen. The author would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme "Geometry, compatibility and structure preservation in computational differential equations" where work on this paper was undertaken. This work was supported by EPSRC grant no EP/K032208/1.

References

[1] M. Ainsworth, G. Andriamaro, and O. Davydov, *Bernstein-Bézier finite elements of arbitrary order and optimal assembly procedures*, SIAM Journal on Scientific Computing, 33 (2011), pp. 3087–3109.

[2] M. Ainsworth and J. Coyle, *Hierarchic finite element bases on unstructured tetrahedral meshes*, International journal for numerical methods in engineering, 58 (2003), pp. 2103–2130.

[3] D. N. Arnold, R. S. Falk, and R. Winther, *Finite element exterior calculus, homological techniques, and applications*, Acta Numerica, 15 (2006), pp. 1–155.

[4] ———, *Geometric decompositions and local bases for spaces of finite element differential forms*, Computer Methods in Applied Mechanics and Engineering, 198 (2009), pp. 1660–1672.

[5] ———, *Finite element exterior calculus: from Hodge theory to numerical stability*, Bulletin of the American Mathematical Society, 47 (2010), pp. 281–354.

[6] A. Bentley, *Explicit construction of computational bases for rtk and bdmk spaces in r3*, Computers & Mathematics with Applications, 73 (2017), pp. 1421–1432.

[7] S. Beuchler, V. Pillwein, and S. Zaglmayr, *Sparsity optimized high order finite element functions for $H(\operatorname{div})$ on simplices*, Numerische Mathematik, 122 (2012), pp. 197–225.

[8] ———, *Sparsity optimized high order finite element functions for $H(\text{curl})$ on tetrahedra*, Advances in Applied Mathematics, 50 (2013), pp. 749–769.

[9] F. Brezzi, J. Douglas, and L. D. Marini, *Two families of mixed finite elements for second order elliptic problems*, Numerische Mathematik, 47 (1985), pp. 217–235.

[10] S. H. Christiansen and A. Gillette, *Constructions of some minimal finite element systems*, ESAIM: Mathematical Modelling and Numerical Analysis, 50 (2016), pp. 833–850.

[11] S. H. Christiansen and F. Rapetti, *On high order finite element spaces of differential forms*, Mathematics of Computation, (2015). electronically published on July 10, 2015.

[12] V. Ervin, *Computational bases for rtk and bdmk on triangles*, Computers & Mathematics with Applications, 64 (2012), pp. 2765–2774.

[13] D. Fuchs and O. Viro, *Topology II: Homotopy and Homology. Classical Manifolds*, Springer, 2004.

[14] A. Gillette and T. Kloefkorn, *Trimmed serendipity finite element differential forms*, Mathematics of Computation, (2018).

[15] J. Gopalakrishnan, L. E. García-Castillo, and L. F. Demkowicz, *Nédélec spaces in affine coordinates*, Computers & Mathematics with Applications, 49 (2005), pp. 1285–1294.

[16] R. Hiptmair, *Higher order Whitney forms*, Progress in Electromagnetics Research, 32 (2001), pp. 271–299.

[17] R. Hiptmair, *Finite elements in computational electromagnetism*, Acta Numerica, 11 (2002), pp. 237–339.

[18] R. C. Kirby, *Low-complexity finite element algorithms for the de Rham complex on simplices*, SIAM Journal on Scientific Computing, 36 (2014), pp. A846–A868.

[19] ———, *Low-complexity finite element algorithms for the de Rham complex on simplices*, SMAI Journal of Computational Mathematics, 4 (2018), pp. 197–224.

[20] J. M. Lee, *Introduction to Smooth Manifolds*, vol. 218 of Graduate Texts in Mathematics, Springer, New York, 2nd ed., 2012.

[21] J.-C. Nédélec, *A new family of mixed finite elements in ? 3*, Numerische Mathematik, 50 (1986), pp. 57–81.

[22] F. Rapetti and A. Bossavit, *Geometrical localisation of the degrees of freedom for Whitney elements of higher order*, Science, Measurement & Technology, IET, 1 (2007), pp. 63–66.

[23] ———, *Whitney forms of higher degree*, SIAM J. Numer. Anal, 47 (2009), pp. 2369–2386.

[24] J. Schöberl and S. Zaglmayr, *High order Nédélec elements with local complete sequence properties*, COMPEL-The international journal for computation and mathematics in electrical and electronic engineering, 24 (2005), pp. 374–384.

UCSD Department of Mathematics, 9500 Gilman Drive MC0112, La Jolla, CA 92093-0112, USA

Email address: mlicht@math.ucsd.edu