ADDITIVITY AND DOUBLE COSET FORMULAE FOR THE MOTIVIC AND ÉTALE BECKER-GOTTLIEB TRANSFER

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ABSTRACT. In this paper, which is a continuation of earlier work by the first author and Gunnar Carlsson, one of the first results we establish is the additivity of the motivic Becker-Gottlieb transfer, the corresponding trace as well as their realizations. We then apply this to derive several important consequences: for example, we settle a conjecture of Morel regarding the assertion that the Euler-characteristic of G/N(T) for a split reductive group scheme G and the normalizer of a split maximal torus N(T) is 1 in the Grothendieck-Witt ring. We also obtain the analogues of various double coset formulae known in the classical setting of algebraic topology.

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1. Introduction

We begin by recalling the basic framework adopted in [CJ19] in order to construct the *motivic and étale* variants of the classical Becker-Gottlieb transfer. Let G denote a linear algebraic group *over a perfect field k of arbitrary characteristic.* Let $\mathbf{Spt}_{\mathrm{mot}}$ ($\mathbf{Spt}_{\mathrm{et}}$, $\mathbf{Spt}_{\mathrm{mot},\mathcal{E}}$, $\mathbf{Spt}_{\mathrm{et},\mathcal{E}}$) denote the category of motivic spectra over k (the corresponding category of spectra on the big étale site of k, the subcategory of $\mathbf{Spt}_{\mathrm{mot}}$ of $\mathcal{E}$-module spectra for a commutative ring spectrum $\mathcal{E}\,\varepsilon\,\mathbf{Spt}_{\mathrm{mot}}$ and the corresponding subcategory of $\mathbf{Spt}_{\mathrm{et}}$ for a commutative ring spectrum $\mathcal{E}\,\varepsilon\,\mathbf{Spt}_{\mathrm{et}}$, respectively). (If S denotes $\mathrm{Spec}\, k$, we may often let $\mathbf{Spt}/\mathrm{S}_{\mathrm{mot}}$ ($\mathbf{Spt}/\mathrm{S}_{\mathrm{et}}$) denote $\mathbf{Spt}_{\mathrm{mot}}$ ($\mathbf{Spt}_{\mathrm{et}}$, respectively) to highlight the base field k.) The corresponding stable homotopy category will be denoted $\mathbf{HSpt}_{\mathrm{mot}}$ ($\mathbf{HSpt}_{\mathrm{et}}$, $\mathbf{HSpt}_{\mathrm{mot},\mathcal{E}}$ and $\mathbf{HSpt}_{\mathrm{et},\mathcal{E}}$, respectively). The homotopy category $\mathbf{HSpt}_{\mathrm{mot}}$ is often denoted $\mathcal{SH}(k)$ in the literature. Our notation of $\mathbf{HSpt}_{\mathrm{mot}}$ hopes to highlight the fact that this is the homotopy category of motivic spectra. In positive characteristic p, we either consider $\mathbf{Spt}_{\mathrm{mot}}[p^{-1}]$, or $\mathbf{Spt}_{\mathrm{mot},\mathcal{E}}$ where $\mathcal{E}$ is any motivic ring spectrum that is ℓ-complete for some prime $\ell \neq p = char(k)$. (In fact $\mathbf{Spt}_{\mathrm{mot}}[p^{-1}]$ identifies with $\mathbf{Spt}_{\mathrm{mot},\mathbb{S}[p^{-1}]}$, where $\mathbb{S}$ denotes the motivic sphere spectrum.) Throughout, $\mathbf{T}$ will denote $\mathbb{P}^1$ pointed by ∞ and $\mathbf{T}^n$ will denote $\mathbf{T}^{\wedge n}$ for any integer $n \geq 0$.

Moreover, it is important for us that the ring-spectrum $\mathcal{E}\,\varepsilon\,\mathbf{Spt}_{\mathrm{mot}}$ ($\mathcal{E}\,\varepsilon\,\mathbf{Spt}_{\mathrm{et}}$) has a lift to an *equivariant ring spectrum* $\mathcal{E}^{\mathrm{G}}$ in the sense of [CJ19, Terminology 3.12]. For example, the usual motivic sphere spectrum $\Sigma_{\mathbf{T}}$ lifts to the equivariant sphere spectrum $\mathbb{S}^{\mathrm{G}}$ defined as in [CJ19, Definition 3.4]. The only other ring spectra we consider will be $\Sigma_{\mathbf{T}}[p^{-1}]$ for $p = char(k)$, and for a fixed prime $\ell \neq p(= char(k)$, $\Sigma_{\mathbf{T},(\ell)}$ which denotes the localization of $\Sigma_{\mathbf{T}}$ at the prime ideal (ℓ) and $\Sigma_{\mathbf{T}}\widehat{\ }_{\ell}$ which denotes the completion of $\Sigma_{\mathbf{T}}$ at the prime ℓ. $\Sigma_{\mathbf{T}}[p^{-1}]$ ($\Sigma_{\mathbf{T},(\ell)}$, $\Sigma_{\mathbf{T}}\widehat{\ }_{\ell}$) lifts to the equivariant spectrum $\mathbb{S}^{\mathrm{G}}[p^{-1}]$ ($\mathbb{S}^{\mathrm{G}}_{(\ell)}$, $\widehat{\mathbb{S}^{\mathrm{G}}}_{\ell}$, respectively).

Let X, Y denote either smooth schemes of finite type over k or unpointed simplicial presheaves on the big Nisnevich site of k provided with actions by G. Le $\mathrm{f}: \mathrm{X} \rightarrow \mathrm{X}$ denote a G-equivariant map. In this case, we first recall from [CJ19, Definition 6.1] (see also Definition 2.5) that the G-equivariant *pre-transfers* are the following maps

$$tr(\mathrm{f})'^{\mathrm{G}}: \mathbb{S}^{\mathrm{G}} \rightarrow \mathbb{S}^{\mathrm{G}} \wedge \mathrm{X}_+, \quad tr(\mathrm{f}_{\mathrm{Y}})'^{\mathrm{G}} = \mathrm{id}_{\mathrm{Y}_+} \wedge tr(\mathrm{f})'^{\mathrm{G}} \tag{1.0.1}$$

where $\mathbb{S}^{\mathrm{G}}$ is the G-equivariant sphere spectrum. (When the group G is trivial, we may replace the G-equivariant sphere spectrum by the usual motivic sphere spectrum.) Then the transfer $tr(\mathrm{f})$ defined on generalized equivariant motivic cohomology theories is obtained by starting with the above G-equivariant pre-transfer, feeding it to a suitable form of the Borel construction, and then performing various modifications to it as discussed in detail in [CJ19, section 6].

(a) $\mathrm{p}: \mathrm{E} \rightarrow \mathrm{B}$ is a G-torsor for the action of a linear algebraic group G with both E and B smooth quasi-projective schemes over k, with B *connected* and

$$\pi_{\mathrm{Y}}: \mathrm{E} \times_{\mathrm{G}} (\mathrm{X} \times \mathrm{Y}) \rightarrow \mathrm{E} \times_{\mathrm{G}} \mathrm{Y}$$

the induced map, where G acts diagonally on $\mathrm{Y} \times \mathrm{X}$. One may observe that, on taking $\mathrm{Y} = \mathrm{Spec}\, \mathrm{k}$ with the trivial action of G, the map π_{Y} becomes $\pi_{\mathrm{Y}}: \mathrm{E} \times_{\mathrm{G}} \mathrm{X} \rightarrow \mathrm{B}$, which is an important special case.

(b) $\mathrm{BG}^{gm,m}$ will denote the m-th degree approximation to the geometric classifying space of the linear algebraic group G (as in [Tot99], [MV99]), $\mathrm{p}: \mathrm{EG}^{gm,m} \rightarrow \mathrm{BG}^{gm,m}$ is the corresponding universal G-torsor and

$$\pi_{\mathrm{Y}}: \mathrm{EG}^{gm,m} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}) \rightarrow \mathrm{EG}^{gm,m} \times_{\mathrm{G}} \times_{\mathrm{G}} \mathrm{Y}$$

is the induced map.

(c) If $\mathrm{p_m}$ ($\pi_{\mathrm{Y,m}}$) denotes the map denoted p (π_{Y}) in (b), here we let $\mathrm{p} = \lim_{\mathrm{m} \rightarrow \infty} \mathrm{p_m}$ and let

$$\pi_{\mathrm{Y}} = \lim_{m \rightarrow \infty} \pi_{\mathrm{Y,m}}: \mathrm{EG}^{gm} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}) = \lim_{\mathrm{m} \rightarrow \infty} \mathrm{EG}^{gm,m} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}) \rightarrow \lim_{\mathrm{m} \rightarrow \infty} \mathrm{EG}^{gm,m} \times_{\mathrm{G}} \mathrm{Y}.$$

Strictly speaking, the above definitions apply only to the case where G is *special* in the sense of Grothendieck (see [Ch]) and when G is *not special*, the above objects will in fact need to be replaced by the derived push-forward of the above objects viewed as sheaves on the big étale site of k to the corresponding big Nisnevich site of k, as discussed in [CJ19, (6.2.7)]. However, we will denote these new objects also by the same notation throughout, except when it is necessary to distinguish between them. For G *not special*, we *will assume the base field is also infinite* to prevent certain unpleasant situations.

Further assumptions on the base field k. We will assume further that k has finite ℓ-cohomological dimension and satisfies the finiteness conditions in [CJ19, (3.0.3)], namely that $\mathrm{H}^{\mathrm{n}}_{\mathrm{et}}(\mathrm{Spec}\, k, \mathrm{Z}/\ell^{\nu})$ is finitely generated in each degree n and vanish for all $n >> 0$. For G *not special*, we *will also assume the base field is infinite* to prevent certain unpleasant situations.

Definition 1.1. (Weak module spectra over commutative ring spectra) Let A denote a commutative ring spectrum in $\mathbf{Spt}_{\mathrm{mot}}$ ($\mathbf{Spt}_{\mathrm{et}}$). Then a spectrum $\mathrm{M} \in \mathbf{Spt}_{\mathrm{mot}}$ ($\mathbf{Spt}_{\mathrm{et}}$) is a weak-module spectrum over A if M is equipped with a pairing $\mu : \mathrm{M} \wedge \mathrm{A} \rightarrow \mathrm{M}$ that is homotopy associative. Such weak-module spectra will be often referred to as module spectra throughout our discussion.

Definition 1.2. Let $\mathrm{M}\,\varepsilon\,\mathbf{Spt}_{\mathrm{mot}}$ ($\mathbf{Spt}_{\mathrm{et}}$). For each prime number ℓ, let $Z_{(\ell)}$ denote the localization of the integers at the prime ideal ℓ and let $Z\hat{_{\ell}} = \lim_{\infty \leftarrow n} Z/\ell^n$. Then we say M *is* $Z_{(\ell)}$*-local (ℓ-complete, ℓ-primary torsion)*, if each $[\mathrm{S}^{1^{\wedge s}} \wedge \mathbf{T}^t \wedge \Sigma_{\mathbf{T}} U_+, M]$ is a $Z_{(\ell)}$-module ($Z\hat{_{\ell}}$-module, $Z\hat{_{\ell}}$-module which is torsion, respectively) as U varies among the objects of the given site, where $[\mathrm{S}^{1^{\wedge s}} \wedge \mathbf{T}^t \wedge \Sigma_{\mathbf{T}} \mathrm{U}_+, \mathrm{M}]$ denotes *Hom* in the stable homotopy category $\mathbf{HSpt}_{\mathrm{mot}}$ ($\mathbf{HSpt}_{\mathrm{et}}$, respectively).

Theorem 1.3. *Let* $\mathrm{f} : \mathrm{X} \rightarrow \mathrm{X}$ *denote a* G*-equivariant map and for each* $m \geq 0$*, let* $\pi_{\mathrm{Y}} : \mathrm{E} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}) \rightarrow \mathrm{E} \times_{\mathrm{G}} \mathrm{Y}$ *denote any one of the maps considered in (a) through (c) above. Let* $\mathrm{f_Y} = \mathrm{id_Y} \times \mathrm{f} : \mathrm{Y} \times \mathrm{X} \rightarrow \mathrm{Y} \times \mathrm{X}$ *denote the induced map.*

Then in case (a), we obtain a map (called the transfer)

$$tr(\mathrm{f_Y}) : \Sigma_{\mathbf{T}}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y})_+ \rightarrow \Sigma_{\mathbf{T}}(\mathrm{E} \times_{\mathrm{G}} (\mathrm{X} \times \mathrm{Y}))_+ \quad (tr(\mathrm{f_Y}) : \mathcal{E} \wedge (\mathrm{E} \times_{\mathrm{G}} \mathrm{Y})_+ \rightarrow \mathcal{E} \wedge (\mathrm{E} \times_{\mathrm{G}} (\mathrm{X} \times \mathrm{Y}))_+)$$

in $\mathbf{HSpt}_{\mathrm{mot}}$ *(*$\mathbf{HSpt}_{\mathrm{mot},\mathcal{E}}$*, respectively) if* $\Sigma_{\mathbf{T}}\mathrm{X}_+$ *is dualizable in* $\mathbf{Spt}_{\mathrm{mot}}$ *(if* $\mathcal{E} \wedge \mathrm{X}_+$ *is dualizable in* $\mathbf{Spt}_{\mathrm{mot},\mathcal{E}}$*, respectively) having the following properties.*

(i) *If* $tr(\mathrm{f_Y})^{\mathrm{m}} : \Sigma_{\mathbf{T}}(\mathrm{EG}^{gm,m} \times_{\mathrm{G}} \mathrm{Y})_+ \rightarrow \Sigma_{\mathbf{T}}(\mathrm{EG}^{gm,m} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}))_+$ ($tr(\mathrm{f_Y})^{\mathrm{m}} : \mathcal{E} \wedge (\mathrm{EG}^{gm,m} \times_{\mathrm{G}} \mathrm{Y})_+ \rightarrow \mathcal{E} \wedge (\mathrm{EG}^{gm,m} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}))_+$) *denotes the corresponding transfer maps in case (b), the maps* $\{tr(\mathrm{f_Y})^{\mathrm{m}} | \mathrm{m}\}$ *are compatible as m varies. The corresponding induced map* $\lim_{m \rightarrow \infty} tr(\mathrm{f_Y})^{\mathrm{m}}$ *will be denoted* $tr(\mathrm{f_Y})$*.*

   For items (ii) through (iv) we will assume any one of the above contexts (a) through (c).

(ii) *If* $\mathrm{h}^{*,\bullet}(\ \ ,\mathcal{E})$ ($\mathrm{h}^{*,\bullet}(\ \ ,\mathrm{M})$*) denotes the generalized motivic cohomology theory defined with respect to the commutative motivic ring spectrum* $\mathcal{E}$ *(a motivic module spectrum* M *over* $\mathcal{E}$*, respectively) then,*

$$tr(\mathrm{f_Y})^*(\pi_{\mathrm{Y}}^*(\alpha).\beta) = \alpha.tr(\mathrm{f_Y})^*(\beta), \quad \alpha \in \mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y}, \mathrm{M}), \beta \in \mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}), \mathcal{E}).$$

   Here $tr(\mathrm{f_Y})^*$ *(*π_{Y}^**) denotes the map induced on generalized cohomology by the map* $tr(\mathrm{f_Y})$ *(*π_{Y}*, respectively). Both* $tr(\mathrm{f_Y})^*$ *and* π_{Y}^* *preserve the degree as well as the weight.*

(iii) *In particular,* $\pi_{\mathrm{Y}}^* : \mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y}, \mathrm{M}) \rightarrow \mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}), \mathrm{M})$ *is split injective if* $tr(\mathrm{f_Y})^*(\pi_{\mathrm{Y}}^*(1))$ *is a unit, where*
   $1 \in \mathrm{h}^{0,0}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y}, \mathcal{E})$ *is the unit of the graded ring* $\mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y}, \mathcal{E})$*.*

(iv) *The transfer* $tr(\mathrm{f_Y})$ *is natural with respect to restriction to subgroups of a given group. It is also natural with respect to change of base fields, assuming taking the dual is compatible with such base-change.*

(v) *If* $\mathrm{h}^{*,\bullet}(\ \ ,\mathrm{M})$ *denotes a generalized motivic cohomology theory, then the map* $tr(\mathrm{f_Y})^* : \mathrm{h}^{*,\bullet}(\mathrm{EG}^{gm} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}), \mathrm{M}) \rightarrow \mathrm{h}^{*,\bullet}(\mathrm{EG}^{gm} \times_{\mathrm{G}} \mathrm{Y}, \mathrm{M})$ *is independent of the choice of a geometric classifying space that satisfies certain basic assumptions (as in* [CJ19, 7.1: Proof of Theorem 1.1]*), and depends only on* X *and the* G*-equivariant map* f*.*

(vi) *Let* $\mathcal{E}$ *denote a commutative ring spectrum in* $\mathbf{Spt}_{\mathrm{et}}$ *which is* ℓ*-complete, in the sense of Definition 1.2 (below), for some prime* $\ell \neq char(k)$*. If* $\mathcal{E} \wedge \mathrm{X}_+$ *is dualizable in* $\mathbf{Spt}_{\mathrm{et},\mathcal{E}}$*, then there exists a transfer* $tr(\mathrm{f_Y})$ *in* $\mathbf{Spt}_{\mathrm{et},\mathcal{E}}$ *satisfying similar properties.*

(vii) *Let* $\mathcal{E}$ *denote a commutative ring spectrum in* $\mathbf{Spt}_{\mathrm{mot}}$ *which is* ℓ*-complete. Let* $\epsilon^* : \mathbf{Spt}_{\mathrm{mot}} \rightarrow \mathbf{Spt}_{\mathrm{et}}$ *denote the map of topoi induced by the obvious map from the étale site of k to the Nisnevich site of k. Then if* $\mathcal{E} \wedge \mathrm{X}_+$ *is is dualizable in* $\mathbf{Spt}_{\mathrm{mot},\mathcal{E}}$ *and* $\epsilon^*(\mathcal{E} \wedge \mathrm{X}_+)$ *is dualizable in* $\mathbf{Spt}_{\mathrm{et},\mathcal{E}}$*, the transfer map* $tr(\mathrm{f_Y})$ *is compatible with étale realizations, and for groups* G *that are special,* $\epsilon^*(tr(\mathrm{f_Y})) = \mathrm{tr}(\epsilon^*(\mathrm{f_Y}))$*.*

Throughout the rest of the paper we will assume that the transfer we consider fits into one of the three basic contexts (a), (b) or (c) above.

While the property (ii) is indeed quite useful, it is also important to be able to understand the composition $\pi^* \circ tr(\mathrm{f})^*$: indeed knowing this composition enables one to determine the image of the map π^* in most cases. When the scheme X is in fact G/H, for a subgroup-scheme H of G, determining this composition is equivalent to obtaining a double coset formula involving the double cosets H\G/H. A main result of this paper is in fact the *Additivity or Mayer-Vietoris property* of the transfer discussed in Theorem 1.12, which implies various key results, such as various double coset formulae. Several of these are discussed in this paper. However, the additivity of the transfer seems to hold only in generalized cohomology theories defined with respect to spectra which have the rigidity property (as discussed in Definition 1.10.)

While the additivity of the transfer clearly implies the additivity of the trace, we are able to provide an independent proof of the additivity of the trace which holds in all generalized cohomology theories. The additivity of the trace then enables us to prove the conjecture of Morel on the motivic Euler-characteristic of $\mathrm{G/N(T)}$. This in turn leads to establishing various splittings in the stable motivic homotopy category, some of which we also discuss in the paper.

Terminology 1.4. First we will consider the case where a given simplicial presheaf X is such that $\Sigma_{\mathbf{T}}\mathrm{X}_+$ is dualizable in $\mathbf{Spt}_{\mathrm{mot}}$. As a matter of notation, when the map f is the identity map on X, we will denote the G-equivariant pre-transfer (transfer) by tr'^{G} (tr, unless we need to emphasize the choice of G, in which case this will be denoted tr^{G}.). It may be important to point out that the context for the G-equivariant pre-transfer itself needs to be expanded a bit: this is discussed in Definition 2.5. Throughout the rest of the discussion, we will adopt this expanded context for the pre-transfer, with C in Definition 2.5 replaced by X throughout.

Moreover, if we need to specify the simplicial presheaf X, we will denote the pre-transfer (transfer) by $tr'^{\mathrm{G}}_{\mathrm{X}}$ (tr_{X} or $tr^{\mathrm{G}}_{\mathrm{X}}$). The composition of the pre-transfer $tr'^{\mathrm{G}}(\mathrm{f})$ with the projection $\mathbb{S}^{\mathrm{G}} \wedge \mathrm{X}_+ \rightarrow \mathbb{S}^{\mathrm{G}}$ will be denoted $\tau^{\mathrm{G}}(\mathrm{f})$ and called the *trace* associated to f. When $f = id_{\mathrm{X}}$, this will be denoted $\tau^{\mathrm{G}}_{\mathrm{X}}$ and called the G-equivariant trace associated to X. When the group G is trivial, we will denote the corresponding pre-transfer (trace) by tr'_{X} (τ_{X}, respectively).

Next we will consider the case where, $\mathcal{E} \in \mathbf{Spt}_{\mathrm{mot}}$ is a commutative ring spectrum and a given simplicial presheaf X is such that $\mathcal{E} \wedge \mathrm{X}_+$ is dualizable in $\mathbf{Spt}_{\mathrm{mot},\mathcal{E}}$. As pointed out earlier, other than the sphere spectrum $\Sigma_{\mathbf{T}}$, we will restrict to the ring spectra $\mathcal{E} = \Sigma_{\mathbf{T}}[p^{-1}]$ for $p = char(k)$, or with ℓ a fixed prime different from $p = char(k)$, the spectrum $\Sigma_{\mathbf{T},(\ell)}$ which denotes the localization of $\Sigma_{\mathbf{T}}$ at the prime ideal (ℓ) and the spectrum $\Sigma_{\mathbf{T}}\widehat{\ }_{\ell}$ which denotes the completion of $\Sigma_{\mathbf{T}}$ at the prime ℓ. Then, $\mathcal{E} = \Sigma_{\mathbf{T}}$ ($\Sigma_{\mathbf{T}}[p^{-1}]$, $\Sigma_{\mathbf{T},(\ell)}$, $\Sigma_{\mathbf{T}}\widehat{\ }_{\ell}$) lifts to the equivariant spectrum $\mathcal{E}^{\mathrm{G}} = \mathbb{S}^{\mathrm{G}}$ ($\mathbb{S}^{\mathrm{G}}[p^{-1}]$, $\mathbb{S}^{\mathrm{G}}_{(\ell)}$, $\widehat{\mathbb{S}^{\mathrm{G}}}_{\ell}$, respectively). The corresponding pre-transfers, transfers and traces will be denoted with a subscript $\mathcal{E}$. That is, $tr(f)'^{\mathrm{G}}_{\mathcal{E}^{\mathrm{G}}}$ ($tr'^{\mathrm{G}}_{\mathcal{E}^{\mathrm{G}}}$, $tr'_{\mathrm{X},\mathcal{E}}$, $\tau^{\mathrm{G}}_{\mathrm{X},\mathcal{E}}$, $\tau_{\mathrm{X},\mathcal{E}}$) will denote the corresponding G-equivariant pre-transfer (the corresponding G-equivariant pre-transfer when $f = id_{\mathrm{X}}$, the corresponding pre-transfer for the simplicial presheaf X, the corresponding G-equivariant trace and the corresponding trace, respectively).

Throughout the paper we will consider the suspension spectra of quasi-projective schemes, or more generally the suspension spectra of simplicial presheaves on the big Nisnevich or étale site of the given base field k. *For the most part, these objects will be unpointed*, but occasionally, we will need to consider objects that are intrinsically pointed, for example schemes like $\mathbb{P}^1$, which is pointed by ∞. We will refer to such objects, throughout the paper, as simplicial presheaves. If these objects are pointed and provided with the action by an algebraic group, we will also assume that the base point remains fixed by the group action.

The additivity theorem for the trace is the following theorem:

Theorem 1.5. *(Mayer-Vietoris and Additivity for the Trace) (i) Let* X *denote a smooth* G*-scheme and let* $i_j : \mathrm{U_j} \rightarrow \mathrm{X}$*,* $j = 1, 2$ *denote the open immersion of two Zariski open subschemes of* X *which are both* G*-stable, with* $\mathrm{X} = \mathrm{U}_1 \cup \mathrm{U}_2$*. Then adopting the terminology above, (i.e. where* $\tau^{\mathrm{G}}_{\mathrm{P}}$ *denotes the* G*-equivariant trace associated to the* G*-simplicial presheaf* P*)*

$$\tau^{\mathrm{G}}_{\mathrm{X}} = \tau^{\mathrm{G}}_{\mathrm{U}_1} + \tau^{\mathrm{G}}_{\mathrm{U}_2} - \tau^{\mathrm{G}}_{\mathrm{U}_1 \cap \mathrm{U}_2}$$

in case $char(k) = 0$*. In case* $char(k) = p > 0$*,*

$$\tau^{\mathrm{G}}_{\mathrm{X},\mathcal{E}^{\mathrm{G}}} = \tau^{\mathrm{G}}_{\mathrm{U}_1,\mathcal{E}^{\mathrm{G}}} + \tau^{\mathrm{G}}_{\mathrm{U}_2,\mathcal{E}^{\mathrm{G}}} - \tau^{\mathrm{G}}_{\mathrm{U}_1 \cap \mathrm{U}_2,\mathcal{E}^{\mathrm{G}}}$$

where $\mathcal{E}^{\mathrm{G}}$ *is any one of* $\mathbb{S}^{\mathrm{G}}[p^{-1}]$*,* $\mathbb{S}^{\mathrm{G}}_{(\ell)}$*,* $\widehat{\mathbb{S}^{\mathrm{G}}}_{\ell}$*, and where* ℓ *is a prime different from* p*.*

(ii) Let $i : \mathrm{Y} \rightarrow \mathrm{X}$ *denote a closed immersion of smooth* G*-schemes with* $j : \mathrm{U} \rightarrow \mathrm{X}$ *denoting the corresponding open complement. Let* $\mathcal{N}$ *denote the normal bundle associated to the closed immersion* i *and let* $\mathrm{Th}(\mathcal{N})$ *denotes its Thom-space. Then adopting the terminology above,*

$$\tau^{\mathrm{G}}_{\mathrm{X}} = \tau^{\mathrm{G}}_{\mathrm{U}} + \tau^{\mathrm{G}}_{\mathrm{X/U}} \text{ and } \tau_{\mathrm{X/U}} = \tau_{\mathrm{Th}(\mathcal{N})} = \tau_{\mathrm{Y}}$$

in case $char(k) = 0$*. In case* $char(k) = p > 0$*,*

$$\tau^{\mathrm{G}}_{\mathrm{X},\mathcal{E}^{\mathrm{G}}} = \tau^{\mathrm{G}}_{\mathrm{U},\mathcal{E}^{\mathrm{G}}} + \tau^{\mathrm{G}}_{\mathrm{X/U},\mathcal{E}^{\mathrm{G}}} \text{ and } \tau_{\mathrm{X/U},\mathcal{E}} = \tau_{\mathrm{Th}(\mathcal{N}),\mathcal{E}} = \tau_{\mathrm{Y},\mathcal{E}}$$

where $\mathcal{E}^{\mathrm{G}}$ *is any one of* $\mathbb{S}^{\mathrm{G}}[p^{-1}]$*,* $\mathbb{S}^{\mathrm{G}}_{(\ell)}$*,* $\widehat{\mathbb{S}^{\mathrm{G}}}_{\ell}$*, and where* $\mathcal{E}$ *is any one of* $\Sigma_{\mathbf{T}}[p^{-1}]$*,* $\Sigma_{\mathbf{T},(\ell)}$*, or* $\widehat{\Sigma_{\mathbf{T}}}_{\ell}$*, with* ℓ *a prime different from* p*.*

(iii) Let $\{\mathrm{S}_\alpha|\alpha\}$ *denote a stratification of the smooth scheme* X *into finitely many locally closed and smooth subschemes* S_α*. Then*

$$\tau_{\mathrm{X}} = \Sigma_\alpha \tau_{\mathrm{S}_\alpha}$$

in case $char(k) = 0$. In case $char(k) = p > 0$,

$$\tau_{\mathrm{X},\mathcal{E}} = \Sigma_\alpha \tau_{\mathrm{S}_\alpha,\mathcal{E}}$$

where $\mathcal{E}$ is any one of $\Sigma_{\mathbf{T}}[p^{-1}]$, $\Sigma_{\mathbf{T},(\ell)}$, or $\widehat{\Sigma_{\mathbf{T}}}_\ell$, with ℓ a prime different from p.

(iv) Let $\mathcal{E}$ denote a commutative ring spectrum in $\mathbf{Spt}_{mot}$, whose presheaves of homotopy groups are all ℓ-primary torsion for a fixed prime $\ell \neq char(k)$, and let $\epsilon^*(\mathcal{E})$ denote the corresponding spectrum in $\mathbf{Spt}_{\mathrm{et}}$. Then the results corresponding to (i) and (ii) hold when the motivic trace $\tau_{\mathrm{Z}}^{\mathrm{G}}$, associated to a smooth scheme Z is replaced by the trace $\tau^{\mathrm{G}}_{\mathrm{Z},\epsilon^*(\mathcal{E})}$. Moreover, the results corresponding to (iii) hold when the motivic trace τ_{Z} associated to a smooth scheme Z is replaced by the trace $\tau_{\mathrm{Z},\epsilon^*(\mathcal{E})}$.

As an important application of the above theorem, we prove the following theorem, which is a conjecture due to Morel: see [Lev18].

Theorem 1.6. *Let G denote a connected split reductive group over a perfect field and let N(T) denote the normalizer of a split maximal torus in G. Then the class, $\tau^*_{\mathrm{G/N(T)}}(1) = 1$ (in particular, is a unit) in the Grothendieck-Witt ring of the base field k, if k is of characteristic 0. If the base field k is of positive characteristic p, then the same conclusion holds in the Grothendieck-Witt ring of k with the prime p inverted. Moreover, in both cases $\tau^*_{\mathrm{G/T}}(1) = |\mathrm{W}|$, where $\mathrm{W} = \mathrm{N(T)/T}$ denotes the Weyl group.*

The above theorem, coupled with the results of [CJ19] enable us to obtain a number of applications, several of which are the splittings discussed in Corollary 1.9. But first we define slice-completed generalized motivic cohomology theories.

Let $\mathrm{M} \,\varepsilon\, \mathbf{Spt}_{\mathrm{mot}}$. Then one has Voevodsky's slice tower (see [Voev00]): $\{\mathrm{f_nM|n}\}$, where $\mathrm{f_{n+1}M}$ is the $n+1$-th connective cover of M. Let $s_{\le n}\mathrm{M}$ be the homotopy cofiber of the map $\mathrm{f_{n+1}M} \to \mathrm{M}$. Then, as shown in [Pel08], the diagram

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & \mathrm{f_{n+1}M} \arrow[r] \arrow[d] & \mathrm{f_nM} \arrow[r] \arrow[d] & \cdots \arrow[d] \\
\cdots \arrow[r, "id"] \arrow[d] & \mathrm{M} \arrow[r, "id"] \arrow[d] & \mathrm{M} \arrow[r, "id"] \arrow[d] & \cdots \arrow[d] \\
\cdots \arrow[r] & s_{\le n}\mathrm{M} \arrow[r] & s_{\le n-1}\mathrm{M} \arrow[r] & \cdots
\end{tikzcd}$$

admits a lifting to $\mathbf{Spt}_{\mathrm{mot}}$.

Definition 1.7. For a smooth scheme Y (smooth ind-scheme $\mathcal{Y} = \{\mathrm{Y_m|m}\}$), we define *the slice completed generalized motivic cohomology spectrum* with respect to a motivic spectrum M to be $\hat{h}(\mathrm{Y,M}) = \underset{\infty\leftarrow \mathrm{n}}{\mathrm{holim}}\mathbb{H}_{\mathrm{Nis}}(\mathrm{Y, s_{\le n}M}) \simeq \mathbb{H}_{\mathrm{Nis}}(\mathrm{Y}, \underset{\infty\leftarrow \mathrm{n}}{\mathrm{holim}}\mathrm{s_{\le n}M})$ ($\hat{h}(\mathcal{Y},\mathrm{M}) = \underset{\infty\leftarrow \mathrm{m}}{\mathrm{holim}}\underset{\infty\leftarrow \mathrm{n}}{\mathrm{holim}}\mathbb{H}_{\mathrm{Nis}}(\mathrm{Y_m, s_{\le nM}}) \simeq \underset{\infty\leftarrow \mathrm{m}}{\mathrm{holim}}\mathbb{H}_{\mathrm{Nis}}(\mathrm{Y_m}, \underset{\infty\leftarrow \mathrm{n}}{\mathrm{holim}}\mathrm{s_{\le n}M})$) where $\mathbb{H}_{Nis}(\mathrm{Y,F})$ and $\mathbb{H}_{Nis}(\mathrm{Y_m,F})$ denote the generalized hypercohomology spectrum with respect to a motivic spectrum F computed on the Nisnevich site. We let $\hat{\mathrm{h}}^{*,\bullet}(\mathrm{Y,M})$ ($\hat{\mathrm{h}}^{*,\bullet}(\mathcal{Y},\mathrm{M})$) denote the homotopy groups of the spectrum $\hat{h}(\mathrm{Y,M})$ ($\hat{h}(\mathcal{Y},\mathrm{M})$, respectively). One may define the completed generalized étale cohomology spectrum of a scheme or an ind-scheme with respect to an S^1-spectrum by using the Postnikov tower in the place of the slice tower.

Remark 1.8. By [Hirsch02, 18.1.8], the homotopy inverse limit $\underset{\infty\leftarrow n}{\mathrm{holim}} s_{\le n}\mathrm{M}$ belongs to $\mathbf{Spt}_{\mathrm{mot}}$. We may, therefore, define the *slice completion* of the spectrum M to be $\mathrm{holim}\, s_{\le n}\mathrm{M}$ (denoted henceforth by $\hat{\mathrm{M}}$) and define M to be slice-complete, if the natural map $\mathrm{M} \to \hat{\mathrm{M}}$ is a weak-equivalence. Therefore, one may see that $\hat{h}(\mathrm{Y,M}) = h(\mathrm{Y},\hat{\mathrm{M}})$ and $\hat{h}(\mathcal{Y},\mathrm{M}) = h(\mathcal{Y},\hat{\mathrm{M}})$. Several important spectra, like the spectrum representing algebraic K-theory and algebraic cobordism, are known to be slice-complete.

Corollary 1.9. *Assume the hypotheses of Theorem 1.6. In addition, assume that the following hypotheses hold for (i) through (iv). Let M denote any motivic spectrum if the base field is of characteristic 0 and let M denote a motivic spectrum in $\mathbf{Spt}_{\mathrm{mot}}[p^{-1}]$ if the base field is of characteristic $p > 0$. Let $\mathrm{p : E \to B}$ denote the map appearing in one of the three cases (a) through (c) considered in Theorem 1.3.*

(i) *Let G denote any linear algebraic group. Let $\pi : \mathrm{E}\underset{\mathrm{G}}{\times}(\mathrm{G/N(T)}) \to \mathrm{B}$ denote the map induced by the projection $\mathrm{G/N(T)} \to \mathrm{Spec\,k}$. Then the corresponding induced map*

$$\pi^* : \mathrm{h}^{\bullet,*}(\mathrm{B,M}) \to \mathrm{h}^{\bullet,*}(\mathrm{E}\underset{\mathrm{G}}{\times}(\mathrm{G/N(T)}),\mathrm{M})$$

is a split monomorphism, where $\mathrm{h}^{*,\bullet}(\quad,\mathrm{M})$ *denotes the generalized motivic cohomology theory defined with respect to the spectrum* M.

(ii) *Let* G *denote any linear algebraic group. Let* Y *denote a* G*-scheme or an unpointed simplicial presheaf provided with a* G*-action. Let* $\mathrm{q}:\mathrm{E}\underset{\mathrm{G}}{\times}(\mathrm{G}\underset{\mathrm{N(T)}}{\times}\mathrm{Y})\rightarrow\mathrm{E}\underset{\mathrm{G}}{\times}\mathrm{Y}$ *denote the map induced by the map* $\mathrm{G}\underset{\mathrm{N(T)}}{\times}\mathrm{Y}\rightarrow\mathrm{Y}$ *sending* $(g,y)\mapsto gy$*. Then, the induced map*

$$\mathrm{q}^*:\mathrm{h}^{\bullet,*}(\mathrm{E}\underset{\mathrm{G}}{\times}\mathrm{Y},\mathrm{M})\rightarrow\mathrm{h}^{\bullet,*}(\mathrm{E}\underset{\mathrm{G}}{\times}(\mathrm{G}\underset{\mathrm{N(T)}}{\times}\mathrm{Y}),\mathrm{M})$$

is also a split injection.

(iii) *Let* $\mathrm{h}^{*,\bullet}$ *denote a generalized motivic cohomology theory defined with respect to a motivic spectrum* M *as in (ii) above. Then the composition of the two maps:*

$$\mathrm{q}^*:\mathrm{h}^{*,\bullet}(\mathrm{E}\underset{\mathrm{G}}{\times}\mathrm{Y})\rightarrow\mathrm{h}^{*,\bullet}(\mathrm{E}\underset{\mathrm{G}}{\times}(\mathrm{G}\underset{\mathrm{N(T)}}{\times}\mathrm{Y}))\ \text{and}$$

$$\mathrm{h}^{*,\bullet}(\mathrm{E}\underset{\mathrm{G}}{\times}(\mathrm{G}\underset{\mathrm{N(T)}}{\times}\mathrm{Y}))\rightarrow\mathrm{h}^{*,\bullet}(\mathrm{E}\underset{\mathrm{G}}{\times}(\mathrm{G}\underset{\mathrm{T}}{\times}\mathrm{Y}))$$

(where the last map is induced by the inclusion $\mathrm{T}\rightarrow\mathrm{N(T)}$*) is injective if* $|\mathrm{W}|$ *(where* $\mathrm{W}=\mathrm{N(T)}/\mathrm{T}$*) is a unit in* $\mathrm{h}^*(\mathrm{BN(T)})$*. In particular, this last condition holds if* $|\mathrm{W}|$ *and* ℓ *are relatively prime, when the homotopy groups of* M *are all* ℓ*-primary torsion.*

(iv) *Let* $\mathcal{E}$ *denote a commutative ring spectrum in* $\mathbf{Spt}_{mot}$*, whose presheaves of homotopy groups are all* ℓ*-primary torsion for a fixed prime* $\ell\neq char(k)$*, and let* $\epsilon^*(\mathcal{E})$ *denote the corresponding spectrum in* $\mathbf{Spt}_{\mathrm{et}}$*. We will further assume that* M *is a module spectrum over* $\epsilon^*(\mathcal{E})$*. Then the results corresponding to (iii) and (iv) hold in* $\mathrm{h}^*(\quad,\mathrm{M})$ *which is the generalized étale cohomology with respect to the étale spectrum* $\epsilon^*(\mathrm{M})$.

We also establish the additivity and Mayer-Vietoris property for the transfer, but *only* for generalized cohomology theories defined with respect to spectra that have *the rigidity property* as discussed in Definition 1.10. This will also provide a second proof of the additivity and Mayer-Vietoris property for the trace, but for generalized cohomology theories defined with respect to spectra that have the rigidity property.

Definition 1.10. (Rigidity) Assume the base field k is infinite. Let M denote a motivic spectrum, so that its presheaves of homotopy groups are all ℓ-primary torsion for a fixed prime ℓ different from the characteristic of the base field k. We will say that M has *the rigidity property* if for any smooth scheme X of finite type over the given base field k, and a point x of X, with residue field $k(x)$ and Hensel local ring at x given by $\mathcal{O}^h_{\mathrm{X},x}$, the map $\mathrm{Spec}\,k(x)\rightarrow\mathrm{Spec}\,\mathcal{O}^h_{\mathrm{X},x}$ induces a weak-equivalence: $\Gamma(\mathrm{Spec}\,\mathcal{O}^h_{\mathrm{X},x},\mathrm{M})\simeq\Gamma(\mathrm{Spec}\,\mathrm{k(x)},\mathrm{M})$.

Let M denote a spectrum in $\mathbf{Spt}_{\mathrm{et}}$ so that its presheaves of homotopy groups are all ℓ-primary torsion for a fixed prime ℓ different from the characteristic of the base field k. We will say that M has *the rigidity property* if for any smooth scheme X of finite type over the given base field k, and a point x of X, with residue field $k(x)$ and strict Hensel local ring at x given by $\mathcal{O}^{sh}_{\mathrm{X},x}$, with residue field $\overline{k(x)}$, the following conditions are satisfied: (i) the map $\mathrm{Spec}\,\overline{k(x)}\rightarrow\mathrm{Spec}\,\mathcal{O}^{sh}_{\mathrm{X},x}$ induces a weak-equivalence: $\Gamma(\mathrm{Spec}\,\mathcal{O}^{sh}_{\mathrm{X},x},\mathrm{M})\simeq\Gamma(\mathrm{Spec}\,\overline{\mathrm{k(x)}},\mathrm{M})$, and (ii) for any inclusion $k\rightarrow K$ of separably closed fields, the induced map $\Gamma(\mathrm{Spec}\,k,\mathrm{M})\rightarrow\Gamma(\mathrm{Spec}\,\mathrm{K},\mathrm{M})$ is a weak-equivalence.

Conventions 1.11. $p\geq 0$ *will denote the characteristic of the base field* k*. Throughout the following theorem, its proof and various applications, that is Theorem 1.13 and Corollaries 1.14 through 1.18, we will fix a commutative motivic ring spectrum* $\mathcal{E}$*, with* $\mathcal{E}^{\mathrm{G}}$ *its lift to a* G*-equivariant spectrum, all chosen as discussed in Terminology 1.4.* M *will always denote module spectra over the given ring spectrum* $\mathcal{E}$.

In order to ensure that the Borel construction for non-special groups (which has to be carried out by pull-back to the étale site) is compatible with the notions of rigidity, one will need to adopt the model structures as discussed in section 4.2. We will assume this implicitly throughout the following theorem and its various applications.

Theorem 1.12. *(Mayer-Vietoris and Additivity for the transfer) (i) Let* X *denote a smooth* G*-scheme and let* $i_j:\mathrm{U_j}\rightarrow\mathrm{X}$*,* $j=1,2$ *denote the open immersion of two Zariski open subschemes of* X *which are both* G*-stable, with* $\mathrm{X}=\mathrm{U}_1\cup\mathrm{U}_2$*. Then adopting the terminology above, (i.e. where* $tr'^{\mathrm{G}}_{\mathrm{P}}$ *denotes the* G*-equivariant pre-transfer associated to the* G*-simplicial presheaf* P *and* $\tau^{\mathrm{G}}_{\mathrm{P}}$ *denotes the corresponding* G*-equivariant trace)*

$$tr'^{\mathrm{G}}_{\mathrm{X}}=i_1\circ tr'^{\mathrm{G}}_{\mathrm{U}_1}+i_2\circ tr'^{\mathrm{G}}_{\mathrm{U}_2}-i_3\circ tr'^{\mathrm{G}}_{\mathrm{U}_1\cap\mathrm{U}_2}\ \text{and}\ \tau^{\mathrm{G}}_{\mathrm{X}}=\tau^{\mathrm{G}}_{\mathrm{U}_1}+\tau^{\mathrm{G}}_{\mathrm{U}_2}-\tau^{\mathrm{G}}_{\mathrm{U}_1\cap\mathrm{U}_2}$$

in case $char(k)=0$*. In case* $char(k)=p>0$*, we obtain*

$$tr'^{\mathrm{G}}_{\mathrm{X},\mathcal{E}^{\mathrm{G}}}=i_1\circ tr'^{\mathrm{G}}_{\mathrm{U}_1,\mathcal{E}^{\mathrm{G}}}+i_2\circ tr'^{\mathrm{G}}_{\mathrm{U}_2,\mathcal{E}^{\mathrm{G}}}-i_3\circ tr'^{\mathrm{G}}_{\mathrm{U}_1\cap\mathrm{U}_2,\mathcal{E}^{\mathrm{G}}}\ \text{and}\ \tau^{\mathrm{G}}_{\mathrm{X},\mathcal{E}^{\mathrm{G}}}=\tau^{\mathrm{G}}_{\mathrm{U}_1,\mathcal{E}^{\mathrm{G}}}+\tau^{\mathrm{G}}_{\mathrm{U}_2,\mathcal{E}^{\mathrm{G}}}-\tau^{\mathrm{G}}_{\mathrm{U}_1\cap\mathrm{U}_2,\mathcal{E}^{\mathrm{G}}}.$$

We also obtain in all characteristics,

$$tr_{\mathrm{X}}=i_1\circ tr_{\mathrm{U}_1}+i_2\circ tr_{\mathrm{U}_2}-i_3\circ tr_{\mathrm{U}_1\cap\mathrm{U}_2}$$

which denote the induced transfers as in any one of the three contexts of Theorem 1.3, with respect to a motivic spectrum M *as in Conventions 1.11.*

(ii) Let $i : \mathrm{Y} \to \mathrm{X}$ *denote a closed immersion of smooth* G*-schemes with* $j : \mathrm{U} \to \mathrm{X}$ *denoting the corresponding open complement. Then adopting the terminology above,*

$$tr'^{\mathrm{G}}_{\mathrm{X}} = j \circ tr'^{\mathrm{G}}_{\mathrm{U}} + tr'^{\mathrm{G}}_{\mathrm{X/U}}, \text{ and } \tau^{\mathrm{G}}_{\mathrm{X}} = \tau^{\mathrm{G}}_{\mathrm{U}} + \tau^{\mathrm{G}}_{\mathrm{X/U}}$$

in case $char(k) = 0$*. In case* $char(k) = p > 0$*, we obtain*

$$tr'^{\mathrm{G}}_{\mathrm{X},\mathcal{E}^{\mathrm{G}}} = j \circ tr'^{\mathrm{G}}_{\mathrm{U},\mathcal{E}^{\mathrm{G}}} + tr'^{\mathrm{G}}_{\mathrm{X/U},\mathcal{E}^{\mathrm{G}}}, \text{ and } \tau^{\mathrm{G}}_{\mathrm{X},\mathcal{E}^{\mathrm{G}}} = \tau^{\mathrm{G}}_{\mathrm{U},\mathcal{E}^{\mathrm{G}}} + \tau^{\mathrm{G}}_{\mathrm{X/U},\mathcal{E}^{\mathrm{G}}}.$$

We also obtain in all characteristics,

$$tr_{\mathrm{X}} = j \circ tr_{\mathrm{U}} + tr_{\mathrm{X/U}}$$

which denote the induced transfers as in any one of the three contexts of Theorem 1.3, with respect to a motivic spectrum M*, as in Conventions 1.11.*

For the remainder of this theorem, we will assume that the base field k *is infinite.*

(iii) Let M *denote a motivic spectrum that has the rigidity property (as discussed in Definition 1.10). Let* $\mathcal{N}$ *denote the normal bundle associated to the closed immersion* i *as in (ii), and let* $\mathrm{Th}(\mathcal{N})$ *denotes its Thom-space. Then we obtain in all characteristics*

$$tr_{\mathrm{X/U}} = tr_{\mathrm{Th}(\mathcal{N})} = i \circ tr_Y,$$

where the following notational convention hold:

- $tr_{\mathrm{X/U}}$*,* $tr_{\mathrm{Th}(\mathcal{N})}$*,* tr_Y *denote the transfers induced on generalized motivic cohomology with respect to the motivic spectrum* M *as in Theorem 1.3 and Terminology 1.4, and*
- *where the corresponding* G*-equivariant pre-transfers are defined with respect to the ring spectrum* $\mathbb{S}^{\mathrm{G}}$ *in characteristic* 0 *and with respect to one of the ring spectra* $\mathbb{S}^{\mathrm{G}}[p^{-1}]$*,* $\mathbb{S}^{\mathrm{G}}_{(\ell)}$*,* $\widehat{\mathbb{S}^{\mathrm{G}}}_{\ell}$*, in case* $char(k) = p > 0$*, and where* ℓ *is a prime different from* p*.*

(iv) Again let M *denote a motivic spectrum that has the rigidity property (as discussed in Definition 1.10). Let* $\{\mathrm{S}_\alpha|\alpha\}$ *denote a stratification of the smooth scheme* X *into finitely many locally closed and smooth* G*-stable subschemes* S_α*. For each* α*, let* $i_\alpha : \mathrm{S}_\alpha \to \mathrm{X}$ *denote the corresponding locally closed immersion. Then one obtains in all characteristics*

$$tr_{\mathrm{X}} = \Sigma_\alpha i_\alpha \circ tr_{\mathrm{S}_\alpha},$$

where the following notational conventions hold:

- tr_{X}*,* tr_{S_α} *denote the transfers induced on generalized motivic cohomology theories with respect to the motivic spectrum* M *as in Theorem 1.3 and Terminology 1.4, and*
- *where the corresponding* G*-equivariant pre-transfers are defined with respect to the ring spectrum* $\mathbb{S}^{\mathrm{G}}$ *in characteristic* 0 *and with respect to one of the ring spectra* $\mathbb{S}^{\mathrm{G}}[p^{-1}]$*,* $\mathbb{S}^{\mathrm{G}}_{(\ell)}$*,* $\widehat{\mathbb{S}^{\mathrm{G}}}_{\ell}$*, in case* $char(k) = p > 0$*, and where* ℓ *is a prime different from* p*.*

(v) Let $\mathcal{E}^{\mathrm{G}}$ *denote a commutative ring spectrum in* $\mathbf{Spt}^{\mathrm{G}}_{mot}$*, whose presheaves of homotopy groups are all* ℓ*-primary torsion for a fixed prime* $\ell \neq char(k)$*, and let* $\epsilon^*(\mathcal{E}^{\mathrm{G}})$ *denote the corresponding spectrum in* $\mathbf{Spt}^{\mathrm{G}}_{\mathrm{et}}$*. Then the results corresponding to (i) through (ii) also hold if* $tr'^{\mathrm{G}}_{\mathrm{Z}}$ *(*$\tau^{\mathrm{G}}_{\mathrm{Z}}$*) is replaced by* $tr'^{\mathrm{G}}_{\mathrm{Z}_+\wedge\epsilon^*(\mathcal{E}^{\mathrm{G}})}$ *(*$\tau^{\mathrm{G}}_{\mathrm{Z}_+\wedge\epsilon^*(\mathcal{E}^{\mathrm{G}})}$*, respectively). Assume next that* M *is a module spectrum over* $\mathcal{E} = (i^* \circ \widetilde{\mathbb{P}} \circ \widetilde{\mathrm{U}})^*(\mathcal{E}^{\mathrm{G}})$ *(which is the non-equivariant spectrum obtained from* $\mathcal{E}^{\mathrm{G}}$ *as in Proposition 2.2) so that* M *has the rigidity property as in Definition 1.10. Then the results corresponding to (i) through (iv) hold for the transfer* tr_Z *in generalized étale cohomology theories defined with respect to the spectrum* $\epsilon^*(\mathrm{M})$*.*

As is shown in [L] and [LMS], the additivity and Mayer-Vietoris property of the transfer can be easily deduced by showing that the corresponding pre-transfer (i.e. the transfer for the trivial group) is additive, or equivalently, has what is often called the Mayer-Vietoris property. We establish such a property, by systematically verifying that the same general strategy carries over to the motivic and étale framework.

In addition, we also verify a multiplicative property of the pre-transfer. As further applications of the additivity and multiplicativity of the pre-transfer, we establish various double coset formulae. The main context in which we consider double coset formulae will be as follows. Let G denote a linear algebraic group, and let H, K denote two closed linear algebraic subgroups. Then, adopting the terminology as in Theorem 1.3, that for a linear algebraic group H, $\mathrm{BH} = \lim_{m\to\infty} \mathrm{BH}^{gm,m}$ and $\mathrm{EH} = \lim_{m\to\infty} \mathrm{EH}^{gm,m}$, we obtain the cartesian square:

$$
\begin{tikzcd}
\mathrm{EK}\underset{\mathrm{K}}{\times}\mathrm{G/H} \arrow[r, "\tilde{p}_K"] \arrow[d, "\tilde{\pi}_H"] & \mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{G/H} \arrow[d, "\pi_H"] \\
\mathrm{BK} \arrow[r, "p_K"] & \mathrm{BG}.
\end{tikzcd}
\tag{1.0.2}
$$

Now a basic assumption we make is that G/H admits a *finite* decomposition $\mathrm{G/H} = \sqcup_i \mathrm{F_i}$, where each $\mathrm{F_i}$ is a locally closed and K-stable smooth subscheme of G/H, where we assume that K acts on the left on G/H and H acts on the right on G. In particular, it follows that each $\mathrm{F_i}$ is a disjoint union of the double-cosets for the left-action of K on G and the right action of H on G.

Theorem 1.13. *Assume the situation as in* (1.0.2).
(i) Let M *denote a motivic spectrum and let* $h^{*,\bullet}(\ \ ,\mathrm{M})$ *denote the generalized cohomology defined with respect to the spectrum* M*. Denoting the maps induced by the transfers*

$$tr^{\mathrm{G}*}: h^{*,\bullet}(\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{G/H},\mathrm{M}) \to h^{*,\bullet}(\mathrm{BG},\mathrm{M}),\quad \mathrm{tr}^{\mathrm{K}*}: h^{*,\bullet}(\mathrm{EK}\underset{\mathrm{K}}{\times}\mathrm{G/H},\mathrm{M}) \to h^{*,\bullet}(\mathrm{BK},\mathrm{M}),\ \textit{and}$$

$$p_{\mathrm{K}}^*: h^{*,\bullet}(\mathrm{BG},\mathrm{M}) \to h^{*,\bullet}(\mathrm{BK},\mathrm{M}),\quad \tilde{p}_{\mathrm{K}}^*: h^{*,\bullet}(\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{G/H},\mathrm{M}) \to h^{*,\bullet}(\mathrm{EK}\underset{\mathrm{K}}{\times}\mathrm{G/H},\mathrm{M})$$

the corresponding pull-backs, we obtain:

$$p_{\mathrm{K}}^* \circ \mathrm{tr}^{\mathrm{G}*} = \mathrm{tr}^{\mathrm{K}*} \circ \tilde{p}_{\mathrm{K}}^*. \tag{1.0.3}$$

(ii) Assume the base field k is infinite. Let M *denote a motivic spectrum that has the rigidity property as in Definition 1.10 and let* $h^{*,\bullet}(\ \ ,\mathrm{M})$ *denote the generalized cohomology defined with respect to the spectrum* M*. Then, the map induced by the transfer* $h^{*,\bullet}(\mathrm{tr}^{\mathrm{K}},\mathrm{M})$ *admits a decomposition as* $\Sigma_j h^{*,\bullet}(\mathrm{i_j}\circ \mathrm{tr}^{\mathrm{K}}_{\mathrm{F_j}},\mathrm{M})$*, where* $tr^{\mathrm{K}*}_{\mathrm{F_j}}: h^{*,\bullet}(\mathrm{EK}\underset{\mathrm{K}}{\times}\mathrm{F_j},\mathrm{M}) \to h^{*,\bullet}(\mathrm{BK},\mathrm{M})$ *is the corresponding transfer and* $i_j: \mathrm{EK}\underset{\mathrm{K}}{\times}\mathrm{F_j} \to \mathrm{EK}\underset{\mathrm{K}}{\times}\mathrm{G/H}$ *is the map induced by the inclusion* $\mathrm{F_j} \to \mathrm{G/H}$.

The above double coset formula itself specializes to provide several interesting applications, examples of which are discussed in the following corollaries.

Corollary 1.14. *(Double coset formulae) Let* $h^{*,\bullet}$ *denote a generalized cohomology theory defined with respect to a motivic spectrum* M*. Assume that the base field k is infinite and that the spectrum* M *has the rigidity property as in Definition 1.10.*

(i) Assume that G *is a connected split reductive group, and that* $\mathrm{K} = \mathrm{H} = \mathrm{T}$ *is a split maximal torus in* G*. Let* N(T) *denote the normalizer of* T *and let* $\mathrm{W} = \mathrm{N(T)/T}$*. Then the right-hand-side above may be written as* $\Sigma_{\bar{g}\,\varepsilon\,\mathrm{W}} C_g$*, where* $C_g: h^{*,\bullet}(\mathrm{BH},\mathrm{M}) \overset{\cong}{\to} h^{*,\bullet}(\mathrm{BH^g},\mathrm{M})$ *is the isomorphism induced by conjugation by g, and where* $\mathrm{H^g} = \mathrm{gHg^{-1}}$.

(ii)Suppose G *is a connected split reductive group, and that* H *is a closed linear algebraic subgroup of* G *of maximal rank and* $\mathrm{K} = \mathrm{T}$ *is a split maximal torus in* G *and* H*. Let* $\mathrm{W_G}$ *(*$\mathrm{W_H}$*) denote the Weyl group of* G *(*H*, respectively). In this case the right-hand-side of (i) may be written as* $\Sigma_{g\,\varepsilon\,\mathrm{W_G/W_H}} p^*(\mathrm{K},\mathrm{H^g}) \circ \mathrm{C_g}$*, where* $C_g: h^{*,\bullet}(\mathrm{BH},\mathrm{M}) \to h^{*,\bullet}(\mathrm{BH^g},\mathrm{M})$ *is as in (ii) and* $p^*(\mathrm{K},\mathrm{H^g}): h^{*,\bullet}(\mathrm{BH^g},\mathrm{M}) \to h^{*,\bullet}(\mathrm{BK},\mathrm{M})$ *is the pull-back induced by the map* $\mathrm{BK} = \mathrm{B(K\cap H^g)} \to \mathrm{BH^g}$.

(iii) Let $\mathcal{E}$ denote a commutative ring spectrum in $\mathbf{Spt}_{mot}$*, whose presheaves of homotopy groups are all ℓ-primary torsion for a fixed prime $\ell \neq char(k)$, and let $\epsilon^*(\mathcal{E})$ denote the corresponding spectrum in* $\mathbf{Spt}_{\mathrm{et}}$*. Assume that* M *is a module spectrum over $\mathcal{E}$ that has the rigidity property as in Definition 1.10. Then the results corresponding to (i) through (iii) also hold for generalized étale cohomology with respect to the spectrum* $\epsilon^*(\mathrm{M})$.

Corollary 1.15. *Let* $h^{*,\bullet}$ *denote a generalized cohomology theory defined with respect to a motivic spectrum* M*. We will further assume that the base field k is infinite and that the spectrum* M *has the rigidity property in Definition 1.10.*

Assume X *is a* G*-scheme or an unpointed simplicial presheaf with* G*-action, for a connected split reductive group* G*, with split maximal torus* T*. Then*

$$h^{*,\bullet}(\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X},\mathrm{M}) \cong h^{*,\bullet}(\mathrm{ET}\underset{\mathbf{T}}{\times}\mathrm{X},\mathrm{M})^{\mathrm{W}}$$

if $|\mathrm{W}|$ *is a unit in the cohomology theory* $h^{*,\bullet}$*. Corresponding results also hold for generalized étale cohomology theories defined with respect to the spectrum* $\epsilon^*(\mathrm{M})\,\varepsilon\,\mathbf{Spt}_{\mathrm{et}}$ *(that is, on the étale site).*

Corollary 1.16. *Assume the base field is separably closed. Let* G *denote a connected reductive group and* T *denote a split maximal torus of* G. *Let* $\mathrm{H}^{*,\bullet}_{\mathrm{mot}}(\quad, \mathrm{Z}/\ell^{\mathrm{n}})$ $(\mathrm{H}^{*}_{\mathrm{et}}(\quad, \mu_{\ell^{\mathrm{n}}}(\bullet)))$ *denote motivic cohomology with* Z/ℓ^n*-coefficients (étale cohomology with respect to the sheaf* $\mu_{\ell^n}(\bullet)$*), where* ℓ *is a prime different from* $char(k)$*. Then the following hold:*

(i) *Assume further that* $|\mathrm{W}|$ *is prime to* ℓ*. If* X *is any smooth* G*-scheme or an unpointed simplicial presheaf with* G*-action, then* $\mathrm{H}^{*,\bullet}_{\mathrm{mot}}(\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X}, \mathrm{Z}/\ell^{\mathrm{n}}) \cong \mathrm{H}^{*,\bullet}_{\mathrm{mot}}(\mathrm{ET}\underset{\mathrm{T}}{\times}\mathrm{X}, \mathrm{Z}/\ell^{\mathrm{n}})^{\mathrm{W}}$*, where* T *denotes a maximal torus in* G.

(ii) *Assume in addition to the hypotheses in (i) that for a fixed integer* $j \geq 0$*, the cycle map induces an isomorphism* $\mathbb{H}^{i,j}_{mot}(\mathrm{X}, \mathrm{Z}/\ell^{\mathrm{n}}) \rightarrow \mathbb{H}^{\mathrm{i}}_{\mathrm{et}}(\mathrm{X}, \mu_{\ell^{\mathrm{n}}}(\mathrm{j}))$ *for all* $i \geq 0$*. Then the induced map* $\mathbb{H}^{i,j}_{\mathrm{G},\mathrm{mot}}(\mathrm{X}, \mathrm{Z}/\ell^{\mathrm{n}}) = \mathbb{H}^{\mathrm{i,j}}_{\mathrm{mot}}(\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X}, \mathrm{Z}/\ell^{\mathrm{n}}) \rightarrow \mathbb{H}^{\mathrm{i}}_{\mathrm{G,et}}(\mathrm{X}, \mu_{\ell^{\mathrm{n}}}(\mathrm{j})) = \mathbb{H}^{\mathrm{i}}_{\mathrm{et}}(\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X}, \mu_{\ell^{\mathrm{n}}}(\mathrm{j}))$ *is also an isomorphism for all* $i \geq 0$ *provided* $|\mathrm{W}|$ *is relatively prime to* ℓ*.*

(iii) *Let* H *denote a closed linear algebraic subgroup of maximal rank in* G *so that* T *is a maximal torus in* H *as well. Let* W_{H} *denote the Weyl group* $\mathrm{N}_{\mathrm{H}}(\mathrm{T})/\mathrm{T}$*, where* $\mathrm{N}_{\mathrm{H}}(\mathrm{T})$ *denotes the normalizer of* T *in* H. *Assume that* $|\mathrm{W}_{\mathrm{H}}|$ *is prime to* ℓ*. Then* $\mathrm{H}^{*,\bullet}_{\mathrm{mot}}(\mathrm{G}/\mathrm{H}, \mathrm{Z}/\ell^{\mathrm{n}}) \cong \mathrm{H}^{*,\bullet}_{\mathrm{mot}}(\mathrm{G}/\mathrm{T}, \mathrm{Z}/\ell^{\mathrm{n}})^{\mathrm{W}_{\mathrm{H}}}$*.*

(iv) *The statements corresponding to those in (i) and (iii) also hold for étale cohomology with* Z/ℓ^n*-coefficients.*

(v) *Therefore, under the assumptions of (iii), the higher cycle map*

$$cycl : \mathrm{H}^{*,\bullet}_{\mathrm{mot}}(\mathrm{G}/\mathrm{H}, \mathrm{Z}/\ell^{\mathrm{n}}) \rightarrow \mathrm{H}^{*,\bullet}_{\mathrm{et}}(\mathrm{G}/\mathrm{H}, \mathrm{Z}/\ell^{\mathrm{n}})$$

is an isomorphism.

(vi) *As a consequence, under the assumptions in (iii), the* ℓ^n*-torsion subgroup of the Brauer group of* G/H *is trivial.*

Remark 1.17. Using similar arguments, it is shown in [IJ20-2], that the cycle map with $\mathbb{Z}/\ell^n$-coefficients from the equivariant motivic cohomology of the semi-stable locus for the action of a split reductive group on certain quasi-projective smooth varieties, to the corresponding equivariant étale cohomology is surjective. This will show the triviality of the ℓ-primary torsion part of the equivariant Brauer group of the semi-stable locus, and hence the triviality of the ℓ-primary torsion part of the Brauer group of the corresponding GIT-quotient for similar values of ℓ.

Corollary 1.18. *Let* X *denote a smooth projective variety over k provided with the action of a connected linear algebraic group* G. *Assume that* X *is also provided with an action by* $\mathbb{G}_m$ *commuting with the action by* G. *Let* M *denote a fibrant motivic spectrum that has the rigidity property. Then, adopting the terminology as in Theorem 1.3, that for a linear algebraic group* G*,* $\mathrm{BG} = \underset{\mathrm{m}\rightarrow\infty}{lim} \mathrm{BG}^{\mathrm{gm,m}}$ *and* $\mathrm{EG} = \underset{\mathrm{m}\rightarrow\infty}{lim} \mathrm{EG}^{\mathrm{gm,m}}$*, one obtains the homotopy commutative diagram*

$$\begin{tikzcd} h(\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X}, \mathrm{M}) \arrow[r, "i^*"] \arrow[d, "tr^*_{\mathrm{X}}"] & h(\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X}^{\mathbb{G}_{\mathrm{m}}}, \mathrm{M}) \arrow[dl, "tr^*_{\mathrm{X}^{\mathbb{G}_{\mathrm{m}}}}"] \\ h(\mathrm{BG}, \mathrm{M}) & \end{tikzcd}$$

where $h(\quad, \mathrm{M}) = \mathrm{Map}(\quad, \mathrm{M})$ *denotes the hypercohomology spectrum with respect to the motivic spectrum* M *and* $i : \mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X}^{\mathbb{G}_{\mathrm{m}}} \rightarrow \mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X}$ *is the map induced by the closed immersion* $\mathrm{X}^{\mathbb{G}_{\mathrm{m}}} \rightarrow \mathrm{X}$*.*

As the next and final example, we consider the stable splittings of $\mathrm{BGL_n}$ as $\bigvee_{i\leq n} \mathrm{BGL_i}/\mathrm{BGL_{i-1}}$ in the motivic (and also étale) stable homotopy framework. Such splittings were originally obtained in [Sn79] and then rederived in [MP]. As a result, we will refer to these splittings as the Snaith-Mitchell-Priddy splittings.

Corollary 1.19. *For each integer* $n \geq 1$*, there exists a splitting*

$$\Sigma_{\mathbf{T}}\mathrm{BGL}_{\mathrm{n},+} \simeq \bigvee_{\mathrm{i}\leq\mathrm{n}} \Sigma_{\mathbf{T}}\mathrm{BGL}_{\mathrm{i},+}/\mathrm{BGL}_{\mathrm{i}-1,+}$$

in $\mathbf{Spt}_{\mathrm{mot}}$*, in case* $char(k) = 0$*. In case* $char(k) = p > 0$*, a corresponding splitting holds on replacing the suspension spectra above with the corresponding suspension spectra with p-inverted. Let* $\mathcal{E}$ *denote a ring spectrum in* $\mathbf{Spt}_{\mathrm{et}}$ *with all its homotopy groups* ℓ*-primary torsion, for some prime* $\ell \neq char(k)$*. Then, a corresponding splitting also holds in* $\mathbf{Spt}_{\mathrm{et},\mathcal{E}}$ *after all the above objects have been smashed with the ring spectrum* $\mathcal{E}$*.*

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aspects of Theorem 5.2. We are also happy to acknowledge [BP, Lemma 3.5 and its proof] as one of the inspirations for this paper.

Overview of the paper. We will presently provide the following overview of the organization of the paper. In the following section, we review the basic results of [CJ19] on the motivic Becker-Gottlieb transfer: this should make the present paper more or less self-contained. This is followed by two key sections where we establish the additivity of the transfer and trace. We discuss the proof of the key Theorem 1.5 in detail in section 3. In addition to invoking key results in motivic homotopy theory, like the purity theorem of Morel-Voevodsky (see [MV99, Theorem 2.23]), and rigidity for $\mathbb{A}^1$-representable cohomology theories, we also introduce the notion of motivic tubular neighborhoods, which enable us to obtain the additivity theorem for the transfer, as in Theorem 1.12, in a general setting. This is discussed in section 4. The last section is all devoted to various applications. In addition to obtaining various splittings making use of the motivic transfer, we obtain a number of double coset formulae in the motivic framework, the analogues of most of which have been known in the classical setting in algebraic topology.

2. G-equivariant spectra, Non-equivariant spectra, Transfer and Trace

2.1. **The G-equivariant spectra.** The G-spectra will be indexed not by the non-negative integers, but by the Thom-spaces of finite dimensional representations of the given linear algebraic group G. We will adopt the terminology and conventions from [CJ19, section 3.2]. Accordingly we let $\mathcal{C} = \mathrm{PSh/S}$ denote the category of pointed simplicial presheaves on the big Nisnevich or étale site of the given base scheme S with the smash product $\wedge^S$ as in [CJ19, (3.0.5)]. $\mathcal{C}^{\mathrm{G}}$ will denote its subcategory consisting of pointed simplicial presheaves with G-actions. $\mathrm{Sph}^{\mathrm{G}}$ will denote the full subcategory of $\mathcal{C}^{\mathrm{G}} = \mathrm{PSh/S}^{\mathrm{G}}$ whose objects are $\{\mathrm{T_V}|\mathrm{V}\}$, and where V varies over all finite dimensional representations of the group G and $\mathrm{T_V}$ denotes its Thom-space.

We also let $\mathrm{USph}^{\mathrm{G}}$ denote the category whose objects are $\{\mathrm{U}(\mathrm{T_V})|\mathrm{T_V}\,\varepsilon\,\mathrm{Sph}^{\mathrm{G}}\}$, where U is the forgetful functor forgetting the G-action, that is, the morphisms between two objects $\mathrm{U}(\mathrm{T_V}) \to \mathrm{U}(\mathrm{T_W})$ will be maps $\mathrm{T_V} \to \mathrm{T_W}$ which are not required to be G-equivariant. We will make $\mathrm{Sph}^{\mathrm{G}}$ ($\mathrm{USph}^{\mathrm{G}}$) an enriched monoidal category, enriched over the category $\mathcal{C}^{\mathrm{G}}$ ($\mathcal{C}$, respectively) as follows. First let $\mathbf{T}^0 = \mathrm{S}^0$, which is the unit for the smash product of pointed simplicial sets. Then for V, W that are G-representations, we let

$$\mathcal{H}om_{\mathcal{C}^{\mathrm{G}}}(\mathrm{T_V}, \mathrm{T_{V\oplus W}}) = \underset{\alpha:\mathrm{V}\to\mathrm{V\oplus W}}{\sqcup}\mathrm{T_W}, \mathrm{W} \neq \{0\}$$
$$= \underset{\alpha:\mathrm{V}\to\mathrm{V}}{\sqcup}\mathrm{S}^0, \mathrm{W} = \{0\} \tag{2.1.1}$$

where the sum varies over all homothety classes of G-equivariant and k-linear injective maps $\mathrm{V} \to \mathrm{V\oplus W}$. Moreover, $\mathrm{S}^0 = (\mathrm{Spec\,k})_+$. This is the enriched internal hom in the $\mathcal{C}^{\mathrm{G}}$-enriched category $\mathrm{Sph}^{\mathrm{G}}$. One defines the $\mathcal{C}$-enriched internal hom in $\mathrm{USph}^{\mathrm{G}}$ by a similar formula as in (2.1.1) where the sum now varies over homothety classes of k-linear injective maps $\mathrm{V} \to \mathrm{V} \oplus \mathrm{W}$, so that the forgetful functor $\mathrm{U} : \mathrm{Sph}^{\mathrm{G}} \to \mathrm{USph}^{\mathrm{G}}$ is a simplicially enriched functor. Then the following is proven in [CJ19, Proposition 3.2].

Proposition 2.1. *With the above definitions, the category* $\mathrm{Sph}^{\mathrm{G}}$ *is a symmetric monoidal* $\mathcal{C}^{\mathrm{G}}$*-enriched category, where the monoidal structure is given by* $\mathrm{T_V}\wedge\mathrm{T_W} = \mathrm{T_{V\oplus W}}$*. (A corresponding result holds for the category* $\mathrm{USph}^{\mathrm{G}}$*.)*

Recall from [CJ19, Definition 3.3] that a G-*equivariant spectrum* will mean a $\mathcal{C}^{\mathrm{G}}$-enriched functor $\mathrm{Sph}^{\mathrm{G}} \to \mathcal{C}^{\mathrm{G}}$. The category of such G-equivariant spectra, $\mathbf{Spt}^{\mathrm{G}}$, is a $\mathcal{C}^{\mathrm{G}}$-enriched category. Paraphrasing this, a G-equivariant spectrum simply means a collection $\mathcal{X}(\mathrm{T_V})\,\varepsilon\,\mathcal{C}^{\mathrm{G}}$, together with a compatible collection of maps $\mathrm{T_W} \wedge \mathcal{X}(\mathrm{T_V}) \to \mathcal{X}(\mathrm{T_{W\oplus V}})$ as W and V vary among all finite dimensional representations of the group G.

A morphism (or map) $\mathcal{X}' \to \mathcal{X}$ between G-equivariant spectra is a $\mathcal{C}^{\mathrm{G}}$-natural transformation: unraveling this definition, one sees that, such a map of spectra is given by a compatible collection of G-equivariant maps $\{\mathcal{X}'(\mathrm{T_V}) \to \mathcal{X}(\mathrm{T_V})|\mathrm{T_V}\,\varepsilon\,\mathrm{Sph}^{\mathrm{G}}\}$ which are compatible with the pairings $\mathrm{T_W} \wedge \mathcal{X}'(\mathrm{T_V}) \to \mathcal{X}'(\mathrm{T_{W\oplus V}})$ and $\mathrm{T_W} \wedge \mathcal{X}(\mathrm{T_V}) \to \mathcal{X}(\mathrm{T_{W\oplus V}})$. The G-equivariant sphere spectrum $\mathbb{S}^{\mathrm{G}}$ will be the spectrum defined by $\mathbb{S}^{\mathrm{G}}(\mathrm{T_V}) = \mathrm{T_V}$, for each finite dimensional representation V of G. At this point one may define a smash product, $\wedge$, on $\mathbf{Spt}^{\mathrm{G}}/\mathrm{S}$ as in [CJ19, Definition 3.5] which will make $\mathbf{Spt}^{\mathrm{G}}/\mathrm{S}$ into a symmetric monoidal category.

2.2. **Non-equivariant spectra.** For us, it is important to consider Spanier-Whitehead duality in the category of non-equivariant spectra. The category of non-equivariant spectra (indexed by the natural numbers $\mathbb{N}$) will be denoted **Spt**. Then the following intermediate categories, denoted $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$ and $\mathbf{USpt}^{\mathrm{G}}$, intermediate between $\mathbf{Spt}^{\mathrm{G}}$ and **Spt** are introduced in [CJ19, section 3.2]. The first category (the second category) will denote the category of $\mathcal{C}$-enriched functors $\mathrm{Sph}^{\mathrm{G}} \to \mathcal{C}$ ($\mathrm{USph}^{\mathrm{G}} \to \mathcal{C}$, respectively). Again, paraphrasing this, an object of the category $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$ is given by $\{\mathcal{X}'(\mathrm{T_V})|\mathrm{T_V}\,\varepsilon\,\mathrm{Sph}^{\mathrm{G}}\}$, provided with a compatible family of structure maps $\mathrm{T_W} \wedge \mathcal{X}'(\mathrm{T_V}) \to \mathcal{X}'(\mathrm{T_{W\oplus V}})$ in PSh, i.e. these maps are no longer required to be G-equivariant. Morphisms between two such objects $\{\mathcal{Y}'(\mathrm{T_V})|\mathrm{T_V}\,\varepsilon\,\mathrm{Sph^G}\}$ and $\{X'(\mathrm{T_V})|\mathrm{T_V}\,\varepsilon\,\mathrm{Sph^G}\}$ are given by compatible collections of maps $\{\mathcal{Y}'(\mathrm{T_V}) \to \mathcal{X}'(\mathrm{T_V})|\mathrm{T_V}\,\varepsilon\,\mathrm{Sph^G}\}$ which are no longer required to be G-equivariant, but compatible with the pairings: $\mathrm{T_W} \wedge \mathcal{Y}'(\mathrm{T_V}) \to \mathcal{Y}'(\mathrm{T_{W\oplus V}})$ and $\mathrm{T_W} \wedge \mathcal{X}'(\mathrm{T_V}) \to \mathcal{X}'(\mathrm{T_{W\oplus V}})$. Objects and morphisms of the $\mathcal{C}$-enriched category $\mathbf{USpt}^{\mathrm{G}}$ have a similar description with $\mathrm{Sph^G}$ replaced by $\mathrm{USph^G}$.

Observe that, now there is a forgetful functor $\tilde{\mathrm{U}}:\mathbf{Spt}^{\mathrm{G}} \to \widetilde{\mathbf{USpt}}^{\mathrm{G}}$, sending a G-equivariant spectrum $\mathcal{X}$ to $\mathrm{For} \circ \mathcal{X}$ where $\mathrm{For}:\mathcal{C}^{\mathrm{G}} \to \mathcal{C}$ is the forgetful functor, forgetting the G-action. One defines a smash product $\wedge$ and an internal hom in these categories just as for $\mathbf{Spt}^{\mathrm{G}}$. One defines a *stable injective model* structure on $\mathbf{USpt}^{\mathrm{G}}$ with respect to which it is shown in [CJ19, Proposition 3.8] that $\mathbf{USpt}^{\mathrm{G}}$ is a symmetric monoidal model category with respect to the smash product.

Let $\mathbf{Spt}$ denote the (usual) category of motivic (or étale) spectra defined as follows. Let $\mathbb{N}$ denote the set of natural numbers. (One may want to observe that for each $n\,\varepsilon\,\mathbb{N}$, there exists a finite dimensional representation V of G with $dim(\mathrm{V}) = \mathrm{n}$.) Then we let $\mathbf{Spt}$ denote the following category: its objects are

$$\mathcal{X}' = \{X_n\,\varepsilon\,\mathrm{PSh/S},\ \text{along with a compatible family of structure maps } \mathbf{T}^{\mathrm{m}} \wedge \mathrm{X_n} \to \mathrm{X_{n+m}}|\mathrm{n},\mathrm{m}\,\varepsilon\,\mathbb{N}\}.$$

(In the motivic setting, $\mathbf{T}$ will denote either $(\mathbb{P}^1,\infty)$ or a variant of it such as $\mathbb{G}_m \wedge \mathrm{S}^1$. In the étale setting, we may take $\mathbf{T}$ to be the same as in the motivic setting, or one may also let it be just S^1.)

Morphisms between two such objects $\mathcal{X}'$ and $\mathcal{Y}'$ are defined as compatible collection of maps $\mathcal{X}'_n \to \mathcal{Y}'_n$, $n\,\varepsilon\,\mathbb{N}$ compatible with suspensions by $\mathbf{T}^m$, $m\,\varepsilon\,\mathbb{N}$. One defines various model structures on this category: see [CJ14, section 3].

In addition to the level-wise projective and injective model structures on the above categories of spectra, one also has the corresponding stable model structures defined in such a way that the fibrant objects are the Ω-spectra where each component space is also a fibrant object in the level-wise model structures: see [CJ19, 3.3] for further details.

For each natural number n, we choose a trivial representation of G of dimension n. We will denote this representation by n and its Thom space by $\mathrm{T_n}$. We will identify $\mathbb{N}$ with the $\mathcal{C}$-enriched subcategory of $\mathrm{USph^G}$ consisting of these objects and where

$$\begin{aligned} \mathcal{H}om_{\mathbb{N}}(\mathrm{T_n}, \mathrm{T_{n+m}}) &= \mathrm{T_m}, \text{if } \mathrm{m} \neq 0 \\ &= \mathrm{S}^0, \text{if } \mathrm{m} = 0. \end{aligned} \tag{2.2.1}$$

Thus, we obtain a $\mathcal{C}$-enriched *faithful functor* $i:\mathbb{N} \to \mathrm{USph^G}$ and the functor i^* defines a simplicially enriched functor $\mathbf{USpt}^{\mathrm{G}} \to \mathbf{Spt}$. The functor i^* admits a left adjoint, which we denote by $\mathbb{P}:\mathbf{Spt} \to \mathbf{USpt}^{\mathrm{G}}$.

To relate the $\mathcal{C}$-enriched categories, $\mathbf{USpt}^{\mathrm{G}}$ and $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$, one first observes that there is a forgetful functor $j:\mathrm{Sph^G} \to \mathrm{USph^G}$ that sends the Thom-space, $\mathrm{T_V}$, of a G-representation V to $\mathrm{T_V}$ but viewing V as just a k-vector space. Therefore, pull-back by j defines the $\mathcal{C}$-enriched functor $j^*:\mathbf{USpt}^{\mathrm{G}} \to \widetilde{\mathbf{USpt}}^{\mathrm{G}}$. One defines a functor $\tilde{\mathbb{P}}$ as the left-adjoint to j^*. Then the following is proven in [CJ19, Proposition 3.10].

Proposition 2.2. *(i) The functors $\mathbb{P}$ and i^* define a Quillen adjunction between the projective stable model structures on $\mathbf{USpt}^{\mathrm{G}}$ and $\mathbf{Spt}$. This is, in fact, a Quillen equivalence.*

(ii) The functors $\tilde{\mathbb{P}}$ and j^* define a Quillen-equivalence between the stable projective model structures on $\mathbf{USpt}^{\mathrm{G}}$ and $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$.

(iii) The functors $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are strict-monoidal functors.

In addition, the following are also proven in [CJ19, section 3.3.5].

2.3. **Equivariant spectra vs. non-equivariant spectra.** We will let $\tilde{\mathrm{U}}$ denote the forgetful functor $\mathbf{Spt}^{\mathrm{G}} \to \widetilde{\mathbf{USpt}}^{\mathrm{G}}$ that forgets the action by G, i.e. sending $\mathcal{X}$ to $\mathrm{For} \circ \mathcal{X}$ where $\mathrm{For}:\mathcal{C}^{\mathrm{G}} \to \mathcal{C}$ is the forgetful functor, forgetting the G-action.

Proposition 2.3. *(See* [CJ19, Proposition 3.13]*.) Let* $\mathrm{M},\mathrm{N}\,\varepsilon\,\mathbf{Spt}^{\mathrm{G}}$*. Then*

$$(i)\ \tilde{\mathrm{U}}(\mathrm{M}) \wedge \tilde{\mathrm{U}}(\mathrm{N}) = \tilde{\mathrm{U}}(\mathrm{M} \wedge \mathrm{N})\ \textit{and}\ (\mathrm{ii})\ \tilde{\mathrm{U}}(\mathcal{H}\mathrm{om}(\mathrm{M},\mathrm{N})) = \mathcal{H}\mathrm{om}(\tilde{\mathrm{U}}(\mathrm{M}), \tilde{\mathrm{U}}(\mathrm{N})).$$

Let $\mathrm{M},\mathrm{N}\,\varepsilon\,\mathbf{Spt}^{\mathrm{G}}$. The fact that one may find functorial cofibrant and fibrant replacements of objects in $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$ shows that one may find a functorial cofibrant replacement $\tilde{\mathrm{M}} \to \tilde{\mathrm{U}}(\mathrm{M})$ in $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$ and a functorial fibrant replacement $\tilde{\mathrm{U}}(\mathrm{N}) \to \hat{\mathrm{N}}$ in $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$. The functoriality of the cofibrant and fibrant replacements, together with the fact that the group action by G is as a presheaf of sections over each object in the site, shows that in fact $\tilde{\mathrm{M}}$, $\tilde{\mathrm{N}}$ and the maps $\tilde{\mathrm{M}} \to \mathrm{M}$, $\mathrm{N} \to \hat{\mathrm{N}}$ all belong to $\mathbf{Spt}^{\mathrm{G}}$. Therefore, it is possible to define

$$\mathrm{M}\overset{\mathrm{L}}{\wedge}\mathrm{N} = \tilde{\mathrm{U}}(\tilde{\mathrm{M}}) \wedge \tilde{\mathrm{U}}(\mathrm{N}), \quad \mathcal{RH}\mathrm{om}(\mathrm{M},\mathrm{N}) = \mathcal{H}\mathrm{om}(\tilde{\mathrm{U}}(\tilde{\mathrm{M}}), \tilde{\mathrm{U}}(\hat{\mathrm{N}})), \quad \mathrm{D}(\mathrm{M}) = \mathcal{RH}\mathrm{om}(\tilde{\mathrm{U}}(\mathrm{M}), \tilde{\mathrm{U}}(\mathbb{S}^{\mathrm{G}})) \tag{2.3.1}$$

with $\mathrm{M}\overset{\mathrm{L}}{\wedge}\mathrm{N}, \mathcal{RH}\mathrm{om}(\mathrm{M},\mathrm{N}), \mathrm{D}(\mathrm{M}) \,\varepsilon\, \mathbf{Spt}^{\mathrm{G}}$. (In fact, since *we choose to work with the injective model structures*, every object is cofibrant and therefore there is no need for any cofibrant replacements.) Therefore, for any simplicial presheaf P, the dual $\mathrm{D}(\mathbb{S}^{\mathrm{G}} \wedge \mathrm{P}_+) = \mathcal{RH}\mathrm{om}(\mathbb{S}^{\mathrm{G}} \wedge \mathrm{P}, \mathbb{S}^{\mathrm{G}})$ will also identify with the dual $\mathrm{D}(\tilde{\mathrm{U}}(\mathbb{S}^{\mathrm{G}} \wedge \mathrm{P}_+))$. In view of the equivalence of the homotopy categories provided by the last theorem, this dual identifies with the dual taken after forgetting the G-actions and in the non-equivariant category of spectra **Spt**.

Similar conclusions will hold when $\mathcal{E}^{\mathrm{G}} \,\varepsilon\, \mathbf{Spt}^{\mathrm{G}}$ is a commutative ring spectrum with the corresponding smash product $\wedge_{\mathcal{E}^{\mathrm{G}}}$ and $\mathcal{H}om_{\mathcal{E}^{\mathrm{G}}}$ considered below. Given a G-equivariant ring spectrum $\mathcal{E}^{\mathrm{G}}$, $\mathcal{E} = i^*(\widetilde{\mathbb{P}}\widetilde{\mathrm{U}}(\mathcal{E}^{\mathrm{G}}))$ will denote the associated non-equivariant ring spectrum. (In this case the dual with respect to the ring spectrum $\mathcal{E}$ will be denoted $\mathrm{D}_{\mathcal{E}}$.)

In case $\mathcal{E}^{\mathrm{G}}$ is a commutative ring spectrum in $\mathbf{Spt}^{\mathrm{G}}$, we will let $\widetilde{\mathbf{USpt}}^{\mathrm{G}}_{\mathcal{E}}$ denote the subcategory of $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$, consisting of module spectra over $\mathcal{E}^{\mathrm{G}}$ and their maps. In this case, the smash product $\wedge$ will be replaced by $\wedge_{\mathcal{E}^{\mathrm{G}}}$ which is defined as

$$\mathrm{M} \wedge_{\mathcal{E}^{\mathrm{G}}} \mathrm{N} = \mathrm{Coeq}(\mathrm{M} \wedge \mathcal{E}^{\mathrm{G}} \wedge \mathrm{N} \rightrightarrows \mathrm{M} \wedge \mathrm{N}), \tag{2.3.2}$$

where the two maps above make use of the module structures on M and N, respectively. The corresponding internal *Hom* will be denoted $\mathcal{H}om_{\mathcal{E}^{\mathrm{G}}}$.

The main G-equivariant ring spectra, other than the sphere spectrum $\mathbb{S}^{\mathrm{G}}$, that enter into the construction of the transfer will be the following:

$$\begin{aligned} &(i) \quad \mathbb{S}^{\mathrm{G}}[p^{-1}] \text{ if the base scheme S is a field of characteristic } p, \text{(ii)} \quad \mathbb{S}^{\mathrm{G}}_{(\ell)} \text{ and} \\ &(iii) \quad \widehat{\mathbb{S}}^{\mathrm{G}}_{\ell}, \text{ where } \ell \text{ is a prime different from the residue characteristics.} \end{aligned} \tag{2.3.3}$$

Definition 2.4. (Co-module structures) Assume further that C is an unpointed simplicial presheaf. Then the diagonal map $\Delta : \mathrm{C}_+ \to \mathrm{C}_+ \wedge \mathrm{C}_+$ together with the augmentation $\epsilon : \mathrm{C}_+ \to \mathrm{S}^0$ defines the structure of an associative co-algebra of simplicial presheaves on C_+. An unpointed simplicial presheaf P will be called a right C-co-module, if it comes equipped with maps $\Delta : \mathrm{P}_+ \to \mathrm{P}_+ \wedge \mathrm{C}_+$ so that the diagrams:

$$\begin{tikzcd} \mathrm{P}_+ \arrow[r, "\Delta"] \arrow[d, "\Delta"] & \mathrm{P}_+ \wedge \mathrm{C}_+ \arrow[d, "id\wedge\Delta"] \\ \mathrm{P}_+ \wedge \mathrm{C}_+ \arrow[r, "\Delta\wedge id"] & \mathrm{P}_+ \wedge \mathrm{C}_+ \wedge \mathrm{C}_+ \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{P}_+ \arrow[d, "\Delta"] \arrow[dr, "id"] & \\ \mathrm{P}_+ \wedge \mathrm{C}_+ \arrow[r, "id\wedge\epsilon"] & \mathrm{P}_+ \wedge \mathrm{S}^0 \end{tikzcd} \tag{2.3.4}$$

commute. *The most common choice of* C *is with* C = P and with the obvious diagonal map $\Delta : \mathrm{P}_+ \to \mathrm{P}_+ \wedge \mathrm{P}_+$ as providing the co-module structure. However, the reason we are constructing the pre-transfer in this generality (see the definition below) is so that we are able to obtain strong additivity results as in Theorem 3.1.

Definition 2.5. (*The* G*-equivariant pre-transfer*) Assume that the unpointed simplicial presheaf P is such that $\mathbb{S}^{\mathrm{G}} \wedge \mathrm{P}_+$ is dualizable in $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$ and is provided with a G-equivariant map $\mathrm{f} : \mathrm{P}_+ \to \mathrm{P}_+$. Assume further that C is an unpointed simplicial presheaf so that P is a right C-co-module. Then the G*-equivariant pre-transfer with respect to* C is defined to be a map $tr(f)'^{\mathrm{G}} : \mathbb{S}^{\mathrm{G}} \to \mathbb{S}^{\mathrm{G}} \wedge \mathrm{C}_+$, which is the composition of the following maps. Let $e : \mathrm{D}(\mathbb{S}^{\mathrm{G}} \wedge \mathrm{P}_+) \wedge \mathbb{S}^{\mathrm{G}} \wedge \mathrm{P}_+ \to \mathbb{S}^{\mathrm{G}}$ denote the evaluation map. Observe that, this map being natural, is automatically G-equivariant. We take the dual of this map to obtain:

$$c = D(e) : \mathbb{S}^{\mathrm{G}} \simeq \mathrm{D}(\mathbb{S}^{\mathrm{G}}) \to \mathrm{D}(\mathrm{D}(\mathbb{S}^{\mathrm{G}} \wedge \mathrm{P}_+) \wedge (\mathbb{S}^{\mathrm{G}} \wedge \mathrm{P}_+)) \xleftarrow{\simeq} \mathrm{D}(\mathbb{S}^{\mathrm{G}} \wedge \mathrm{P}_+) \wedge (\mathbb{S}^{\mathrm{G}} \wedge \mathrm{P}_+) \xrightarrow{\tau} (\mathbb{S}^{\mathrm{G}} \wedge \mathrm{P}_+) \wedge \mathrm{D}(\mathbb{S}^{\mathrm{G}} \wedge \mathrm{P}_+). \tag{2.3.4}$$

Here τ denotes the obvious flip map interchanging the two factors and c denotes the co-evaluation. The reason that taking the double dual yields the same object up to weak-equivalence is because we are in fact taking the dual in the setting discussed above. Observe that all the maps that *go in the left-direction are weak-equivalences.* All the maps involved in the definition of the co-evaluation map are *natural maps and therefore automatically* G*-equivariant.*

To complete the definition of the pre-transfer, one simply composes the co-evaluation map with the following composite map:

$$\begin{aligned}(2.3.5)\qquad &(\mathbb{S}^{\mathrm{G}} \wedge \mathrm{P}_+) \wedge \mathrm{D}(\mathbb{S}^{\mathrm{G}} \wedge \mathrm{P}_+) \xrightarrow{\tau} \mathrm{D}(\mathbb{S}^{\mathrm{G}} \wedge \mathrm{P}_+) \wedge (\mathbb{S}^{\mathrm{G}} \wedge \mathrm{P}_+) \xrightarrow{\mathrm{id}\wedge \mathrm{f}} \mathrm{D}(\mathbb{S}^{\mathrm{G}} \wedge \mathrm{P}_+) \wedge (\mathbb{S}^{\mathrm{G}} \wedge \mathrm{P}_+)\\ &\xrightarrow{id\wedge\Delta} \mathrm{D}(\mathbb{S}^{\mathrm{G}} \wedge \mathrm{P}_+) \wedge (\mathbb{S}^{\mathrm{G}} \wedge \mathrm{P}_+) \wedge (\mathbb{S}^{\mathrm{G}} \wedge \mathrm{C}_+) \xrightarrow{\mathrm{e}\wedge \mathrm{id}} \mathbb{S}^{\mathrm{G}} \wedge (\mathbb{S}^{\mathrm{G}} \wedge \mathrm{C}_+) \simeq \mathbb{S}^{\mathrm{G}} \wedge \mathrm{C}_+.\end{aligned}$$

The corresponding G-*equivariant trace*, $\tau(f)'_{\mathrm{G}}$ is defined as the composition of the above pre-transfer $tr(f)'^{\mathrm{G}}$ with the projection sending C_+ to S^0_+.

The first step in the corresponding *transfer* is to apply the construction $\mathrm{E}\underset{\mathrm{G}}{\times}$ to the pre-transfer, where $\mathrm{E} \to \mathrm{B}$ is a G-torsor, for the linear algebraic group G.

When $\mathrm{C} = \mathrm{P}$ and $f = id_{\mathrm{P}}$, the pre-transfer (trace) will be denoted $tr'^{\mathrm{G}}_{\mathrm{P}}$ ($\tau^{\mathrm{G}}_{\mathrm{P}}$, respectively). In the non-equivariant case, the corresponding maps will be denoted tr'_{P} (τ'_{P}, respectively).

Remark 2.6. Observe that now the trace maps identify with the following composite maps:

$$\tau^{\mathrm{G}}_{\mathrm{P}} : \mathbb{S}^{\mathrm{G}} \xrightarrow{c} \mathbb{S}^{\mathrm{G}} \wedge \mathrm{P}_+ \wedge \mathrm{D}(\mathbb{S}^{\mathrm{G}} \wedge \mathrm{P}_+) \xrightarrow{\tau} \mathrm{D}(\mathbb{S}^{\mathrm{G}} \wedge \mathrm{P}_+) \wedge \mathbb{S}^{\mathrm{G}} \wedge \mathrm{P}_+ \xrightarrow{\mathrm{e}} \mathbb{S}^{\mathrm{G}} \text{ and}$$

$$\tau_{\mathrm{P}} : \mathbb{S} \xrightarrow{c} \mathrm{P}_+ \wedge \mathrm{D}(\mathbb{S} \wedge \mathrm{P}_+) \xrightarrow{\tau} \mathrm{D}(\mathbb{S} \wedge \mathrm{P}_+) \wedge \mathbb{S} \wedge \mathrm{P}_+ \xrightarrow{\mathrm{e}} \mathbb{S}.$$

Definition 2.7. If $\mathcal{E}$ denotes *any commutative ring spectrum* in $\mathbf{Spt}^{\mathrm{G}}$, one may replace the sphere spectrum $\mathbb{S}^{\mathrm{G}}$ everywhere by $\mathcal{E}^{\mathrm{G}}$ and define the pre-transfer and trace similarly, provided the unpointed simplicial presheaf P is such that $\mathcal{E}^{\mathrm{G}} \wedge \mathrm{P}_+$ is dualizable in $\widetilde{\mathbf{USpt}}^{\mathrm{G}}_{\mathcal{E}^{\mathrm{G}}}$ and is provided with a G-equivariant map $\mathrm{f} : \mathrm{P}_+ \to \mathrm{P}_+$. These will be denoted $tr(f)'^{\mathrm{G}}_{\mathcal{E}}$, $tr'^{\mathrm{G}}_{\mathcal{E}}$, $tr'_{\mathrm{P},\mathcal{E}}$, $\tau^{\mathrm{G}}_{\mathrm{P},\mathcal{E}}$, $\tau_{\mathrm{P},\mathcal{E}}$, etc.

Definition 2.8. $\mathbf{Spt}^{\mathrm{G}}_{\mathrm{mot}}$, $\widetilde{\mathbf{USpt}}^{\mathrm{G}}_{mot}$ and $\mathbf{USpt}^{\mathrm{G}}_{mot}$ ($\mathbf{Spt}^{\mathrm{G}}_{\mathrm{et}}$, $\widetilde{\mathbf{USpt}}^{\mathrm{G}}_{et}$ and $\mathbf{USpt}^{\mathrm{G}}_{et}$) will denote the corresponding category of spectra defined on the Nisnevich site (on the étale site, respectively).

3. The additivity of the trace: Proof of Theorem 1.5.

Let

$$(3.0.6)\qquad \mathrm{U} \xrightarrow{\mathrm{j}} \mathrm{X} \xrightarrow{\mathrm{k}} \mathrm{X}/\mathrm{U} = \mathrm{Cone(j)} \to \mathrm{S}^1 \wedge \mathrm{U}_+$$

denote a cofiber sequence where both U and X are unpointed simplicial presheaves, with j a cofibration. Now a key point to observe is that all of U, X and X/U have the structure of right X-co-modules. The right X-co-module structure on X is given by the diagonal map $\Delta : \mathrm{X}_+ \to \mathrm{X}_+ \wedge \mathrm{X}_+$, while the right X-co-module structure on U is given by the map $\Delta : \mathrm{U}_+ \xrightarrow{\Delta} \mathrm{U}_+ \wedge \mathrm{U}_+ \xrightarrow{\mathrm{id}\wedge \mathrm{j}_+} \mathrm{U}_+ \wedge \mathrm{X}_+$, where $j : \mathrm{U} \to \mathrm{X}$ is the given map. The right X-co-module structure on X/U is obtained in view of the commutative square

$$(3.0.7)\qquad \begin{tikzcd} \mathrm{U} \arrow[r, "(id\times j)\circ\Delta"] \arrow[d, "j"] & \mathrm{U}\times\mathrm{X} \arrow[d, "j\times id"] \\ \mathrm{X} \arrow[r, "\Delta"] & \mathrm{X}\times\mathrm{X} \end{tikzcd}$$

which provides the map

$$(3.0.8)\qquad \mathrm{X}/\mathrm{U} \to (\mathrm{X}\times\mathrm{X})/(\mathrm{U}\times\mathrm{X}) \cong (\mathrm{X}/\mathrm{U}) \wedge \mathrm{X}_+.$$

If one assumes that all the simplicial presheaves in (3.0.6) are provided with the action by a linear algebraic group G so that all the maps in (3.0.6) are also G-equivariant, one may see also that the all the above co-module structures are compatible with the actions by G.

We begin with the following results, which are variants of [LMS, Theorem 7.10, Chapter III and Theorem 2.9, Chapter IV] adapted to our contexts.

Theorem 3.1. *Let* $\mathrm{U} \xrightarrow{\mathrm{j}} \mathrm{X} \xrightarrow{\mathrm{k}} \mathrm{X}/\mathrm{U} = \mathrm{Cone(j)} \to \mathrm{S}^1 \wedge \mathrm{U}_+$ *denote a cofiber sequence as in* (3.0.6) *where all the above simplicial presheaves are provided with actions by a linear algebraic group* G *so that all the above maps are* G*-equivariant. Let* $f : \mathrm{U} \to \mathrm{U}$, $g : \mathrm{X} \to \mathrm{X}$ *denote two* G*-equivariant maps so that the diagram*

$$\begin{tikzcd} U_+ \arrow[r, "j"] \arrow[d, "f"'] & X_+ \arrow[d, "g"] \\ U_+ \arrow[r, "j"] & X_+ \end{tikzcd}$$

commutes. Let $h : X/U \to X/U$ *denote the corresponding induced map. Then, with the right* X*-co-module structures discussed above, one obtains the following commutative diagram:*

$$\begin{tikzcd} U_+ \arrow[r, "j_+"] \arrow[d, "\Delta"] & X_+ \arrow[r, "k_+"] \arrow[d, "\Delta"] & X/U \arrow[r, "l"] \arrow[d, "\Delta"] & S^1 \wedge U_+ \arrow[d, "S^1\Delta"] \\ U_+ \wedge X_+ \arrow[r, "j_+\wedge id"] & X_+ \wedge X_+ \arrow[r, "k_+\wedge id"] & (X/U) \wedge X_+ \arrow[r, "l\wedge id"] & S^1 \wedge U_+ \wedge X_+ \end{tikzcd} \tag{3.0.9}$$

Assume further that the $\mathbb{S}^G$*-suspension spectra of all the above simplicial presheaves are dualizable in* $\widetilde{\mathbf{USpt}}^G_{mot}$*. Then*

$$tr'^G(g) = tr'^G(f) + tr'^G(h) \quad (\text{see Definition 2.5}), \text{ and } \tau^G(g) = \tau^G(f) + \tau^G(h).$$

Moreover if $tr(f)$*,* $tr(g)$ *and* $tr(h)$ *denote the induced transfer maps in any one of the three basic contexts (a), (b) or (c) of Theorem 1.3, then*

$$tr(g) = tr(f) + tr(h).$$

Let $\mathcal{E}$ *denote a commutative ring spectrum in* $\widetilde{\mathbf{USpt}}^G_{mot}$ *or* $\widetilde{\mathbf{USpt}}^G_{et}$*. In the latter case, we will further assume that* $\mathcal{E}$ *is* ℓ*-complete for some prime* $\ell \neq char(k)$*. Then the corresponding results also hold if the smash products of the above simplicial presheaves with the ring spectrum* $\mathcal{E}$ *are dualizable in* $\widetilde{\mathbf{USpt}}^G_{mot,\mathcal{E}}$ *and* $\widetilde{\mathbf{USpt}}^G_{et,\mathcal{E}}$*.*

Theorem 3.2. *Let* $F = F_1 \sqcup_{F_3} F_2$ *denote a pushout of unpointed simplicial presheaves on the big étale or the big Nisnevich site of the base scheme, with the corresponding maps* $F_3 \to F_2$*,* $F_3 \to F_1$ *and* $F_j \to F$*, for* $j = 1, 2, 3$*, assumed to be cofibrations (i.e. injective maps of presheaves). Assume further the following:*

(i) *all the above simplicial presheaves are provided with compatible actions by the linear algebraic group* G *making all the maps above* G*-equivariant and*

(ii) *the* T*-suspension of spectra of all the above simplicial presheaves are dualizable in* $\widetilde{\mathbf{USpt}}^G_{mot}$*.*

Let $i_j : F_j \to F$ *denote the inclusion* $F_j \to F$*,* $j = 1, 2, 3$ *as well as the induced map on the Borel constructions. Then*

(1) $tr'^G_F = i_1 \circ tr'^G_{F_1} + i_2 \circ tr'^G_{F_2} - i_3 \circ tr'^G_{F_3}$ *and* $\tau^G_F = \tau^G_{F_1} + \tau^G_{F_2} - \tau^G_{F_3}$*,*
where tr'^G_F *and* $tr'^G_{F_j}$*,* $j = 1, 2, 3$ *(*τ^G_F*,* τ_{F_j}*,* $j = 1, 2, 3$*) denote the* G*-equivariant pre-transfer maps (*G*-equivariant trace maps, respectively) with equality holding in* $\mathbf{H}\widetilde{\mathbf{USpt}}^G_{mot}$*, which denotes the corresponding homotopy category. Moreover,* $tr_F = i_1 \circ tr_{F_1} + i_2 \circ tr_{F_2} - i_3 \circ tr_{F_3}$ *which denote the corresponding transfer maps in any one of the three basic contexts as in Theorem 1.3 with equality holding in* $\mathbf{H}\widetilde{\mathbf{USpt}}^G_{mot}$*.*

(2) In particular, taking $F_2 = *$*, and* $F = \mathrm{Cone}(F_3 \to F_1)$*, we obtain:*
$tr'^G_F = i_1 \circ tr'^G_{F_1} - i_3 \circ tr'^G_{F_3}$ *and* $\tau^G_F = \tau^G_{F_1} - \tau^G_{F_3}$ *as well as* $tr_F = i_1 \circ tr_{F_1} - i_3 \circ tr_{F_3}$ *which denote the corresponding transfer maps in any one of the three basic contexts as in Theorem 1.3 with equality holding in* $\mathbf{H}\widetilde{\mathbf{USpt}}^G_{mot}$*.*

Let $\mathcal{E}$ *denote a commutative ring spectrum in* $\widetilde{\mathbf{USpt}}^G_{mot}$ *or* $\widetilde{\mathbf{USpt}}^G_{et}$*. In the latter case, we will further assume that* $\mathcal{E}$ *is* ℓ*-complete for some prime* $\ell \neq char(k)$*. Then the corresponding results also hold if the smash products of the above simplicial presheaves with the ring spectrum* $\mathcal{E}$ *are dualizable in* $\mathbf{H}\widetilde{\mathbf{USpt}}^G_{mot,\mathcal{E}}$ *and* $\mathbf{H}\widetilde{\mathbf{USpt}}^G_{et,\mathcal{E}}$*, which denotes the corresponding homotopy category.*

Our next goal is to provide proofs of these two theorems. We will explicitly discuss only the case where the ring spectrum $\mathcal{E}^G$ ($\mathcal{E}$) is the equivariant sphere spectrum $\mathbb{S}^G$ (the sphere spectrum $\Sigma_{\mathbf{T}}$, respectively), as proofs in the other cases follow along the same lines. The additivity of the trace follows readily from the additivity of the pre-transfer, as it is obtained by composing with the with the projection $\mathbb{S}^G \wedge X_+ \to \mathbb{S}^G$. Proposition 3.5 below shows that the additivity of the transfer also follows from the additivity of the pre-transfer. Therefore, in what follows, we will only discuss the additivity of the pre-transfer. One may observe also that the discussion in section 2.3, especially Proposition 2.3, (2.3.1), (2.3.2) reduce the treatment of the equivariant case to the non-equivariant case.

Therefore, it suffices to consider the non-equivariant case, but making sure the key arguments, and constructions are canonical (or natural), so that they are automatically G-equivariant for the action of a linear algebraic group.

Since this is discussed in the topological framework in [LMS, Theorem 7.10, Chapter III and Theorem 2.9, Chapter IV], our proof amounts to verifying carefully and in a detailed manner that the same arguments there carry over to our framework. This is possible, largely because the arguments in the proof of [LMS, Theorem 7.10, Chapter III and Theorem 2.9, Chapter IV] depend only on a theory of Spanier-Whitehead duality in a symmetric monoidal triangulated category framework and [DP84] shows that the entire theory of Spanier-Whitehead duality works in such general frameworks. Nevertheless, it seems prudent to show explicitly that at least the key arguments in [LMS, Theorem 7.10, Chapter III and Theorem 2.9, Chapter IV] carry over to our framework.

The very first observation is that the hypotheses of Theorem 3.1 readily implies the commutativity of the diagram:

$$\begin{tikzcd} \mathrm{U}_+ \arrow[r, "j_+"] \arrow[d, "f"'] & \mathrm{X}_+ \arrow[r, "k_+"] \arrow[d, "g"'] & \mathrm{X/U} \arrow[r, "l"] \arrow[d, "h"'] & \mathrm{S}^1 \wedge \mathrm{U}_+ \arrow[d, "\mathrm{S}^1 f"'] \\ \mathrm{U}_+ \arrow[r, "j_+"] & \mathrm{X}_+ \arrow[r, "k_+"] & \mathrm{X/U} \arrow[r, "l"] & \mathrm{S}^1 \wedge \mathrm{U}_+ \end{tikzcd}$$

Next we proceed to verify the commutativity of the diagram (3.0.9). Since the first square clearly commutes, it suffices to verify the commutativity of the second square. This follows readily in view of the following commutative square of pairs:

$$\begin{tikzcd} (\mathrm{X}, \phi) \arrow[r] \arrow[d, "\Delta"] & (\mathrm{X}, \mathrm{U}) \arrow[d, "\Delta"] \\ (\mathrm{X} \times \mathrm{X}, \phi) \arrow[r] & (\mathrm{X} \times \mathrm{X}, \mathrm{U} \times \mathrm{X}) \end{tikzcd}$$

Observe, as a consequence that we have verified the hypotheses of [LMS, Theorem 7.10, Chapter III] are satisfied in the case of Theorem 3.1.

The next step is to observe that the F_{i}, $i = 1, 2, 3$ (F) in our Theorem 3.2, correspond to the F_{i} (F, respectively) in [LMS, Theorem 2.9, Chapter IV]. Now observe that

$$\mathrm{F}_3 \to \mathrm{F}_1 \sqcup \mathrm{F}_2 \to \mathrm{F} \to \mathrm{S}^1 \wedge \mathrm{F}_{3,+} \tag{3.0.10}$$

is a distinguished triangle. Moreover as $\mathrm{F}_1 \sqcup \mathrm{F}_2$ has a natural map (which we will call k) into F, there is a commutative diagram:

$$\begin{tikzcd} (\mathrm{F}_1 \sqcup \mathrm{F}_2) \arrow[r, "k"] \arrow[d, "\Delta"] & \mathrm{F} \arrow[d, "\Delta"] \\ (\mathrm{F}_1 \sqcup \mathrm{F}_2)_+ \wedge \mathrm{F}_+ \arrow[r, "k \wedge id"] & \mathrm{F}_+ \wedge \mathrm{F}_+. \end{tikzcd}$$

Then the distinguished triangle (3.0.10) provides the commutative diagram:

$$\begin{tikzcd} (\mathrm{F}_1 \sqcup \mathrm{F}_2) \arrow[r, "k"] \arrow[d, "\Delta"] & \mathrm{F} \arrow[r] \arrow[d, "\Delta"] & \mathrm{S}^1 \wedge \mathrm{F}_{3,+} \arrow[r] \arrow[d, "\Delta"] & \mathrm{S}^1 \wedge (\mathrm{F}_1 \sqcup \mathrm{F}_2)_+ \arrow[d, "\Delta"] \\ (\mathrm{F}_1 \sqcup \mathrm{F}_2)_+ \wedge \mathrm{F}_+ \arrow[r, "k \wedge id"] & \mathrm{F}_+ \wedge \mathrm{F}_+ \arrow[r] & (\mathrm{S}^1 \wedge \mathrm{F}_{3,+}) \wedge \mathrm{F}_+ \arrow[r] & (\mathrm{S}^1 \wedge (\mathrm{F}_1 \sqcup \mathrm{F}_2)_+) \wedge \mathrm{F}_+. \end{tikzcd}$$

so that the hypotheses of [LMS, Theorem 7.10, Chapter III] are satisfied with X, Y and Z there equal to $(\mathrm{F}_1 \sqcup \mathrm{F}_2)$, F and $\mathrm{S}^1 \wedge \mathrm{F}_{3,+}$. These arguments, therefore reduce the proof of Theorem 3.2 to that of Theorem 3.1.

Therefore, what we proceed to verify is that, then the proof of [LMS, Theorem 7.10, Chapter III] carries over to our framework. This will then complete the proof of Theorem 3.1. A key step of this amounts to verifying that the big commutative diagram given on [LMS, p. 166] carries over to our framework. One may observe that this big diagram is broken up into various sub-diagrams, labeled (I) through (VII) and that it suffices to verify that each of these sub-diagrams commutes up to homotopy. Moreover, one may observe that the maps that make up each of these sub-diagrams are natural maps and therefore are G-equivariant, so that one may feed each of these sub-diagrams into the Borel construction. This will prove that additivity holds for the induced trace and transfer on Borel-style generalized cohomology theories.

For this, it seems best to follow the terminology adopted in [LMS, Theorem 7.10, Chapter III]: therefore we will let U_+ (X_+ and X/U) in Theorem 3.1 be denoted X (Y and Z, respectively) for the remaining part of the proof of Theorem 3.1. Let $k : \mathrm{X} \to \mathrm{Y}$ ($i : \mathrm{Y} \to \mathrm{Z}$ and $\pi : \mathrm{Z} \to \mathrm{S}^1 \wedge \mathrm{X}$) denote the corresponding maps $j_+ : \mathrm{U}_+ \to \mathrm{X}_+$ ($k_+ : \mathrm{X}_+ \to \mathrm{X/U}$, and the map $l : \mathrm{X/U} \to \mathrm{S}^1 \wedge \mathrm{U}_+$) as in Theorem 3.1. Then the very first step in this direction is to verify that the three squares

(3.0.11)
$$\begin{tikzcd}
\mathrm{DY} \wedge \mathrm{X} \arrow[r, "id\wedge k"] \arrow[d, "\mathrm{Dk}\wedge\mathrm{id}"] & \mathrm{DY} \wedge \mathrm{Y} \arrow[d, "e"] \\
\mathrm{DX} \wedge \mathrm{X} \arrow[r, "e"] & \mathbb{S}^{\mathrm{G}}
\end{tikzcd}, \quad
\begin{tikzcd}
\mathrm{DZ} \wedge \mathrm{Y} \arrow[r, "id\wedge i"] \arrow[d, "\mathrm{Di}\wedge\mathrm{id}"] & \mathrm{DZ} \wedge \mathrm{Z} \arrow[d, "e"] \\
\mathrm{DY} \wedge \mathrm{Y} \arrow[r, "e"] & \mathbb{S}^{\mathrm{G}}
\end{tikzcd}, \text{ and }
\begin{tikzcd}
\mathrm{D}(\mathrm{S}^1 \wedge \mathrm{X}) \wedge \mathrm{Z} \arrow[r, "id\wedge\pi"] \arrow[d, "\mathrm{D}\pi\wedge\mathrm{id}"] & \mathrm{D}(\mathrm{S}^1 \wedge \mathrm{X}) \wedge (\mathrm{S}^1 \wedge \mathrm{X}) \arrow[d, "e"] \\
\mathrm{DZ} \wedge \mathrm{Z} \arrow[r, "e"] & \mathbb{S}^{\mathrm{G}}
\end{tikzcd}$$

commute up to homotopy. (The homotopy commutativity of these squares is a formal consequence of Spanier-Whitehead duality: see [Sw75, pp. 324-325] for proofs in the classical setting.) As argued on [LMS, page 167, Chapter III], the composite $e \circ (\mathrm{D}\pi \wedge \mathrm{i}) : \mathrm{D}(\mathrm{S}^1 \wedge \mathrm{X}) \wedge \mathrm{Y} \to \mathbb{S}^{\mathrm{G}}$ is equal to $e \circ ((id \wedge \pi) \circ (id \wedge i)$ and is therefore the trivial map. Therefore, if j denotes the inclusion of $\mathrm{DZ} \wedge \mathrm{Z}$ in the cofiber of $\mathrm{D}\pi \wedge \mathrm{i}$, one obtains the induced map $\bar{e} : (\mathrm{DZ} \wedge \mathrm{Z})/(\mathrm{D}(\mathrm{S}^1 \wedge \mathrm{X}) \wedge \mathrm{Y}) \to \mathbb{S}^{\mathrm{G}}$ so that the triangle

(3.0.12)
$$\begin{tikzcd}
\mathrm{DZ} \wedge \mathrm{Z} \arrow[r, "e"] \arrow[d, "j"] & \mathbb{S}^{\mathrm{G}} \\
(\mathrm{DZ} \wedge \mathrm{Z})/(\mathrm{D}(\mathrm{S}^1 \wedge \mathrm{X}) \wedge \mathrm{Y}) \arrow[ur, "\bar{e}"] &
\end{tikzcd}$$

homotopy commutes. This provides the commutative triangle denoted (I) in [LMS, p. 166] there and the commutative triangle denoted (II) there commutes by the second and third commutative squares in (3.0.11). The duals of (I) and (II) are the triangles denoted (I*) and (II*) (on [LMS, p. 166]) and therefore, they also commute.

Next we briefly consider the homotopy commutativity of the remaining diagram beginning with the squares labeled (III), (IV) and (V) in [LMS, p. 166]. Since the maps denoted δ are weak-equivalences, it suffices to show that these squares homotopy commute when the maps denoted δ^{-1} are replaced by the corresponding maps δ going in the opposite direction. Such maps δ appearing there are all special instances of the following natural map: $\delta : \mathrm{DB} \wedge \mathrm{A} \to \mathrm{D}(\mathrm{DA} \wedge \mathrm{B})$, for two spectra A and B in **Spt**. The homotopy commutativity of the squares (III), (IV) and (V) are reduced therefore to the naturality of the above map in the arguments A and B. The commutativity of the triangle labeled (VI) follows essentially from the definition of the maps there. Finally the homotopy commutativity of the square (VII) is reduced to the following lemma, which is simply a restatement of [LMS, Lemma 7.11, Chapter III]. These will complete the proof for the additivity property for the trace. To complete the proof for the additivity of the transfer tr, one only needs to expand the bottom most square (II) in the diagram in [LMS, p. 166] as indicated there. These complete the proofs of Theorems 3.1 and 3.2.

Lemma 3.3. *Let* $\mathrm{f} : \mathrm{A} \to \mathrm{X}$ *and* $\mathrm{g} : \mathrm{B} \to \mathrm{Y}$ *be maps in* **Spt** *and let* $i : \mathrm{X} \to \mathrm{Cone(f)}$ *and* $j : \mathrm{Y} \to \mathrm{Cone(g)}$ *be the inclusions into their cofibers. Then the boundary map* $\delta : \mathrm{S}^{-1}\mathrm{Cone}(\mathrm{i} \wedge \mathrm{j}) \to \mathrm{Cone}(\mathrm{f} \wedge \mathrm{g})$ *in the cofiber sequence* $Cone(\mathrm{f} \wedge \mathrm{g}) \to \mathrm{Cone}((\mathrm{i} \circ \mathrm{f}) \wedge (\mathrm{j} \circ \mathrm{g})) \to \mathrm{Cone}(\mathrm{i} \wedge \mathrm{j})$ *is the sum of the two composites:*

$$\mathrm{S}^{-1}\mathrm{Cone}(\mathrm{i} \wedge \mathrm{j}) \overset{\mathrm{S}^{-1}\mathrm{Cone}(\mathrm{i}\wedge\mathrm{id})}{\to} \mathrm{Cone}(\mathrm{id}_{\mathrm{Cone(f)}} \wedge \mathrm{j}) = \mathrm{Cone(f)} \wedge \mathrm{B} \cong \mathrm{Cone}(\mathrm{f} \wedge \mathrm{id_B}) \overset{\mathrm{Cone}(\mathrm{id}\wedge\mathrm{g})}{\to} \mathrm{Cone}(\mathrm{f} \wedge \mathrm{g})$$

$$\mathrm{S}^{-1}\mathrm{Cone}(\mathrm{i} \wedge \mathrm{j}) \overset{\mathrm{S}^{-1}\mathrm{Cone}(\mathrm{id}\wedge\mathrm{j})}{\to} \mathrm{S}^{-1}\mathrm{Cone}(\mathrm{i} \wedge \mathrm{id}_{\mathrm{Cone(g)}}) = \mathrm{A} \wedge \mathrm{Cone(g)} \cong \mathrm{Cone}(\mathrm{id_A} \wedge \mathrm{g}) \overset{\mathrm{Cone}(\mathrm{f}\wedge\mathrm{id})}{\to} \mathrm{Cone}(\mathrm{f} \wedge \mathrm{g})$$

Proposition 3.4. *(Multiplicative property of the pre-transfer and trace) Assume* $\mathrm{F_i}$, $i = 1, 2$ *are simplicial presheaves provided with actions by the group* G. *Let* $\mathrm{f_i} : \mathrm{F_i} \to \mathrm{F_i}$, $i = 1, 2$ *denote a* G*-equivariant map. Let* $\mathrm{F} = \mathrm{F}_{1+} \wedge \mathrm{F}_{2+}$ *and let* $\mathrm{f} = \mathrm{f}_{1+} \wedge \mathrm{f}_{2+}$. *Then*

$$tr'_{\mathrm{F}}(\mathrm{f}) = \mathrm{tr}'_{\mathrm{F}_{1+}}(\mathrm{f}_{1+}) \wedge \mathrm{tr}'_{\mathrm{F}_{2+}}(\mathrm{f}_{2+}) \textit{ and}$$
$$\tau_{\mathrm{F}}(\mathrm{f}) = \tau_{\mathrm{F}_{1+}}(\mathrm{f}_{1+}) \wedge \tau_{\mathrm{F}_{2+}}(\mathrm{f}_{2+}).$$

Proof. A key point is to observe is that the evaluation $e_{\mathrm{F}} : D(\mathrm{F}) \wedge \mathrm{F} \to \mathbb{S}^{\mathrm{G}}$ is given by starting with $e_{\mathrm{F}_{1+}} \wedge e_{\mathrm{F}_{2+}} : D(\mathrm{F}_{1+}) \wedge \mathrm{F}_{1+} \wedge \mathrm{D}(\mathrm{F}_{2+}) \wedge \mathrm{F}_{2+} \to \mathbb{S}^{\mathrm{G}} \wedge \mathbb{S}^{\mathrm{G}} \simeq \mathbb{S}^{\mathrm{G}}$ and by precomposing it with the map $D(\mathrm{F}) \wedge \mathrm{F} = \mathrm{D}(\mathrm{F}_{1+} \wedge \mathrm{F}_{2+}) \wedge \mathrm{F}_{1+} \wedge \mathrm{F}_{2+} \overset{\tau}{\to} \mathrm{D}(\mathrm{F}_{1+}) \wedge \mathrm{F}_{1+} \wedge \mathrm{D}(\mathrm{F}_{2+}) \wedge \mathrm{F}_{2+}$, where τ is the obvious map that interchanges the factors. Similarly the co-evaluation map $c : \mathbb{S}^{\mathrm{G}} \simeq \mathbb{S}^{\mathrm{G}} \wedge \mathbb{S}^{\mathrm{G}} \overset{c_{\mathrm{F}_{1+}} \wedge c_{\mathrm{F}_{2+}}}{\to} \mathrm{F}_{1+} \wedge \mathrm{D}(\mathrm{F}_{1+}) \wedge \mathrm{F}_{2+} \wedge \mathrm{D}(\mathrm{F}_{2+})$ provides the co-evaluation map for F. The multiplicative property of the pre-transfer follows readily from the above two observations as well as from the definition of the pre-transfer as in Definition 2.5. In view of the definition of the trace as in Definition 2.5, the multiplicative property of the trace follows from the multiplicative property of the pretransfer. ☐

Let B denote a smooth quasi-projective scheme over the given base scheme *Spec k*. Let $\mathrm{E} \rightarrow \mathrm{B}$ denote a G-torsor (in the given topology, which could be either the Zariski, Nisnevich, or étale) for the action of a linear algebraic group G.

Proposition 3.5. *(i) Given* G*-equivariant pointed simplicial presheaves* $\mathcal{P}_1, \mathcal{P}_2$ *on Spec k, one obtains a natural map*

$$[\mathcal{P}_1, \mathcal{P}_2] \rightarrow [\mathrm{E}\underset{\mathrm{G}}{\times}\mathcal{P}_1, \mathrm{E}\underset{\mathrm{G}}{\times}\mathcal{P}_2]$$

where the $[\quad,\quad]$ *on the left denotes the Hom in the homotopy category of* G*-equivariant pointed simplicial presheaves and the* $[\quad,\quad]$ *on the right denotes the Hom in the homotopy category of simplicial presheaves over* B *as in* [CJ19, section 3]. *Moreover the quotient construction* $\mathrm{E}\underset{\mathrm{G}}{\times}\mathcal{P}_i$ *is carried out as in* [CJ19, section 6.2]*, that is, when* G *is special as a linear algebraic group, the quotient is taken on the Zariski (or Nisnevich) site of* E*, while when* G *is not special, it is taken on the étale site and then followed by a derived push-forward to the Nisnevich site. (For more details on this, see the discussion in subsection 4.2.)*

(ii) Given pointed simplicial presheaves $\mathcal{Q}_i$*,* $i = 1, 2, 3, 4$*, over* B*, we obtain a pairing:*

$$[\mathcal{Q}_1, \mathcal{Q}_2] \times [\mathcal{Q}_3, \mathcal{Q}_4] \rightarrow [\mathcal{Q}_1\underset{\mathrm{B}}{\wedge}\mathcal{Q}_2, \mathcal{Q}_3\underset{\mathrm{B}}{\wedge}\mathcal{Q}_4]$$

where the $[\quad,\quad]$ *denotes the Hom in the homotopy category of simplicial presheaves over* B.

Proof. (i) We may assume $\mathcal{P}_1$ is cofibrant and $\mathcal{P}_2$ fibrant in the given model structure, so that $[\mathcal{P}_1, \mathcal{P}_2] = \pi_0(\mathrm{Map}(\mathcal{P}_1, \mathcal{P}_2))$ where Map denotes the simplicial mapping space. Then the assertion in (i) is clear.

For (ii), we may assume $\mathcal{Q}_1, \mathcal{Q}_3$ are cofibrant and $\mathcal{Q}_2, \mathcal{Q}_4$ are fibrant in the given model structure, so that $[\mathcal{Q}_1, \mathcal{Q}_2] = \pi_0(\mathrm{Map}(\mathcal{Q}_1, \mathcal{Q}_2))$ and $[\mathcal{Q}_3, \mathcal{Q}_4] = \pi_0(\mathrm{Map}(\mathcal{Q}_3, \mathcal{Q}_4))$, where Map denotes the corresponding simplicial mapping space. Then the assertion in (ii) is also clear. □

Proof of Theorem 1.5. We will explicitly discuss only the case where the ring spectrum $\mathcal{E}^{\mathrm{G}}$ ($\mathcal{E}$) is the equivariant sphere spectrum $\mathbb{S}^{\mathrm{G}}$ (the sphere spectrum $\Sigma_{\mathbf{T}}$, respectively), as proofs in the other cases follow along the same lines.

First we will consider (i), namely the Mayer-Vietoris sequence. For this, one begins with the stable cohomotopy cofiber sequence $\mathbb{S}^{\mathrm{G}} \wedge (\mathrm{U}_1 \cap \mathrm{U}_2)_+ \rightarrow \mathbb{S}^{\mathrm{G}} \wedge (\mathrm{U}_1 \sqcup \mathrm{U}_2)_+ \rightarrow \mathbb{S}^{\mathrm{G}} \wedge (\mathrm{X})_+$ and then applies Theorem 3.2(1) to it. This proves (i).

Next one recalls the stable homotopy cofiber sequence (see [MV99, Theorem 2.23])

$$\mathbb{S}^{\mathrm{G}} \wedge \mathrm{U}_+ \rightarrow \mathbb{S}^{\mathrm{G}} \wedge \mathrm{X}_+ \rightarrow \mathbb{S}^{\mathrm{G}} \wedge (\mathrm{X}/\mathrm{U}) \simeq \mathbb{S}^{\mathrm{G}} \wedge \mathrm{Th}(\mathcal{N}) \tag{3.0.12}$$

in the stable motivic homotopy category over the base scheme. The first statement in (ii) follows by applying Theorem 3.1 to the stable homotopy cofiber sequence in (3.0.12).

We proceed to establish the remaining statement in (ii). Observe that this is in the non-equivariant framework, so that we will work in **Spt**. First we will consider the case where the normal bundle $\mathcal{N}$ is trivial, mainly because this is an important special case to consider. When the normal bundle is trivial, we observe that $\mathrm{X}/\mathrm{U} \simeq \mathrm{Th}(\mathcal{N}) \simeq \mathbf{T}^{\mathrm{c}} \wedge \mathrm{Y}_+$. Next, the observation (see [CJ19, Corollary 2.4(ii)]) that

$$\begin{aligned} \mathrm{D}(\mathbf{T}^{\mathrm{c}} \wedge \mathrm{Y}_+) \wedge \Sigma_{\mathbf{T}}(\mathbf{T}^{\mathrm{c}} \wedge \mathrm{Y}_+) &\simeq \mathcal{RH}om(\Sigma_{\mathbf{T}}(\mathbf{T}^{c} \wedge \mathrm{Y}_+), \Sigma_{\mathbf{T}}\mathbf{T}^{\mathrm{c}} \wedge \mathrm{Y}_+) \\ &\simeq \mathcal{RH}om(\Sigma_{\mathbf{T}}(\mathrm{Y}_+), \Sigma_{\mathbf{T}}\mathrm{Y}_+) \simeq \mathrm{D}(\Sigma_{\mathbf{T}}\mathrm{Y}_+) \wedge \Sigma_{\mathbf{T}}\mathrm{Y}_+ \end{aligned} \tag{3.0.13}$$

along with Remark 2.6 shows one may identify the trace $\tau_{\mathrm{X}/\mathrm{U}}$ with the map τ_{Y}. In (3.0.13), we follow the terminology of [CJ19], where $\mathcal{RH}om$ denotes the derived functor of the internal $\mathcal{H}om$-functor in the model category **Spt** and $\wedge$ denotes the derived functor of the corresponding smash product. The general case, when the normal bundle $\mathcal{N}$ is not necessarily trivial, follows from Proposition 3.6 discussed below. These complete the proof of all statements in (ii).

Next we consider the statement in (iii). This will follow from the second statement in (ii) using ascending induction on the number of strata. However, as this induction needs to be handled carefully, we proceed to provide an outline of the relevant argument. We will assume that the stratification of X defines the following increasing filtrations:

(a) $\phi = \mathrm{X}_{-1} \subseteq \mathrm{X}_0 \subseteq \cdots \subseteq \mathrm{X}_{\mathrm{n}} = \mathrm{X}$ where each X_{i} is closed and the strata $\mathrm{X}_{\mathrm{i}} - \mathrm{X}_{\mathrm{i}-1}$, $i = 0, \cdots, n$ are smooth (regular).

(b) $\mathrm{U}_0 \subseteq \mathrm{U}_1 \subseteq \cdots \subseteq \mathrm{U}_{\mathrm{n}-1} \subseteq \mathrm{U}_{\mathrm{n}} = \mathrm{X}$, where each U_{i} is open in X (and therefore smooth (regular)) with $\mathrm{U}_{\mathrm{i}} - \mathrm{U}_{\mathrm{i}-1} = \mathrm{X}_{\mathrm{n}-\mathrm{i}} - \mathrm{X}_{\mathrm{n}-\mathrm{i}-1}$, for all $i = 0, \cdots n$. Now observe that each $\mathrm{U}_{\mathrm{k}} \rightarrow \mathrm{X}$ is an open immersion while each $\mathrm{X}_{\mathrm{k}} - \mathrm{X}_{\mathrm{k}-1} \rightarrow \mathrm{X} - \mathrm{X}_{\mathrm{k}-1}$ is a closed immersion.

We now apply Theorem 1.5(ii) with $\mathrm{U} = \mathrm{U_{n-1}}$, and $\mathrm{Y} = \mathrm{U_n} - \mathrm{U_{n-1}} = \mathrm{X_0} - \mathrm{X_{-1}} = \mathrm{X_0}$, the closed stratum. Since X is now smooth(regular) and so is Y, the hypotheses of Theorem 1.5(ii) are satisfied. This provides us

$$\tau_{\mathrm{X}} = \tau_{\mathrm{U_{n-1}}} + \tau_{\mathrm{X/U_{n-1}}} \text{ and } \tau_{\mathrm{X/U_{n-1}}} = \tau_{\mathrm{X_0}} \tag{3.0.14}$$

Next we replace X by $\mathrm{U_{n-1}}$, U by $\mathrm{U_{n-2}}$ and Y by $\mathrm{X_1} - \mathrm{X_0}$. Since $\mathrm{X_1} - \mathrm{X_0}$ is smooth (regular), Theorem 1.5(ii) now provides us

$$\tau_{\mathrm{U_{n-1}}} = \tau_{\mathrm{U_{n-2}}} + \tau_{\mathrm{X_1 - X_0}}. \tag{3.0.15}$$

Substituting these in (3.0.14), we obtain

$$\tau_{\mathrm{X}} = \tau_{\mathrm{U_{n-2}}} + \tau_{\mathrm{X_1 - X_0}} + \tau_{\mathrm{X_0}}.$$

Clearly this may be continued inductively to deduce statement (iii) in Theorem 1.5 from Theorem 1.5(ii). (iv) follows readily from [CJ19, Proposition 8.1] where it is shown that the trace τ_{X} pulls-back to the corresponding trace defined on the étale site. □

Proposition 3.6. *Assume the situation of Theorem 1.5(ii). Then there exists a natural map*

$$\mathcal{RH}om(\Sigma_{\mathbf{T}}\mathrm{Y}_+, \Sigma_{\mathbf{T}}\mathrm{Y}_+) \to \mathcal{RH}\mathrm{om}(\Sigma_{\mathbf{T}}\mathrm{Th}(\mathcal{N}), \Sigma_{\mathbf{T}}\mathrm{Th}(\mathcal{N})) \tag{3.0.16}$$

which is a weak-equivalence. Here $\mathcal{RH}om$ denotes the internal hom in **Spt***.*

Proof. In view of the adjunction between the internal hom, $\mathcal{H}om$ and the smash product, $\wedge$, we will first show that there is a natural map

$$Hom(K \wedge \Sigma_{\mathbf{T}}\mathrm{Y}_+, \Sigma_{\mathbf{T}}\mathrm{Y}_+) \to \mathrm{Hom}(\mathrm{K} \wedge \Sigma_{\mathbf{T}}\mathrm{Th}(\mathcal{N}), \Sigma_{\mathbf{T}}\mathrm{Th}(\mathcal{N})), \tag{3.0.17}$$

where Hom denotes the external hom in the category **Spt** and for every $\mathrm{K}\,\varepsilon\,\mathbf{Spt}$. The adjunction between $\wedge$ and the internal hom, will then show this induces a natural map

$$Hom(K, \mathcal{H}om(\Sigma_{\mathbf{T}}\mathrm{Y}_+, \Sigma_{\mathbf{T}}\mathrm{Y}_+)) \to \mathrm{Hom}(\mathrm{K}, \mathcal{H}\mathrm{om}(\Sigma_{\mathbf{T}}\mathrm{Th}(\mathcal{N}), \Sigma_{\mathbf{T}}\mathrm{Th}(\mathcal{N}))$$

for all $K\,\varepsilon\,\mathbf{Spt}$ and therefore a natural map

$$\mathcal{H}om(\Sigma_{\mathbf{T}}\mathrm{Y}_+, \Sigma_{\mathbf{T}}\mathrm{Y}_+) \to \mathcal{H}\mathrm{om}(\Sigma_{\mathbf{T}}\mathrm{Th}(\mathcal{N}), \Sigma_{\mathbf{T}}\mathrm{Th}(\mathcal{N})).$$

By making use of the the injective model structures on **Spt** (as in [CJ19, 3.3.5]), we may assume that every object is cofibrant, and therefore, the above map will then induce a natural map (by taking fibrant replacements):

$$\mathcal{RH}om(\Sigma_{\mathbf{T}}\mathrm{Y}_+, \Sigma_{\mathbf{T}}\mathrm{Y}_+) \to \mathcal{RH}\mathrm{om}(\Sigma_{\mathbf{T}}\mathrm{Th}(\mathcal{N}), \Sigma_{\mathbf{T}}\mathrm{Th}(\mathcal{N})). \tag{3.0.18}$$

Moreover, the fact the above map is a weak-equivalence will follow by showing that, on working locally on the Zariski topology of Y, we reduce to the case where the normal bundle $\mathcal{N}$ is in fact trivial, where the calculation reduces to the one given in (3.0.13) above. Moreover, the compatibility of the map above with the G-action will follow from its naturality.

Therefore, it suffices to show that there is natural map as in (3.0.17). Recall that the Thom-space $\mathrm{Th}(\mathcal{N})$ is defined as the pushout:

$$\begin{tikzcd} Proj(\mathcal{N}) \arrow[r] \arrow[d] & Proj(\mathcal{N} \oplus 1) \arrow[d] \\ Spec\, k \arrow[r] & \mathrm{Th}(\mathcal{N}). \end{tikzcd} \tag{3.0.19}$$

One may observe that this pushout may be also obtained in two stages, by taking the pushout of the first diagram and then the second in:

$$\begin{tikzcd} Proj(\mathcal{N}) \arrow[r] \arrow[d] & Proj(\mathcal{N} \oplus 1) \arrow[d] & \mathrm{Y} \arrow[r] \arrow[d] & \mathrm{S}(\mathcal{N} \oplus 1) \arrow[d] \\ \mathrm{Y} \arrow[r] & \mathrm{S}(\mathcal{N} \oplus 1), & Spec\, k \arrow[r] & \mathrm{Th}(\mathcal{N}). \end{tikzcd} \tag{3.0.20}$$

Next let $\{\mathrm{U_i} | \mathrm{i} = 1, \cdots, \mathrm{n}\}$ denote a Zariski open cover of Y so that $\mathcal{N}$ is trivial on this cover, that is, $\mathcal{N}_{|U_i} = U_i \times \mathbb{A}^c$, for each i. Here we assume that c is the codimension of Y in X. Let $\mathcal{U} = \sqcup_i \mathrm{U_i}$ and $\mathcal{U}_\bullet = cosk_0^{\mathrm{Y}}(\mathcal{U})$, so that in degree n, $\mathcal{U}_n = (\mathcal{U} \underset{\mathrm{Y}}{\times} \cdots \underset{\mathrm{Y}}{\times} \mathcal{U})$, which is the n-fold fibered product of $\mathcal{U}$ with itself over Y. Observe that now one has an isomorphism of simplicial presheaves

$$\phi : \mathcal{U}_\bullet \underset{Spec\, k}{\times} \mathbf{T}^c \overset{\cong}{\to} \mathrm{S}(\mathcal{N} \oplus 1)_{|\mathcal{U}_\bullet}, \tag{3.0.21}$$

where $\mathrm{S}(\mathcal{N}\oplus 1)_{|\mathcal{U}_\bullet}$ denotes the pull-back of $\mathrm{S}(\mathcal{N}\oplus 1)$ to $\mathcal{U}_\bullet$. This isomorphism defines an isomorphism of simplicial objects of spectra:

$$\Sigma_{\mathbf{T}}\phi_+ : \Sigma_{\mathbf{T}}\mathcal{U}_{\bullet,+}\wedge\mathbf{T}^c_+ = \Sigma_{\mathbf{T}}(\mathcal{U}_\bullet \underset{Spec\, k}{\times} \mathbf{T}^c)_+ \overset{\cong}{\to} \Sigma_{\mathbf{T}}(\mathrm{S}(\mathcal{N}\oplus 1)_{|\mathcal{U}_\bullet,+}. \tag{3.0.22}$$

Now consider the cosimplicial simplicial sets $Hom(K\wedge\Sigma_{\mathbf{T}}\mathcal{U}_{\bullet,+}, \Sigma_{\mathbf{T}}\mathcal{U}_{\bullet,+})$ and $Hom(K\wedge\Sigma_{\mathbf{T}}\mathrm{S}(\mathcal{N}\oplus 1)_{|\mathcal{U}_\bullet}, \Sigma_{\mathbf{T}}\mathrm{S}(\mathcal{N}\oplus 1)_{|\mathcal{U}_\bullet})$. Sending an $f: K\wedge\Sigma_{\mathbf{T}}\mathcal{U}_{n,+}\to\Sigma_{\mathbf{T}}\mathcal{U}_{m,+}$ to $\Sigma_{\mathbf{T}}\phi_+(f\wedge id_{\mathbf{T}^c,+})$ (which denotes the induced map $K\wedge\Sigma_{\mathbf{T}}\mathrm{S}(\mathcal{N}\oplus 1)_{|\mathcal{U}_\bullet,+}\to\Sigma_{\mathbf{T}}\mathrm{S}(\mathcal{N}\oplus 1)_{|\mathcal{U}_\bullet,+}$ defined by making use of the isomorphism $\Sigma_{\mathbf{T}}\phi_+$) defines a map

$$Hom(K\wedge\Sigma_{\mathbf{T}}\mathcal{U}_{\bullet,+}, \Sigma_{\mathbf{T}}\mathcal{U}_{\bullet,+}) \to Hom(K\wedge\Sigma_{\mathbf{T}}\mathrm{S}(\mathcal{N}\oplus 1)_{|\mathcal{U}_\bullet,+}, \Sigma_{\mathbf{T}}\mathrm{S}(\mathcal{N}\oplus 1)_{|\mathcal{U}_\bullet,+})$$

which one may verify is compatible with the cosimplicial simplicial structure on either side. Moreover, one also obtains a commutative diagram

$$\begin{tikzcd} K\wedge\Sigma_{\mathbf{T}}\mathcal{U}_{n,+} \arrow[r, "f"] \arrow[d, "s"] & \Sigma_{\mathbf{T}}\mathcal{U}_{m,+} \arrow[d, "s"] \\ K\wedge\Sigma_{\mathbf{T}}\mathrm{S}(\mathcal{N}\oplus 1)_{|\mathcal{U}_{n,+}} \arrow[r, "\phi(f\times id_{\mathbf{T}^c})"] & \Sigma_{\mathbf{T}}\mathrm{S}(\mathcal{N}\oplus 1)_{|\mathcal{U}_{m,+}}, \end{tikzcd} \tag{3.0.23}$$

where s denotes the canonical section. Therefore, collapsing the section s defines a map

$$Hom(K\wedge\Sigma_{\mathbf{T}}\mathcal{U}_{\bullet,+}, \Sigma_{\mathbf{T}}\mathcal{U}_{\bullet,+}) \to Hom(K\wedge\Sigma_{\mathbf{T}}\mathrm{Th}(\mathcal{N}_{|\mathcal{U}_\bullet}), \Sigma_{\mathbf{T}}\mathrm{Th}(\mathcal{N}_{|\mathcal{U}_\bullet}))$$

of cosimplicial simplicial sets. Since this is functorial in K, it follows that this defines a map of cosimplicial simplicial spectra of internal homs:

$$\mathcal{RH}om(\Sigma_{\mathbf{T}}\mathcal{U}_{\bullet,+}, \Sigma_{\mathbf{T}}\mathcal{U}_{\bullet,+}) \to \mathcal{RH}om(\Sigma_{\mathbf{T}}\mathrm{Th}(\mathcal{N}_{|\mathcal{U}_\bullet}), \Sigma_{\mathbf{T}}\mathrm{Th}(\mathcal{N}_{|\mathcal{U}_\bullet})).$$

The proof of the proposition may now be completed by observing the weak-equivalences:

$$\mathcal{RH}om(\Sigma_{\mathbf{T}}\mathrm{Y}_+, \Sigma_{\mathbf{T}}\mathrm{Y}_+) \simeq \underset{\Delta}{\mathrm{holim}}\,\underset{\Delta}{\mathrm{hocolim}}\mathcal{RH}\mathrm{om}(\Sigma_{\mathbf{T}}\mathcal{U}_{\bullet,+}, \Sigma_{\mathbf{T}}\mathcal{U}_{\bullet,+}) \text{ and}$$

$$\mathcal{RH}om(\Sigma_{\mathbf{T}}\mathrm{Th}(\mathcal{N}), \Sigma_{\mathbf{T}}\mathrm{Th}(\mathcal{N})) \simeq \underset{\Delta}{\mathrm{holim}}\,\underset{\Delta}{\mathrm{hocolim}}\mathcal{RH}\mathrm{om}(\Sigma_{\mathbf{T}}\mathrm{Th}(\mathcal{N}_{|\mathcal{U}_\bullet}), \Sigma_{\mathbf{T}}\mathrm{Th}(\mathcal{N}_{|\mathcal{U}_\bullet})).$$

Here we are making use of the weak-equivalences $\underset{\Delta}{\mathrm{hocolim}}\Sigma_{\mathbf{T}}\mathcal{U}_{\bullet,+}\simeq\Sigma_{\mathbf{T}}\mathrm{Y}_+$ (see for example, [DHI04]) and

$$\begin{aligned} \underset{\Delta}{\mathrm{hocolim}}\Sigma_{\mathbf{T}}\mathrm{Th}(\mathcal{N}_{|\mathcal{U}_\bullet}) &\simeq \underset{\Delta}{\mathrm{hocolim}}(\Sigma_{\mathbf{T}}S(\mathcal{N}\oplus 1)_{|\mathcal{U}_\bullet}/s(\Sigma_{\mathbf{T}}(\mathcal{U}_{\bullet,+}))) \\ &\simeq \underset{\Delta}{\mathrm{hocolim}}(\Sigma_{\mathbf{T}}S(\mathcal{N}\oplus 1)_{|\mathcal{U}_\bullet})/s(\underset{\Delta}{\mathrm{hocolim}}\Sigma_{\mathbf{T}}(\mathcal{U}_{\bullet,+})) \\ &\simeq \Sigma_{\mathbf{T}}S(\mathcal{N}\oplus 1)/\Sigma_{\mathbf{T}}\mathrm{Y}_+ \simeq \Sigma_{\mathbf{T}}\mathrm{Th}(\mathcal{N}) \end{aligned} \tag{3.0.24}$$

where s is the section considered in (3.0.23). Finally we observe that the homotopy colimit in the left argument pulls out of the $\mathcal{RH}om(\quad,\quad)$ as a homotopy inverse limit, while the homotopy colimit in the right argument pulls out as a homotopy colimit, since the left argument of the $\mathcal{RH}om(\quad,\quad)$ is a compact object.

□

4. **Proof of Theorem 1.12**

Once again, we will explicitly discuss only the case where the ring spectrum $\mathcal{E}^{\mathrm{G}}$ ($\mathcal{E}$) is the equivariant sphere spectrum $\mathbb{S}^{\mathrm{G}}$ (the sphere spectrum $\Sigma_{\mathbf{T}}$, respectively), as proofs in the other cases follow along the same lines. First we will consider (i), namely the Mayer-Vietoris sequence. For this, one begins with the stable cohomotopy cofiber sequence $\mathbb{S}^{\mathrm{G}}\wedge(\mathrm{U}_1\cap\mathrm{U}_2)_+\to\mathbb{S}^{\mathrm{G}}\wedge(\mathrm{U}_1\sqcup\mathrm{U}_2)_+\to\mathbb{S}^{\mathrm{G}}\wedge(\mathrm{X})_+$ and then applies Theorem 3.2(1) to it. This proves (i).

Next we will consider (ii). One recalls the stable homotopy cofiber sequence (see [MV99, Theorem 2.23])

$$\mathbb{S}^{\mathrm{G}}\wedge\mathrm{U}_+\to\mathbb{S}^{\mathrm{G}}\wedge\mathrm{X}_+\to\mathbb{S}^{\mathrm{G}}\wedge(\mathrm{X}/\mathrm{U})\simeq\mathbb{S}^{\mathrm{G}}\wedge\mathrm{Th}(\mathcal{N}) \tag{4.0.25}$$

in the stable motivic homotopy category over the base scheme. The statement in (ii) follows by applying Theorem 3.1 to the stable homotopy cofiber sequence in (4.0.25).

Next we consider (iii). For this, one first needs the technique of deformation to the normal-bundle: see [Verd, section 2], or [MV99, Lemma 2.26]. Here one considers first the blow-up of $\mathrm{X}\times\mathbb{A}^1$ along $\mathrm{Y}\times\{0\}$. As G has only the trivial action on $\mathbb{A}^1$, and Y is G-stable, it follows that the above blow-up, $Bl_{\mathrm{Y}\times\{0\}}(\mathrm{X}\times\mathbb{A}^1)$, the deformation space $Bl_{\mathrm{Y}\times\{0\}}(\mathrm{X}\times\mathbb{A}^1)-\mathrm{Bl}_{\mathrm{Y}}(\mathrm{X})$ as well as the normal bundle $\mathcal{N}$ and its Thom space $\mathrm{Th}(\mathcal{N})$ all have induced actions by G. Moreover the deformation consists in sending the general fiber, which sits over $\mathbb{A}^1-\{0\}$ to the special fiber which sits over 0 in $\mathbb{A}^1$. However, as shown below in Proposition 4.11, one needs to supplement this with the technique of *motivic tubular neighborhoods.*

A key step here is to show the following. Let M denote a motivic spectrum that has the rigidity property as in Definition 1.10, let Map denote the simplicial mapping space functor, and $E \to B$ is a G-torsor for a fixed linear algebraic group G that acts on X, U and Y. Let $\Delta_{X/U}$ denote the diagonal map (see 3.0.8):

$$\Delta_{X/U} : \mathbb{S}^G \wedge (X/U) \to \mathbb{S}^G \wedge (X/U) \wedge (\mathbb{S}^G \wedge X_+). \tag{4.0.26}$$

Then the map

$$\begin{aligned} &\mathrm{Map}(\Delta_{X/U}, M) \quad (\mathrm{Map}(E\underset{G}{\times}\Delta_{X/U}, M)) \text{ factors through the map} \\ &\mathrm{Map}(\mathrm{id} \wedge (\mathbb{S}^G \wedge \mathrm{i}), M) \quad (\text{the map } \mathrm{Map}(E\underset{G}{\times}(\mathrm{id} \wedge (\mathbb{S}^G \wedge \mathrm{i}), M)), \text{respectively}), \end{aligned} \tag{4.0.27}$$

where $id \wedge (\mathbb{S}^G \wedge i)$ is the map : $\mathbb{S}^G \wedge (X/U) \wedge (\mathbb{S}^G \wedge Y_+) \to \mathbb{S}^G \wedge (X/U) \wedge (\mathbb{S}^G \wedge X_+)$. Here the quotient construction $E\underset{G}{\times}$ is carried out as in [CJ19, section 6.2], that is, when G is special as a linear algebraic group, the quotient is taken on the Zariski (or Nisnevich) site of E, while when G is not special, it is taken on the étale site and then followed by a derived push-forward to the Nisnevich site. (For more details on this, see the discussion in subsection 4.2.)

Once the above factorization of the diagonal maps in (4.0.27) is established, the definition of the pre-transfer and transfer as in Definition 2.5 shows that $\mathrm{Map}(E\underset{G}{\times} tr'^G_{X/U}, M)$ identifies with $\mathrm{Map}(E\underset{G}{\times}(\mathrm{i} \circ tr'^G_Y), M)$ and therefore, that one obtains the identification $tr_{X/U} = i \circ tr_Y$, which denote the transfers induced in generalized motivic cohomology with respect to the spectrum M. This is worked out in detail in Proposition 4.11(i) through (iii).

One can also show, at least non-equivariantly, that $\mathrm{Map}(\Delta_{X/U}, M)$ also identifies with $\mathrm{Map}(\Delta'_{X/U}, M)$, where $\Delta'_{X/U}$ denotes the composite map

(4.0.28)

$$\Delta'_{X/U} : \mathbb{S}^G \wedge (\mathrm{Th}(\mathcal{N}))_+ \to \mathbb{S}^G \wedge (\mathrm{Th}(\mathcal{N}))_+ \wedge (\mathbb{S}^G \wedge E(\mathcal{N})_+) \overset{\mathrm{id}\wedge r}{\to} \mathbb{S}^G \wedge (\mathrm{Th}(\mathcal{N}))_+ \wedge (\mathbb{S}^G \wedge Y_+) \overset{\mathrm{id}\wedge i}{\to} \mathbb{S}^G \wedge (\mathrm{Th}(\mathcal{N}))_+ \wedge (\mathbb{S}^G \wedge X_+).)$$

This is discussed in Proposition 4.11(iv).

The statements in Theorem 1.12(iv) now follow from the statements (ii) and (iii) using ascending induction on the number of strata. However, as this induction needs to be handled carefully, we proceed to provide an outline of the relevant argument. We will assume that the stratification of X defines the following increasing filtrations:

(a) $\phi = X_{-1} \subseteq X_0 \subseteq \cdots \subseteq X_n = X$ where each X_i is closed and the strata $X_i - X_{i-1}$, $i = 0, \cdots, n$ are smooth (regular).

(b) $U_0 \subseteq U_1 \subseteq \cdots \subseteq U_{n-1} \subseteq U_n = X$, where each U_i is open in X (and therefore smooth (regular)) with $U_i - U_{i-1} = X_{n-i} - X_{n-i-1}$, for all $i = 0, \cdots n$. We let $j_k : U_k \to X$ and $j'_k : U_k \to U_{k+1}$ denote the corresponding open immersions while $i'_k : X_k - X_{k-1} \to X - X_{k-1}$ denotes the corresponding closed immersion and $i_k : X_k - X_{k-1} \to X$ denotes the corresponding locally closed immersion.

We now apply Theorem 1.12(ii) with $U = U_{n-1}$, and $Y = U_n - U_{n-1} = X_0 - X_{-1} = X_0$, the closed stratum. Since X is now smooth(regular) and so is Y, the hypotheses of Theorem 1.12(ii) are satisfied. This provides us

$$\begin{gathered} tr'^G_X = j_{n-1} \circ tr'^G_{U_{n-1}} + tr'^G_{X/U_{n-1}}, \\ tr_X = j_{n-1} \circ tr_{U_{n-1}} + tr_{X/U_{n-1}}, \tau^G_X = \tau^G_{U_{n-1}} + \tau^G_{X/U_{n-1}} \text{ and } \tau^G_{X/U_{n-1}} = \tau^G_{X_0}. \end{gathered} \tag{4.0.29}$$

Similarly applying Theorem 1.12(iii) with $U = U_{n-1}$, and $Y = U_n - U_{n-1} = X_0 - X_{-1} = X_0$, we obtain:

$$tr_{X/U_{n-1}} = i_0 \circ tr_{X_0}. \tag{4.0.30}$$

Next we replace X by U_{n-1}, U by U_{n-2} and Y by $X_1 - X_0$. Since $X_1 - X_0$ is smooth (regular), Theorem 1.12(ii) and (iii) now provide us

$$tr_{U_{n-1}} = j'_{n-2} \circ tr_{U_{n-2}} + i'_1 \circ tr_{X_1 - X_0}. \tag{4.0.31}$$

Substituting these in (4.0.29), we obtain

$$tr_X = j_{n-2} \circ tr_{U_{n-2}} + i_1 \circ tr_{X_1 - X_0} + i_0 \circ tr_{X_0}.$$

Clearly this may be continued inductively to deduce statement (iv) in Theorem 1.12 from Theorem 1.12(ii) and (iii).

Finally, the proof of (v) follows from the compatibility of the pre-transfer with étale realization as shown in [CJ19, Proposition 8.1 and Corollary 8.2]. Therefore, it is immediate that one obtains the corresponding statements for the G-equivariant pre-transfer and the G-equivariant trace. The corresponding statements for the transfer in (i) and (ii) then follow readily from these statements for the G-equivariant pre-transfer. In order to obtain the corresponding statements for the transfer in (iii) and (iv), one needs to invoke Proposition 4.11(iii)'. For groups that are special, one may also see these more directly, as $\epsilon^*(tr(\mathrm{f})) = tr(\epsilon^*(\mathrm{f}))$: see [CJ19, Corollary 8.2]. Therefore, for groups that are special, we may also obtain the corresponding results for the étale version of the transfer by simply applying pull-back functor ϵ^* to the étale site. □

4.1. **Motivic Tubular Neighborhoods and Henselization along a closed smooth subscheme.** We devote this small section to a discussion of gadgets we call motivic tubular neighborhoods: we model this on the étale tubular neighborhoods that have been around since [Cox]. Given a smooth scheme X, a *rigid Nisnevich cover* of X is a map of schemes $\mathrm{U} \to \mathrm{X}$, which is a Nisnevich cover, and in addition, U is a disjoint union of pointed étale separated maps $\mathrm{U_x}, \mathrm{u_x} \to \mathrm{X}, \mathrm{x}$, where each $\mathrm{U_x}$ is *connected*, and as x varies over points of X, so that the above map induces an isomorphism of residue fields $k(\mathrm{u_x}) \cong \mathrm{k}(\mathrm{x})$. Given two rigid Nisnevich covers $\mathrm{U} \to \mathrm{X}$ and $\mathrm{V} \to \mathrm{Y}$, the *rigid product* $\mathrm{U}\overset{\mathrm{R}}{\times}\mathrm{V}$ is given as the disjoint union of $(\mathrm{U_x} \times \mathrm{V_y})_0$ which is the connected component of $\mathrm{U_x} \times \mathrm{V_y}$ indexed by the point $\mathrm{x} \times \mathrm{y}$ of $\mathrm{X} \times \mathrm{Y}$.

Given a scheme X, a *hypercover* $\mathrm{U}_\bullet$ of X is a simplicial scheme $\mathrm{U}_\bullet$ together with an étale map $\mathrm{U}_\bullet \to \mathrm{X}$, so that (i) $\mathrm{U}_0 \to \mathrm{X}$ is a Nisnevich cover and (ii) the induced map $\mathrm{U_t} \to (\mathrm{cosk}^{\mathrm{X}}_{\mathrm{t}-1}\mathrm{U}_\bullet)_\mathrm{t}$ is a Nisnevich cover, for each $t \geq 1$. Such a hypercover is a *rigid hypercover* if the given maps in (i) and (ii) are rigid Nisnevich covers. One may readily show that the category of rigid Nisnevich hypercovers of a given scheme X is a left directed category: see [Cox, section 1]. This category will be denoted HRR(X).

Definition 4.1. (Motivic tubular neighborhoods) Let Y denote a closed smooth subscheme of a smooth scheme X. We define the *(rigid) motivic tubular neighborhood* of Y in X to be the inverse system of simplicial schemes $\mathrm{N}^{\mathrm{Y}}_\bullet(\mathrm{U}_\bullet)$ for which there exists a rigid Nisnevich hypercover $\mathrm{U}_\bullet$ of X so that $\mathrm{N}^{\mathrm{Y}}_{\mathrm{n}}(\mathrm{U}_\bullet) = \sqcup_{\mathrm{i}}\mathrm{U}_{\mathrm{n,i}}$, where the sum runs over $\mathrm{U}_{\mathrm{n,i}}$, which are connected components of $\mathrm{U_n}$ with the property that $\mathrm{U}_{\mathrm{n,i}}\underset{\mathrm{X}}{\times}\mathrm{Y} \neq \phi$. One may readily see that the motivic tubular neighborhood of Y in X is a left-directed category. This will be denoted $t_{\mathrm{X/Y}}$. See [Cox, section 1] for similar definitions of étale tubular neighborhoods.

Remark 4.2. The inverse system of all rigid Nisnevich neighborhoods of Y in X corresponds to the Henselizaton of X along Y.

Next we will provide the following Lemma, whose proof is skipped as it follows exactly as in [Cox, Lemma 1.2].

Lemma 4.3. *Given a* $\mathrm{V}_\bullet$ *in* $t_{\mathrm{X/Y}}$, *and a separated étale map* $\phi : \mathrm{W} \to \mathrm{V_n}$, *and so that the induced map* $\mathrm{W}\underset{\mathrm{X}}{\times}\mathrm{Y} \to \mathrm{V_n}\underset{\mathrm{X}}{\times}\mathrm{Y}$ *is a Nisnevich cover, there is a map* $\phi_\bullet : \mathrm{W}_\bullet \to \mathrm{V}_\bullet$ *in* $t_{\mathrm{X/Y}}$, *so that* ϕ_n *factors through the map* ϕ.

Next we recall that the main model structure we use on the category $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$ is defined as follows. First we start with the injective model structure on the category PSh/S of pointed simplicial presheaves, where cofibrations (weak-equivalences) are section-wise cofibrations (weak-equivalences, respectively) of pointed simplicial sets, and fibrations are defined by the right lifting property with respect to maps that are trivial cofibrations. Then we localize this model structure by inverting maps that are both stalk-wise weak-equivalences and also maps of the form $\mathbb{A}^1 \times \mathrm{U} \to \mathrm{U}$, for any U in the site. We will call the resulting category, the category of *motivic spaces.* Recall that the objects of the category $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$ are PSh/S-enriched functors $\mathrm{Sph}^{\mathrm{G}} \to \mathrm{PSh/S}$, where PSh/S is provided with the above model structure. We start with the *level-wise injective model structure* on this category, where the cofibrations (weak-equivalences) are maps $\phi : \mathcal{X}' \to \mathcal{X}$ for which the induced map $\phi(\mathrm{T_V}) : \mathcal{X}'(\mathrm{T_V}) \to \mathcal{X}(\mathrm{T_V})$ is a cofibration (weak-equivalence, respectively) for every $\mathrm{T_V}$. Finally we obtain the corresponding stable model structure, where the fibrant objects are the Ω-spectra. A map between two fibrant spectra $\mathrm{M}' = \{\mathrm{M}'(\mathrm{T_V})|\mathrm{V}\} \to \mathrm{M} = \{\mathrm{M}(\mathrm{T_V})|\mathrm{V}\}$ is a weak-equivalence if and only if the map $\mathrm{M}'(\mathrm{T_V}) \to \mathrm{M}(\mathrm{T_V})$ is a weak-equivalence for each V, in the level-wise injective model structure, which implies (in view of the above discussion) that it is a stalk-wise weak-equivalence of $\mathbb{A}^1$-localized simplicial presheaves.

It is shown in [CJ19, Proposition 3.10] that one has a Quillen equivalence between the model category **Spt** of motivic spectra (i.e. sequence of motivic spaces $\{\mathrm{M_n}|\mathrm{n} \geq 0\}$ together with compatible suspensions $\mathrm{T} \wedge \mathrm{M_n} \to \mathrm{M_{n+1}}$) and the model category $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$. At this point it is convenient to introduce the category of motivic S^1-spectra: this will be sequences $\{\mathrm{M_n}|\mathrm{n} \geq 0\}$ of motivic spaces, together with a compatible family of structure maps $\mathrm{S}^1 \wedge \mathrm{M_n} \to \mathrm{M_{n+1}}$, $n \geq 0$. We will put the *level-wise injective* model structure on this category, where cofibrations and weak-equivalences are defined level-wise and fibrations defined by the right-lifting property with respect to trivial cofibrations. This category will be denoted $\mathbf{Spt}_{\mathrm{S}^1}$. A motivic Ω*-bi-spectrum* $\mathcal{S}$ is given by a sequence $\{\mathcal{S}_n|n \geq 0\}$ of motivic S^1-spectra, together with compatible weak-equivalences $\{\mathcal{S}_n \to \Omega_{\mathbf{T}}(\mathcal{S}_{n+1})|n \geq 0\}$. One may define motivic bi-spectra similarly, by just relaxing the condition that the maps $\mathcal{S}_n \to \Omega_{\mathbf{T}}(\mathcal{S}_{n+1})$ are weak-equivalences. With suitable model structures, the above categories of spectra are all Quillen-equivalent. (See

[Lev08, section 8], where such spectra are called by a slightly different name. The relationship between various categories of spectra is discussed there.)

A useful observation for us is the following: given a motivic S^1-spectrum $\mathrm{M} = \{\mathrm{M_n} | \mathrm{n} \geq 0\}$, so that (i) each $\mathrm{M_n}$ is $\mathbb{A}^1$-local and stalk-wise fibrant, one may obtain a fibrant replacement (i.e. fibrant in the injective model structure on $\mathbf{Spt}_{\mathrm{S}^1}$) by applying the canonical Godement resolution $\mathrm{G}^\bullet$ (which produces a cosimplicial object) and then taking its homotopy inverse limit): the resulting motivic spectrum will be denoted $\mathrm{G(M)}$.

Proposition 4.4. *(i) Given any motivic* S^1*-spectrum* $\mathrm{M} = \{\mathrm{M_n} | \mathrm{n} \geq 0\}$*, there exists a spectral sequence*

$$E_2^{s,t} = \underset{\alpha}{colim}\mathrm{H^s}(\Gamma(\mathrm{U}^\alpha_\bullet, \pi_\mathrm{t}(\mathrm{M}))) \Rightarrow \pi_{-\mathrm{s+t}}(\underset{\Delta}{holim}\, \underset{\alpha}{colim}\Gamma(\mathrm{U}^\alpha_\bullet, \mathrm{M}))$$

where the colimit is taken over all rigid Nisnevich hypercovers of the given smooth scheme X*. This spectral sequence converges strongly as* $E_2^{s,t} = \mathrm{H^s_{Nis}}(\mathrm{X}, \mathrm{a}\pi_\mathrm{t}(\mathrm{M})) = 0$ *for all* $s > dim(X)$*, where* $a\pi_t(\mathrm{M})$ *denotes the associated abelian sheaf.*

(ii) If M *is a motivic* S^1*-spectrum as in (i) and is replaced by its fibrant replacement in the injective model structure on* $\mathbf{Spt}_{\mathrm{S}^1}$ *(for example, its Godement resolution* $\mathrm{G(M)}$*), then the map from the spectral sequence*

$$E_2^{s,t} = \mathrm{H^s}(\Gamma(\mathrm{U}^{\alpha_0}_\bullet, \pi_\mathrm{t}(\mathrm{G(M)}))) \Rightarrow \pi_{-\mathrm{s+t}}(\underset{\Delta}{holim}\,\Gamma(\mathrm{U}^{\alpha_0}_\bullet, \mathrm{GM}))$$

for a fixed rigid Nisnevich hypercover $\mathrm{U}^{\alpha_0}_\bullet$*, to the spectral sequence in (i), is an isomorphism.*

Proof. (i) The existence of the spectral sequence and the identification of the E_2-terms follow readily in view of the discussion in [Th85, Proposition 1.16]. At this point, one knows that cohomology on any site with respect to an abelian presheaf may be computed using hypercoverings in that site: see [SGA4]. Therefore, the $E_2^{s,t}$-term in (i) identifies with $\mathrm{H^s_{Nis}}(\mathrm{X}, \mathrm{a}\pi_\mathrm{t}(\mathrm{M}))$, where $a\pi_t(\mathrm{M})$ denotes the sheaf associated to the presheaf $\pi_t(\mathrm{M})$. Finally one uses the fact that the Nisnevich site of the smooth scheme X has cohomological dimension given by $dim(\mathrm{X})$ to complete the proof of (i).

(ii) We will assume that $\mathcal{S}$ is replaced by $\mathrm{G(M)} = \underset{\Delta}{\mathrm{holim}}\, \mathrm{G}^\bullet(\mathrm{M})$. Then the $E_2^{s,t}$-term of the spectral sequence in (ii) identifies with $\mathrm{H^s}(\Gamma(\mathrm{U}^{\alpha_0}_\bullet, \pi_\mathrm{t}(\mathrm{G(M)}))) \cong \mathrm{H^s}(\Gamma(\mathrm{U}^{\alpha_0}_\bullet, \mathrm{G}^\bullet\pi_\mathrm{t}(\mathrm{M}))) \cong \mathrm{H^s_{Nis}}(\mathrm{X}, \mathrm{a}\pi_\mathrm{t}(\mathrm{M}))$. Thus these $E_2^{s,t}$-terms do not depend on the choice of the rigid Nisnevich hypercover, and also identifies with the $E_2^{s,t}$-term of the spectral sequence in (i). Since both spectral sequences converge strongly, we obtain an isomorphism on the abutments, thereby proving (ii). $\square$

Proposition 4.5. *Let* M *denote a motivic spectrum so that there exists a prime* $\ell \neq char(k)$*, so that the homotopy groups of the spectrum* M *are all* ℓ*-primary torsion. Then* M *has the rigidity property (as in Definition 1.10) in the following cases:*

(i) *The base field is infinite and* M *defines an orientable motivic cohomology theory, that is, one that has a theory of Chern classes.*
(ii) *The base field* k *is algebraically or quadratically closed and of characteristic* 0*: there are no further restrictions on the motivic spectrum* M.
(iii) *The base field* k *is infinite, non-real and of characteristic* 0*: there are no further restrictions on the motivic spectrum* M.
(iv) *The base field* k *is infinite, non-real, of characteristic different from* 2 *and the prime* $\ell \neq 2$*: there are no further restrictions on the motivic spectrum* M.

In particular, the spectrum representing algebraic K-theory with finite coefficients prime to the characteristic has the rigidity property.

Proof. The fact that one has the above rigidity property for the spectrum representing algebraic K-theory follows from Gabber's theorem which holds for all Hensel pairs: see [Gab, Theorem 1]. The remaining statements need to be deduced from what is in the literature on rigidity: statements (i) ((ii) and (iii)) when x is a k-rational point of a smooth variety is stated in [PY02, Theorem 1.13], [HY07, Theorem 0.3, Corollary 0.4], as well as [Y04, Theorem 1.5] and [Y11, Corollary 2.6]. Observe that, in Definition 1.10, one does not require the point x to be a k-rational point. Therefore, we proceed to show that the above rigidity property can be deduced from the corresponding statement for the case x is a k-rational point.

Next let Y denote the closure of the given point x and let $y = x$, but viewed as a point of Y. By replacing X and Y by open subschemes, we may assume without loss of generality that Y is smooth and y denotes the generic point of Y. The local structure discussed below in Lemma 4.12 shows that there is a Zariski open neighborhood $\mathrm{U_y}$ of y in X and an étale map $q_y : \mathrm{U_y} \rightarrow \mathbb{A}^\mathrm{n}$, (where $n = dim_k(\mathrm{X})$), so that $\mathrm{U_y} \cap \mathrm{Y} = \mathrm{q_y^{-1}}(\mathbb{A}^{\mathrm{n-c}} \times \{0\})$, (where $c = codim_\mathrm{X}(\mathrm{Y})$). Moreover, there is then a smaller open $\mathrm{V_y}$ in $\mathrm{U_y}$ which is a Nisnevich neighborhood of $\mathrm{U_y} \cap \mathrm{Y}$ in $\mathrm{U_y}$ and also of $(\mathrm{U_y} \cap \mathrm{Y}) \times \{0\}$ in $(\mathrm{U_y} \cap \mathrm{Y}) \times \mathbb{A}^\mathrm{c}$, in the sense that the conditions in Lemma 4.12(ii) are satisfied.

Then, since Nisnevich neighborhoods of the form $\mathrm{W_y} \times_{\mathrm{k}} \mathrm{W}'_0$, where $\mathrm{W_y}$ is a Nisnevich neighborhood of y in Y and W'_0 is Nisnevich neighborhood of 0 in $\mathbb{A}^c$, are cofinal in the system of all Nisnevich neighborhoods of the point $y \times 0$ in $\mathrm{Y} \times \mathbb{A}^c$, we obtain

$$\mathcal{O}^h_{\mathrm{X},x} \cong \mathcal{O}^h_{\mathrm{Y},y} \otimes_k \mathcal{O}^h_{\mathbb{A}^c,0}. \tag{4.1.1}$$

Since y is the generic point of Y, clearly $\mathcal{O}^h_{\mathrm{Y},y} \cong k(y)$. Thus $\mathcal{O}^h_{\mathrm{X},x} \cong k(y) \otimes_k \mathcal{O}^h_{\mathbb{A}^c,0}$. At this point, we may consider the scheme $\mathit{Spec}\, k(y) \underset{\mathit{Spec}\, k}{\times} \mathbb{A}^c$: clearly $y \underset{\mathit{Spec}\, k}{\times} 0$ is a $k(y)$ rational point of the scheme $\mathit{Spec}\, k(y) \underset{\mathit{Spec}\, k}{\times} \mathbb{A}^c$.

It is observed on [HY07, p. 441] that any generalized orientable motivic cohomology theory is normalized with respect to any field. Therefore, [HY07, Theorem 0.3 and Corollary 0.4] apply to prove the statement in (i). The field $k(y)$ and the fraction field of $k(y)\underset{k}{\otimes}\mathcal{O}^h_{\mathbb{A}^c,0}$ satisfy the hypotheses of [Y11, Corollary 2.6], which proves the statements in (ii) and (iii). The statement in (iv) also follows from [Y11, Corollary 2.6], once the restriction that the field of fractions of the Hensel ring be perfect is removed. An analysis of the proof of [Y11, Corollary 2.6] shows that this condition is put in because of the restriction that the field be perfect in Morel's theorem as in [Mo4, Theorem 6.2.2]. By [BH, Theorem 10.12], the above assumption is no longer needed in the above Theorem of Morel. □

Proposition 4.6. *Let* M *denote a motivic spectrum. If the homotopy groups of* M *are all ℓ-primary torsion, for some prime $\ell \neq char(k)$, then the slices of* M *have the rigidity property. In particular, if the spectrum* M *has the rigidity property as in Definition 1.10, then all its slices have the rigidity property.*

Proof. We observe from [Pel11, Theorem 2.4] that the slices of any motivic spectrum are orientable. Therefore, in view of Proposition 4.5(i), it suffices to show that if the spectrum M is such that its homotopy groups are all ℓ-primary torsion for some prime $\ell \neq char(k)$, then the same property holds for its slices. For this one needs to recall the construction of the $\mathbb{P}^1$-slices of a motivic spectrum as in [Lev08, sections 8, 9]. First one shows that the $\Omega_{\mathbf{T}}$-spectrum, $\widehat{\mathrm{M}}$, associated to M also has its homotopy groups all ℓ-primary torsion: this follows readily from the fact that $\widehat{\mathrm{M}}_n = \lim_{m\to\infty} \Omega^m_{\mathbf{T}} \mathrm{M_{n+m}}$. It follows that $\widehat{\mathrm{M}}_n$ has its homotopy groups all ℓ-primary torsion, for each fixed n. Next one constructs a bi-spectrum by taking the S^1-suspension spectrum, $\Sigma_{\mathrm{S}^1}\widehat{\mathrm{M}}_n$, of each $\widehat{\mathrm{M}}_n$: $\{\Sigma_{\mathrm{S}^1}\widehat{\mathrm{M}}_n | n \geq 0\}$. One may see readily that each of the S^1-suspension spectra, $\Sigma_{\mathrm{S}^1}\widehat{\mathrm{M}}_n$ also has its homotopy groups all ℓ-primary torsion. Finally one applies the construction of the slices as in [Lev08, 8.3] in terms of the slices of the S^1-spectra, $\Sigma_{\mathrm{S}^1}\widehat{\mathrm{M}}_n$. Thus, we reduce to showing that if the homotopy groups of the S^1-spectrum $\mathcal{X}$ are all ℓ-primary torsion, then its S^1-slices also have their homotopy groups all ℓ-primary torsion: this is clear from the explicit construction of such slices as in [Lev08, 2.1]. □

Lemma 4.7. *Let* M *denote a motivic ring spectrum whose homotopy groups are all ℓ-primary torsion, for a fixed prime $\ell \neq char(k)$. Then if* M *has the rigidity property as in Definition 1.10, its pull-back $\epsilon^*(\mathrm{M})$ to the étale site also has the rigidity property*

Proof. Clearly for every Hensel ring R with residue field K, the map $\Gamma(\mathrm{Spec\,R}, \mathrm{M}) \to \Gamma(\mathrm{Spec\,K}, \mathrm{M})$ is a weak-equivalence, since M has the rigidity property (on the Nisnevich site). The fact that, if $\mathrm{K}_1 \subseteq \mathrm{K}_2$ is an extension of algebraically closed fields, then the induced map $\Gamma(\mathrm{Spec\,K}_1, \mathrm{M}) \to \Gamma(\mathrm{Spec\,K}_2, \mathrm{M})$ is a weak-equivalence is shown in [Y04, Theorem 1.10]. To see that the same holds when K_1 and K_2 are only separably closed, one observes that for any purely inseparable field extension $\mathrm{K} \subseteq \mathrm{K}'$ of fields containing the base field k, $\Gamma(\mathrm{Spec\,K}, \mathrm{M}) \simeq \Gamma(\mathrm{Spec\,K}', \mathrm{M})$ as the homotopy groups of M are all ℓ primary torsion and the degree of the field extension is prime to ℓ. □

Next we recall the definition of *Hensel pairs* for affine schemes from [St, Definition 15.11.1]. Accordingly an (affine) Hensel pair is given by a pair (A, I) where A is a commutative ring with 1 and I is an ideal in A contained in the Jacobson radical of A, so that for any monic polynomial $f \in \mathrm{A[T]}$ and factorization $\bar{\mathrm{f}} = \bar{\mathrm{g}}.\bar{\mathrm{h}}$ with $\bar{\mathrm{g}}, \bar{\mathrm{h}} \in (\mathrm{A/I})[\mathrm{T}]$, which is monic and generating the unit ideal in $\mathrm{A/I[T]}$, there exists a factorization $\mathrm{f} = \mathrm{g.h}$ in $\mathrm{A[T]}$ with $\mathrm{g, h}$ monic and $\bar{\mathrm{g}}$ (and $\bar{\mathrm{h}}$) the image of g (f) in $\mathrm{A/I[T]}$.

It is important to view the above *affine Hensel pair as the affine scheme given by* $(\mathit{Spec}\,(\mathrm{A/I}), \mathrm{A})$, that is, the underlying topological space is $\mathit{Spec}(\mathrm{A/I})$ and the structure sheaf is the one defined by the ring A.

Definition 4.8. (Hensel pairs and Henselization) Let X be a given scheme and Y a closed subscheme of X. Then (X, Y) is a *Hensel pair* if for every affine open cover $\{\mathrm{U_i}|\mathrm{i}\}$ of X, $(\mathrm{U_i}, \mathrm{U_i} \cap \mathrm{Y})$ is Hensel pair as in [St, Definition 15.11.1]. Given a scheme X and a closed subscheme Y of X, the Henselization of X along Y is the scheme obtained by gluing the Henselization of the affine schemes $\{\mathrm{U_i}|\mathrm{i}\}$ along the closed subschemes $\mathrm{U_i} \cap \mathrm{Y}$. This will be denoted $\mathrm{X^h_Y}$.

It is important to view the scheme $\mathrm{X^h_Y}$ as given by the underlying space Y and provided with the structure sheaf $\mathcal{O}^h_{\mathrm{X,Y}}$ which is obtained by the gluing $(\mathrm{Y} \cap \mathrm{U_i}, \mathcal{O}^{\mathrm{h}}_{\mathrm{U_i,U_i\cap Y}})$ for any affine open cover $\{\mathrm{U_i}|\mathrm{i}\}$ of the scheme X. (See [Cox, p. 213]

Lemma 4.9. *If*

$$\begin{tikzcd} \mathrm{Y'} \arrow[r] \arrow[d] & \mathrm{X'} \arrow[d] \\ \mathrm{Y} \arrow[r] & \mathrm{X} \end{tikzcd}$$

is a cartesian square where the horizontal maps are closed immersions, one obtains an induced map $\mathrm{X'^h_{Y'}} \to \mathrm{X^h_Y}$.

Proof. This is skipped as it is an easy exercise to complete from the definition of Henselization: one may in fact reduce to the case where all the schemes are affine. □

The following *rather subtle point* is the main thrust of the following result. Given a presheaf M on the *big Nisnevich site* over k, there is no apriori reason for the cohomology with respect to M for the Henselization of a given scheme X along a closed sub-scheme Y to be isomorphic to the cohomology of Y with respect to M. This issue does not arise if M is a sheaf on the *small* Nisnevich site of X. The assumption that M has the rigidity property, then does ensure that the above cohomologies are isomorphic.

Corollary 4.10. *Let* $i : \mathrm{Y} \to \mathrm{X}$ *denote a closed immersion of smooth schemes. (i) Then for any motivic* $\mathrm{S^1}$*-spectrum* $\mathrm{M} = \{\mathrm{M_n}|\mathrm{n}\}$ *in* $\mathbf{Spt}_{\mathrm{S^1}}$*, one obtains a weak-equivalence*

$$\underset{\Delta}{holim}\ \underset{\alpha}{colim}\Gamma(\mathrm{N^Y(U^\alpha_\bullet)}, \mathrm{M}) \simeq \mathbb{H}_{\mathrm{Nis}}(\mathrm{X^h_Y}, \mathrm{M})$$

where $\mathrm{U^\alpha_\bullet}$ *varies among all hypercoverings of* X*,* $\mathrm{X^h_Y}$ *denotes the Henselization of the scheme* X *along* Y*, and* $\mathbb{H}_{\mathrm{Nis}}(\mathrm{X^h_Y}, \mathrm{M})$ *denotes the spectrum* $\Gamma(\mathrm{X^h_Y}, G\mathrm{M})$*, with* $G\mathrm{M}$ *denotes a fibrant replacement of* M*.*

(ii) If in addition, the spectrum M *has the rigidity property in Definition 1.10, then one obtains the weak-equivalence:*

$$\mathbb{H}_{\mathrm{Nis}}(\mathrm{X^h_Y}, \mathrm{M}) \simeq \mathbb{H}_{\mathrm{Nis}}(\mathrm{Y}, \mathrm{M}).$$

(iii) If X *and* Y *are provided with the action of a linear algebraic group* G *with the map* i G*-equivariant, the same conclusions also hold for any spectrum* $\mathrm{M}\,\varepsilon\,\widetilde{\mathbf{USpt}}^{\mathrm{G}}$.

(iv) Corresponding results also hold for hypercohomology computed on the étale site, provided the base field k has finite ℓ-cohomological dimension for some prime $\ell \neq char(k)$ and the homotopy groups of the spectrum M *are ℓ-primary torsion.*

Proof. Throughout this proof, we will adopt the following notational conventions. For a smooth scheme Z, we let $\mathrm{M_{|Z}}$ denote the restriction of M to the *small* Nisnevich site of the scheme Z. Given a presheaf P on the small Nisnevich site of the scheme X, we will let $i^{-1}(\mathrm{P})$ denote the restriction of P to the small Nisnevich site of the closed subscheme Y.

As in Proposition 4.4, one obtains spectral sequences:

$$E_2^{s,t}(1) = \underset{\alpha}{\mathrm{colim}}\mathrm{H^s}(\Gamma(\mathrm{N^Y(U^\alpha_\bullet)}, \pi_{\mathrm{t}}(\mathrm{M_{|X}})) \Rightarrow \pi_{-s+t}(\underset{\Delta}{\mathrm{holim}}\ \underset{\alpha}{\mathrm{colim}}\Gamma(\mathrm{N^Y(U^\alpha_\bullet)}, \mathrm{M_{|X}})) \text{ and}$$
$$E_2^{s,t}(2) = \mathrm{H^s}(\mathrm{Y}, \pi_{\mathrm{t}}(\mathrm{i^{-1}M_{|X}})) \Rightarrow \pi_{-s+t}\mathbb{H}(\mathrm{Y}, \mathrm{i^{-1}M_{|X}}). \tag{4.1.2}$$

Since every scheme that appears in the simplicial scheme $\mathrm{N^Y(U^\alpha_\bullet)}$ in each degree belongs to the small Nisnevich site of X, one may identify the first spectral sequence with

$$E_2^{s,t}(1) = \underset{\alpha}{\mathrm{colim}}\mathrm{H^s}(\Gamma(\mathrm{N^Y(U^\alpha_\bullet)}, \pi_{\mathrm{t}}(\mathrm{M})) \Rightarrow \pi_{\mathrm{-s+t}}(\underset{\Delta}{\mathrm{holim}}\ \underset{\alpha}{\mathrm{colim}}\Gamma(\mathrm{N^Y(U^\alpha_\bullet)}, \mathrm{M})).$$

In addition, there is also a third spectral sequence:

$$E_2^{s,t}(3) = \mathrm{H^s}(\mathrm{Y}, \pi_{\mathrm{t}}(\mathrm{M})) \cong \mathrm{H^s}(\mathrm{Y}, \pi_{\mathrm{t}}(\mathrm{M_{|Y}})) \Rightarrow \pi_{-s+t}\mathbb{H}(\mathrm{Y}, \mathrm{M}) \cong \pi_{\mathrm{-s+t}}\mathbb{H}(\mathrm{Y}, \mathrm{M_{|Y}}). \tag{4.1.3}$$

We reduce to showing there are natural maps of these spectral sequences that inducing an isomorphism at the E_2-terms, and that all three of these spectral sequences converge strongly. First making use of Lemma 4.3, we obtain the identification:

$$\underset{\alpha}{\mathrm{colim}}\mathrm{H^s}(\Gamma(\mathrm{N^Y(U^\alpha_\bullet)}, \pi_{\mathrm{t}}(\mathrm{M_{|X}})) \cong \underset{\alpha}{\mathrm{colim}}\mathrm{H^s_{Nis}}(\mathrm{N^Y(U^\alpha_\bullet)}, \pi_{\mathrm{t}}(\mathrm{M_{|X}})),$$

where the term on the left (right) denotes the cohomology of the co-chain complex $\Gamma(\mathrm{N^Y}(\mathrm{U}^\alpha_\bullet), \pi_t(\mathrm{M}_{|\mathrm{X}})$ (the Nisnevich hypercohomology of $\mathrm{N^Y}(\mathrm{U}^\alpha_\bullet)$ with respect to the abelian sheaf $\pi_t(\mathrm{M}_{|\mathrm{X}})$, respectively). Observe that $\mathrm{N^Y}(\mathrm{U}^\alpha_\bullet)\underset{\mathrm{X}}{\times}\mathrm{Y}$ is a Nisnevich hypercover of Y, so that one obtains a natural map

$$\underset{\alpha}{\mathrm{colim}}\mathrm{H}^{\mathrm{s}}_{\mathrm{Nis}}(\mathrm{N^Y}(\mathrm{U}^\alpha_\bullet), \pi_{\mathrm{t}}(\mathrm{M}_{|\mathrm{X}})) \rightarrow \underset{\alpha}{\mathrm{colim}}\mathrm{H}^{\mathrm{s}}_{\mathrm{Nis}}(\mathrm{N^Y}(\mathrm{U}^\alpha_\bullet)\underset{\mathrm{X}}{\times}\mathrm{Y}, \pi_{\mathrm{t}}(\mathrm{M}_{|\mathrm{X}})) \cong \mathrm{H}^{\mathrm{s}}_{\mathrm{Nis}}(\mathrm{Y}, \pi_{\mathrm{t}}(\mathrm{M}_{|\mathrm{X}})).$$

This provides a map between the first two spectral sequences. To prove that this map will be an isomorphism at the E_2-terms, exactly the same arguments as in the proof of [Cox, Theorem 1.3] carry over from the étale framework to the Nisnevich framework, we are considering. Now it is clear that $E_2^{s,t} = 0$, for all $s > dim_k(\mathrm{Y})$, so that both these spectral sequences converge strongly providing the required isomorphism at the abutments. Observe that the stalk of $\pi_t(\mathrm{M}_{|\mathrm{X}})$ at a point $y\,\varepsilon\,\mathrm{Y}$ identifies with $\pi_t(\Gamma(Spec\,\mathcal{O}^h_{\mathrm{X},y}, \mathrm{M}))$, so that we obtain the identification

$$\mathrm{H}^{\mathrm{s}}_{\mathrm{Nis}}(\mathrm{Y}, \pi_{\mathrm{t}}(\mathrm{i}^{-1}\mathrm{M}_{|\mathrm{X}})) \cong \mathrm{H}^{\mathrm{s}}_{\mathrm{Nis}}(\mathrm{X}^{\mathrm{h}}_{\mathrm{Y}}, \pi_{\mathrm{t}}(\mathrm{M}_{|\mathrm{X}})). \tag{4.1.4}$$

This proves the first statement. Next we consider the second statement. Observe that there is a map from the second spectral sequence in (4.1.2) to the spectral sequence in (4.1.3). As both spectral sequences converge strongly, it suffices to show that the obvious map of sheaves $\pi_t(\mathrm{M}_{|\mathrm{X}}) \rightarrow \pi_t(\mathrm{M}_{|\mathrm{Y}})$ is an isomorphism stalk-wise at every point of Y. As observed above, the stalk of $\pi_t(\mathrm{M}_{|\mathrm{X}})$ at a point $y\,\varepsilon\,\mathrm{Y}$ identifies with $\pi_t(\Gamma(Spec\,\mathcal{O}^h_{\mathrm{X},y}, \mathrm{M}))$. The stalk of $\pi_t(\mathrm{M}_{|\mathrm{Y}})$ at the same point y identifies with $\pi_t(\Gamma(Spec\,\mathcal{O}^h_{\mathrm{Y},y}, \mathrm{M}))$. By the assumed rigidity property of M, both of the above groups identify with $\pi_t(\Gamma(Spec\,k(y), \mathrm{M}))$. Therefore, the required isomorphism follows from the assumed rigidity property of the spectrum M and the isomorphism in (4.1.4). This completes the proof of the second statement. The third statement now follows in view of the Quillen-equivalence of model categories between $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$ and the model category of motivic spectra established in [CJ19, Proposition 3.10].

Next we consider the statement in (iv). One can see that essentially the same spectral sequences exist on the étale site: their strong convergence is guaranteed by the assumption that the base field k has finite ℓ-cohomological dimension. Now, the main point is to show that one obtains a weak-equivalence

$$\mathbb{H}_{et}(\mathrm{X}^{\mathrm{h}}_{\mathrm{Y}}, \epsilon^*(\mathrm{M})) \simeq \mathbb{H}_{\mathrm{et}}(\mathrm{Y}, \epsilon^*(\mathrm{M})). \tag{4.1.5}$$

For a smooth scheme Z, we let $\epsilon^*(\mathrm{M})_{|\mathrm{Z}}$ denote the restriction of $\epsilon^*(\mathrm{M})$ to the *small* étale site of the scheme Z. Given a presheaf P on the small étale site of the scheme X, we will let $i^{-1}(\mathrm{P})$ denote the restriction of P to the small étale site of the closed subscheme Y. As the space underlying the scheme $\mathrm{X}^{\mathrm{h}}_{\mathrm{Y}}$ is just the space underlying the scheme Y, the left-hand-side of (4.1.5) identifies with $\mathbb{H}_{et}(\mathrm{Y}, \mathrm{i}^{-1}(\epsilon^*(\mathrm{M})_{|\mathrm{X}}))$. The right-hand-side of (4.1.5) identifies with $\mathbb{H}_{et}(\mathrm{Y}, \epsilon^*(\mathrm{M})_{|\mathrm{Y}})$. Now the stalk of $i^{-1}(\epsilon^*(\mathrm{M})_{|\mathrm{X}})$ at a geometric point $\bar{y}$, corresponding to a point $y \in \mathrm{Y}$, is given by $\Gamma(\mathrm{Spec}\,(\mathcal{O}^{sh}_{\mathrm{X,y}}), \mathrm{M})$ while the stalk of $\epsilon^*(\mathrm{M})_{|\mathrm{Y}}$ at the same geometric point $\bar{y}$ is given by $\Gamma(\mathrm{Spec}\,(\mathcal{O}^{sh}_{\mathrm{Y,y}}), \mathrm{M})$. By the assumed rigidity property of M, both of these identify with $\Gamma(\mathrm{Spec}\,(\overline{k(y)}), \mathrm{M})$. This then provides the required weak-equivalence of the étale hypercohomology spectra in (4.1.5), as the corresponding spectral sequences that compute the homotopy groups of the hypercohomology spectra converge strongly. □

4.2. **More on rigidity.** One may let $\{\mathrm{Y} \rightarrow \mathrm{X}^{\mathrm{h}}_{\mathrm{Y}}\}$ denote the family of Henselizations of smooth schemes X along a closed smooth subscheme Y. Now one may enlarge the generating trivial cofibrations on the stable motivic homotopy category $\mathbf{Spt}_{\mathrm{mot}}$ by including the $\mathbf{T}$-suspension spectra of the above family of maps among the generating trivial cofibrations. In the resulting model category, one can see that the fibrant objects are exactly the fibrant spectra in $\mathbf{Spt}_{\mathrm{mot}}$ that have the rigidity property. We will denote the corresponding model category of motivic spectra by $\mathbf{Spt}_{\mathrm{mot,r}}$. Let $\epsilon^* : \mathbf{Spt}_{\mathrm{mot,r}} \rightarrow \mathbf{Spt}_{\mathrm{et}}$ denote the pull-back to the étale site. In order that ϵ^* be a left-Quillen functor, it is clear that we need to enlarge the generating trivial cofibrations on $\mathbf{Spt}_{\mathrm{et}}$ by adding maps of the form: $\{\epsilon^*(\Sigma_{\mathbf{T}}\mathrm{Y}) \rightarrow \epsilon^*(\Sigma_{\mathbf{T}}\mathrm{X}^{\mathrm{h}}_{\mathrm{Y}})\}$. We will denote the resulting model category of étale spectra by $\mathbf{Spt}_{\mathrm{et,r}}$. In a similar manner, on incorporating the action of a linear algebraic group G, we obtain the left-Quillen functor on the corresponding model categories:

$$\epsilon^* : \widetilde{\mathbf{USpt}}^{\mathrm{G}}_{mot,r} \rightarrow \widetilde{\mathbf{USpt}}^{\mathrm{G}}_{et,r} \tag{4.2.1}$$

with its right adjoint given by $\mathrm{R}\epsilon_*$.

Proposition 4.11. *Assume the situation as in Theorem 1.12.*

(i) Then one obtains the commutative square:

$$\begin{tikzcd}
\mathrm{X^h_Y/(X^h_Y - Y)} \arrow[r] \arrow[d, "\Delta^h"] & \mathrm{X/U} \arrow[d, "\Delta"] \\
\mathrm{(X^h_Y/(X^h_Y - Y)) \wedge X^h_{Y,+}} \arrow[r] & \mathrm{X/U \wedge X_+}
\end{tikzcd}$$

where the top horizontal map induces a motivic stable equivalence on the associated suspension spectra and Δ^h, Δ *denote the corresponding diagonal maps. Taking the smash product with the sphere spectrum* $\mathbb{S}^{\mathrm{G}}$, *one obtains a similar commutative square, where each term above is replaced by the smash product with* $\mathbb{S}^{\mathrm{G}}$.

(ii) Let $p : \mathrm{E} \to \mathrm{B}$ *denote a* G*-torsor for a linear algebraic group* G. *Then the commutative diagram in (i) induces the commutative diagram:*

$$\begin{tikzcd}
\mathrm{E}\underset{\mathrm{G}}{\times}(\mathbb{S}^{\mathrm{G}} \wedge (\mathrm{X^h_Y/(X^h_Y - Y)})) \arrow[r] \arrow[d, "\mathrm{E}\underset{\mathrm{G}}{\times}(\mathbb{S}^{\mathrm{G}}\wedge\Delta^h)"] & \mathrm{E}\underset{\mathrm{G}}{\times}(\mathbb{S}^{\mathrm{G}} \wedge (\mathrm{X/U})) \arrow[d, "\mathrm{E}\underset{\mathrm{G}}{\times}(\mathbb{S}^{\mathrm{G}}\wedge\Delta)"] \\
\mathrm{E}\underset{\mathrm{G}}{\times}(\mathbb{S}^{\mathrm{G}} \wedge (\mathrm{X^h_Y/(X^h_Y - Y) \wedge X^h_{Y,+}})) \arrow[r] & \mathrm{E}\underset{\mathrm{G}}{\times}(\mathbb{S}^{\mathrm{G}} \wedge (\mathrm{X/U \wedge X_+}))
\end{tikzcd}$$

so that the map in the top row is again a weak-equivalence. Here the quotient construction $\mathrm{E}\underset{\mathrm{G}}{\times}$ *is carried out as follows: when* G *is special as a linear algebraic group, the quotient is taken on the big Zariski (or the big Nisnevich) site, while when* G *is not special, it is taken after applying* ϵ^* *in the model category* $\widetilde{\mathbf{USpt}}^{\mathrm{G}}_{et,r}$ *(on the big étale site) and then followed by the derived push-forward* $\mathrm{R}\epsilon_*$, *to the model category* $\widetilde{\mathbf{USpt}}^{\mathrm{G}}_{mot,r}$ *(on the big Nisnevich site).*

(iii) Let M *denote a fibrant motivic spectrum that has the rigidity property as in Definition 1.10. Then, denoting by* $\mathrm{Map}(\quad, \mathrm{M})$ *the simplicial mapping space of maps to* M *on the Nisnevich site, one obtains the weak-equivalence*

$$\mathrm{Map}(\mathrm{E}\underset{\mathrm{G}}{\times}(\mathbb{S}^{\mathrm{G}} \wedge (\mathrm{X^h_Y/(X^h_Y - Y) \wedge X^h_{Y,+}})), \mathrm{M}) \simeq \mathrm{Map}(\mathrm{E}\underset{\mathrm{G}}{\times}(\mathbb{S}^{\mathrm{G}} \wedge (\mathrm{X^h_Y/(X^h_Y - Y) \wedge Y_+})), \mathrm{M}) \tag{4.2.2}$$

so that the map $\mathrm{Map}(\mathrm{E}\underset{\mathrm{G}}{\times}(\mathbb{S}^{\mathrm{G}} \wedge \Delta), \mathrm{M})$ *factors through the map* $\mathrm{Map}(\mathrm{id} \wedge \mathrm{i}, \mathrm{M})$ *where the map* $id \wedge i$ *denotes the map* $\mathrm{E}\underset{\mathrm{G}}{\times}(\mathbb{S}^{\mathrm{G}} \wedge (\mathrm{X/U \wedge Y_+})) \to \mathrm{E}\underset{\mathrm{G}}{\times}(\mathbb{S}^{\mathrm{G}} \wedge (\mathrm{X/U \wedge X_+}))$. *Taking* $\mathrm{E} = \mathrm{B} \times \mathrm{G} \to \mathrm{B}$, *one obtains:*

$$\mathrm{Map}((\mathbb{S}^{\mathrm{G}} \wedge (\mathrm{X^h_Y/(X^h_Y - Y) \wedge X^h_{Y,+}})), \mathrm{M}) \simeq \mathrm{Map}((\mathbb{S}^{\mathrm{G}} \wedge (\mathrm{X^h_Y/(X^h_Y - Y) \wedge Y_+})), \mathrm{M}).$$

(iii)' Let M *denote a fibrant motivic spectrum that has the rigidity property as in Definition 1.10. Then corresponding results as in (iii) hold for the spectrum* $\epsilon^*(\mathrm{M})$ *when the quotients appearing above are replaced by the quotients in the étale topology of the corresponding sheaves pulled back to the étale site.*

(iv) Moreover $\mathrm{Map}((\mathbb{S}^{\mathrm{G}} \wedge \Delta), \mathrm{M})$ *also identifies with* $\mathrm{Map}((\mathbb{S}^{\mathrm{G}} \wedge \Delta'), \mathrm{M})$, *where* Δ' *denotes the composite map*

$$\Delta' : (Th(\mathcal{N}))_+ \to (Th(\mathcal{N}))_+ \wedge E(\mathcal{N})_+ \overset{id\wedge r}{\to} (Th(\mathcal{N}))_+ \wedge \mathrm{Y}_+ \overset{\mathrm{id}\wedge \mathrm{i}}{\to} (\mathrm{Th}(\mathcal{N}))_+ \wedge \mathrm{X}_+. \tag{4.2.3}$$

Here $\mathrm{E}(\mathcal{N})$ *denotes the total space of the normal bundle* $\mathcal{N}$, *and r (i) is the map induced by the obvious retraction* $\mathrm{E}(\mathcal{N}) \to \mathrm{Y}$ *(i is the corresponding zero-section, respectively). The same identification holds when both are viewed as maps in* $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$.

Proof. The commutative square in (i) follows readily from the cartesian square:

$$\begin{tikzcd}
\mathrm{Y} \arrow[r, "id"] \arrow[d] & \mathrm{Y} \arrow[d] \\
\mathrm{X^h_Y} \arrow[r] & \mathrm{X}
\end{tikzcd} \tag{4.2.4}$$

Next one may recall from [MV99, Lemma 2.27] the weak-equivalences:

$$\mathrm{X/U} \simeq \mathrm{Th}(\mathcal{N}) \simeq \mathrm{X^h_Y/(X^h_Y - Y)}. \tag{4.2.5}$$

(In fact, the normal bundle $\mathcal{N}$ pulls back to the normal bundle associated to the closed immersion Y in $\mathrm{X^h_Y}$.) This proves that the map in the top row of the square in (i), namely that the map $\mathrm{X^h_Y/(X^h_Y - Y) \to X/(X - Y)}$ is a weak-equivalence. Therefore, these complete the proofs of the statements in (i).

Next we consider the statements in (ii). The functoriality of the Henselization as in Lemma 4.9 shows that the action of the group G on X and Y induces an action by G on X^h_Y. This observation, along with the observations in (i) provide the commutative square in (ii). The fact that the top row in the corresponding square is also a weak-equivalence follows from the fact that the top row of the square in (i) is also a weak-equivalence, making use of the appropriate quotient construction utilized there, as discussed in (ii).

When the group G is special, the torsor $E \to B$ trivializes on a Zariski open cover. Therefore, the weak-equivalence in (4.2.2) follows from the above observations, in view of the weak-equivalence provided by Proposition 4.10 as the spectrum M is assumed to have the rigidity property. In general, the torsor $E \to B$ only trivializes on an étale cover. Then one makes use of the quotient construction in $\mathbf{Spt}_{et,r}$ after applying ϵ^*. The discussion in the subsection 4.2 shows that this preserves weak-equivalences and so does the right derived functor $R\epsilon_*$ in (4.2.1). The second statement in (iii) is clear.

Next we consider the statement in (iii)'. This follows readily, since $\epsilon^*(M)$ also has the rigidity property, the torsor $E \to B$ is locally trivial in the étale topology and in view of Corollary 4.10(iv). Next we consider the statements in (iv). In order to identify the two diagonal maps $\Delta_{X/U}$ and $\Delta'_{X/U}$, we need to first observe that the required conclusion is true when $X = Y \times \mathbb{A}^c$ for any positive integer c and Y is imbedded in X as the 0-section. In this case, the total space of the normal bundle $\mathcal{N}$ is $Y \times \mathbb{A}^c$, so that both the diagonal maps are given by

$$\Sigma_{\mathbf{T}}((Y \times \mathbb{A}^c)/(Y \times \mathbb{A}^c - Y \times \{0\}) \to \Sigma_{\mathbf{T}}((Y \times \mathbb{A}^c)/(Y \times \mathbb{A}^c - Y \times \{0\}) \wedge \Sigma_{\mathbf{T}}((Y \times \mathbb{A}^c)_+$$

which clearly factors through the map

$$id \wedge \Sigma_{\mathbf{T}} i : \Sigma_{\mathbf{T}}((Y \times \mathbb{A}^c)/(Y \times \mathbb{A}^c - Y \times \{0\})) \wedge \Sigma_{\mathbf{T}} Y_+ \to \Sigma_{\mathbf{T}}((Y \times \mathbb{A}^c)/(Y \times \mathbb{A}^c - Y \times \{0\})) \wedge \Sigma_{\mathbf{T}}(Y \times \mathbb{A}^c)_+.$$

What remains then in the general case is to reduce it to the above case by making use of Lemma 4.12 and an argument as in the proof of [MV99, Theorem 2.23 and Proposition 2.24].

The next step is to consider the case when X is replaced by a Zariski open subset of the form U_y, which contains a point $y \,\varepsilon\, Y$. For the following discussion, we will adopt the terminology used in Lemma 4.12. Observe that in this case, V_y is a Nisnevich neighborhood of y that maps to both U_y and $(U_y \cap Y) \times \mathbb{A}^c$. Moreover, this case we will presently observe that the three maps

$$\Delta_1 : \Sigma_{\mathbf{T}}(U_y/(U_y \cap U)) \to \Sigma_{\mathbf{T}}(U_y/(U_y \cap U)) \wedge \Sigma_{\mathbf{T}}(U_y)_+,$$

$$\Delta_2 : \Sigma_{\mathbf{T}}(V_y/(V_y \underset{U_y}{\times} (U_y - (U_y \cap Y)))) \to \Sigma_{\mathbf{T}}(V_y/(V_y \underset{U_y}{\times} (U_y - (U_y \cap Y)))) \wedge \Sigma_{\mathbf{T}}(V_y)_+ \text{ and}$$

$$\Delta_3 : \Sigma_{\mathbf{T}}((U_y \cap Y) \times \mathbb{A}^c/((U_y \cap Y) \times (\mathbb{A}^c - \{0\})) \to \Sigma_{\mathbf{T}}((U_y \cap Y) \times \mathbb{A}^c/((U_y \cap Y) \times (\mathbb{A}^c - \{0\})) \wedge \Sigma_{\mathbf{T}}((U_y \cap Y) \times \mathbb{A}^c)_+$$

identify in the motivic stable homotopy category. As in the discussion above, it suffices to show that these maps identify on running over rigid Nisnevich hypercoverings of U_y, V_y and $(U_y \cap Y) \times \mathbb{A}^c$ as well as those of $U_y \cap U$, $(V_y \underset{U_y}{\times} (U_y - (U_y \cap Y)))$ and of $((U_y \cap Y) \times (\mathbb{A}^c - \{0\})$. Since $V_y \underset{U_y}{\times} Y \cong U_y \cap Y$ and $V_y \underset{(U_y \cap Y) \times \mathbb{A}^c}{\times} ((U_y \cap Y) \times \{0\}) \cong (U_y \cap Y)$, it follows readily that while doing so, the sources and the targets of the above diagonal maps all identify and so do the corresponding diagonal maps.

Finally we take a Zariski open cover of the scheme X, take its Cech nerve by taking repeated intersections and apply the above argument to each such Zariski open set in the place of the U_y above and complete the argument as in the proof of [MV99, Theorem 2.23 and Proposition 2.24]. We skip the remaining details. ☐

Lemma 4.12. *Let $i : Y \to X$ denote a closed immersion of smooth schemes of finite type over k of pure codimension c and X is of pure dimension n. Then the following hold.*

(i) *For every point $y \,\varepsilon\, Y$, there exists a Zariski neighborhood U_y of y in X and an étale map $q_y : U_y \to \mathbb{A}^n$, so that one has the cartesian square:*

$$\begin{tikzcd} U_y \cap Y \arrow[r] \arrow[d, "q'_y"] & U_y \arrow[d, "q_y"] \\ \mathbb{A}^{n-c} \arrow[r] & \mathbb{A}^n. \end{tikzcd}$$

(ii) *For every point $y \,\varepsilon\, Y$, there exists a commutative square*

$$\begin{tikzcd} V_y \arrow[r] \arrow[d, "q_{V_y}"] & U_y \arrow[d, "q_y"] \\ (U_y \cap Y) \times \mathbb{A}^c \arrow[r, "p_y"] & \mathbb{A}^n \end{tikzcd}$$

so that $\mathrm{V_y} \underset{(\mathrm{U_y}\cap \mathrm{Y})\times \mathbb{A}^c}{\times} ((\mathrm{U_y}\cap \mathrm{Y})\times\{0\}) \cong (\mathrm{U_y}\cap\mathrm{Y})$ *and* $\mathrm{V_y}\underset{\mathrm{U_y}}{\times}(\mathrm{U_y}\cap\mathrm{Y}) \cong \mathrm{U_y}\cap\mathrm{Y}$ *i.e.* $\mathrm{V_y}$ *is a Nisnevich neighborhood of* $(\mathrm{U_y}\cap\mathrm{Y})\times\{0\}$ *in* $(\mathrm{U_y}\cap\mathrm{Y})\times\mathbb{A}^c$ *and that* $\mathrm{V_y}$ *is a Nisnevich neighborhood of* $\mathrm{U_y}\cap\mathrm{Y}$ *in* $\mathrm{U_y}$.

Proof. For each $y\,\varepsilon\,\mathrm{Y}$, one knows by [EGA, IV, 17.12.2] that there exists a Zariski open neighborhood $\mathrm{U_y}$ of y in X and an étale map $q_y : \mathrm{U_y}\to\mathbb{A}^n$ so that one obtains the first cartesian square in the lemma.

Let $p_y = q'_y \times id : (\mathrm{U_y}\cap\mathrm{Y})\times\mathbb{A}^c \to \mathbb{A}^{n-c}\times\mathbb{A}^c = \mathbb{A}^n$ denote the induced map. Let $\mathrm{V'_y}$ be defined by the cartesian square:

$$\begin{tikzcd} \mathrm{V'_y} \arrow[r] \arrow[d, "q'_{U_y}"] & \mathrm{U_y} \arrow[d, "q_y"] \\ (\mathrm{U_y}\cap\mathrm{Y})\times\mathbb{A}^c \arrow[r, "p_y"] & \mathbb{A}^n. \end{tikzcd}$$

i.e. $\mathrm{V'_y} = \mathrm{U_y}\underset{\mathbb{A}^n}{\times}((\mathrm{U_y}\cap\mathrm{Y})\times\mathbb{A}^c)$. Now one may observe the commutative diagram

$$\begin{tikzcd} (\mathrm{U_y}\cap\mathrm{Y})\underset{\mathbb{A}^{n-c}}{\times}(\mathrm{U_y}\cap\mathrm{Y}) \arrow[r] \arrow[d] & (\mathrm{U_y}\cap\mathrm{Y}) \arrow[r, "i"] \arrow[d, "q'_y"] & \mathrm{U_y} \arrow[d, "q_y"] \\ (\mathrm{U_y}\cap\mathrm{Y}) \arrow[r, "q'_y"] & \mathbb{A}^{n-c} \arrow[r] & \mathbb{A}^n \end{tikzcd}$$

where both the squares are cartesian, which provides the isomorphism: $(\mathrm{U_y}\cap\mathrm{Y})\underset{\mathbb{A}^{n-c}}{\times}(\mathrm{U_y}\cap\mathrm{Y}) \cong \mathrm{U_y}\underset{\mathbb{A}^n}{\times}(\mathrm{U_y}\cap\mathrm{Y})$. We call this scheme $\mathrm{W'_y}$. Then one observes the isomorphism:

$$\begin{aligned} \mathrm{V'_y}\underset{(\mathrm{U_y}\cap\mathrm{Y})\times\mathbb{A}^c}{\times}(\mathrm{U}\cap\mathrm{Y})\times\{0\} &\cong \mathrm{U_y}\underset{\mathbb{A}^n}{\times}((\mathrm{U_y}\cap\mathrm{Y})\times\mathbb{A}^c)\underset{(\mathrm{U_y}\cap\mathrm{Y})\times\mathbb{A}^c}{\times}(\mathrm{U_y}\cap\mathrm{Y})\times\{0\} \\ &\cong \mathrm{U_y}\underset{\mathbb{A}^n}{\times}(\mathrm{U_y}\cap\mathrm{Y})\times\{0\} = (\mathrm{U_y}\cap\mathrm{Y})\underset{\mathbb{A}^{n-c}}{\times}(\mathrm{U_y}\cap\mathrm{Y}) = \mathrm{W'_y}. \end{aligned} \tag{4.2.6}$$

Next one observes that the map $q'_y : (\mathrm{U_y}\cap\mathrm{Y})\to\mathbb{A}^{n-c}$ is étale, which implies the diagonal map $\Delta : (\mathrm{U_y}\cap\mathrm{Y})\to(\mathrm{U_y}\cap\mathrm{Y})\underset{\mathbb{A}^{n-c}}{\times}(\mathrm{U_y}\cap\mathrm{Y})$ is an open immersion. Let $\mathrm{Z_y}$ denote $(\mathrm{U_y}\cap\mathrm{Y})\underset{\mathbb{A}^{n-c}}{\times}(\mathrm{U_y}\cap\mathrm{Y}) - \Delta(\mathrm{U_y}\cap\mathrm{Y})$, which is therefore closed in $\mathrm{V'_y}$ by (4.2.6). Let $\mathrm{V_y} = \mathrm{V'_y} - \mathrm{Z_y}$ and $\mathrm{U_y}\cap\mathrm{Y} = \mathrm{W'_y} - \mathrm{Z_y}$. Then one may see that, with the above choice of $\mathrm{V_y}$, one obtains the commutative square in (ii). □

5. Applications of the Additivity (and Multiplicativity) of the trace and transfer

We begin by discussing the following Proposition, which seems to be rather well-known. (See for example, [Th86, Proposition 4.10] or [BP, (3.6)].)

Proposition 5.1. *Let* T *denote a split torus acting on a smooth scheme* X *all defined over the given perfect base field* k.

Then the following hold.

X *admits a decomposition into a disjoint union of finitely many locally closed,* T*-stable subschemes* $\mathrm{X_j}$ *so that*

$$\mathrm{X_j} \cong (\mathrm{T}/\Gamma_\mathrm{j})\times\mathrm{Y_j}. \tag{5.0.7}$$

Here each Γ_j *is a sub-group-scheme of* T*, each* $\mathrm{Y_j}$ *is a scheme of finite type over* k *which is also regular and on which* T *acts trivially with the isomorphism in* (5.0.7) *being* T*-equivariant.*

Proof. One may derive this from the generic torus slice theorem proved in [Th86, Proposition 4.10], which says that if a split torus acts on a reduced separated scheme of finite type over a perfect field, then the following are satisfied:

1. there is an open subscheme U which is regular and stable under the T-action
2. a geometric quotient U/T exists, which is a regular scheme of finite type over k
3. U is isomorphic as a T-scheme to $\mathrm{T}/\Gamma\times\mathrm{U}/\mathrm{T}$ where Γ is a diagonalizable subgroup scheme of T and T acts trivially on U/T.

(See also [BP, (3.6)] for a similar decomposition.) □

Next we consider the following theorem.

Theorem 5.2. *Under the assumption that the base field* k *is of characteristic* 0*, the following hold, where* τ_X *denotes the trace associated to the scheme* X*:*

(i) $\tau_{\mathbb{G}_m} = 0$ *in the Grothendieck-Witt ring of the base field* k *and if* T *is a split torus,* $\tau_{\mathrm{T}} = 0$ *in the Grothendieck-Witt ring of the base field* k.

(ii) *Let* T *denote a split torus acting on a smooth scheme* X. *Then* X^{T} *is also smooth, and* $\tau_{\mathrm{X}} = \tau_{\mathrm{X}^{\mathrm{T}}}$ *in the Grothendieck-Witt ring.*

If the base field is of positive characteristic, the corresponding assertions hold with the Grothendieck-Witt ring replaced by the Grothendieck-Witt ring with the prime p inverted.

Assume M *denotes a motivic spectrum that has the rigidity property as in Definition 1.10 and that* $\mathrm{T} = \mathbb{G}_{\mathrm{m}}$. *Then if* $\mathrm{Map}(\quad, \mathrm{M})$ *denotes the simplicial mapping space functor,*

(iii) $\mathrm{Map}(\mathrm{j} \circ tr'_{\mathbb{G}_m}, \mathrm{M})$ *is trivial, where* $j : \mathbb{G}_m \to \mathbb{A}^1$ *in the open immersion.*

(iv) Let T *act on a smooth scheme* X *so that for each* T*-orbit* $\mathrm{T}/\Gamma_{\mathrm{j}}$ *with the orbit* $\mathrm{T}/\Gamma_{\mathrm{j}} \cong \mathbb{G}_{\mathrm{m}}$*, the locally closed immersion* $\mathrm{T}/\Gamma_{\mathrm{j}} \times \mathrm{Y}_{\mathrm{j}} \to \mathrm{X}$ *as in* (5.0.7) *factors through a map* $\mathbb{A}^1 \times \mathrm{Y}_{\mathrm{j}} \to \mathrm{X}$. *Then* $\mathrm{Map}(tr'_{\mathrm{X}}, \mathrm{M}) \simeq \mathrm{Map}(\mathrm{i} \circ tr'_{\mathrm{X}^{\mathrm{T}}}, \mathrm{M})$*, where* $i : \mathrm{X}^{\mathrm{T}} \to \mathrm{X}$ *is the inclusion.*

(v) Moreover, under the assumptions of (iv), if G *is a linear algebraic group acting on the scheme* X *commuting with the action of a split torus* T *so that the decomposition of* X *in* (5.0.7) *is* G*-stable, and* $\mathrm{E} \to \mathrm{B}$ *is a* G*-torsor, then* $tr^*_{\mathrm{X}} = \mathrm{Map}(\mathrm{E}\underset{\mathrm{G}}{\times} tr'^{\mathrm{G}}_{\mathrm{X}}, \mathrm{M}) \simeq \mathrm{tr}^*_{\mathrm{X}^{\mathbb{G}_{\mathrm{m}}}} \circ \mathrm{i}^* = \mathrm{Map}(\mathrm{i} \circ (\mathrm{E}\underset{\mathrm{G}}{\times} tr'^{\mathrm{G}}_{\mathrm{X}^{\mathrm{T}}}), \mathrm{M})$*, where* $i : \mathrm{E}\underset{\mathrm{G}}{\times}\mathrm{X}^{\mathrm{T}} \to \mathrm{E}\underset{\mathrm{G}}{\times}\mathrm{X}$ *is the inclusion.*

Proof. First observe from Definition 2.5, that the trace τ_{X} associated to any smooth scheme X is a map $\Sigma_{\mathbf{T}} \to \Sigma_{\mathbf{T}}$: as such, we will identify τ_{X} with the corresponding class $\tau^*_{\mathrm{X}}(1)$ in the Grothendieck Witt-ring of the base field. We will only consider the proofs when the base field is of characteristic 0, since the proofs in the positive characteristic case are entirely similar. However, it is important to point out that in positive characteristics p, it is important to invert p: for otherwise, one no longer has a theory of Spanier-Whitehead duality.

(i) and (iii). We observe that the scheme $\mathbb{A}^1$ is the disjoint union of the closed point $\{0\}$ and $\mathbb{G}_m$. If $i_1 : \{0\} \to \mathbb{A}^1$ and $j_1 : \mathbb{G}_m \to \mathbb{A}^1$ are the corresponding immersions, Theorems 1.5(ii) and (iii) and 1.12(ii) and (iii) show that

$$\tau_{\mathbb{A}^1} = \tau_{\{0\}} + \tau_{\mathbb{G}_m} \text{ and } \mathrm{Map}(tr'_{\mathbb{A}^1}, \mathrm{M}) = \mathrm{Map}(\mathrm{i}_1 \circ tr'_{\{0\}}, \mathrm{M}) + \mathrm{Map}(\mathrm{j}_1 \circ \mathrm{tr}'_{\mathbb{G}_{\mathrm{m}}}, \mathrm{M}) \tag{5.0.8}$$

However, by $\mathbb{A}^1$-contractibility, $\tau_{\mathbb{A}^1} = \tau_{\{0\}}$ and $tr'_{\mathbb{A}^1} = i_1 \circ tr'_{\{0\}}$. One may readily see this from the definition of the pre-transfer as in Definition 2.5, which shows that both the pre-transfer $tr'_{\mathrm{P}} = tr'_{\mathrm{P}}(id)$ and hence the corresponding trace, $\tau_{\mathrm{P}} = \mathrm{p} \circ \mathrm{tr}_{\mathrm{P}}$ depend on P only up to its class in the motivic stable homotopy category. Therefore, $\tau_{\mathbb{G}_m} = 0$ and $\mathrm{Map}(\mathrm{j} \circ tr'_{\mathbb{G}_m}, \mathrm{M})$ is trivial. Since T is a split torus, we may assume $\mathrm{T} = \mathbb{G}^{\mathrm{n}}_{\mathrm{m}}$ for some positive integer n. Then the multiplicative property of the trace and pre-transfer (see Proposition 3.4) prove that $\tau_{\mathrm{T}} = 0$. These complete the proof of statements (i) and (iii).

Therefore, we proceed to prove the statement in (ii) and (iv). First, we invoke Proposition 5.1 to conclude that X^{T} is the disjoint union of the schemes X_{j} for which $\Gamma_j = \mathrm{T}$.

Let $i_j : \mathrm{X}_{\mathrm{j}} \cong (\mathrm{T}/\Gamma_{\mathrm{j}}) \times \mathrm{Y}_{\mathrm{j}} \to \mathrm{X}$ denote the locally closed immersion. Next observe that the additivity of the trace proven in Theorem 1.5, the additivity of pre-transfer proven in Theorem 1.12, and the multiplicativity of the pre-transfer and trace proven in Proposition 3.4 along with the decomposition in Proposition 5.0.7 show that

$$\begin{gathered} \tau_{\mathrm{X}} = \Sigma_j \tau_{\mathrm{X}_{\mathrm{j}}} = \Sigma_j (\tau_{\mathrm{T}/\Gamma_{\mathrm{j}}+}) \wedge \tau_{\mathrm{Y}_{\mathrm{j}+}} \text{ and} \\ \mathrm{Map}(tr'_{\mathrm{X}}, \mathrm{M}) \simeq \Sigma_{\mathrm{j}} \mathrm{Map}(\mathrm{i}_{\mathrm{j}} \circ \mathrm{tr}'_{\mathrm{X}_{\mathrm{j}}}, \mathrm{M}) = \Sigma_j \mathrm{Map}(\mathrm{i}_{\mathrm{j}} \circ (tr'_{\mathrm{T}/\Gamma_{\mathrm{j}}+} \wedge tr'_{\mathrm{Y}_{\mathrm{j}+}}), \mathrm{M}). \end{gathered} \tag{5.0.9}$$

Now statements (i) and (iii) in the theorem along with the assumptions in (iv) prove that the j-th summand on the right-hand-sides are trivial unless $\Gamma_j = \mathrm{T}$. But, then X^{T} is the disjoint union of such X_{j}. Finally the additivity of the trace and pre-transfer in Theorems 1.5 and 1.12 applied once more to X^{T} proves the sum of the non-trivial terms on the right-hand-side is $\tau_{\mathrm{X}^{\mathrm{T}}}$ for the first equation and is given by $\mathrm{Map}(\mathrm{i} \circ tr'_{\mathrm{X}^{\mathrm{T}}}, \mathrm{M})$ for the second equation. These prove the statements in (ii) and (iv).

Finally, we consider the last statement. In view of the assumption that the actions by the linear algebraic group G and the split torus T on the scheme X commute and that the decomposition of X into the schemes X_{j} as in (5.0.7) is stable by the action of G, the weak-equivalence $\mathrm{Map}(tr'_{\mathrm{X}}, \mathrm{M}) \simeq \mathrm{Map}(\mathrm{i} \circ tr'_{\mathrm{X}^{\mathrm{T}}}, \mathrm{M})$ obtained in (iv) implies the weak-equivalence $\mathrm{Map}(\mathrm{E}\underset{\mathrm{G}}{\times} tr'^{\mathrm{G}}_{\mathrm{X}}, \mathrm{M}) \simeq \mathrm{Map}(\mathrm{i} \circ (\mathrm{E}\underset{\mathrm{G}}{\times} tr'^{\mathrm{G}}_{\mathrm{X}^{\mathrm{T}}}), \mathrm{M})$ in (v), in view of Theorem 1.12(iii) and (iv). □

Remark 5.3. Here we provide an explanation of the condition in Theorem 5.2(iv). The first observation is that then the origin in $\mathbb{A}^1$ corresponds to a fixed point x for the $\mathbb{G}_m$-action on X. The corresponding $\mathbb{G}_m$-orbit is then contained in a *slice* at x. The condition in Theorem 5.2(iv) may now be interpreted as saying the fixed point x is *an attractive fixed point* for the $\mathbb{G}_m$-action on X, in the sense that all the weights for the induced $\mathbb{G}_m$ action on the Zariski tangent space T_{x} at x lie in an open half-plane. (See [BJ, 2.2, 2.3] for further details.)

Corollary 5.4. *Let* X *denote a smooth scheme provided with the action of a linear algebraic group* G*. Assume that that* X *is also provided with an action by* $\mathbb{G}_m$ *commuting with the action by* G *and that the hypotheses in Theorem 5.2(v) hold with* $\mathrm{T} = \mathbb{G}_{\mathrm{m}}$*. Let* M *denote a fibrant motivic spectrum that has the rigidity property. Then, adopting the terminology as in Theorem 1.3, that for a linear algebraic group* G, $\mathrm{BG} = \underset{\mathrm{m}\to\infty}{\mathit{lim}}\, \mathrm{BG}^{\mathrm{gm,m}}$ *and* $\mathrm{EG} = \underset{\mathrm{m}\to\infty}{\mathit{lim}}\, \mathrm{EG}^{\mathrm{gm,m}}$*, one obtains the homotopy commutative diagram*

$$\begin{tikzcd}
h(\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X}, \mathrm{M}) \arrow[r, "i^*"] \arrow[d, "tr_{\mathrm{X}}^*"] & h(\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X}^{\mathbb{G}_{\mathrm{m}}}, \mathrm{M}) \arrow[dl, "tr^*_{\mathrm{X}^{\mathbb{G}_{\mathrm{m}}}}"] \\
h(\mathrm{BG}, \mathrm{M}) &
\end{tikzcd}$$

where $h(\quad, \mathrm{M}) = \mathrm{Map}(\quad, \mathrm{M})$ *denotes the hypercohomology spectrum with respect to the motivic spectrum* M *and* $i : \mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X}^{\mathbb{G}_{\mathrm{m}}} \to \mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X}$ *is the map induced by the closed immersion* $\mathrm{X}^{\mathbb{G}_{\mathrm{m}}} \to \mathrm{X}$*.*

Proof. We will show that there is a corresponding commutative diagram, when BG and EG are replaced by their finite dimensional approximations $\mathrm{BG}^{\mathrm{gm,m}}$ and $\mathrm{EG}^{\mathrm{gm,m}}$. Therefore let m denote a fixed non-negative integer and let $\mathrm{BG}^{\mathrm{gm,m}}$ denote the approximation of $\mathrm{BG}^{\mathrm{gm}}$ to degree m and let $\mathrm{EG}^{\mathrm{gm,m}}$ denote its universal principal G-bundle.

Next we observe that X admits the decomposition $\mathrm{X} = (\mathrm{X} - \mathrm{X}^{\mathbb{G}_{\mathrm{m}}}) \sqcup \mathrm{X}^{\mathbb{G}_{\mathrm{m}}}$ and that this decomposition is stable under the action of G (as the action of G and $\mathbb{G}_m$ are assumed to commute). Moreover, $\mathrm{X} - \mathrm{X}^{\mathbb{G}_{\mathrm{m}}} \cong \mathbb{G}_{\mathrm{m}} \times \mathrm{Y}$, where Y in fact denotes the geometric quotient $(\mathrm{X} - \mathrm{X}^{\mathbb{G}_{\mathrm{m}}})/\mathbb{G}_{\mathrm{m}}$. By Theorem 5.2(v), one obtains the identification of the G-equivariant transfers

$$tr_{\mathrm{X}}^* = \mathrm{Map}(\mathrm{EG}^{\mathrm{gm,m}}\underset{\mathrm{G}}{\times} tr'^{\mathrm{G}}_{\mathrm{X}}, \mathrm{M}) = \mathrm{Map}(\mathrm{EG}^{\mathrm{gm,m}}\underset{\mathrm{G}}{\times}(i \circ tr'^{\mathrm{G}}_{\mathrm{X}^{\mathbb{G}_{\mathrm{m}}}}), \mathrm{M}) = tr^*_{\mathrm{X}^{\mathbb{G}_{\mathrm{m}}}} \circ i^*.$$

Finally, taking the homotopy inverse limit as $m \to \infty$ provides the homotopy commutative triangle in the corollary. □

Remarks 5.5. (i) A result analogous to the last corollary and Corollary 1.18 is proven for the classical Becker-Gottlieb transfer in [Beck74, Lemma 1]. There it is used to show that the transfer maps stabilize for infinite families of classifying spaces of compact Lie groups, such as $\{\mathrm{BO(2n)} \to \mathrm{BN(T_n)}|n\}$. Work in progress in [JP20-1] has the goal of producing similar results in the motivic framework.

(ii) One may observe that proofs of the statements analogous to the ones in Theorem 1.12(ii) and (iii) are easier to obtain in the topological context. In the topological framework, the only suspension that one needs to consider is the suspension by the simplicial sphere S^1, and in this case it is shown in [LMS, Chapter IV, Theorem 2.10] that this simplicial suspension simply amounts to multiplying the transfer and the trace by a sign. The proof of the analogous result in [MP, (1.1) Theorem] strongly uses the fact that the boundary of an n-cell (or a disk-bundle) is a simplicial sphere (sphere bundle) along which the n-cell (the disk bundle) is glued. Clearly these arguments do not carry over as such into the motivic context. One may also see from [MP, (1.4) Corollary] that for complex varieties, the additivity theorem they obtain for the transfer agrees with the ones we obtain in Theorem 1.12(iii). In particular, the additivity theorem we obtain will provide corresponding additivity theorems on taking the étale or Betti-realization, assuming the compatibility of the transfer with realizations: see [CJ19, section 8]. Therefore, it does seem the deformation to the normal cone argument we have used along with the use of the motivic tubular neighborhood is a replacement for the purely topological constructions that occur in the proofs of the corresponding additivity theorems in the topological contexts.

Proof of Theorem 1.6 and Corollary 1.9. We will first consider the case where the base field is of characteristic 0. Then we observe that since G/N(T) is the variety of all split maximal tori in G, T has an action on G/N(T) (induced by the left translation action of T on G) so that there is exactly a single fixed point, namely the coset eN(T). i.e. $(\mathrm{G/N(T)})^{\mathrm{T}} = \{e\mathrm{N(T)}\} = \{\mathrm{Spec\,k}\}$. (To prove this assertion, one may reduce to the case where the base field is algebraically closed, since the formation of fixed point schemes respects change of base fields as shown in [Fog, p. 33, Remark (3)]. See also [BP, Lemma 3.5].) Therefore, by Theorem 5.2(ii),

$$\tau_{\mathrm{G/N(T)}} = \tau_{(\mathrm{G/N(T)})^{\mathrm{T}}} = \tau_{\mathrm{Spec}\, k} = id_{\Sigma_{\mathbf{T}}},$$

which is the identity map of the motivic sphere spectrum. Therefore, $\tau^*_{\mathrm{G/N(T)}}(1) = 1$. The motivic stable homotopy group $\pi_{0,0}(\Sigma_{\mathbf{T}})$ identifies with the Grothendieck-Witt ring by [Mo4]. This completes the proof of the statement on $\tau_{\mathrm{G/N(T)}}$ in Theorem 1.6 in this case. In case the base field is of positive characteristic p, one observes that $\Sigma_{\mathbf{T}}\mathrm{G/N(T)}_+$ will be dualizable only in $\mathbf{Spt}_{\mathrm{mot}}[p^{-1}]$. But once the prime p is inverted the same arguments as before carry over proving the corresponding statement. Observe that if B denotes a Borel subgroup containing the split maximal torus T, $\tau^*_{G/T}(1) = \tau^*_{G/B}(1) = \tau^*_{(G/B)^T}(1) = |W|$, which proves the last statement in Theorem 1.6.

Now we consider the proof of Corollary 1.9. (i) and (ii) follow readily from the Theorem 5.2 in view of the stable splittings in the motivic homotopy theory worked out in [CJ19, Theorem 1.5]. The key point to observe here is that the composition $tr^*(\pi^*(1)) = \tau^*_{G/N(T)}(1)$ in case (i) and $tr^*(q^*(1)) = \tau^*_{G/N(T)}(1)$ in case (ii). One also needs to observe the isomorphism of G-schemes: $G \times_{N(T)} Y \cong G/N(T) \times Y$ and $G \times_T Y \cong G/T \times Y$, so that [CJ19, Theorem 1.5] readily applies to the situations considered in (ii) as well as in (iii). Recall from [CJ19, Proposition 8.1] that the traces in étale case are obtained by taking the étale realization of the traces in the motivic context. Therefore, corresponding results for étale cohomology as in (iv) follow similarly.

□

5.1. **Double Coset formulae.** In this section, we establish various double coset formulae, the analogues of which have been known in the setting of group cohomology for finite groups and also for compact Lie groups. We will explicitly consider only the motivic framework, since the corresponding results in the étale framework may be established by entirely similar arguments.

Proof of Theorem 1.13 (i) follows readily from the naturality of the transfer map established in [CJ19, Theorem 7.1]. Next we consider (ii). This follows readily from Theorem 1.12(iv). □

Proof of Corollary 1.14. First we will consider (i). In this case we first observe that the homogeneous space G/T admits a decomposition into the double cosets $T\backslash G/T$ which will identify with affine spaces over each of the Bruhat-cells. i.e. One begins with the Bruhat decomposition $G = \sqcup_{w \,\varepsilon\, W} BwB^-$, where B is a Borel subgroup containing the given maximal torus T and B^- is its opposite Borel subgroup. Then $T\backslash G/T = \sqcup_{w\,\varepsilon\, W} R_u(B)wR_u(B^-)$ where $R_u(B)$ ($R_u(B^-)$) denotes the unipotent radical of B (B^-, respectively). Now we invoke Theorem 1.13(ii).

Since both $R_u(B)$ and $R_u(B^-)$ are affine spaces, it suffices to consider the double cosets corresponding to each $w \,\varepsilon\, W$. Observe that the strata are the affine spaces $R_u(B)wR_u(B^-)$, and hence the normal bundles to these strata are trivial. Moreover, each of the strata $R_u(B)wR_u(B^-)$ has a fixed (k-rational) point for the action of T, which corresponds to the origin of the affine space $R_u(B)wR_u(B^-)$. We will denote this k-rational point by 0_w. The corresponding transfer sends $\Sigma_{\mathbf{T}} BT^{gm,m}_+$ to $\Sigma_{\mathbf{T}} BT^{gm,m}_+ \simeq \Sigma_{\mathbf{T}} ET^{gm,m} \underset{T}{\times} 0_w$ by the map induced by sending $Spec\, k$ to the coset $\tilde{w}T$ in G/T, where $\tilde{w} \,\varepsilon\, N(T)$ is the element corresponding to w. This in fact corresponds to the self-map of $\Sigma_{\mathbf{T}} BT^{gm,m}_+$ induced by the automorphism of T defined by conjugation by w. (See, for example, the discussion in [BM, (3.5) Theorem].) This proves (i).

The proof of (ii) is similar. First we consider the case where H is a parabolic subgroup, with Levi-factor L. Then one knows that there is a set of simple roots I, among the basis of simple roots Δ, so that $L = L_I = Z_G((\cap_{\alpha\,\varepsilon\, I} Ker(\alpha))^o)$ and $H = P_I$, the corresponding parabolic subgroup. The Weyl group W_H is then generated by the simple reflections $s_\alpha, \alpha \,\varepsilon\, I$. In this case, one obtains a decomposition of G as $\sqcup_{w\,\varepsilon\, W_G/W_H} BwH$ and therefore, the double coset decomposition $T\backslash G/H = \sqcup_{w\,\varepsilon\, W_G/W_H} R_u(B)w$. In case H is actually a Borel subgroup, W_H is trivial.

One may observe that the conjugate H^g will be, in general, distinct from H, as the normalizer of a parabolic subgroup is itself. Nevertheless, since $g \,\varepsilon\, W_G$, conjugation by g sends the given maximal torus $K = T$ to itself, though it will induce an automorphism of K. Therefore, $K = K \cap H^g$ and the restriction homomorphism $p(K, H^g) = p(K \cap H^g, H^g) : h^{*,\bullet}(BH^g, E) \to h^{*,\bullet}(BK, E)$. In this case the origin of the affine space that corresponds to the cell $R_u(B)w$ is the coset $\tilde{w}H$ in G/H, with $\tilde{w} \,\varepsilon\, N(T)$ is the element corresponding to w. Then the corresponding transfer is induced by the map $BK \to BH^{\tilde{w}} \to BH = EH \underset{H}{\times} 0_w$, where the last map is the one induced by the map on H sending H to itself by conjugation by $\tilde{w}$. This proves (ii) in the case H is a parabolic subgroup.

The case $H = L = L_I$ a Levi subgroup reduces to the above case, since we may start with the decomposition of G as $\sqcup_{w\,\varepsilon\, W_G/W_H} BwP_I$ and hence a double coset decomposition $T\backslash G/L = \sqcup_{w\,\varepsilon\, W_G/W_H} R_u(B)wR_u(P_I)$. The general case, where H is a closed linear algebraic subgroup of G with maximal rank comes intermediate between the case where H is a Levi-subgroup and a parabolic subgroup and may be handled in a similar manner. We skip the proof of (iii) which is similar. □

Proof of Corollary 1.15. Let $\pi : ET \underset{T}{\times} X \simeq EG \underset{G}{\times} G \underset{T}{\times} X \to EG \underset{G}{\times} X$ denote the map induced by the map $G \underset{T}{\times} X \to X$, sending $(g, x) \mapsto gx$ and let the corresponding transfer be denoted tr. Then the first step is to observe that the map

$$\pi^* : \hat{h}^{*,\bullet}(EG \underset{G}{\times} X) \to \hat{h}^{*,\bullet}(ET \underset{\mathbf{T}}{\times} X)$$

is a split injection since |W| is a unit in the given generalized cohomology theory. Then Corollary 1.14(ii) shows that the image of the last map identifies with the W-invariant part of $\hat{\mathrm{h}}^{*,\bullet}(\mathrm{ET}\underset{\mathbf{T}}{\times}\mathrm{X})$. This is a standard argument, but for the sake of completeness, we will provide further details.

Then, since $\chi(\mathrm{G/N(T)}) = \tau^*_{\mathrm{G/N(T)}}(1) = 1$ and $\chi(\mathrm{N(T)/T}) = \tau^*_{\mathrm{N(T)/T}}(1) = |\mathrm{W}|$, one sees that $\chi(\mathrm{G/T}) = \tau^*_{\mathrm{G/T}}(1) = |\mathrm{W}|$. Therefore, we obtain:

$$tr^* \circ \pi^*(\alpha) = |\mathrm{W}|\alpha, \alpha\, \varepsilon\, \hat{\mathrm{h}}^{*,\bullet}(\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X}). \tag{5.1.1}$$

Therefore, since |W| is assumed to be a unit, the map π^* is injective. Next let $\alpha\,\varepsilon\,\hat{\mathrm{h}}^{*,\bullet}(\mathrm{ET}\underset{\mathrm{T}}{\times}\mathrm{X})^{\mathrm{W}}$. Observe from Proposition 4.6, that the slices of the motivic spectrum M also have the rigidity property: therefore, the corresponding slice-completed generalized cohomology theory $\hat{\mathrm{h}}^{*,\bullet}$ also has the rigidity property. Then, by Corollary 1.14(ii), we obtain:

$$\pi^* \circ tr^*(\alpha) = \Sigma_{g\,\varepsilon\,\mathrm{W}} C_g(\alpha) = |\mathrm{W}|\alpha. \tag{5.1.2}$$

Then (5.1.1) and (5.1.2) along with the assumption that |W| is a unit show that $\hat{\mathrm{h}}^{*,\bullet}(\mathrm{ET}\underset{\mathrm{T}}{\times}\mathrm{X})^{\mathrm{W}} \subseteq \mathrm{Image}(\pi^*)$. Finally, one may observe that $C_g \circ \pi^* = \pi^*$, for all $g\,\varepsilon\,\mathrm{W}$, which shows that the image of π^* is contained in $\hat{\mathrm{h}}^{*,\bullet}(\mathrm{ET}\underset{\mathrm{T}}{\times}\mathrm{X})^{\mathrm{W}}$. □

Proof of Corollary 1.16. The statement in (i) is clear from Corollary 1.15. Next we will consider the statement in (ii). Clearly one obtains a corresponding statement in étale cohomology as well. Next, invoking the transfer, one observes the isomorphisms:

$$\mathbb{H}^{*,\bullet}_{\mathrm{G,mot}}(\mathrm{X}, \mathrm{Z}/\ell^{\mathrm{n}}) \cong \mathbb{H}^{*,\bullet}_{\mathrm{T,mot}}(\mathrm{X}, \mathrm{Z}/\ell^{\mathrm{n}})^{\mathrm{W}} \text{and}$$
$$\mathbb{H}^{*,\bullet}_{\mathrm{G,et}}(\mathrm{X}, \mathrm{Z}/\ell^{\mathrm{n}}) \cong \mathbb{H}^{*,\bullet}_{\mathrm{T,et}}(\mathrm{X}, \mathrm{Z}/\ell^{\mathrm{n}})^{\mathrm{W}}.$$

Therefore, we reduce to the case where G is replaced by a split torus T. At this point, we observe that a choice of $\mathrm{BT}^{\mathrm{gm,m}} = \Pi_{\mathrm{i=1}}^{\mathrm{n}}\mathbb{P}^{\mathrm{m}}$, if $\mathrm{T} = \mathbb{G}_{\mathrm{m}}^{\mathrm{n}}$.

Observe that $\mathrm{ET}^{\mathrm{gm,m}} \to \mathrm{BT}^{\mathrm{gm,m}}$ is a Zariski locally trivial torsor for the action T as $\mathrm{T} = \mathbb{G}_{\mathrm{m}}^{\mathrm{n}}$ is a split torus and hence is special as a linear algebraic group in the sense of Grothendieck: see [Ch]. Taking $n = 1$, we see that $\pi^m : \mathrm{E}\mathbb{G}_m^{gm,m} \to \mathrm{B}\mathbb{G}_{\mathrm{m}}^{\mathrm{gm,m}}$ is such a torsor, so that there is a Zariski open cover $\{\mathrm{U_j}|\mathrm{j} = 1, \cdots, \mathrm{N}\}$ so that $\pi^m_{\mathrm{U_j}}$ is of the form $\mathrm{U_j} \times \mathbb{G}_{\mathrm{m}}$, $j = 1, \cdots, \mathrm{N}$.

Let $\{\mathrm{V_0}, \cdots \mathrm{V_m}\}$ denote the open cover of $\mathbb{P}^m$ obtained by letting $\mathrm{V_i}$ be the open subscheme where the homogeneous coordinates x_i, $i = 0, \cdots, m$ on $\mathbb{P}^m$ is non-zero. Without loss of generality, we may assume the $\mathrm{U_j}$ refine open cover $\{\mathrm{V_i}|\mathrm{i} = 0, \cdots \mathrm{m}\}$. Finally the observation that the Picard groups of affine spaces are trivial, shows that one may in fact take $\mathrm{N} = \mathrm{m}$ and $\mathrm{U_j} = \mathrm{V_j}$, $j = 0, \cdots, m$. Now one may take an open cover of $\Pi_{i=1}^n \mathbb{P}^m$ by taking the product of the affine spaces that form the open cover of each factor $\mathbb{P}^m$. We will denote this open cover of $\Pi_{i=1}^n \mathbb{P}^m$ by $\{\mathrm{W}_\alpha|\alpha\}$.

Let $p : \mathrm{ET}^{\mathrm{gm,m}}\underset{\mathrm{T}}{\times}\mathrm{X} \to \mathrm{BT}^{\mathrm{gm,m}}$ denote the obvious map, and let ϵ denote the map from the étale site to the Nisnevich site. Let $\mathbb{Z}/\ell^n(i)$ denote the motivic complex of weight i on the Nisnevich site of $\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X}$. Then one obtains the identification (see [Voev11], [HW]):

$$\mathbb{Z}/\ell^n(i) = \tau_{\le i}\mathrm{R}\epsilon_*\epsilon^*(\mathbb{Z}/\ell^{\mathrm{n}}(\mathrm{i})). \tag{5.1.3}$$

Therefore, on applying Rp_*, we obtain the natural maps:

$$\begin{aligned} Rp_*(\mathbb{Z}/\ell^n(i)) &\overset{\simeq}{\to} Rp_*(\tau_{\le i}\mathrm{R}\epsilon_*\epsilon^*(\mathbb{Z}/\ell^{\mathrm{n}}(\mathrm{i}))) \to Rp_*\mathrm{R}\epsilon_*\epsilon^*(\mathbb{Z}/\ell^{\mathrm{n}}(\mathrm{i})) \\ &\cong Rp_*\mathrm{R}\epsilon_*\mu_{\ell^{\mathrm{n}}}(\mathrm{i})) \cong \mathrm{R}\epsilon_*\mathrm{R}p_*\mu_{\ell^{\mathrm{n}}}(\mathrm{i})). \end{aligned} \tag{5.1.4}$$

On taking the sections over each Zariski open set in the above cover W_α of $\mathrm{BT}^{\mathrm{gm,m}}$, we obtain a quasi-isomorphism, since affine-spaces are contractible for motivic cohomology and also étale cohomology with respect to $\mu_{\ell^n}(j)$ and the base field is separably closed with $\ell \ne char(k)$, with $\ell \ne char(k)$. Now a Mayer-Vietoris argument using the above open cover of $\mathrm{BT}^{\mathrm{gm,m}}$ completes the proof. Since the iterated intersections of the affine spaces forming the open cover of $\mathrm{BT}^{\mathrm{gm,m}}$ are products of an affine space with a split torus, one may invoke the derived Kunneth formula for motivic cohomology as in [J01] to reduce to the case where the split torus above is 1-dimensional. Then a localization sequence for the pair $(\mathbb{A}^n \times \mathbb{A}^1, \mathbb{A}^n \times \{0\})$ shows that the map in (5.1.4) induces a quasi-isomorphism on taking sections over iterated intersections of the sets W_α. This completes the proof of (ii).

Next we will prove (iii) by observing the sequence of isomorphisms:

$$\mathrm{H}^{*,\bullet}_{\mathrm{mot}}(\mathrm{G/H}, \mathrm{Z}/\ell^{\mathrm{n}}) \cong \mathrm{H}^{*,\bullet}_{\mathrm{mot}}(\mathrm{EH}\underset{\mathrm{H}}{\times}\mathrm{G}, \mathrm{Z}/\ell^{\mathrm{n}}) \cong \mathrm{H}^{*,\bullet}_{\mathrm{mot}}(\mathrm{ET}\underset{\mathrm{T}}{\times}\mathrm{G}, \mathrm{Z}/\ell^{\mathrm{n}})^{\mathrm{W_H}} \cong \mathrm{H}^{*,\bullet}_{\mathrm{mot}}(\mathrm{G/T})^{\mathrm{W_H}}.$$

The first and last isomorphisms follow from the fact that EH and ET are both $\mathbb{A}^1$-acyclic. The statement in (i) provides the second isomorphism. This proves (iii) and clearly the same proof carries over to étale cohomology.

The Bruhat decomposition shows that G/B has a stratification into strata that are affine spaces. Moreover G/B is projective and smooth. Therefore, [J01, Theorem 1.2] shows that the higher cycle map for G/B is an isomorphism. Since G/T is an affine space bundle over G/B, this isomorphism extends to G/T as well. The Bruhat decomposition of G/B shows that there is a natural action by the Weyl group $\mathrm{W_H}$ on G/B and hence on G/T. On viewing the cycle map as induced by the map $\mathbb{Z}/\ell^n(i) \to \mathrm{R}\epsilon_*\epsilon^*\mathbb{Z}/\ell^{\mathrm{n}}(\mathrm{i}))$, it becomes clear that it is compatible with the action of $\mathrm{W_H}$ on G/B. This proves (v).

Finally to see the last statement on Brauer groups, we make use of the short-exact sequence

$$0 \to \mathrm{H}^{2,1}_{\mathrm{mot}}(\mathrm{G/H}), \mathbb{Z}/\ell^{\mathrm{n}}) \to \mathrm{H}^2_{\mathrm{et}}(\mathrm{G/H}, \mu_{\ell^{\mathrm{n}}}(1)) \to \mathrm{Br}'(\mathrm{G/H})_{\ell^{\mathrm{n}}} \to 0 \text{ and} \tag{5.1.3}$$

where $\mathrm{Br}'(\mathrm{G/H}) = \mathrm{H}^2_{\mathrm{et}}(\mathrm{G/H}, \mathbb{G}_{\mathrm{m}})_{\mathrm{tor}}$ which is the torsion subgroup of $\mathrm{H}^2_{\mathrm{et}}(\mathrm{G/H}, \mathbb{G}_{\mathrm{m}})$. (See, for example, [IJ20-1, (8)].) Since the map $\mathrm{H}^{2,1}_{\mathrm{mot}}(\mathrm{G/H}), \mathbb{Z}/\ell^{\mathrm{n}}) \to \mathrm{H}^2_{\mathrm{et}}(\mathrm{G/H}, \mu_{\ell^{\mathrm{n}}}(1))$ is the cycle map, which has been observed to be an isomorphism, it follows that $\mathrm{Br}'(\mathrm{G/H})_{\ell^{\mathrm{n}}} \cong 0$. This completes the proof of Corollary 1.16. □

Proof of Corollary 1.18. In view of the assumption that X is projective, we invoke the Bialynicki-Birula decomposition of X into finitely many locally closed subschemes X^+_α, so that each X^+_α is an affine space bundle on X_α, which is a connected component of the fixed point scheme $\mathrm{X}^{\mathbb{G}_{\mathrm{m}}}$. (See [dB01, Theorem 2.1], [B-B].) Since the actions of G and $\mathbb{G}_m$ commute, and G is connected, each connected component of the fixed point scheme $\mathrm{X}^{\mathbb{G}_{\mathrm{m}}}$ is stable by G. Therefore, it follows that each of the X^+_α is also stable by G.

In view of the assumed rigidity property for the spectrum M, Theorem 1.12(iv) shows that

$$tr^*_{\mathrm{X}} = \mathrm{Map}(\mathrm{EG}^{\mathrm{gm,m}}\underset{\mathrm{G}}{\times} {tr'}^{\mathrm{G}}_{\mathrm{X}}, \mathrm{M}) = \Sigma_\alpha \mathrm{tr}^*_{\mathrm{X}^+_\alpha} = \Sigma_\alpha \mathrm{Map}(\mathrm{EG}^{\mathrm{gm,m}}\underset{\mathrm{G}}{\times}(i^+_\alpha \circ tr'_{\mathrm{X}^+_\alpha}), \mathrm{M}).$$

In view of the observation that each $\mathrm{X}^+_\alpha \to \mathrm{X}_\alpha$ is an affine-space bundle, the last sum identifies with

$$\Sigma_\alpha \mathrm{Map}(\mathrm{EG}^{\mathrm{gm,m}}\underset{\mathrm{G}}{\times}(i_\alpha \circ tr'_{\mathrm{X}_\alpha}), \mathrm{M}) = \mathrm{Map}(\mathrm{EG}^{\mathrm{gm,m}}\underset{\mathrm{G}}{\times}(i \circ tr'_{\mathrm{X}^{\mathbb{G}_{\mathrm{m}}}}), \mathrm{M}) = \mathrm{tr}^*_{\mathrm{X}^{\mathbb{G}_{\mathrm{m}}}} \circ \mathrm{i}^*,$$

where $i^+_\alpha : \mathrm{X}^+_\alpha \to \mathrm{X}$, $i_\alpha : \mathrm{X}_\alpha \to \mathrm{X}$ and $i : \mathrm{X}^{\mathbb{G}_{\mathrm{m}}} \to \mathrm{X}$ are the locally closed immersions. Now one takes the colimit as $m \to \infty$, to complete the proof. □

Remark 5.6. An example of the situation considered in the above corollary is the following. Let $\mathrm{X} = \mathrm{GL_{n+1}}/\mathrm{B_{n+1}}$, which is the variety of all Borel subgroups in GL_{n+1}. Let $\mathbb{G}_m$ denote the 1-parameter subgroup imbedded in GL_{n+1} as the diagonal matrices of the form $\mathrm{I_n} \times \mathbb{G}_{\mathrm{m}}$, with $\mathbb{G}_m$ appearing in the $(n+1, n+1)$-position. Then consider the action of this $\mathbb{G}_m$ by conjugation on X. Then let $\mathrm{G} = \mathrm{GL_n}$ acting by conjugation on X: then the actions by G and $\mathbb{G}_m$ commute. (See [JP20-1] for more on this.)

Proof of Corollary 1.19: the motivic analogue of the Snaith-Mitchell-Priddy splittings. We will explicitly consider only the case where $char(k) = 0$. We will fix a positive integer m and consider the finite degree approximations of all classifying spaces of degree m. However, as m will be fixed throughout our discussion, we will omit the superscript m and gm so that BG (EG) will mean $\mathrm{BG^{gm,m}}$ ($\mathrm{EG^{gm,m}}$, respectively), for any linear algebraic group G. The proof begins with cartesian square:

$$\begin{tikzcd} \mathrm{E} \arrow[r] \arrow[d] & \mathrm{BGL_i} \times \mathrm{BGL_j} \arrow[d, "m_{i,j}"] \\ \mathrm{BGL_r} \times \mathrm{BGL_s} \arrow[r, "m_{r,s}"] & \mathrm{BGL_{i+j=r+s}} \end{tikzcd} \tag{5.1.4}$$

In this situation, we let the transfer $tr_{i,j} : \Sigma_{\mathbf{T}}(\mathrm{BGL_{i+j,+}}) \to \Sigma_{\mathbf{T}}(\mathrm{BGL_{i,+}} \wedge \mathrm{BGL_{j,+}})$. This fits in the framework of Theorem 1.13(i), by taking $\mathrm{G} = \mathrm{GL_{i+j=r+s}}$, $\mathrm{H} = \mathrm{GL_i} \times \mathrm{GL_j}$, $\mathrm{K} = \mathrm{GL_r} \times \mathrm{GL_s}$. Then E admits a decomposition into components each of which is of the form, as a, b vary so that $a \leq r, b \leq s$ and $a + b = i$:

$$\mathrm{EGL_r}\underset{\mathrm{GL_r}}{\times}\mathrm{GL_r}/(\mathrm{GL_a} \times \mathrm{GL_{r-a}}) \times \mathrm{EGL_s}\underset{\mathrm{GL_s}}{\times}\mathrm{GL_s}/(\mathrm{GL_b} \times \mathrm{GL_{b-s}}).$$

Therefore, by the formula in Theorem 1.13(i), which holds for all motivic spectra, applies to provide the identification in the stable motivic homotopy category:

$$tr_{i,j} \circ m_{r,s} = \bigvee_{a+b=i, a\leq r, b\leq s} n_{w_{a,b}} c_{w_{a,b}} \circ (tr_{a,r-a} \wedge tr_{b,s-b}) \tag{5.1.4}$$

where $c_{w_{a,b}}: \mathrm{BGL_a} \times \mathrm{BGL_{r-a}} \times \mathrm{BGL_b} \times \mathrm{BGL_{s-b}} \rightarrow \mathrm{BGL_a} \times \mathrm{BGL_b} \times \mathrm{BGL_{r-a}} \times \mathrm{BGL_{s-b}} \rightarrow \mathrm{BGL_i} \times \mathrm{BGL_j}$ is the obvious map switching the two inner factors. $n_{w_{a,b}}$ is a non-negative integer depending on the multiplicity of the above components.

Next one defines $\overline{\mathrm{BGL}}_n = \mathrm{BGL_n}/\mathrm{BGL_{n-1}}$ and $f_{i,j}: \Sigma_{\mathbf{T}}\mathrm{BGL_{i+j,+}} \overset{\mathrm{tr_{i,j}}}{\rightarrow} \Sigma_{\mathrm{T}}\mathrm{BGL_{i,+}} \wedge \Sigma_{\mathbf{T}}\mathrm{BGL_{j,+}} \overset{\pi_{\mathrm{i,j}}}{\rightarrow} \Sigma_{\mathbf{T}}\overline{\mathrm{BGL}}_{\mathrm{j},+}$. Note that $f_{n,0}: \Sigma_{\mathbf{T}}\mathrm{BGL_{n,+}} \rightarrow \Sigma_{\mathbf{T}}\mathrm{S}^0$ is the augmentation and $f_{0,n}: \Sigma_{\mathbf{T}}\mathrm{BGL_{n,+}} \rightarrow \Sigma_{\mathbf{T}}\overline{\mathrm{BGL}}_{\mathrm{n},+}$ is the projection. By composing the maps on the two sides of (5.1.4) with $\pi_{i,j}$, we obtain:

$$f_{i,j} \circ m_{r,s} = \bigvee_{a+b=i, a\le r, b\le s} n_{w_{a,b}} \overline{m}_{r-a,s-b} \circ (f_{a,r-a} \wedge f_{b,s-b}) \tag{5.1.5}$$

where $\overline{m}_{r-a,s-b}: \Sigma_{\mathbf{T}}\overline{\mathrm{BGL}}_{r-a,+} \wedge \Sigma_{\mathbf{T}}\overline{\mathrm{BGL}}_{s-b,+} \rightarrow \Sigma_{\mathbf{T}}\overline{\mathrm{BGL}}_{r-a+s-b,+} = \Sigma_{\mathbf{T}}\overline{\mathrm{BGL}}_{j,+}$ denotes the induced map.

Now observe that the maps $f_{n-j,j}: \Sigma_{\mathbf{T}}\mathrm{BGL_{n,+}} \rightarrow \Sigma_{\mathbf{T}}\overline{\mathrm{BGL}}_{\mathrm{j},+}$ define the map

$$\Pi_{0\le j\le n} f_{n-j,j}: \Sigma_{\mathbf{T}}\mathrm{BGL_{n,+}} \rightarrow \Pi_{0\le \mathrm{j}\le \mathrm{n}}\Sigma_{\mathbf{T}}\overline{\mathrm{BGL}}_{\mathrm{j},+} \simeq \bigvee_{0\le \mathrm{j}\le \mathrm{n}} \Sigma_{\mathbf{T}}\overline{\mathrm{BGL}}_{\mathrm{j},+} \tag{5.1.6}$$

It suffices to show that this map is a weak-equivalence. For this, we will adopt the argument given in [MP, Proof of Theorem 4.2]. Let $g_j: \mathrm{BGL_j} \rightarrow \mathrm{BGL_n}$, for $n = i+j$, denote the map induced by the inclusion of $\mathrm{GL_j}$ into the last $j \times j$ block in $\mathrm{GL_n}$. Now it suffices to show that the composition $\bar{g}_{j,+} = f_{n-j,j} \circ \Sigma_{\mathbf{T}} g_{j,+}$ is the projection $\Sigma_{\mathbf{T}}\mathrm{BGL_{j,+}} \rightarrow \Sigma_{\mathbf{T}}\overline{\mathrm{BGL}}_{\mathrm{j},+}$, since then the map in (5.1.6) would be a filtration preserving map that induces a weak-equivalence on the associated graded objects.

Therefore, we proceed to show that, the composition $\bar{g}_{j,+} = f_{n-j,j} \circ \Sigma_{\mathbf{T}} g_{j,+}$ is the projection $\Sigma_{\mathbf{T}}\mathrm{BGL_{j,+}} \rightarrow \Sigma_{\mathbf{T}}\overline{\mathrm{BGL}}_{\mathrm{j},+}$. We will take $r = i$, $s = j$ in (5.1.5) and then pre-compose the map there with the map $\mathrm{S}^0 \wedge \Sigma_{\mathbf{T}}\mathrm{BGL_{j,+}} \rightarrow \Sigma_{\mathbf{T}}\mathrm{BGL_{i,+}} \wedge \Sigma_{\mathbf{T}}\mathrm{BGL_{j,+}}$. Then the left-hand-side yields $\bar{g}_{j,+}$, while the right-hand-side yields a finite sum of terms of the form:

$$\Sigma_{\mathbf{T}}\mathrm{S}^0 \wedge \Sigma_{\mathbf{T}}\mathrm{BGL_{j,+}} \rightarrow \Sigma_{\mathbf{T}}\mathrm{BGL_{i,+}} \wedge \Sigma_{\mathbf{T}}\mathrm{BGL_{j,+}} \overset{\mathrm{tr_{a,i-a}} \wedge \mathrm{tr_{b,j-b}}}{\rightarrow} \tag{5.1.7}$$

$$\Sigma_{\mathbf{T}}(\mathrm{BGL_{a,+}} \wedge \mathrm{BGL_{i-a,+}} \wedge \mathrm{BGL_{b,+}} \wedge \mathrm{BGL_{s-b,+}}) \overset{\pi_{i-a}\wedge\pi_{j-b}}{\rightarrow} \Sigma_{\mathbf{T}}\overline{\mathrm{BGL}}_{i-a,+} \wedge \Sigma_{\mathbf{T}}\overline{\mathrm{BGL}}_{j-b,+} \overset{\overline{m}_{i-a,j-b}}{\rightarrow} \Sigma_{\mathbf{T}}\overline{\mathrm{BGL}}_{j,+}.$$

If $i > a$, then the above map $\Sigma_{\mathbf{T}}\mathrm{S}^0 \rightarrow \Sigma_{\mathbf{T}}\overline{\mathrm{BGL}}_{\mathrm{i-a},+}$ will factor through $\Sigma_{\mathbf{T}}\mathrm{BGL}_{i-a-1,+}$, so that S^0 maps to the base point in $\overline{\mathrm{BGL}}_{i-a}$, and therefore the above map will be trivial. Therefore, the only non-trivial summand above will be a map of the form:

$$\Sigma_{\mathbf{T}}\mathrm{S}^0 \wedge \Sigma_{\mathbf{T}}\mathrm{BGL_{j,+}} \rightarrow \Sigma_{\mathbf{T}}\mathrm{BGL_{i,+}} \wedge \Sigma_{\mathbf{T}}\mathrm{BGL_{j,+}} \overset{\epsilon\wedge\pi}{\rightarrow} \Sigma_{\mathbf{T}}\mathrm{S}^0 \wedge \overline{\mathrm{BGL}}_{\mathrm{j},+} \cong \Sigma_{\mathbf{T}}\overline{\mathrm{BGL}}_{\mathrm{j},+}.$$

i.e. The composition in (5.1.7) will be trivial for all terms except when $j = b$, in which case it is

$$\mathrm{S}^0 \wedge \Sigma_{\mathbf{T}}\mathrm{BGL_{j,+}} \rightarrow \Sigma_{\mathrm{T}}\mathrm{BGL_{i,+}} \wedge \Sigma_{\mathbf{T}}\mathrm{BGL_{j,+}} \overset{\epsilon\wedge\pi}{\rightarrow} \mathrm{S}^0 \wedge \Sigma_{\mathbf{T}}\overline{\mathrm{BGL}}_{+} \cong \Sigma_{\mathbf{T}}\overline{\mathrm{BGL}}_{\mathrm{j},+}.$$

This identifies with the projection $\Sigma_{\mathbf{T}}\mathrm{BGL_{j,+}} \rightarrow \Sigma_{\mathbf{T}}\overline{\mathrm{BGL}}_{\mathrm{j},+}$ thereby completing the proof of the corollary in the motivic setting, when all the classifying spaces have been replaced by a fixed finite degree approximation, to order m. One may simply take the (homotopy) colimit as $m \rightarrow \infty$ to obtain the corresponding statement for the infinite classifying spaces. The proof of the corresponding statement in the étale setting is similar, and is therefore skipped.

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