Computational Challenges in Partial Differential Equations

20 January to 4 July 2003

Report from the Organisers: M Ainsworth (Strathclyde), CM Elliott (Sussex), E Süli (Oxford)

Scientific Background

The study of partial differential equations (PDEs) is a fundamental area of mathematics which links important strands of pure mathematics to applied and computational mathematics. Indeed PDEs are ubiquitous in almost all of the applications of mathematics where they provide a natural mathematical description of phenomena in the physical, natural and social sciences.

Partial differential equations and their solutions exhibit rich and complex structures. Unfortunately, closed analytical expressions for their solutions can be found only in very special circumstances, and these are mostly of limited theoretical and practical interest. Thus, scientists and mathematicians have naturally been led to seek techniques for the approximation of solutions. Indeed, the advent of digital computers has stimulated the incarnation of Computational Mathematics, much of which is concerned with the construction and mathematical analysis of numerical algorithms for the approximate solution of PDEs.

The most powerful and generally applicable algorithms for the approximate solution of partial differential equations rely on the concept of *discretisation*, whereby the PDE under consideration is replaced by a finite-dimensional problem. The transition from the partial differential equation to the discrete model is a non-trivial mathematical problem, and the selection of an appropriate finite-dimensional representation is rarely a matter of arbitrary choice: physical properties behind the mathematical model (such as energy and mass conservation, positivity, total-variation-boundedness, dispersion and dissipation) have to be borne in mind, as well as issues of resolution of relevant scales, complete and guaranteed control of the discretisation error, in addition to concerns about the efficiency and reliability of the resulting algorithm. The study of these mathematical questions represents the focus of the field of Numerical Analysis of Partial Differential Equations.

Simultaneously, the development of high speed digital computers has led to consideration of very large systems of PDEs in science and engineering and to scientific demands for highly accurate approximations. In addition, in recent years the applied analysis of nonlinear partial differential equations has progressed in parallel with their theoretical numerical analysis and the development of appropriate numerical algorithms, leading to a very important and intellectually rich area of applied mathematics referred to as Computational Partial Differential Equations (CPDEs).

This programme focussed on some of the most exciting and promising mathematical ideas in these fields, and those branches of PDE theory that provide a source of physically relevant and mathematically hard problems to stimulate future developments. The three main themes were:

• Cognitive algorithms: adaptivity, feed-back and *a posteriori* error control

• Hierarchical modelling: multi-scale mathematical models and variational multi-scale algorithms

• Nonlinear degenerate PDEs and problems with interfaces

Structure of the Programme

Apart from the nine events described below we had a seminar programme. Because of the large number of talks during the events and also in the parallel programme, we restricted the number of seminars to about one per week. Most of the long-term participants delivered a lecture or seminar during the programme.
There were 57 long-term (21 UK) as well as 125 short-term (approximately 56 UK) participants. The meetings also attracted others who were not formally participants in the programme. The large UK participation is a reflection of the current strength of and interest in the field in the UK. The three organisers M Ainsworth, CM Elliott and E Süli together with J Barrett were in residence throughout the programme. Other participants staying for substantial periods, partially supported by EPSRC fellowships, were C Carstensen, PB Monk, C Johnson, R Nochetto and C Schwab. The Rothschild Visiting Professor was F Brezzi. These contributions added greatly to the cohesion of the programme throughout the six months.

The Newton Institute's Junior Membership scheme supported the participation of 11 young researchers. Other sources of funding for young researchers were the EU (for attendance at the Euroworkshop) and the EPSRC Computational Engineering Mathematics programme.

The parallel programme at the Institute, *Nonlinear Hyperbolic Waves in Phase Dynamics and Astrophysics*, had a number of activities concerning computational aspects of hyperbolic and related systems. This led to several discussions between participants of the two programmes and to mutual attendance of seminars and lectures.

Workshops and Conferences

Mathematical Challenges in Scientific and Engineering Computation
Workshop, 20-24 January 2003
Organisers: M Ainsworth (Strathclyde), CM Elliott (Sussex) and E Süli (Oxford)

In the opening lecture R Rannacher (Heidelberg) discussed the issues of error control and adaptivity in numerical computation, and presented applications to numerical approximation of chemically reacting flows. These issues were taken up in the context of free boundary problems by R Nochetto (Maryland), who observed that many of the difficulties revolve around identification of an appropriate notion of residual for such problems, and showed how error control is accomplished for a range of examples including nonlinear degenerate parabolic equations, mean curvature problems and surface diffusion. The day concluded with an open discussion on the role of computational mathematics in complex industrial simulations chaired by J Ockendon (Oxford).

Problems involving interfaces formed the theme of the second day opened by O Pironneau (Paris VI), who described his recent work on the sensitivity of shocks in solutions of Euler equations which are important in computations of flutter. A variety of physical problems where curvature-dependent evolution of interfaces plays a lead role, including diffusion-induced grain boundary motion, along with an overview of the principal mathematical models was presented by CM Elliott (Sussex). Diffuse interface, or phase field, models and their application to solidification were discussed by A Wheeler (Southampton), while the use of simulation at the atomistic level was advocated by A Sutton (Oxford).
Atomistic and multi-scale modelling of materials were the topic on the following day, where E Tadmor (Technion) discussed the problems of dealing with widely varying length-scales found in materials science. He described the "quasi-continuum" approach whereby different local levels of resolution are identified using an adaptive approach. B Engquist (Princeton) continued the theme in his talk on the Heterogeneous Multiscale Method, in joint work with Weinan E (Princeton) who later described some of the new mathematics underlying the method. The importance of using constitutive laws at the molecular level was highlighted in the talk of T McLeish (Leeds), who gave two examples in polymer processing where difference in the structure of the polymer at the molecular level is responsible for completely different macroscopic behaviour.

Thursday was devoted to interaction with the EPSRC Computational Engineering Mathematics (CEM) programme (which provided an additional £10,000 of financial support for the meeting) and consisted of two lectures by overseas experts: R Glowinski (Houston) and T Chan (UCLA). These were followed by a number of short talks based on projects supported through the CEM initiative presented by the principal investigators, and a poster session comprised of other projects supported under CEM. The day concluded with presentations by EPSRC personnel from the engineering and high-performance computing programmes, paving the way for an open forum discussion on the topics raised.
The final day witnessed a change of pace as, firstly, A Quarteroni (EPFL, Lausanne) discussed the applications and advantages of numerical modelling of the cardiovascular system, a problem requiring coupled, heterogeneous PDE models of the entire system. PB Monk (Delaware) gave an overview of numerical techniques used in solving inverse problems and described a new technique, the linear sampling method, developed in collaboration with D Colton. The meeting closed with talks by R Davies (Aberystwyth) on PDEs in the food industry and issues of mesh generation in computational modelling by N Weatherill (Swansea).

The organisers prepared an article which appeared in the June 2003 issue of SIAM News that described the meeting and its context in the six- month programme at the Newton Institute.

Computational Challenges in Micromagnetics and Superconductivity
Hewlett-Packard Event, 13-14 February 2003
Organisers: M Ainsworth (Strathclyde), CM Elliott (Sussex) and PB Monk (Delaware)

One of the aims of this two-day meeting was to identify the major applications in each area where progress in computational techniques for PDEs is needed and will have a significant impact. Leading experts, including physicists, engineers, numerical analysts and theoretical PDE analysts delivered survey lectures on their area of expertise.
The speakers were selected with the aim to provide a comprehensive overview of the current state of the field, with particular emphasis on the interactions between theory, applications and practice.

J Chapman (Glasgow) described experimental results in micromagnetism obtained using Lorentz microscopy showing the existence of free vortices in magnetic permalloy films and reversal magnetisation effects in multi-layer structures.

T Schrefl (Vienna) discussed the different length- and time-scales that arise in micromagnetic simulations, and demonstrated the performance of adaptive finite element schemes for some three-dimensional applications. A Prohl (Zürich) discussed the numerical analysis of microstructure with applications to micromagnetism. G Carbou (Bordeaux) presented results on the existence, uniqueness and regularity of solutions of the Landau-Lifschitz equations. PB Monk (Delaware) concluded the day with a presentation on his recent collaborative work on the numerical modelling using an eddy current micromagnetic model.

The second day focussed on mathematical models for superconductivity. M McCulloch (Oxford), in his talk *Using PDEs to develop the application of superconductors*, described the importance of computational methods for PDEs in engineering applications and in particular the use of superconductors in electric motors. J Chapman (Oxford) discussed in his talk *Macroscopic models of superconductivity* a hierarchy of models from the Ginzburg-Landau equations to Bean's critical state model. Q Du (HKUST) described numerical computations associated with *Quantized vortices: from Ginzburg-Landau to Gross-Pitaevskii*. V Styles (Sussex) discussed critical state computations in her talk *A finite element approximation of a variational formulation of Bean's model for superconductivity*. Finally, L Prigozhin (Ben Gurion) gave an overview of his research on the *Solution of critical-state problems in superconductivity*.

Computational PDEs: Giving Industry the Edge
Joint Workshop with the Smith Institute,
4-5 March 2003
Organisers: CM Elliott (Sussex), RE Hunt (Newton Institute), R Leese (Smith Institute), J Ockendon (Oxford) and E Süli (Oxford)

The challenges of industry call for new mathematics of the highest scientific quality. The Isaac Newton Institute and the Smith Institute for Industrial Mathematics and System Engineering held this joint workshop in order to bring together world-class researchers and industrialists to work on real problems. The meeting presented six current Smith Institute projects involving collaboration between industry and University mathematicians in the areas of food, electromagnetism and violent mechanics. The industrial collaborators described the background and the problems being addressed. The academic researchers discussed the development of mathematical models together with the resulting PDE problems. The Computational Overview sessions illustrated some of the current numerical approaches to the areas of interest, and there were chaired discussions on the critical computational issues in the projects. Topics considered included bread crusting, scraped surface heat exchangers, microwaving, inertia-dominated free surfaces, droplet impact on water layers, electromagnetic compatibility and shaped charge mechanics.

Multiscale Modelling, Multiresolution and Adaptivity
Workshop, 7-11 April 2003
Organisers: M Ainsworth (Strathclyde), W Dahmen (Aachen), C Schwab (Zürich), and E Süli (Oxford)

This meeting was devoted to identifying new algorithmic paradigms which will lead to computationally affordable numerical algorithms for the approximate solution of multiscale problems, with focus on recent developments. This was achieved by bringing together leading experts from applied mathematics, scientific applications and various branches of scientific computation, who work on different aspects of multiscale modelling. The meeting stimulated interactions and cross-fertilisation between the various subject areas involved, and resulted in an assessment by leading researchers of the state-of-the-art in the field, the identification of key problems and obstacles to progress, and the indication of promising directions for future research. Fifteen leading scientists working in the field were invited to present plenary lectures.

M Ainsworth (Strathclyde) discussed the use of high-order finite element methods for the approximation of Maxwell's equations. He presented families of hierarchic basis functions for the Galerkin discretisation of the space H(curl; W). The conditioning and dispersive behaviour of the elements was discussed along with approximation theory.
F Brezzi (Pavia) highlighted the fact that residual-free bubbles proved to be a powerful technique to deal with subscale phenomena and to have optimal stabilising effects. He also discussed, through several examples of PDEs, the amount of computational effort required to solve the subgrid problem.

C Canuto (Torino) used the properties of wavelet bases to design an adaptive descent algorithm for solving a convex minimisation problem in a function space of Hilbert type. He proved the convergence of the algorithm and discussed its optimality in the framework of best N-term approximation results due to A Cohen, W Dahmen and R DeVore.
C Carstensen (Vienna) considered three applic-ations for a nonconvex minimisation problem model: the examples included phase transitions, optimal design tasks and micromagnetics. Through the three examples at hand, he showed that relaxation theory provides auxiliary problems (relaxed models) which are convex but not uniformly or even strictly convex. Their numerical simulation in the quasiconvexified approach to computational microstructures is relatively easy; he recommended the use of this approach, since even the Young measures generated in the nonconvex model can be recovered.

A Cohen (Paris) discussed the convergence analysis of a class of adaptive schemes for evolution equations. These schemes combined adaptive mesh refinement with wavelets based on a framework introduced by A Harten. The analysis mainly focussed on the difficult case of nonlinear hyperbolic conservation laws.
W Dahmen (Aachen) spoke on the adaptive application of (linear and nonlinear) operators in wavelet coordinates. He showed that there are two principal steps: first, the reliable

prediction of significant wavelet coefficients of the output of such adaptive applications from those of the input, and second, the accurate and efficient computation of the significant output coefficients. Some of the main conceptual ingredients of such schemes were discussed. Moreover, it was also highlighted how this approach leads to adaptive solution schemes for variational problems that can be shown to have asymptotically optimal complexity.

R DeVore (South Carolina) discussed two results that have proven to be important in the analysis of convergence rates for adaptive finite element methods. The first is how to bound the number of additional subdivisions needed to remove hanging nodes in newest vertex bisection. Such a result is necessary for deriving computational complexity estimates in adaptive methods. The second result is how to generate a near-best adaptive approxi- mation to a given function in linear time. This result was used in coarsening routines to keep control of the number of cells in adaptively generated triangular partitions.

Weinan E (Princeton) gave an extensive survey of various multiscale methods for a range of physical applications that exhibit multiscale behaviour, and then discussed the numerical analysis of multiscale methods for several classes of multi-physics problems. B Engquist (Princeton and Stockholm) talked about the heterogeneous multiscale method and discussed its application to the numerical solution of stiff ordinary differential equations. He argued that the computational complexity of these techniques is, in many cases, lower than that of traditional methods. He illustrated this point through analysis and numerical examples.
T Hou (Caltech) showed that many problems of fundamental and practical importance contain multiple-scale solutions. He introduced a dynamic multiscale method for approximating nonlinear PDEs with multiscale solutions. The main idea is to construct semi-analytic multiscale solutions local in space and time, and use them to construct the coarse grid approximation to the global multiscale solution. He showed that the method provides an effective multiscale numerical method for computing two-phase and incompressible Euler and Navier-Stokes equations with multiscale solutions. Finally, he introduced a new class of numerical methods to solve the stochastically forced Navier-Stokes equations, and demonstra-ted that the method can be used to compute accurately high order statistical quantities more efficiently than the traditional Monte Carlo method.

C Johnson (Chalmers, Göteborg) presented a framework for adaptive computational modelling with applications to reaction-diffusion in laminar and incompressible flow. He estimated, using a duality argument, the total computational error in different outputs *a posteriori*, with contributions from both discretisation in space/time and subgrid modelling of unresolved scales. He considered subgrid models based on extrapolation or local resolution of subgrid scales, and presented computational results for laminar and turbulent Couette flow.

R Nochetto (Maryland) proposed an algorithm for saddle point problems consisting of two nested iterations, the outer iteration being an Uzawa algorithm to update the scalar variable and the inner iteration being an elliptic AFEM for the vector variable. He showed linear convergence in terms of the outer iteration-counter provided the elliptic AFEM guarantees an error reduction rate together with a reduction rate of data-oscillation (information missed by the underlying averaging process), and applied this idea to the Stokes system without relying on the discrete inf-sup condition. He finally assessed the complexity of the elliptic AFEM, and provided consistent computational evidence that the resulting meshes were quasi-optimal.

R Rannacher (Heidelberg) presented a systematic approach to error control and mesh adaptivity in the numerical solution of optimisation problems with PDE constraints. By the Lagrangian formalism the optimisation problem was reformulated as a saddle-point boundary value problem which was discretised by a Galerkin finite element method. The accuracy of this discretisation was controlled by residual-based a posteriori error estimates. The main features of this method were illustrated by examples from optimal control of fluids and parameter estimation.

C Schwab (ETH Zürich) showed that elliptic problems with multiple scales in bounded domains in $\mathbf{R}^d$ can be reformulated as elliptic problems with a single scale in high-dimensional domains. Standard finite element methods for such problems exhibit poor accuracy due to lack of scale-resolution. Schwab showed how a two-scale finite element method in a higher dimensional space can be developed to overcome this problem. He further showed how sparsification of the finite element space and hierarchical finite element spaces can be used to reduce computational complexity.

J Xu (Penn State) reported on some recent studies on using multigrid ideas in grid adaptation. The results presented included gradient and Hessian recovery schemes by using averaging and smoothing (as in multigrid), interpolation error estimates for both isotropic and anisotropic grids and multilevel techniques for global grid movement and local grid refinement.

The meeting included a further 12 (contributed) talks, mostly by young researchers (under the age of 35) working in the field, and four one-hour coordinated discussion sessions (on Adaptivity, Wavelets and multiresolution for PDEs, Multiscale methods for PDEs, and on Multigrid and multilevel methods); the meeting was attended by 79 participants from academia and industry. The discussion sessions were productive, lively, and, on a number of occasions, quite controversial. The workshop was a very successful event in bringing together leading researchers working in the field.

INTERPHASE 2003: Numerical Methods for Free Boundary Problems Conference, 14-18 April 2003
Organisers: CM Elliott (Sussex), J Barrett (Imperial), G Dziuk (Freiburg) and R Nochetto (Maryland)

INTERPHASE 2003 was the latest in a series of meetings on numerical methods for free boundary problems. The title is a word-play on the ideas that free boundary problems may be interfaces, that many applications involve multiple phases and that the interdisciplinary nature of the workshop brings together computational mathematicians, analysts, modellers and scientists. The meeting was held over four days in which there were 21 talks and much time for discussion. There were 57 participants including well known experts in the field and many young researchers. The topics ranged from image processing, differential geometry, phase transitions and material science, cosmology, mesh generation and flow in porous media, to fluid drops.

Numerical Analysis of Maxwell Equations on Non-Smooth Domains Workshop, 30 April 2003
Organisers: M Ainsworth (Strathclyde) and C Schwab (Zürich)

The numerical analysis of Maxwell equations on non-smooth domains poses a number of computational challenges due to the variational setting of the problem and the presence of edge and corner singularities in the solutions. M Dauge (Rennes) gave two survey talks on the structure of the singularities in three dimensions. M Costabel (Rennes) compared various alternative variational settings for the time-harmonic Maxwell equations and outlined recent theoretical results on the convergence of Maxwell eigenvalues using a regularised variational formulation. D Boffi began by surveying the theory for the numerical approximation of eigenvalues for non-coercive operators, and then showed how the general theory applies to the Maxwell equations and highlighted the importance of the discrete compactness property.

The First European Finite Element Fair
Workshop, 8-9 May 2003
Organisers: M Ainsworth (Strathclyde), C Carstensen (Vienna), CM Elliott (Sussex), C Schwab (Zürich) and E Süli (Oxford)

The European Finite Element Fair (EFEF) was the inaugural meeting of what is planned to become an annual series of completely informal small workshops throughout Europe with equal initial conditions for each speaker. The idea of EFEF is to provide a platform for high-level discussions on current research on finite element approximation, in the broadest sense, of PDEs. The format is based on a long series of such meetings held in the USA. A few, but strict, rules were applied in order to distinguish it from existing workshops and minisymposia in the field.

• Provided that he or she was present throughout the meeting, each participant was invited to talk.

• Based on the number of speakers the time available was divided into slots for the talks. The order of the speakers was determined through random choice, by drawing names out of a hat. Speakers could not request a specific time to talk.

• Each speaker had to introduce himself or herself, the title and topic, and was expected to leave sufficient time, within the allocated time-slot, for discussion. Speakers had to prepare a talk that could be trimmed to various lengths.

There were 34 participants at the meeting.

Partial Differential Equations and Computational Material Science
Spitalfields Day, 13 May 2003
Organisers: CM Elliott (Sussex) and M Luskin (Minneapolis)

Three lectures of a general nature were presented and were well attended by Institute participants from both parallel programmes and academics from within and without Cambridge. The aim was to present exciting new areas of research in materials science requiring significant computational mathematics based on PDEs.
A follow-up day on 14 May 2003, entitled *Partial Differential Equations and Computational Material Science and Solid Mechanics,* was more specialised and focussed on the interaction between the micro and macro scales.

Mathematics of Finite Elements and Applications
Satellite Conference at Brunel University,

21-24 June 2003
Organiser: JR Whiteman (Brunel)

The Newton Institute played a prominent role in the latest of this long-standing series of conferences. The Rothschild Visiting Professor, F Brezzi, gave the Isaac Newton Lecture: *Stabilising sub-grids and their construction*. The Director of the Institute, Sir John Kingman, gave the after-dinner speech and there was a Newton minisymposium running throughout the conference. In addition the following long term participants in the programme gave invited lectures: M Ainsworth, C Carstensen, C Schwab and E Süli.

Outcome and Achievements

At the date of printing there are over 35 preprints associated with the programme in the Newton Institute Preprint Series. The slides, overheads and computer presentations of a large number of lectures in the programme may be found on the web pages of the Institute. There was a great deal of interaction and collaborative research during the programme and we discuss this below under non-mutually exclusive headings.

Nonlinear PDEs and Free Boundary Problems

Diffuse Interface and Phase Field Methods
Z Chen, A Schmidt and R Nochetto made substantial progress in completing a project on A*daptive FEM for diffuse interface models*.
D Kessler, R Nochetto and A Schmidt derived *a posteriori* error bounds to use in *Error control for the Allen-Cahn equation*. J Barrett and CM Elliott began a project to derive fully discrete stable schemes for Cahn-Hilliard fluids entitled *Finite element approximation of Cahn-Hilliard, Navier- Stokes and Hele-Shaw systems*. J Barrett, H Garcke and R Nürnberg developed an approach to surface diffusion with elasticity via the coupling of elasticity equations to a degenerate Cahn-Hilliard equation.

Geometric Flows, Surface Diffusion and Level Set Methods
K Deckelnick and CM Elliott completed a project on *Uniqueness and error analysis for Hamilton- Jacobi equations with discontinuities*. This involves viscosity solutions and has applications to solving the level set eikonal equations in discontinuous media. There was significant activity associated with surface diffusion: K Deckelnick, G Dziuk and CM Elliott derived an algorithm and proved optimal order error bounds for *Fully discrete second order splitting finite element methods for anisotropic surface diffusion for graphs*; G Dziuk and CM Elliott developed level set methods; and P Morin and R Nochetto improved their algorithm for parametric evolution of surfaces by introducing volume preserving mesh regularisation. R Davies and CM Elliott derived a level set model for the proving of shaped bread rolls. O Lakkis and R Nochetto completed their work *A posteriori error analysis for the mean curvature of graphs.* K Mikula completed a long manuscript, *Computational solution, applications and analysis of some geometrical evolution equations*, with a major application oriented towards image processing.

Nonlinear PDEs
R Nochetto, A Schmidt, K Siebert and A Veeser collaborated on *a posteriori* error analysis for *Semilinear elliptic PDEs with free boundaries*.

J Barrett, C Schwab and E Süli initiated a major project on the analysis and numerical analysis of kinetic models for polymeric fluids. J Barrett,
X Feng and A Prohl collaborated on the numerical analysis of Erickson's model for nematic liquid crystals.
Pani collaborated with Süli on the error analysis of discontinuous Galerkin finite element methods and derived optimal error bounds in the L2 norm. He also contributed to the Newton Institute workshop at the MAFELAP conference.

Superconductivity
L Prigozhin made substantial progress on his work P*enetration of nonuniform magnetic field fluctu-ations into type II superconductors and the AC losses*. CM Elliott, D Kay and V Styles developed an analysis of a degenerate Stefan formulation of a critical state model in *Finite element analysis of a current density-electric field formulation of Bean's model for superconductivity*.

Evolving Surfaces
G Dziuk and CM Elliott developed an elegant compact formulation for transport and diffusion on evolving surfaces. This has applications to, for example, the transport of surfactants on fluid surfaces. A finite element algorithm based on evolving triangulations was developed together with stability and error analysis.

Microstructure
C Carstensen developed an approach to the convergence of adaptive finite elements for degenerate convex problems.

Adaptivity and *A Posteriori* Error Control

R Nochetto, E Bäensch, O Lakkis, P Morin, K Siebert, A Schmidt and A Veeser continued their very active collaboration on the theory and implementation of adaptive finite element methods. Areas where significant progress has been made include error control for the Allen-Cahn equation, adaptive finite element methods for diffuse interface models, *a posteriori* error analysis for mean curvature on graphs and convergence of adaptive Raviart-Thomas finite element methods. W Dörfler and M Ainsworth developed a new *a posteriori* error estimator based on the nonconforming Crouzeix-Raviart element for the Stokes equations. C Schwab, P Houston and E Süli devised a new regularity estimation strategy for *hp*-adaptive finite element methods, E Süli and J Robson developed the *a posteriori* error analysis of finite element approximation of non-Newtonian flows of power-law type, while B Guo and M Ainsworth collabo-rated on the *a posteriori* error analysis of *h*- and *hp*-version finite element methods. B Guo completed a paper on the *hp*-version of the boundary element method. F Brezzi, LD Marini and E Süli developed an error analysis of the discontinuous Galerkin finite element method for hyperbolic problems which sheds new insight on the dissipation properties of these methods. P Houston and TJ Barth developed a new post-processing technique for adaptive finite element approximations of hyperbolic problems. C Carstensen and A Veeser collaborated on the convergence of adaptive finite element methods and developed a general technique of proof of convergence. ADC Hill worked on the *a posteriori* error analysis of finite element methods for nonlinear parabolic PDEs.

Multiscale and Multilevel Algorithms

W Dahmen, A Cohen and R DeVore continued their collaboration on multiscale methods and the stability of solutions to conservation laws in metrics other than the usual ones. W Dahmen also started a new collaboration with Kunoth during their stay at the Institute on the *a posteriori* error analysis of wavelet-based methods in the framework of dual-weighted residual type estimators. F Brezzi made considerable advances on the error analysis of a class of multiscale finite element methods based on residual-free bubbles.
W McLean and M Ainsworth continued their collaboration on multilevel preconditioners for first-kind boundary integral equations with weakly singular kernels.
R Duran and R Rodriguez collaborated on the development and analysis of numerical methods for folded elastic plates. P Plechac and MAK Katsoulakis collaborated on coarse-graining of multiscale discretisation algorithms, while R Scheichl worked on the numerical linear algebra of PDEs which arise from fluid flows in porous media.

Computational Wave Propagation

M Costabel and M Dauge completed a manuscript on the regularity of solutions of Maxwell's equations in non-homogeneous media, and, in collaboration with D Boffi, made significant progress on proving the discrete compactness property for higher order edge elements.
IG Graham completed a paper on the computation of diffraction coefficients in high frequency acoustic scattering. PB Monk completed a manuscript on the ultra-weak variational formulation for wave propagation in elastic media and another on computational micromagnetics, and initiated a project on discontinuous Galerkin methods for Maxwell's equations with P Houston. M Ainsworth completed analyses of the dispersive and dissipative properties of higher order elements for standard and discontinuous Galerkin methods and initiated a new collaboration on this topic with PB Monk. D Silvester developed software that will form part of his forthcoming book with
A Wathen and HC Elman, and initiated a new collaboration with M Ainsworth and PB Monk on preconditioners for the Helmholtz equation.

Published Books

During the course of the programme, C Johnson completed a major book on mathematical analysis (for Springer-Verlag) in which constructive and computational/algorithmic aspects play a vital role, and E Süli completed the undergraduate textbook *An Introduction to Numerical Analysis* (Cambridge University Press, 2003).

Conclusion

Based on the reaction of the participants, the organisers feel that this has been an extremely exciting and fruitful programme in which a great deal has been achieved and in which a substantial number of new projects and collaborations has been initiated. We wish to express our sincere gratitude to all members of staff of the Isaac Newton Institute for their help and support throughout the programme, and for providing such a friendly, pleasant and stimulating working environment.