Combinatorics and Statistical Mechanics
Final report

Organisers: Professor P. J. Cameron (Queen Mary, London); Professor B. Jackson (Queen Mary, London); Professor A. Scott (Oxford); Professor A. Sokal (NYU/UCL); Professor D. G. Wagner (Waterloo).

Background

The aim of the programme was to stimulate the growing interactions among combinatorialists, probabilists, computer scientists and theoretical physicists concerned broadly with probability theory or statistical mechanics on graphs and other structures such as matroids, set partitions and constraint satisfaction problems. In particular, increasing computer power has greatly widened the class of examples which can be examined, and methods from physics are increasingly used in combinatorics.

The programme ran from 14 January to 4 July 2008.

Structure

The programme included six workshops, as follows:

14–18 January: *Introductory Cross-Disciplinary Symposium*

21–25 January: *Zeros of Graph Polynomials*

25–28 March: *Markov-chain Monte Carlo Methods*

7–11 April: *Combinatorial Identities and their Applications in Statistical Mechanics*

21–25 April: *Statistical-Mechanics and Quantum-Field Theory Methods in Combinatorial Enumeration*

23–27 June: *Combinatorial and Probabilistic Inequalities*

In addition there were a number of medium and long term participants, including Professor C. Thomassen (the Rothschild Professor) and Professor S. Janson (Microsoft research fellow). Regular seminars were held during weeks when no workshop ran, and the participants interacted in the excellent surroundings of the Newton Institute.

At the Director's suggestion, there was a cross-programme seminar with partipants of the SCH programme. This was very successful, and revealed

a number of points of contact between the two programmes; for example, several participants of our programme are interested in convergence rates for Markov chains, while participants in the other programme use Markov chain methods for exploring high-dimensional datasets. See further comments below on collaborative research resulting from this seminar and the presence of the participants of the SCH programme.

The workshop on combinatorial identities was dedicated to Pierre Leroux, who helped to organise it and died shortly before. One participant described it as "a moving celebration of Pierre Leroux and his work".

Outcome and achievements

A number of areas saw significant advances as a result of the programme. There is no space in the report to discuss all of these. Here are a few selected examples.

- At the Director's suggestion we had a working seminar on the algebraic number theory properties of chromatic roots. The general questions asked were, which algebraic numbers arise as chromatic roots, and what is the typical behaviour of Galois groups of chromatic polynomials. We formulated some conjectures, made progress towards their solution, and found a number of remarkable examples. This involved the Director and a number of long-term participants (especially Cameron, Dong, Farr, Jackson, Morgan, Sellers, Sokal, Wagner) and also V. Dokchitser, a number theorist from DPMMS, Cambridge.

- Jackson, Noble, and Wagner have made significant progress on a 10-year old conjecture of Merino and Welsh, answering a related question of Jerrum in the process. The conjecture concerns values of the Tutte polynomial at specific points; for graphs, these count spanning trees, acyclic orientations and totally cyclic orientations. Cameron and Thomassen attacked the Merino–Welsh conjecture from another point of view and obtained new estimates for the numbers involved.

- There was significant work on negative correlation, negative association and log-concavity by Borcea, Brandan, Kahn, Neiman, van den Berg, Wagner and Chaiken, which has continued since the end of the meeting. In particular, conjectures of Markström and Welsh were disproved.

- Fernandez, Procacci, Salas, Kotecky, Severini, Sokal, Shrock, Faris and Jackson worked on cluster expansions and produced several new ap-

proaches, including a general set-up for cluster expansions for systems with two-body attractive interactions; cluster expansions for not necessarily positive interactions; and the meaning of cluster expansions in the theory of species.

- Royle and Sokal found a bound on the chromatic roots of series-parallel graphs in terms of maxmaxflow (a bound that is also surprisingly close to sharp). Also, Royle found examples of graphs with real flow roots above 4, disproving a conjecture of Welsh, and that he is working with Jacobsen and Salas to try to show that there can be roots approaching 5 from below.

Among many other successful research collaborations, we note work on spin models; computing Ottaviani's invariant; using improved bounds on the Mayer expansion to prove existence of mean field phase transition near the Kac limit (using insights from the Identities workshop); multipermutation solutions of the set-theoretic Yang–Baxter equation; counting defective parking functions; matroids defined by root systems; computational complexity of partition functions and counting hypergraph homomorphisms; the connection between Ashkin–Teller model partition functions and generalised Tutte-Whitney polynomials; the subgraph enumerating polynomial; the asymptotic upper matching conjecture; extending the "entropy method" for counting graph homomorphisms to non-bipartite graphs; zero-free regions for the Potts model; complex-temperature phase diagrams and associated partition function zeros; chromatic zeros of three-dimensional lattices; Cayley-type identities (new identities and new proofs of traditional identities by methods of fermionic integration, hopefully giving meaning to a "fractional" Cayley-type identity); Capelli identities (a proof technique which extends the results beyond Weyl algebras); and invariant theory for OSP(1|2) and OSP(1|2m) and relationship with counting spanning forests.

In addition, the participants recorded many discussions which may later lead to research or possibly collaboration; many pursued their own research as well, often assisted by these discussions.

As a result of the cross-programme session with SCH, Penman worked with S. Marron on a problem arising from DNA, which boils down to "coding with an involution"; Severini worked with M. Pontil on ground states of Hamiltonians; Banks, Said, Wagner, and Wegman worked on relations between agent-based models in statistics and interacting particle systems in statistical mechanics.

Publications

Publication times in the different areas represented in the programme are widely different, and final details of the publications resulting from the seminar will not be available for some time. It is likely that most of the collaborations mentioned above will lead to publications, and many participants also wrote papers on their own research during the programme.

Ellis-Monaghan is working on a survey paper on graph and matroid polynomials containing information important for statistical mechanics; she took much benefit from the programme.

Miscellaneous comments

As well as the traditional workshop dinners, regular 5-a-side football matches were arranged for programme participants.

The participants' comments were extremely positive. In particular, the interdisciplinary nature of the programme was much appreciated. Among the comments are the following:

- The high quality of the environment and facilities enabled me to perform research particularly effectively.
- A fantastic opportunity to meet great mathematicians and physicists and learn new techniques and results.
- A terrific workshop. The support provided by the Institute will have a lasting effect on my research.
- My stay has been successful beyond my expectation. The organisers got together very strong scientists from neighbouring subjects: statistical physics, probability, combinatorics, knot theory ... I gained a huge amount of positive input.