Model Theory and Applications to Algebra and Analysis

17 January to 15 July 2005

Report from the Organisers:
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Left to right: A Pillay, Z Chatzidakis, HD Macpherson and A Wilkie

Background

Model theory is a branch of mathematical logic dealing with 'definability' in various forms. As well as having a rich internal development including stability theory and its generalisations, there have been interactions with and applications to other areas of mathematics for many years. The model-theoretic study of valued fields such as the *p*-adics has a long history, from the Ax–Kochen solution of a problem of Artin, through Denef's proof of the rationality of certain Poincaré series, to current work around motivic and definable integration. The model-theoretic study of differential fields (fields equipped with a derivation) also goes back a long way, but the past 12 years have seen a new level of applications to diophantine geometry. The model-theoretic notion of *o*-minimality, which was isolated and developed twenty years ago or so, has led to very close interactions with real analytic geometers.

The main purpose of the programme was to bring together model theorists in these and other 'applied' topics, with mathematicians from the related areas, so as to solve existing problems as well as open up and explore new areas of research. We also welcomed the presence of model theorists working on the 'purer' side of the subject (although the work of many leading figures typically straddles the pure/applied divide, and in fact each aspect reinforces and feeds into the other).

Structure of the Programme

The programme had three workshops, two at the Newton Institute (both funded by Marie Curie Actions through the EU) and a satellite conference at the University of East Anglia. There was also a one-day Spitalfields Meeting.

Throughout the programme there was regular seminar activity, typically with 6–8 hours of seminars per week. Some of these were part of extended series (for example a series by Wilkie on *o*-minimal diophantine applications of a theorem of Gromov, a series by Macintyre on the model theory of elliptic functions, and several coordinated lectures on compact complex spaces). Other seminars were single talks on current research, in some cases reporting results obtained during the programme.

In total, the programme had 68 long-stay and 41 short-stay visitors. At any given time there were some 20–25 visitors present. The organisers encouraged groups of participants with a specific common interest to come around the same time; thus there was emphasis on groups of finite Morley rank in March, and on *o*-minimality in May.

There were visits, both for workshops and at other times, from specialists in other fields who had previously shown interest in model theory. These included, among others, D Bertrand, Bost, Fesenko, Gabrielov, Khovanskii, M Kim, Pink,

Poonen, Y Raynaud, Roessler, Rolin, MF Singer, Timmesfeld, JS Wilson and Yomdin. PhD students came frequently for seminars, for example from Oxford and Leeds.

Workshops

Groups of Finite Morley Rank

Spitalfields Day, 16 March 2005

The theme was an intensive and coordinated programme, instigated around 1988, to prove the 'Algebraicity Conjecture', that every simple group of finite Morley rank is an algebraic group over an algebraically closed field. Many ideas from the classification of finite simple groups have been used. Altinel (Lyon 1) gave an overview of the programme. Cherlin (Rutgers) and Jaligot (Lyon 1) reported on recent progress. Zilber (Oxford) discussed a possible strategy for refuting the conjecture, and Borovik (Manchester) described interactions between groups of finite Morley rank and finite groups, via 'Black Box' groups. These lectures, beautifully presented, were for the non-specialist. More detailed expositions were given in other seminars during March.

An Introduction to Recent Applications of Model Theory

Workshop, 29 March–8 April 2005

This workshop had some 120 participants, many of them PhD students and postdoctoral researchers (some of whom gave talks or presented posters). The workshop had a strong training component. It featured six tutorial series of up to five lectures each, sometimes linked to more advanced lectures, on recent research. These series were reinforced by lecture notes provided by the lecturers, as well as evening discussion sessions.

The tutorial series were somewhat ambitious and covered key areas of current research on the applied side of model theory: *Stability, differential fields and compact complex manifolds*; *Model theory of algebraically closed valued fields*; *Model theory for metric structures*; *Analogues of Hilbert's Tenth Problem*; *Operations on constructible functions*; and *Zariski-type structures*. We were fortunate in attracting the most prominent researchers for both the tutorials and other lectures. There were also several contributions from people outside model theory. Among the high points were fascinating talks by Fesenko on possible new interactions between model theory and number theory, and by Roessler on the use of model-theoretic results in diophantine and algebraic geometry.

Pure Model Theory

Satellite Workshop at the University of East Anglia, 4–8 July 2005

This satellite meeting was sponsored by EPSRC and the London Mathematical Society. There were three tutorials, given by Ben-Yaacov (on continuous model theory), Poizat (on Hrushovski constructions) and Newelski (on topological methods in model theory). Additional talks covered the state-of-the-art in stable and 'simple' theories, but also covered exciting new developments around *o*-minimality, the independence property, and continuous model theory.

Model Theory, Algebraic and Analytic Geometry

Workshop, 11–15 July 2005

This workshop, coming at the end of the programme, consisted of 24 one-hour lectures by leading experts who presented the latest results in model theory and its applications, many of which had been obtained at the Newton Institute during the preceding 6 months. There were 110 participants, plus a reserve list of 8 disappointed applicants.

The main themes were *o*-minimality, from model theoretic, real analytic and computational viewpoints, the interaction of model theory with diophantine geometry in the spirit of Hrushovski's famous work, rigid analytic geometry and 'definable groups'. The latter cropped up as a unifying element in several talks: groups defined over valued fields, in *o*-minimal structures, in the theory of compact complex spaces, as well as in their own right in the theory of small profinite groups and groups of finite Morley rank. A further theme, unexpected by the organisers at the time the programme was drawn up but a good example of work done during the period itself, concerned first-order definability questions over the class of finitely generated (rather than all) fields.

Even though, as requested by the organisers, most of the talks reported on up-to-the-minute research, the presentations were of such a high standard that they could all be appreciated by the large number of students and young researchers attending the meeting. The replies to the Institute's questionnaires provided ample evidence for this.

Outcome and Achievements

The overall consensus of participants and organisers was that the programme was an overwhelming success. The Newton Institute provided a very supportive environment both for quiet contemplation and intense collaborations. We will describe below some of the more explicit and tangible advances and breakthroughs that were achieved during the programme. There were, of course, other less tangible benefits which were frequently indicated in participants' reports, including inspiring conversations, often with specialists from other fields, possibly leading to later collaborations; greater understanding of adjacent fields of research, through the seminars and workshops; the opportunity for young researchers to broaden their knowledge and build contacts; and the completion or revision of earlier work, often assisted by the presence of collaborators.

Continuous Model Theory

Through excellent tutorials in two of the workshops, some current foundational work by Ben-Yaacov, Henson and their collaborators was successfully disseminated and picked up by other researchers. This was partly the aim of the organisers of the March–April workshop in asking Henson and Berenstein to prepare a tutorial on Henson's 'logic for metric structures'. Henson's logic and various related generalisations of classical first order model theory have now been unified in an attractive framework: continuous logic for metric structures. New research projects were initiated, including connections between the continuous model theory of probability spaces and the Macpherson–Steinhorn 'measurable' supersimple structures.

o-minimality

The broad topic of *o*-minimality, namely ordered structures with good topological behaviour of definable sets, was in evidence throughout the programme. A striking success was that a conjecture of Pillay relating groups definable in saturated *o*-minimal structures to compact Lie groups via an intrinsic 'standard part map' was proved by Hrushovski, Peterzil and Pillay early in the programme, and further conjectures and results around a new notion of 'compact domination' were developed later. Early in the programme Wilkie obtained an important representation theorem for definable sets in arbitrary *o*-minimal expansions of real closed fields, generalising work of Yomdin and Gromov, and, with Pila, applied his theorem to *o*-minimal diophantine questions. The theorem was also used in parts of the Hrushovski–Peterzil–Pillay work mentioned above. Lipshitz and Robinson presented a new *o*-minimal structure in the second July workshop, and their construction was almost immediately used by Hrushovski and Peterzil to solve negatively an outstanding open question of van den Dries in the general theory of *o*-minimality. Moosa, Speissegger and Starchenko made new connections between *o*-minimality and compact complex manifolds, by characterising the compactness of an irreducible component of the space of cycles of a compact complex space in terms of uniform definability in *o*-minimal expansions of $\mathbb{R}_{\text{an}}$. The presence of the real geometer Jean-Philippe Rolin for various periods of time, and his collaborations with Salma Kuhlmann, Speissegger and others, were also very important for this aspect of our programme.

Motivic Integration

Some of the main players in motivic integration, such as Loeser and Cluckers, were present during various parts of the programme. While at the Newton Institute, Hrushovski completed the writing of a major paper (with Kazhdan) on the topic. This paper gives an alternative 'geometric' approach to that of Denef–Loeser, and is likely to have a big impact.

Groups of Finite Morley Rank

The tightly-knit group of researchers working on the conjecture that simple groups of finite Morley rank are algebraic groups made tremendous advances during their time at the Institute. So-called 'degenerate type' groups, those with no infinite 2-Sylow subgroups, present a potential

source of counterexamples. A collaborative effort involving Borovik, Burdges and Cherlin led to a proof that degenerate-type groups of finite Morley rank have no involutions, a result with substantial impact on the overall project.

In a somewhat separate development, Hrushovski and Wagner proved a striking model-theoretic result on counting and dimensions with applications to counting the number of points in intersections of subvarieties of simple algebraic groups with finite subgroups, as well as counting the number of 'rational' points of definable subsets of 'bad fields'.

Number Theory and Geometry

Long- and short-stay visits by D Bertrand, Bost, M Kim, Pink, Poonen, Roessler and MF Singer had a big impact on the programme. Bertrand and Pink discovered the connections between Zilber's conjecture on intersections in tori (related to his study of the complex exponential function), recent conjectures and work of Bombieri–Masser–Zannier, and generalisations of the André–Oort conjecture. In fact, Pink reported on these connections in his talk in the second July workshop. Influenced by discussions with Bertrand and Bost, Pillay proved that the nonlinear version of the Grothendieck–Katz conjecture can be reduced to the linear version modulo a conjecture around 'nonorthogonality to the constants' in the model theory of differential fields. The 'elliptic Schanuel conjecture' emerged as another common theme, and discussions with Bertrand impacted upon the work of Jonathan Kirby, a graduate student from Oxford who is now extending his Ax-theorems for the Weierstrass function to the case of elliptic curves with non-constant j-invariant. Through discussions with Singer, Chatzidakis found a unification of the two different treatments of difference Galois theory.

Model Theory and Noncommutative Geometry

There were considerable discussions around the connections between model-theoretic constructions and examples on the one hand, and noncommutative geometry on the other. These speculations, fuelled originally by Zilber, grabbed the imagination of many participants and visitors. David Evans was able to answer some of Zilber's

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Participants at the workshop on 'Pure Model Theory'

concrete questions related to definability in some of these 'noncommutative geometric objects' such as the Heisenberg group and Hrushovski's non-algebraic 2-cover of the projective line. Zilber himself completed a long paper on noncommutative geometry and new stable structures.

Other Results

We would like to mention a couple of additional beautiful and surprising results which were proved during the programme. Both relate to studying the expressive power of first order logic when ranging over objects with some finite nature (finitely generated fields and finite groups respectively). Bjorn Poonen proved that (i) there is a first order sentence separating the class of finitely generated fields of characteristic zero from the class of all fields of positive characteristic, and (ii) for each n the property that n elements are algebraically independent over the prime field is first order definable within the class of all finitely generated fields.

JS Wilson proved that there is a first order sentence σ in the language of groups such that, for any finite group G, σ holds of G if and only if G is soluble.

Publications

The four organisers will be the editors of a volume (probably, in fact, two volumes, and probably published by Cambridge University Press) based on the programme. This will not be a Proceedings volume, but rather an attempt to reflect the most important current activity in the field through about 20–25 commissioned articles by participants. Most of these will be related to seminars given at the Newton Institute, and some will be survey articles.