New Contexts for Stable Homotopy Theory

2 September to 20 December 2002

Report from the Organisers: JPC Greenlees (Sheffield), HR Miller (MIT), F Morel (Jussieu), VP Snaith (Southampton)

Scientific Background

Algebraic topology started in the late 19th century with the work of Henri Poincaré. In the beginning its objective was to study geometric objects, such as smooth manifolds, which arise in connection with the differential equations of physics, by converting the high-dimensional geometry into more accessible algebra using various cohomology theories. Stable homotopy theory is the ultimate context in which to perform the type of conversion from geometrical to algebraic data which Poincaré began. The cohomology theories themselves are represented by geometric objects; only very recently have really satisfactory models been found, opening a wide range of new possibilities for exploiting algebraic ideas in topology, geometry, number theory and physics. The programme explored several of these new avenues.
Algebraic topology was applied with astounding success by Lefschetz, Hodge and others to tackle problems in complex algebraic geometry. The development of cohomology theories which behaved in positive characteristic like the classical cohomology used by Lefschetz was a major achievement of the 20th century. The Grothendieck school flourished in the 1960s and 1970s and left a legacy of powerful new techniques, famous successes (such as Deligne's proof of the Weil Conjectures) and a series of mysterious cohomological problems concerning Grothendieck's nebulous notion of a 'motive' (or motif).

In the 1990s, techniques of homotopy theory were adapted to the algebraic geometry context to form motivic homotopy theory by Morel, Voevodsky and others. Results in algebraic geometry can be exploited through structures familiar from algebraic topology. These ideas led to the construction of motivic cohomology and to Voevodsky's proof of the Milnor conjecture. This exploitation of motivic homotopy theory in number theory and algebraic topology was spurred on by the programme, which also played an important role in spreading the developing body of knowledge. The motivic world is only one of several new manifestations of stable homotopy theory which made important progress during the programme.

Another subject benefitting from the new structures of stable homotopy theory is elliptic cohomology. This covers a large and rapidly growing field, involving a conjunction of algebra, geometry, analysis and topology, of a depth and power unseen since the development of *K*-theory. The first elliptic genus to be widely studied was introduced by E Witten in connection with conformal field theory, as the equivariant index of a putative signature operator on the free loop space of a manifold. He provided a physical proof of its rigidity, establishing at the outset the connection with differential geometry.

A number of attempts to find geometric cycles for this theory were discussed and developed during the programme. So far the greatest success has been through homotopy theory. Quillen's discovery that the formal group law of complex cobordism is universal opened a new era for algebraic topology, because the theory of formal groups provides a remarkably accurate image of the stable homotopy category. It is natural to ask whether there is a functorial way to associate a complex orientable cohomology theory to a formal group, lifting parts of the category of formal groups to the category of spectra itself, not merely to the homotopy category. New techniques for doing this using the new models for stable homotopy theory were discussed during the programme. One notable success was the construction by Hopkins and Miller of a lift of the moduli stack of elliptic curves to the stable category. This is an elliptic analogue of Atiyah's K-theory with reality, and by taking the homotopy inverse limit of this diagram one obtains the spectrum tmf of topological modular forms. New calculations, variations and consequences of this remarkable object were central to several parts of the programme.

This programme at the Newton Institute furthered this synergistic development by bringing together the practitioners of stable homotopy in all its current diverse disguises - from arithmetic to physics to topological modular forms - to investigate possible new applications of their current techniques.

Structure of the Programme

The programme was structured around three workshops and a number of related lecture series: it started and finished with intensive workshop activity, separated by a period for sustained research and collaboration amongst the longer term participants. The programme opened in earnest with the very well attended two week NATO Advanced Study Institute *Axiomatic, enriched and motivic homotopy theory*, which provided introductions to several of the programme themes. Many participants stayed on for the workshop *K-theory and arithmetic*. During October and November, regular seminar series and three Newton Institute Seminars continued the themes these opened up. The final two weeks were taken up with an EU workshop, *Elliptic and chromatic phenomena*, giving a suitably climactic finish before Christmas.

Meetings and Workshops

Axiomatic, Enriched and Motivic Homotopy Theory
NATO Advanced Study Institute, 9-20 September 2002
Directors: JPC Greenlees and IB Zhukov Organisers: PG Goerss, JF Jardine, F Morel and VP Snaith

This first workshop was a two week NATO Advanced Study Institute attended by 90 participants from 17 countries. It was arranged around a series of 15 introductory minicourses

by experts, designed to make the remainder of the programme accessible to newcomers. These were supplemented by a handful of stand-alone lectures, a problem session and a poster session. Several new collaborations were born during the ASI.
The ASI came about because of a remarkable conjunction in two related subjects. The subject of stable homotopy theory has been transformed in the last ten years by key technical advances making distant dreams into reality, but the fact that its methods have also been used in recent spectacular progress in motivic homotopy theory was a quite separate development. The ASI brought together principal exponents of both themes, and it was gratifying to see how effective it was at encouraging substantial two-way interaction between them. This mathematical cohesion is illustrated by the manner in which workshops in the programme often referred directly to one another. For example, (i) the talks of Hornbostel, Jardine, Hesselholt, Kahn, Levine, Morel, Rognes, Snaith, Toen, Totaro, Weibel and Vishik from the motivic homotopy week of the NATO workshop were all cited during the *K*-theory workshop; and (ii) a similar relation is true of lectures of Goerss, Madsen and Strickland which surfaced again in the EU workshop on elliptic cohomology and chromatic phenomena. Particularly notable and stimulating was the number of originators of results who were present, many of whom had never given lectures in the UK before.

The objective of the ASI was to survey recent developments and outline research perspectives in stable homotopy theory, the homotopy theory of structured ring spectra and motivic homotopy theory. A major effort was made to encourage communication between those with experience from the various different themes.
The lectures at the ASI highlighted the directions of research in which major breakthroughs have been achieved in recent years. These can be broadly grouped into six areas:

• Abstract homotopy theory and models (Dwyer, Jeff Smith, Schwede, Strickland)

• Operads and structured ring spectra (McClure, May, Richter, Smirnov)

• Chromatic and geometric applications of structured ring spectra (Goerss, Madsen, Rognes)

• Foundations of motivic homotopy theory (Jardine, Weibel, Toen)

• Motivic homotopy theory (Hornbostel, Levin, Morel, Totaro, Vishik)

• *K*-theory (Bloch, Carlsson, Hesselholt, Snaith)

We are confident that the positive impact of this ASI will be felt in the mathematical community for many years to come.

K-theory and Arithmetic
Workshop, 30 September - 4 October 2002
Organisers: S Lichtenbaum and VP Snaith

This workshop attracted over 80 participants and concentrated on aspects of interplay between algebraic *K*-theory, arithmetic and algebraic geometry. Particular emphasis was placed upon applications of the recently developed homotopy theory of geometric and motivic categories. In addition to lectures on current results, a number of expository lectures were scheduled to provide researchers and graduate students in related areas with an opportunity to learn about these new techniques. Topics of current interest in this area include: Beilinson-Soulé conjectures, Bloch-Kato conjecture, Beilinson-Borel regulators, Kato- Parshin-Saito higher class field theory, Lichtenbaum-Quillen conjecture, Milnor *K*-theory, motivic cohomology, Brumer-Coates-Sinnott conjectures, polylogarithms, modularity and special values of L-functions.

Elliptic Cohomology and Chromatic Phenomena
Euroworkshop, 9-20 December 2002
Organisers: HR Miller and DC Ravenel

Elliptic cohomology is the third of a sequence of natural probes into the nature of high dimensional objects bringing together ideas from geometry, arithmetic and analysis. Ordinary cohomology dominated the first half of the twentieth century, and K-theory dominated the second half. As Ed Witten has said, elliptic cohomology is a piece of twenty-first century mathematics that happened to commence in the twentieth. It has been most highly developed within the homotopy theory, but early discoveries were made by mathematical physicists studying conformal field theory. The formal structure of elliptic cohomology is becoming well understood, but its geometric foundations are still mysterious. Hints have come from physics (conformal field theory), algebraic geometry (stacks and 2-vector spaces) and representation theory (moonshine and vertex operator algebras), as well as traditional algebraic topology (structured ring spectra, equivariant cohomology).

This workshop attracted over 75 participants from 13 nations, bringing together top researchers in mathematical physics, homotopy theory, algebraic geometry and representation theory to investigate this mysterious object. Significant progress was reported, and discussions at the conference resulted in further progress.
The first week was focussed sharply on bringing together a wide range of perspectives on elliptic genera and elliptic cohomology. There were 19 talks during this first week. The quality of exposition was simply excellent. The high degree of audience participation and the animated conversation, often between new acquaintances, made it clear that the hoped for cross-fertilisation was occurring. While each talk represented a different and important perspective, we might pick out the following sessions as shedding the most new light specifically on the meaning of elliptic cohomology.

• Mike Hopkins described the homotopy theoretical construction of elliptic cohomology, along with the ensuing theory of topological modular forms and the generalized Witten genus from string manifolds.

• John Rognes described joint work with Nils Baas and others giving a 2-vector space model for the Waldhausen *K*-theory of topological *K*-theory, and gave evidence of its relationship with elliptic cohomology.

• Stephan Stolz described his work with Peter Teichner, providing an operator algebra approach to the notion of string structure, and a corresponding candidate for elliptic objects.

• Charles Rezk described explicit formulae for homotopy theoretic logarithms in terms of power operations, resulting in a homotopy theoretic extension of the theory of Hecke operators.

• Vassily Gorbunov described the construction of a geometric object, a sheaf of vertex operator algebras, associated with a string structure.

• Burt Totaro described the universal character of elliptic cohomology in relation to complex projective varieties.

• Matthew Ando described how to explain the phenomenon of Witten rigidity from the perspective of elliptic cohomology.

The second week of this workshop was more narrowly homotopy theoretic, but the focus was broadened to encompass the next steps, beyond elliptic cohomology. From a homotopy theoretic perspective, ordinary cohomology, *K*-theory and elliptic cohomology form the first three of an infinite sequence of probes into the nature of high dimensional geometry.

The following were among the highlights of the ten talks. Hans-Werner Henn described the status of joint work with Paul Goerss, Mark Mahowald and Charles Rezk on the structure of the sphere spectrum from the elliptic perspective. Jorge Devoto described a speculative approach to a geometric theory of K3-cohomology. Andy Baker described the obstruction theory associated to the construction of ring structures on certain higher analogues of *K*-theory. Doug Ravenel described part of the theory of abelian varieties which seems related to these higher *K*-theories.

Outcomes, Achievements and Publications

The programme was the foremost international event in the subject in 2002, and one of special significance as it brought together several strands at seminal stages of their development. Some of this will be reflected in the proceedings volume for the NATO ASI *Axiomatic, Enriched and Motivic Homotopy Theory*, to be published by Kluwer.
Here we summarise some of the work that took place during the programme and was directly related to one of the main themes of the programme. Although there have been many new collaborations, it is especially notable how many participants benefitted from less formal interaction with those of more distant expertise. Which of these contacts will lead to new collaborations, and what the fruit of these will be, we will only know in a few years' time. Some of the areas where significant work was done during the programme are as follows. Inevitably, there are many omissions. There are already preprints reflecting this work, several of them in the Newton Institute preprint series.

• Algebra of rings up to homotopy (Galois and étale maps, duality and finiteness conditions, centres)

• Elliptic and chromatic phenomena (localisation and comodules, small resolutions of local spheres, 2-vector bundles, generalised Mumford conjecture)

• Equivariant cohomology theories (elliptic cohomology, sigma orientation, rational torus-equivariant cohomology)

• Algebraic groups and cohomology theories (arithmetic Jacobians in the non-equivariant case and rational Jacobians in the equivariant case)

• *K*-theory (Weil-étale topology for number fields and zeta functions, additive dilogarithm, de Rham-Witt complex, Galois modules, the Vandiver conjecture)

• Algebraic and motivic homotopy theory (transfer functors, Balmer-Witt groups of classifying spaces, motivic stable homotopy groups, higher stacks)

• Abstract homotopy theory (operads and higher categories, sequence operads and stable *p*-adic homotopy theory, Franke equivalence, cubical sets, algebraic cobordism)

This programme brought together a remarkable collection of researchers from a wide range of areas including parts of homotopy theory, K-theory and elliptic cohomology. It organised, disseminated and consolidated the existing state of the subjects. Many new collaborations were formed, new dreams born, and new agenda set. The beneficial effects of this programme will appear and ripen over the coming years.
