Symmetric Functions and Macdonald Polynomials

8 January to 6 July 2001
Report from the Organisers: P Hanlon (Michigan), IG Macdonald (QMW), AO Morris (Aberystwyth)

Scientific Background

The overall programme was in the general area of symmetric functions and the representation theory of Hecke algebras and the symmetric groups and its generalisations, reflection groups and, in particular, Weyl groups. The importance of this subject area has been long recognised due to its applicability in a wide range of mathematical and scientific disciplines. The subject is also central to algebraic combinatorics and has a particularly distinguished history in the UK with such names as MacMahon, Young, Littlewood (DE) and Richardson regarded as the leading innovators in the field, culminating in more recent times with the breakthroughs by Ian Macdonald.

The programme was focused on Macdonald polynomials around which a great deal of these recent developments have been centred. It was in the 1980's that a series of conjectures were formulated, now known as the Macdonald constant term conjectures. These conjectures predicted the constant term of certain power series indexed by parameters related to semisimple Lie algebras. During the following two decades, these conjectures and related problems concerning Macdonald polynomials generated more activity amongst those working along the interface between algebra and combinatorics than any other specific problem.

The conjectures (and others like them) when they first appeared seemed to be isolated curiosities and it was not clear what (if anything) lay behind them. That became clear a few years later with the introduction of Macdonald polynomials. These are polynomials $\mathrm{P}_{l}(q,t)$ in several variables, depending on parameters q and t, indexed by the dominant weights l for a root system F (partitions l of n for the more classical case of symmetric groups). The Macdonald constant term conjectures, in their more general form, predict the value of áP_{l},P_{l}ñ. It was believed that there is some deep algebraic, topological or geometric phenomenon which has these constant term conjectures as a consequence.

The $P_{l}(q,t)$ for the root system A_{n-1} coincide with a number of well-studied symmetric functions under suitable evaluation or deformation of the parameters q,t, for example, the spherical functions for the classical symmetric spaces, the Hall-Littlewood symmetric functions and the Jack symmetric functions. Attempts to interpret the Macdonald polynomials come in different forms - for example there have been attempts to interpret the transition functions between the Macdonald polynomials and the Schur functions. These transition functions $K_{l,m}(q,t)$ are polynomials in q and t : the (q,t)-Kostka-Foulkes polynomials.

In the intervening years, several different explanations have been discovered coming from different directions. The first method, conceived by the Dutch mathematicians Eric Opdam and Gert Heckman, proves the constant term identities inductively making use of special operators on polynomial operators on polynomial rings called *shift operators*. Ivan Cherednik showed that these shift operators are realised by the action of special elements in double affine Hecke algebras and proceeded to give the first proof. The second approach, pioneered by Phil Hanlon, interprets the two sides of the conjectures as two ways to compute the Euler characteristic of the homology of a certain Lie superalgebra. According to this approach, the conjectures are trying to tell what the homology is and the approach leads to a series of new conjectures called the strong Macdonald conjectures.

The third approach to the Macdonald polynomials (for the root system A_{n-1}), developed by Adriano Garsia and Mark Haiman, involves the study of the diagonal action of the symmetric group S_n on the ring of polynomials in two set of variables $\{x_1,¼,x_n,y_1,¼,y_n\}$. The Macdonald polynomials are interpreted in terms of analogues of classical results about the harmonic/invariant structure of the polynomial ring in one set of variables $\{x_1,¼,x_n\}$. The Garsia-Haiman conjectures concern the S_n-module V_l obtained by applying all partial derivatives in the x_i and y_i to a certain polynomial v_l. A particularly intriguing form of their conjecture states that V_l, viewed as an ungraded S_n-module, is equal to the regular representation. In particular, this contains their celebrated n! conjecture. The fourth approach seeks to interpret the Macdonald polynomials as spherical functions on an appropriately defined quantum group.

Each of these approaches involves the development of a deep theory and collection of conjectures and results which provide an interesting interpretation of the Macdonald polynomials and have the original Macdonald constant term conjectures as a consequence. What is remarkable is that all these theories deal with quite different mathematical constructs so that on the face of it, these approaches are unrelated. The goal of the programme was to unify and further develop these disparate approaches to the Macdonald polynomials with the ideal but not easily attainable outcome of a single theory which would encompass all the above approaches. At the same time, the algebraic combinatorics and representation theory of Hecke algebras and quantum affine algebras and their relations with the Macdonald polynomials and Kostka-Foulkes polynomials have become increasingly significant in mathematical physics. For example, q,t deformations of Virasoro algebras and W-algebras are related to Macdonald polynomials, and the combinatorics of crystals are related to Kostka-Foulkes polynomials. With other equally significant applications in mathematical physics and different approaches to the same subject, one of the main aims of the programme was to provide a unique opportunity for an interdisciplinary exchange of ideas.

Organisation

The organisation was shared between the three organisers, with Phil Hanlon mainly responsible for the organisation ahead of the programme and Alun Morris for the day to day organisation during the programme. In addition, the programme benefited from the excellent workshop organised by Jean-Yves Thibon (Principal Organiser) in April and the stimulating NATO ASI organised by Sergey Fomin (NATO Countries Organiser) at the conclusion of the programme in late June to early July.

During the two workshops, Satellite Meeting and the final NATO meeting there was a full programme of lectures scheduled to leave time for participants to interact with each other. In the intervening weeks, seminars were on the whole restricted to Tuesday and Thursday afternoons with on average three to four one and half hour sessions per week. This left participants with time to continue with their own research work and to carry on more informal discussions.

Participation

The programme was fortunate to have around 55 long term participants staying for periods from three or so weeks to a full six months. There was a similar number who attended for a shorter term. Two of the organisers, IG Macdonald and AO Morris, were present for the whole programme as were S Kumar and A Ram whose active participation contributed strongly to the success of the programme. We were also fortunate that a number of other leading figures in the field spent long periods of three or four months at the Newton Institute. Participants were uniform in their praise for the unique opportunities offered at the Institute; both old and new collaborations flourished and all concerned benefited through interaction with different aspects of the subject.

Participants were in great demand as Visiting Lecturers at other UK Universities during their visit to the Newton Institute; the breadth of the field is illustrated by the different subject seminars which were visited.

Meetings and Workshops

Euroworkshop on *Conjectures, Recent Results and Open Problems Related to the Macdonald Polynomials,* 8 - 12 January 2001

Organisers: P Hanlon, IG Macdonald and AO Morris

The first meeting was designed to launch the programme with the aim of introducing the subject to those parts of the European Community, especially the UK, where the subject is not strongly represented at present. Its title indicated its importance in setting goals for the overall programme. A series of lectures was given by five lecturers who are acknowledged to be the leading authorities and the main innovators in this field: IG Macdonald (QMW London), A Garsia (UC San Diego), P Hanlon (Michigan), E Opdam (Amsterdam) and JR Stembridge (Michigan).

There were a number of participants who were new to the subject and the responses, especially from the younger participants, indicated strongly that the programme had achieved its main aims.

Euroconference on *Applications of the Macdonald Polynomials*, 17 - 20 April 2001

Organisers: J-Y Thibon, B Leclerc, ML Nazarov and M Noumi

The second workshop was devoted to applications of Macdonald polynomials. The programme was mainly, but not exclusively, devoted to applications to Harmonic Analysis and Integrable Quantum Systems. The main aim was to introduce mathematicians working in combinatorics, algebra or group theory to methods coming from physics or analysis. The speakers and participants included most of those who had historically been involved in the subject and the world leading specialists in this field.

The programme included two introductory lectures by P Forrester (Melbourne) on the applications of Jack polynomials in Quantum Statistical Mechanics and Random Matrix Theory and three lectures by members of the team which discovered the deformed Virasoro algebra: H Awata (Nagoya), S Odake (Shinshu) and J Shiraishi (Tokyo). The remaining talks were devoted to the announcement of new results. Apart from the themes mentioned above, several talks discussed interesting problems about quantum groups, affine Lie algebras, differential equations and algebraic combinatorics.

The lectures clearly demonstrated the wide range of applicability of the Macdonald polynomials in physics and harmonic analysis and gave a strong impulse for further research.

Satellite Meeting at Gregynog Hall, University of Wales on *The Heritage of I Schur's 1901 Dissertation*, 2 - 5 June 2001

Organisers: AO Morris, C Bessenrodt, S Donkin, GD James, A Kerber and JB Olsson

The meeting celebrated the centenary of I Schur's Dissertation which has been so influential in the development of representation theory. Also, it was in this work that Schur functions first appeared especially in their relationship to the representation theory of general linear groups and symmetric groups. Schur functions were the precursors of the Macdonald polynomials. JA Green in his work on the characters of general linear groups, where what are now called Hall-Littlewood polynomials (a generalisation of Schur functions) first appeared, and also in his influential Springer Lecture Notes where Schur's work was presented in modern terminology has made an equally significant contribution. This meeting served to celebrate his contribution and also his 75th birthday. The main aim was to bring Newton Institute participants into contact with those who work in contemporary representation theory.

The meeting turned out to be especially successful not only on the research front but also because participants enjoyed the beautiful venue provided at Gregynog Hall. The meeting was funded jointly by a Conference grant from the London Mathematical Society and by the Newton Institute. A highlight of the meeting was the announcement that the 2001 De Morgan Medal will be awarded to JA Green.

NATO Advanced Study Institute: *Symmetric Functions 2001: Surveys of Developments and Perspectives*, 25 June - 6 July 2001

Organisers: S Fomin, G Olshanski, S Kerov (deceased), P Hanlon, IG Macdonald and A Okounkov

The NATO ASI consisted of 13 mini-courses containing 38 one-hour lectures distributed over 10 working days. These mini-courses surveyed recent developments, including those that occurred during the rest of the programme, and indicated directions of future research in various fields for which fundamental questions can be stated in the language of symmetric functions. These courses brought insight from such areas as algebraic geometry, mathematical physics, invariant theory, combinatorics, statistical mechanics, classical representation theory, random matrices, special functions and orthogonal polynomials, Lie theory and quantum groups. On one day the work and contribution of Sergei Kerov, who was one of the two original co-directors of the ASI but who sadly died a year earlier, was commemorated and celebrated.

The ASI was also supported financially by a grant from the EC and from EPSRC funds. The meeting was regarded as a highly successful one by all concerned and in particular by the 102 participants. It was a most worthy event to celebrate the culmination of the Newton Institute programme on Symmetric Functions and Macdonald Polynomials.

Outcome and Achievements

A month or so after the end of the programme is far too early to properly assess the achievements of the programme. Many participants in their individual reports indicated that new ideas had arisen in the formal and informal discussions which would be the basis for their research in the future, some of it in collaboration with other participants. These reports commented almost without exception that the participants had benefited immensely from the exceptionally stimulating environment of the Newton Institute and from interaction with other participants. In comparing with other well-known mathematical institutes, one participant stated “never before have I been so thoroughly pleased with the facilities, staff, location, quality of the scientific program and general comfortable atmosphere.”

The outcomes and general proceedings may have been rather different from those outlined in the original proposal three years earlier. The subject has been extremely active in the intermediary period with some problems solved, but generally with the subject moving in new directions. However, as a perceptive referee of the original proposal had said, “Bringing together the experts listed as potential participants would likely result in a variety of new results, since they come from different backgrounds, but do speak a common language of symmetric functions and Macdonald polynomials.” In the event, this proved to be exactly what happened.

Shortly in advance of the commencement of the programme, M Haiman had, amongst other things, announced the proof of his and A Garsia’s celebrated n! conjecture which also establishes the Macdonald positivity conjecture. During the programme he gave a number of lectures on this subject and the connections between symmetric functions and Macdonald polynomials, combinatorics and the geometry of Hilbert schemes. He completed the second in a series of four papers on this subject and did considerable preparatory work on the remaining two papers. Far more important however was that he, like others, took full advantage from the environment at the Newton Institute to take stock after his proofs of these major theorems.

The research activities of S Fomin and A Zelevinsky centred on their ongoing long-term project on Cluster Algebras. This project aims at constructing an explicit algebraic/combinatorial framework for the study of dual canonical basis and total positivity in semisimple groups. Two papers were completed and two more are forthcoming. In addition, both interacted in a positive way with a number of other participants; for example, a further paper in the series will involve R Marsh. Also, both were involved in discussions with M Noumi on the relationship between their approach to totally positive matrices and his approach to birational Weyl group actions.

During his two short visits, not only did M Noumi give a number of lectures to the April workshop and final ASI, but he also had discussions with G Heckman and M Olshanetsky on moduli spaces related singularity theory and nonlinear integrable systems, and with B Leclerc and J-Y Thibon on representation theoretic aspects of Macdonald polynomials and related combinatorics.

Although the French group based at Marne-le-Vallee already meet regularly, they were able to further develop their collaborative work at the Institute. A Lascoux, B Leclerc, J-Y Thibon together with G Duchamp and F Hivert co-operated on non-symmetric and multi-parameter Macdonald polynomials, R-matrices and Yang-Baxter equations. J-Y Thibon, F Hivert and A Lascoux were able to propose a definition of noncommutative and quasi-symmetric analogues of the classical Macdonald polynomials, which followed earlier similar work on Hall-Littlewood polynomials. However, far more important possibly is the ambitious project about some new connections between affine Hecke algebras and Macdonald polynomials. The initial extensive computer calculations involving R-matrices which were completed are very promising of some significant new results.

A Ram was present for the full programme and not only had a fruitful six months on his own account but also had a major impact on the work of other participants; equally though he took full advantage of the expertise of others readily at hand. He completed three papers during the programme, all of them in collaboration, with C Kirillov,

A Shepler (who visited the Institute) and R Orellana. These involved the representations of graded Hecke algebras, the classification of graded Hecke algebras for complex reflection groups, and affine braids, Jantzen filtrations and the category O respectively. In addition considerable progress was made on three other projects: Kostka-Foulkes polynomials, category O for quantum groups at roots of 1 and branching rules for affine Hecke algebra representations. He also lectured at ten other venues in the UK and three on continental Europe.

Significant progress was made by J Stembridge during his four month visit in his research programme of understanding the combinatorial structure of root systems, Coxeter groups and Weyl characters. In particular he was able to finish writing two papers, the first of which provides a minimal set of axioms for the most important features of Weyl characters, providing easy access to tensor product and branching rules. The second paper provides a interesting new insight into the combinatorial structure of the Bruhat ordering of any finite Weyl group. He also had fruitful discussions with A Schilling during her short visit on their long-term project on crystal bases and combinatorics.

RP Stanley, the programme's Rothschild Professor, was only able to be present for about five weeks but during that time was heavily involved in the programme. His main research activities were in three different areas: the enumeration of chains in the Bruhat order of the symmetric group, the Frobenius rank of a skew partition and Kerov's character polynomial of a k-cycle. The first of these involved a new approach via Schubert polynomials which he was able to discuss with the leading authorities who were present,

J Stembridge, A Lascoux and IG Macdonald. The second was inspired by a lecture by M Nazarov and resulted in joint work with C Bessenrodt. The third took advantage of the presence of G Olshanski, A Okounkov and A Vershik.

C Bessenrodt also continued with her work on a number of problems concerning the representation theory of the symmetric groups and related groups and the associated combinatorics - including Schur functions and Schur Q-functions. For example, she together with JB Olsson worked on the problem of finding zeros on elements of prime order for the characters of symmetric and alternating groups which led to a purely combinatorial question of classifying partitions which are of maximal p-weight for all primes p. The question has been answered for n sufficiently large and n £ 9×10^8. She also obtained a similar answer for the covering groups. A number of questions involving Kronecker products of characters of these groups were also considered; in particular, she obtained a classification of multiplicity-free outer products of Schur Q-functions.

E Vallejo's research involved the difficult problem of the inner product of two Schur functions. He was able to show that ás_l * s_m, s_n ñ is equal to the number of 3-dimensional matrices of zeros and ones and plane sum vectors l, m and n which satisfy some easily checked properties. He also collaborated with A Ram on the decomposition into irreducible modules of the restriction of an irreducible module from the affine Hecke algebra to the finite Hecke algebra.

M Nazarov and V Tarasov had already proved a conjecture of V Chari and A Pressley on the irreducibility of a tensor product of irreducible finite-dimensional modules over the Yangian $Y(gl_N)$ of the general linear Lie algebra gl_N for a wide class of irreducible modules; while together at the Institute they worked on proving this in general taking advantage of discussions with many other participants.

In advance of the programme, B Sagan and M Rosas had worked on what turned out to be related problems. One was working in the context of symmetric functions in noncommuting variables and the other in the context of MacMahon symmetric functions; however, in both contexts a definition of an analogue of Schur functions was lacking. They were able to jointly obtain a combinatorial definition and explored further some of their properties; this will eventually result in a joint paper.

C Dunkl interacted strongly with many participants during his four month visit. However, his greatest achievement was the successful extension jointly with E Opdam, who was also a participant, of the Dunkl operators to complex reflection groups. These are likely to be as influential as those earlier defined for real reflection groups. This work was a real blend of algebra and analysis, another example of the sort of interaction encouraged by the programme.

AO Morris completed some work on root systems and Macdonald representations for complex reflection groups and benefited from discussions with them. C Dunkl also worked on a problem in generalised spherical harmonics with octahedral symmetry - the aim being to construct orthogonal bases for these objects. Typical of other interactions was the following: R Stanley conjectured a summation formula for a certain terminating hypergeometric series, C Dunkl provided a nice analytic proof which

T Koornwinder was then able to connect with the problem of evaluating hypergeometric sums by computer algebra and which then led to more examples and generalisations.

M Rössler took full advantage of contact with C Dunkl and others to advance her work on special functions related with root systems and Dunkl operators. She studied in particular the integral kernel of the Dunkl transform and its asymptotic properties. Also, she commenced work on the theory of trigonometric Dunkl operators and Cherednik operators and the corresponding heat equation.

T Koornwinder also worked on the limit of Macdonald polynomials to the Heckman-Opdam Jacobi functions as q tends to 1. He was able to prove this for the root system A_n and for $t=q^k$ with k a non-negative integer. Contact with T Koornwinder and C Dunkl turned out to be very valuable to P Forrester in his research into Jack and Macdonald polynomials during his short visit.

S Sahi during his visit wrote a joint paper with A Dvorsky describing a construction of a class of small unitary representations of semisimple Lie algebras which arise as conformal groups of real semisimple Jordan algebras. The intention is that this should lead to a generalisation of the theta-correspondence and of Howe’s theory of dual reductive pairs. He also described his work on Jack polynomials - an interesting outcome is a new combinatorial formula for Jack polynomials which is new even for Schur functions.

E Sklyanin worked in collaboration with V Kuznetsov to construct a Q-operator for Jack polynomials - this is an operator whose eigenfunctions are Jack polynomials and whose eigenvalues are the separated polynomials arising from the separation of variables in Jack polynomials. Discussions with T Koornwinder in particular helped M Olshanetsky to further develop his work on unitary representations of the quantum Lorentz group and the theory of Bessel-Jackson functions.

M Wachs was able to complete work on a paper with P Hanlon on a conjecture closely related to the strong Macdonald conjectures - the property M conjecture for the Heisenberg Lie algebra. The work resulted in a new conjecture. A number of lectures were given during the programme on the strong Macdonald conjecture; in particular

C Teleman elucidated his recent proof in conjunction with I Grojnowski and S Fishel. A great deal of discussion of work in this area involved the above and S Kumar and J-L Loday. In fact, during his six months on the programme S Kumar made a major contribution to the discussions in general and especially so in the weekly seminars.

Using the q-deformed algebra $U'_q(so_n)$, A Klimyk was able to derive some properties of the Macdonald polynomials $P_\lambda(x;q,q^{1/2})$ and $P_\lambda(x;q,q^2)$. He gave a group theoretical explanation of the relation $P_{\lambda_1} P_{\lambda_2} = \sum_\lambda f^{\lambda}_{\lambda_1\lambda_2} P_\lambda$ for these Macdonald polynomials. He solved the problem of the positivity of these coefficients and also solved the problem relating the vanishing of these coefficients to the vanishing of the corresponding Littlewood-Richardson coefficients. B Srinivasan made further progress in her efforts to obtain a better understanding of the Green functions for Macdonald polynomials.

A Sergeev constructed a superanalogue SL of the Calogero operator depending on a complex parameter k which is related to the root system of the Lie superalgebra $gl(n|m)$. For generic m and n, the superanalogues of the Jack polynomials constructed by A Kerov, A Okounkov and G Olshanskii are eigenfunctions of *SL*. He also showed that Schur Q-functions may be interpreted as bispherical functions of certain algebras involving the queer algebra $q(n)$, from which he characterised Schur Q-functions as common eigenfunctions of an algebra of differential operators.

IG Macdonald was able almost to complete his book on affine Hecke algebras and orthogonal polynomials during the six month programme. As this work was the inspiration for a great deal of what has occurred in the subject, the book will be eagerly awaited by all the participants.

Some other younger UK mathematicians such as A Cox, R Green, S Perkins, C Wensley and U Onn (from Israel) whose research interests may be regarded as peripheral to the main theme of the programme seemed to benefit from their short stays - all lectured and had valuable discussions with the other participants. Also, they made considerable progress in their research with publications resulting.

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