THE SELBERG CLASS OF $L$-FUNCTIONS: NON-LINEAR TWISTS

Selberg (1950’s; 1989) $\rightarrow$ Selberg class $S$:

(i) (ordinary Dirichlet series) $F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ absolutely convergent for $\sigma > 1$

(ii) (analytic continuation) $(s - 1)^m F(s)$ entire function of finite order for some integer $m \geq 0$

(iii) (functional equation) $\Phi(s) = \omega \overline{\Phi(1 - s)}$ where $\overline{f(s)} := f(\overline{s})$

\[ \Phi(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j)F(s) \]

$r \geq 0$, $Q > 0$, $\lambda_j > 0$, $\Re \mu_j \geq 0$, $|\omega| = 1$

(iv) (Ramanujan conj.) $a(n) \ll n^\epsilon \quad \forall \epsilon > 0$

(v) (Euler product) $\log F(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}$ with $b(n) = 0$ unless $n = p^m$ with $m \geq 1$, and $b(n) \ll n^{\vartheta}$ for some $\vartheta < \frac{1}{2}$

extended Selberg class $S^\#$: axioms (i) - (iii)
Examples: \( L(s, \chi), L_K(s, \chi) \), suitably normalized \( L_f(s) \), Artin \( L \)-functions (mod. Artin conj.), automorphic \( L \)-functions (mod. Ramanujan conj.)

Axioms → standard analytic properties: Euler product \( F(s) = \prod_p F_p(s) \) non-vanishing for \( \sigma > 1 \); critical strip and line, trivial and non-trivial zeros; \( N_F(T) \sim c_FT \log T \); polynomial growth on vertical strips, Lindelöf \( \mu \)-function,...

\[ - \text{ Structure of } S \]

What is in \( S \)? **Main Conjecture:**

\[ S = \text{class of automorphic } L \text{-functions} \]

(if true, very deep: Langlands ...)

**degree** of \( F(s) \): \( d_F = 2 \sum_{j=1}^r \lambda_j \) (invariant)

\( (d_\zeta = 1, d_{L(,\chi)} = 1, d_Lf = 2, d_{\zeta K} = [K : Q], ...) \)

\[ S_d = \{ F \in S : d_F = d \} \]
**Conj. 1.** *(general converse theorem):* for \( d \in \mathbb{N} \)
\[
S_d = \{ \text{automorphic } L\text{-functions of degree } d \}
\]

**Conj. 2.** *(degree conjecture):* for \( d \notin \mathbb{N} \)
\[
S_d = \emptyset
\]

**Remark.** Conj.2 expected for \( S^\# \) as well, but **false** if

*ordinary D-series \( \rightarrow \) general D-series:

\[
D(\lambda, \mu, Q, \omega) = \text{vector space of general D-series satisfying } (i) - (iii); \text{ using Hecke's theory}
\]

**Th.** *(K-P)* \( D(\lambda, \mu, Q, \omega) \) has uncountable basis

**State of art:** conjectures 1 and 2 true for
\[
0 \leq d < 5/3
\]

Precisely:

\[
d = 0: \quad S_0 = \{1\} \quad \text{(Conrey-Ghosh 1993)}
\]

\[
0 < d < 1: \quad S_d = \emptyset \quad \text{(Richert 1957, Bochner 1958, C-G 1993, Molteni 1999, K-P 2002,200?)}
\]
\[ d = 1: \ S_1 = \{ L(s + i\theta, \chi) \} \quad \text{(K-P 1999)} \]

\[ 1 < d < 5/3: \ S_d = \emptyset \quad \text{(K-P 2002)} \]

- **Linear twists**

Main tool for \( d \geq 1 \): **linear twists** \((e(x) = e^{2\pi ix})\)

\[
F(s, \alpha) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} e(-n\alpha)
\]

To study **analytic properties**: for \( N, \alpha > 0 \) by Mellin + functional equation

\[
F_N(s, \alpha) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} e(-n\alpha)e^{-n/N}
\]

\[ = R_N(s, \alpha) + \omega Q^{1-2s} \sum_{n=1}^{\infty} \frac{a(n)}{n^{1-s}} H_K \left( \frac{n}{Q^2 \left( \frac{1}{N} + 2\pi i\alpha \right)}, s \right) \]

where

\[
H_K(z, s) = \frac{1}{2\pi i} \int_{(-K-\frac{1}{2})} \prod_{j=1}^{r} \frac{\Gamma(\lambda_j(1-s) + \mu_j - \lambda_j w)}{\Gamma(\lambda_j s + \mu_j + \lambda_j w)} \times \Gamma(w)z^w \, dw,
\]
the hypergeometric functions. For $s$ fixed studied by Braaksma (1963); behaviour depends on value of

$$\mu = 2 \sum_{j=1}^{r} \lambda_j - 1 = d_F - 1$$

$\mu = 0$ ($d_F = 1$) simpler case; $\mu > 0$ ($d_F > 1$) more complicated by "exponential part".

Development of 2-variables theory; since want $N \to \infty$, main interest for $H_K(-iy, s)$, $y = \frac{n}{2\pi Q^2 \alpha}$.

Let conductor $q_F$ and shift $\theta_F$ (invariants) be

$$q_F = (2\pi)^{d_F} Q^2 \prod_{j=1}^{r} \lambda_j^{2 \lambda_j}, \quad \theta_F = \Im(2 \sum_{j=1}^{r} (\mu_j - \frac{1}{2}));$$

critical value: $n_\alpha = q_F d_F^{-d_F} \alpha^{d_F}; a(n_\alpha) = 0$ if $n_\alpha \notin \mathbb{N}$

**Case** $d_F = 1$:

**Th.1.** (K-P) Let $F \in S_1^\#$ and $\alpha > 0$. Then $F(s, \alpha)$ is entire if $a(n_\alpha) = 0$, while if $a(n_\alpha) \neq 0$ then $F(s, \alpha)$ has at most simple poles at $s_k = 1 - k - i\theta_F$ ($k = 0, 1, ...$), with non-vanishing residue at $s = s_0$
**Case**  $1 < d_F < 2$: let

$$
\kappa = \frac{1}{d - 1} \quad A = (d-1)q_F^{-\kappa} \quad s^* = \kappa \left(s + \frac{d}{2} - 1 + i\theta_F\right)
$$

**Th.2. (K-P)** Let $1 < d < 2$, $F \in S^\#_d$ and $\alpha > 0$. Then

$$
F(s, \alpha) = e^{as + b} \sum_{n=1}^{\infty} \frac{a(n)}{n^{s^*}} e\left(A\left(\frac{n}{\alpha}\right)^\kappa\right) + G(s, \alpha)
$$

where $G(s, \alpha)$ is holomorphic for $\sigma^* > \sigma_a(F) - \kappa$

**Remark:** $\sigma^* > \sigma$ for $\sigma > \frac{1}{2}$ and $1 < d < 2$, hence "overconvergence": suspicious!

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**Non-linear twists**

For $F \in S^\#_d$ with $d > 0$, **non-linear twist** ($\alpha > 0$)

$$
F(s, \alpha) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} e\left(-n^{1/d} \alpha\right)
$$

Theorem 1 is special case of a general result for non-linear twists:
Th.3. (K-P) Let \( d > 0 \), \( F \in S_{d}^{\#} \) and \( \alpha > 0 \). Then \( F(s, \alpha) \) is entire if \( a(n\alpha) = 0 \), while if \( a(n\alpha) \neq 0 \) then \( F(s, \alpha) \) has at most simple poles at \( s_k = \frac{d+1}{2d} - \frac{k}{d} - i\frac{\theta_F}{d} \) \((k = 0, 1, ...)\), with non-vanishing residue at \( s = s_0 \).

Bounds on vertical strips, uniform for \( F(s) \) in suitable families \( \mathcal{F} \) (roughly, bounded degree and \( \mu \)-coefficients) can also be obtained.

Applications of Theorem 3

i) non-linear exponential sums: \( \phi(u) \) smooth with compact support, \( F \in S_{d}^{\#} \)

\[
S_{F}(x, \alpha) = \sum_{n=1}^{\infty} a(n) e(-n^{1/d}\alpha) \phi\left( \frac{n}{x} \right).
\]

Then asymptotic expansion, uniform for \( F \in \mathcal{F} \), of type

\[
S_{F}(x, \alpha) = \sum_{k} c_k(F, \alpha) x^{s_k} + O(x^{-A})
\]

( extending results by Iwaniec-Luo-Sarnak in degree 2, different method)
ii) Ω-results: $F \in S_d^\#$ with $d \geq 1$

$$A_F(x) = \sum_{n \leq x} a(n).$$

Then

$$A_F(x) = \text{res}_{s=1} F(s) \frac{x^s}{s} + \Omega(x \frac{d-1}{2d})$$

Remark. Exponent caused by pole of $F(s, \alpha)$ (with suitable $\alpha$) at $s = s_0$; possibly same result obtainable by Voronoi-type arguments

iii) $S_d^\# = \emptyset$ for $0 < d < 1$: pole of $F(s, \alpha)$ at $s = s_0$ has real part $> 1$ if $0 < d < 1$, contradiction

iv) characterization of $\zeta(s)$: let $F \in S_d$ with $d \geq 1$. If

$$\sum_{n=1}^{\infty} \frac{a(n) - 1}{n^s}$$

converges for $\sigma > \frac{1}{5} - \delta$, then $F(s) = \zeta(s)$

Remark. Similarly for $L(s, \chi)$; uses $S_d = \emptyset$ for $1 < d < 5/3$