THE PROPAGATION OF QUANTUM INFORMATION THROUGH SPIN (& OTHER) SYSTEMS

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OUTLINE

- Introduction
- Basic description of proposals & model Hamiltonian
- QI approach
- QI transfer in symmetric ferromagnetic models
- Extensions
  (i) multiple wavepackets
  (ii) decreased control
- Antiferromagnetic models (time permitting)
- Conclusions & future directions
INTRODUCTION

- Motivation: a functioning quantum computer will probably require a means of moving quantum information (qubits) from one place to another (flying qubits).
- Various proposals exist, such as using photonic channel.
- Problem may be that interfacing is difficult?

We are therefore interested in a method to communicate qubits over mesoscopic distances like “quantum wires” on a “quantum chip”.


Use a locally-interacting quantum spin chain and put state in one end and let natural dynamics transfer quantum information to other end!

For latest literature see Burgarth & Bose, quant-ph/0406112 and references therein.
**BASIC PROPOSAL & PROBLEM**

- We are given access to a locally interacting spin system:

\[
H = \frac{1}{2} \sum_{j,k=1}^{N} J_{j,k} \sigma_j \cdot \sigma_k + h \sigma_j z
\]

(topology of network encoded here)

- (Results pertain to much Class, but we fix, for concreteness the model to be Quantum Heisenberg model on ring)

\[ N \text{ qubits in system} \]

- Alice & Bob have only access to a small part of system, possibly only one qubit each.

**TASK:** Communicate quantum state of single qubit from Alice to Bob using only natural dynamics of system, i.e. with no fast local control on every qubit.
QUANTUM INFO. THEORISTS APPROACH: ABSTRACTIFY!

- Encoder:
  - Acts via local control on Alice's part of system (Unitary $U_A$)

- Channel:
  - Natural (non-Markovian)
  - Evolution of spin system for some pre-agreed time (Unitary $e^{-i\mathbf{H}t}$)

- Decoder:
  - Acts via local control on Bob's part of system (Unitary $U_B$)

- We have now: (i) the task
  (ii) the channel

- We need figure of merit to determine if task is completed successfully

FIGURE OF MERIT: AVERAGE FIDELITY

$$F(\mathbf{H}, N, \Lambda, T, U_A, U_B) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \langle \psi_0 | \rho_T | \psi_0 \rangle$$

Where $\mathbf{H}$ is Hamiltonian, $N$ is no. of qubits, $\Lambda$ is size (in sites) of Alice/Bob's accessibility region, $T$ is the time A&B wait, $U_A, U_B$ are Alice (Bob's) encoding (decoding) operation, $\rho_T$ is the state of chain Bob sees after decoding his part of chain and the integral is an avg. over all input states.
SPECIFICS OF MODEL

- Recall:

\[ H = \frac{1}{2} \sum_{j,k} \sigma_j \cdot \sigma_k + h \sigma_j^z \]  
(RING TOPOLOGY)

- Important fact:

\[ [H, \sum_{j=1}^{N} \sigma_j^z] = 0 \]

- So \( H \) & \( S_{\text{rot}}^z \) can be simultaneously diagonalised.

- This breaks Hilbert space into direct sum:

\[ P_0 = \{ 100\ldots0x0\ldots01 \} \]

\[ P_1 = \{ 110\ldots0x10\ldots01, 1010\ldots0x010\ldots01, \ldots \} \]

&c.

where \( \text{HP}_0 \subseteq P_0 \) 3 \( \text{HP}_1 \subseteq P_1 \) &c.

Call this "vacuum"  Call this single particle space, &c.
MORE SPECIFICS

- We assume system is initialised to
  \[ |0\rangle = |00...0\rangle \quad \text{("vacuum")} \]

- We also assume Alice & Bob only perform operations which move system between "vacuum" & single particle space
  \[ |1\rangle = |10...0\rangle \]
  \[ |a\rangle = |010...0\rangle \]
  \[ |n\rangle = |0...01\rangle \]

- So dynamics of system only occur in 0- & 1-particle space
  i.e. general state is assumed to always be expressible:
  \[ |\Psi(t)\rangle = c_0 |00\rangle + c_1 \sum_{j=1}^{N} e^{i\epsilon_j(t)} |j\rangle \quad \text{(up to overall phase)} \]

where
\[ |c_0|^2 + |c_1|^2 = 1, \quad \sum_{j=1}^{N} |e^{i\epsilon_j(t)}|^2 = 1 \]

**Why is this?**

\[ (e^{-iH_0t}) |\Psi(0)\rangle = (e^{-iH_0t}) (c_0 |00\rangle + c_1 \sum_{j=1}^{N} e^{i\epsilon_j(0)} |j\rangle) \]

\[ -c_0 e^{-iH_{10}t} |00\rangle + c_1 \sum_{j=1}^{N} e^{i\epsilon_j(0)} e^{-iH_{j10}t} |j\rangle \]

where \( H_{10} = \langle 0 | H_{10} | 0 \rangle \); \( H_{j10} = \langle j | H_{10} | 0 \rangle \)

So
\[ \epsilon_j(t) = \langle j | e^{-iH_{10}t} | k \rangle e_k(0) = \sum_{k=1}^{N} [e^{-iH_{j1k}t}] e_k(0) \]

\[ \sum_{j=1}^{N} |e^{i\epsilon_j(t)}|^2 = 1 \]
SPECIAL PROPERTIES OF SYSTEM STATE

- We can always write system state as (assuming $C = 0$):

\[
|\psi(t)\rangle = \sqrt{1 - C_B(t)} \left[ |\phi_B\rangle \otimes \phi^B_{\overline{B}} + \sqrt{C_B(t)} |\phi^B_{\overline{B}}\rangle \right]
\]

where $B$ means Bob's part of system & $\overline{B}$ means rest of system and

\[
C_B(t) = \sum_{j \in B} |e_j(t)|^2
\]

\[
|\phi_B\rangle = \frac{1}{\sqrt{1 - C_B(t)}} \sum_{j \in B} e_j(t) |j\rangle
\]

\[
|\phi^B_{\overline{B}}\rangle = \frac{1}{\sqrt{C_B(t)}} \sum_{j \in \overline{B}} e_j(t) |j\rangle
\]

- Note that (\#) is a two-term Schmidt decomposition because $\langle \phi|\phi\rangle_B = \langle \phi|\phi\rangle_{\overline{B}} = 0$

When Alice wants to send $|\psi\rangle = |0\rangle_B + |1\rangle_B$ she encodes this as (this is most general code in $\otimes_B 1$-particle space)

\[
|\psi(0)\rangle = |\phi^0_B\rangle + c \sum_{j \in \overline{B}} e_j(0) |j\rangle
\]

where $|e_j(0)| = 0$ when $j \notin J$ of Alice's accessibility region
PROPERTIES (Cont.)

- After time $T$ has elapsed, this state evolves to
  \[
  |\Psi(T)\rangle = c_i \sqrt{1-C_0(T)} \left[ \langle \Omega_2| \langle 0_2 + I_0 \rangle (\langle 0_2 + C_0(T) \rangle |\psi\rangle \right]
  \]
  where $|\psi\rangle$ is partition of system into Bob's $A+B$ and Bob's region $B$.
  \[
  C_0(T) = \sum_{j \neq B} |\psi_j(T)|^2.
  \]

- Bob decodes this state optimally using
  \[
  U_B|\Omega_2\rangle = |\Omega_2\rangle
  \]
  \[
  U_B|\Omega'_2\rangle = |N_2\rangle
  \]
  i.e. qubit is concentrated into qubit $N_2$.

- After decoding, Bob sees the state:
  \[
  \rho = \left( \begin{array}{cc}
  C_0 + I_0 & \sqrt{C_0(T)} C_0^* \\
  \sqrt{C_0(T)} C_0^* C_1 & I_1 C_0^* C_0
  \end{array} \right)
  \]
  with respect to basis $|0\rangle_{N_2}, |1\rangle_{N_2}$ of qubit $N_2$.

- We can write Bob's state as: $\rho = \rho_T(14\times41)$; where
  \[
  \rho_T(14\times41) = M_0 \rho T(14\times41) M_0^* + M_1 \rho T(14\times41) M_1^*
  \]
  & $M_0, M_1$ are Kraus operators of amplitude damping channel:
  \[
  M_0 = \left( \begin{array}{cc}
  1 & \frac{\sqrt{1-C_0(T)}}{2} \\
  0 & \frac{-\sqrt{1-C_0(T)}}{2}
  \end{array} \right)
  \]
  \[
  M_1 = \left( \begin{array}{cc}
  0 & \sqrt{1-C_0(T)} \\
  0 & 0
  \end{array} \right)
  \]

Note: $\rho_T(\rho_{T1}(14\times21)) \neq \rho_{T1}(14\times21)$
AVERAGE FIDELITY & GRAPH OF STATE

The average fidelity of transmitted state can be computed as:

\[ F = \frac{1}{2} + \frac{1}{6} \sqrt{C_0(t)^2 + \frac{1}{6} C_2(t)} \]

which is monotonically increasing with \( C_0(t) \).

To visualise this quantity we introduce the graph of a state \( |\phi\rangle \) in subspace \( P_i \).

The graph of a state \( |\phi\rangle \) is the bar graph where we plot \( |\langle j|t\rangle|^2 \) above site \( j \) where

\[ |\phi(t)\rangle = \sum_{j=1}^{N} c(j|t) |j\rangle. \]

EXAMPLE

(i) \[ |\phi(t)\rangle = |127\rangle \]

\[ \begin{array}{cccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array} \]

Area under curve = 1

(ii) \[ |\phi(t)\rangle = \sqrt{\frac{N}{2}} \sum_{k=1}^{N} e^{\frac{2\pi i j k}{N}} |k\rangle \]

\[ \begin{array}{cccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array} \]

Area under graph = 1

Obviously graph doesn't represent phase information
FIDELITY & GRAPH (Cont.)

- In terms of the graph of system state $|\psi(t)\rangle$ the quantity $(\rho_{AC})$ is the area under graph in Bob's region

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<tr>
<td>Alices Region</td>
<td>Bob's region</td>
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- We have now reduced communication to following classical problem:

Alice needs to design a complex scalar wavepacket such that the power $(\rho_{AC})$ of the wave in Bob's region after time $T$ is maximised.

That is, Alice needs to choose a complex no.s $e_j(0); j \in A$ such that $[e^{-iH't}]_{jk} e_k(0) = e_j(T)$ has maximum area/power in Bob's region $B$.

For Heisenberg model on ring/chain this is a well known problem:
REDUCTION TO PARTICLE IN BOX

For the case where the channel is a ring then

\[
H_{jk}^{(\ell)} = \begin{pmatrix}
-2 & 1 & 0 & 1 \\
1 & -2 & 1 & -1 \\
0 & 1 & -2 & 1 \\
1 & 0 & -2 & -1 \\
1 & -2 & 1 & -1 \\
\end{pmatrix} = -\nabla^2_{\text{ring}}
\]

Discrete approximant to Laplacian of free ring

This means that

\[
\left[ e^{-iH_{jk}^{(\ell)}t} \right]_{jk} \psi_{\ell j}(t) = e^{-i\nabla^2_{\text{ring}}t} \psi_{\ell j}(0)
\]

Which is (discrete approximation) to the propagator for the Schrödinger equation of a free particle on ring. (Free rotor)

This is a dispersive linear equation; what scalar wavepackets "hold together" best?

Answer: Coherent states:

\[
|\alpha\rangle = \frac{1}{\sqrt{\pi}a} \int dx \ e^{-\frac{x^2}{2a^2}} |\alpha\rangle
\]

for continuous variables

(\alpha_\ell can be complex)
DYNAMICS OF COHERENT STATES

- Inspired by solution in continuous variable case we take for Alice's encoded state the discretisation of a coherent state truncated outside a region $A$:

$$\left| \tilde{\Psi} \right> = \left| \tilde{G}(x_a, k_0, A) \right> = \frac{1}{\sqrt{\Pi}} \sum_{j=0}^{\infty} e^{-\frac{(x_j-x_a)^2}{2\sigma^2}} e^{2\pi i k_0 x_j} |j\rangle$$

- In continuous variable case, a coherent state propagates to a coherent state with a larger variance (strictly not a coherent state, but still Gaussian):

$$\begin{array}{c}
\text{Free particle:} \\
\sqrt{L(0)} \to \text{evolution} \to \sqrt{L(t)}
\end{array}$$

where

$$\frac{L(t)}{L(0)} = \left[ 1 + \left( \frac{\mathcal{L}(k_0) t}{L(k_0)} \right)^2 \right]^\frac{1}{2}$$

where $\mathcal{L}(k_0) = c \hbar k_0$ is the dispersion relation.

(Recall that eigenstates of free particle on ring are momentum states $\psi_n(x) = e^{i k_0 x}$; $\mathcal{L}(k_0) = c \hbar k_0$ are eigenvalues of these eigenstates.)
DISCRETISED VERSION

The eigenstates of \( H^{(n)} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \) are Fourier modes (spin waves):

\[
|\psi(k)\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{-i\frac{2\pi j k}{N}} |1k\rangle
\]

with eigenvalues

\[
\omega(k) = 2 - 2 \cos(\frac{2\pi k}{N})
\]

This is the discrete version of dispersion relation.

Mini Summary:

1. Task: Send qubit from A to B
2. Task takes place in single particle & vacuum subspace
3. Figure of merit: Average fidelity is proportional to area under graph of system state \( \rho \) vs. \( \theta \)
4. This translates task into problem of designing wavepacket which is localised & doesn't disperse too much
5. For ring/chain the single particle dynamics are discretisation of particle in box/on ring
6. Continuous variable solution: coherent state; suggest solution for discrete case.
HOW SMALL CAN WE MAKE SIGNAL?

- Intuition: the thinner the initial signal the more localised it is in position space hence (uncertainty principle for Fourier analysis on groups) the more delocalised the state is in momentum space.
- In other words, we expect that there is a compromise between localisation of initial signal & subsequent dispersion.

- How much will a wave packet spread by the time it reaches Bob? Solution comes from Gaussian broadening equation for free particle:

\[ S(N) = \frac{L(N)}{L(0)} = \left[ 1 + \left( \frac{A}{NL(0)} \right)^2 \right]^{1/2} \]

Inserting dispersion relation: \[ \Delta x(t) = \frac{2\Delta P}{P} = \frac{2\hbar t}{m} \]

- Solving this equation we find that a packet of width of \( \sqrt{N}L(0) \) sites will only spread by a constant factor which doesn't depend on N as it reaches Bob.

This is a vanishing proportion of \( \sqrt{N} \) in limit \( N \to \infty \).
Heisenberg model on ring of $N=100$ sites where Alice sends the state "11" by encoding it as a 1 at 50th site. (Equivalent to the encoding scheme of Bose, 2002.)
Graph of state (2)

Same system as before except Alice now encodes "1\% = 117" as a Gaussian $\mathcal{G}(50)$ centered at site 50 with a width of 10 sites.
SOME EXTENSIONS & NEW RESULTS

1. **Different models:** All previous discussion applies to any system interacting locally on a chain or ring which commutes with $S^z_{\text{tot}}$.

2. **Different topologies:** It is straightforward to apply coherent state signal states to 2D & higher dimensional Heisenberg $XXZ$ models. Because the signal states stay local many parties can communicate with each other even in $10$.

3. **Reducing control requirements:** Haselgrove, quant-ph/0404152, has shown how to implement our protocol using only two spins for Alice & two spins for Bob. This method requires good control of only coupling between these two respective qubits.

4. **Arbitrary coupling & topologies:** In quant-ph/0404152, Haselgrove also showed how to calculate optimal signal packets for arbitrary interaction patterns $J_{jk}$ & $h_j$:

$$H = \frac{1}{2} \sum_{j,k} J_{jk} s_j^x s_k^x + h_j s_j^z$$
EXTENSIONS (Cont...)

5 Higher spin sectors. All protocols currently proposed require that the signal propagates completely to Bob’s region before another can be sent. Intuitively it is clear that if Alice waits until signal has left her region she should be able to send another immediately. Proving this is slightly involved: we require 2nd quantised formalism. We have recently shown that a Heisenberg, XY chain can support $\geq N^2$ signals at once (as opposed to just one). A pleasing consequence of the 2nd quantised formalism analysis is that we have been able to extend our protocols to interacting Fermi & Bose gases & even local quantum field theories.
EVEN MORE EXTENSIONS...

6) Antiferromagnetic sector: Using a scattering theory argument we have developed a communication protocol to use strongly correlated ground states as a dynamic resource. (Recall a basic assumption of all communication protocols for spin systems use facts that "vacuum state" $|0\rangle = |00\cdots0\rangle$ is uncorrelated)
CONCLUSIONS & FUTURE DIRECTIONS

- We have shown that general locally interacting Heisenberg & XY systems are useful for communicating quantum information with only polynomial (\(\text{poly}(N)\)) overhead.

- We have argued the correct approach to this problem is to introduce some encoding (just like for fibre optic channels) to minimize dispersion errors.

Future Directions

- Errors (?) What is a good error model?
- How to encode against such errors? (Error filtration?)
- What about thermal errors? What is threshold for quantum communication?
- Are protocols robust?