Valuing Correlation-Dependent Derivatives Using a Structural Approach

John Hull

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Background

- Hull and White (2001): Structural model for jointly simulating defaults on several names
- Gregory and Laurent, Andersen and Sidenius, etc: Copula survival time model for CDOs
- Hull, Predescu and White: Revisit structural model

Hull and White (Journal of Derivatives, 2001)

- Jointly simulate “default indicator variables” for several companies
- Default occurs when barrier is hit
- Computationally very slow.

The One-Factor Gaussian Copula Model of Survival Time (Gregory and Laurent)

- N issuers
- The time to default of the i^{th} issuer is t_i
- The cumulative probability distribution of t_i is Q_i

Model (continued)

- Define x_i so that $N(x_i)=Q_i(t_i)$ and assume that
$$x_i = a_i M + \sqrt{1-a_i^2}\, Z_i$$
where M and the Z_i have independent standard normal distributions with the Z_i identically distributed
- The a_i's define a one-factor correlation structure between the x_i's ($\rho_{ij}=a_i a_j$)
- When quoting prices the market assumes that Q_i and a_i are the same for all issuers

Extensions

- More than one factor
- Other copulas (Clayton, Archimedian, double t…)
- Negative correlation between M and recovery rate
- Random a's possibly dependent on M

Advantages of Copula Survival time model

- Computationally fast
- Large number of issuers can be handled
- When Q_i and a_i are assumed to be the same for all issuers there is a one-to-one correspondence between prices and pairwise correlations

Disadvantages

- The realization of a single factor governs the default environment in all future time periods
- No model for evolution of credit spreads or correlations
- No underlying economic rationale
- No way of knowing what the copula correlations should be

A Factor-Based Structural Model

$$dV_i(t) = \mu_i V_i(t) + \sigma_i V_i(t) dX_i(t)$$

where $V_i(t)$ is the value of the assets of company i at time t

$$dX_i(t) = \alpha_i dF(t) + \sqrt{1-\alpha_i^2}\, dU_i(t)$$

where X_i, F, and U_i follow Wiener processes

with $X_i(0) = F_i(0) = U_i(0) = 0.$

It follows that

$$\ln V_i(t) - \ln V_i(0) = (\mu_i - \sigma_i^2/2)t + \sigma_i X_i(t)$$

with

$$X_i(t) = \alpha_i F(t) + \sqrt{1-\alpha_i^2}\, U_i(t)$$

Model continued

Company i defaults when $V_i < B_i$ for the first time

This is when $X_i < \beta_i + \gamma_i t$ for the first time where

$$\beta_i = \frac{\ln(B_i) - \ln(V_i(0))}{\sigma_i}$$

$$\gamma_i = -\frac{\mu_i - \sigma_i^2/2}{\sigma_i}$$

Examples

σ_i = 10% and μ_i=4%

- CDX IG: 5-year CDS spread = 60.53 bps

β_i= -3.02911 and γ_i=0.35

- iTraxx IG: 5-year CDS spread = 40.58 bps

β_i= -3.34575 and γ_i=0.35

CDS Spreads for Examples

Maturity (yrs)	CDS Spread for CDX (bps)	CDS Spread for iTraxx (bps)
1	4.80	1.44
2	30.08	14.94
3	48.36	28.41
4	57.16	36.47
5	60.53	40.54

Analytic results for Implementation of the Model

- For linear barrier, the probability of default in any interval conditional on the value of F at a particular time
- For linear barrier, the probability of default of default conditional on the value of F at a number of discrete points in time
- Extend results to a piecewise linear barrier
- We can extend results so that we condition on F and α at discrete points in time

Two Models

- Quasi Structural Model: When calculating cash flows between (t_1, t_2), condition on a value of F at time t_2 and then integrate over this value
- Full Structural Model: When calculating all cash flows, condition on a path for F (and α) and then integrate over the path

CDX Results For Quasi Structural Model and Gaussian Copula

Quasi Structural model

Range		Correlation						Average for Market
Start	End	0.0	0.1	0.2	0.3	0.4	0.5	
0%	3%	61.5	45.9	34.4	24.6	15.9	8.0	40.7
3%	7%	229.1	420.6	471.4	470.0	444.5	403.5	319.7
7%	10%	0.4	70.7	161.0	218.7	241.2	239.8	122.9
10%	15%	0.0	10.1	52.2	98.5	145.9	187.8	45.9
15%	30%	0.0	0.2	5.0	19.6	38.9	62.4	13.1

Gaussian Copula Model

Range		Correlation						Average for Market
Start	End	0.0	0.1	0.2	0.3	0.4	0.5	
0%	3%	61.5	47.5	38.0	30.0	22.9	16.5	40.7
3%	7%	229.1	389.1	424.3	423.8	407.2	381.1	319.7
7%	10%	0.4	70.1	141.4	184.1	208.2	221.2	122.9
10%	15%	0.0	11.8	48.4	85.1	115.4	134.4	45.9
15%	30%	0.0	0.4	5.4	17.2	32.3	50.2	13.1

iTraxx Results For Quasi Structural and Gaussian Copula Model

Quasi Structural model

Range		Correlation						Average for Market
Start	End	0.0	0.1	0.2	0.3	0.4	0.5	
0%	3%	40.2	30.8	22.3	14.8	8.0	1.8	26.9
3%	6%	63.9	227.5	299.0	318.6	308.8	280.1	160.5
6%	9%	0.1	35.1	98.1	148.9	182.6	196.4	67.9
9%	12%	0.0	5.2	34.9	67.8	98.2	132.0	42.2
12%	22%	0.0	0.3	5.6	20.5	40.5	59.1	19.7

Gaussian Copula Model

Range		Correlation						Average for Market
Start	End	0.0	0.1	0.2	0.3	0.4	0.5	
0%	3%	40.2	31.3	24.1	17.9	12.3	7.1	26.9
3%	6%	63.9	212.1	267.5	284.7	284.1	274.0	160.5
6%	9%	0.1	35.8	87.9	124.5	145.5	153.1	67.9
9%	12%	0.0	6.2	32.3	60.6	86.0	108.1	42.2
12%	22%	0.0	0.4	5.9	18.0	32.4	47.4	19.7

Full Structural Model Results And Gaussian Copula Results for CDX

Full Structural Model

Range		Correlation						Average for Market
Start	End	0.0	0.1	0.2	0.3	0.4	0.5	
0%	3%	61.5	48.1	38.9	31.2	24.4	18.1	40.7
3%	7%	229.1	382.9	421.6	423.7	410.3	388.4	319.7
7%	10%	0.4	64.3	134.3	178.7	204.7	218.5	122.9
10%	15%	0.0	10.0	43.6	79.2	108.2	129.6	45.9
15%	30%	0.0	0.3	4.5	14.8	29.2	45.8	13.1

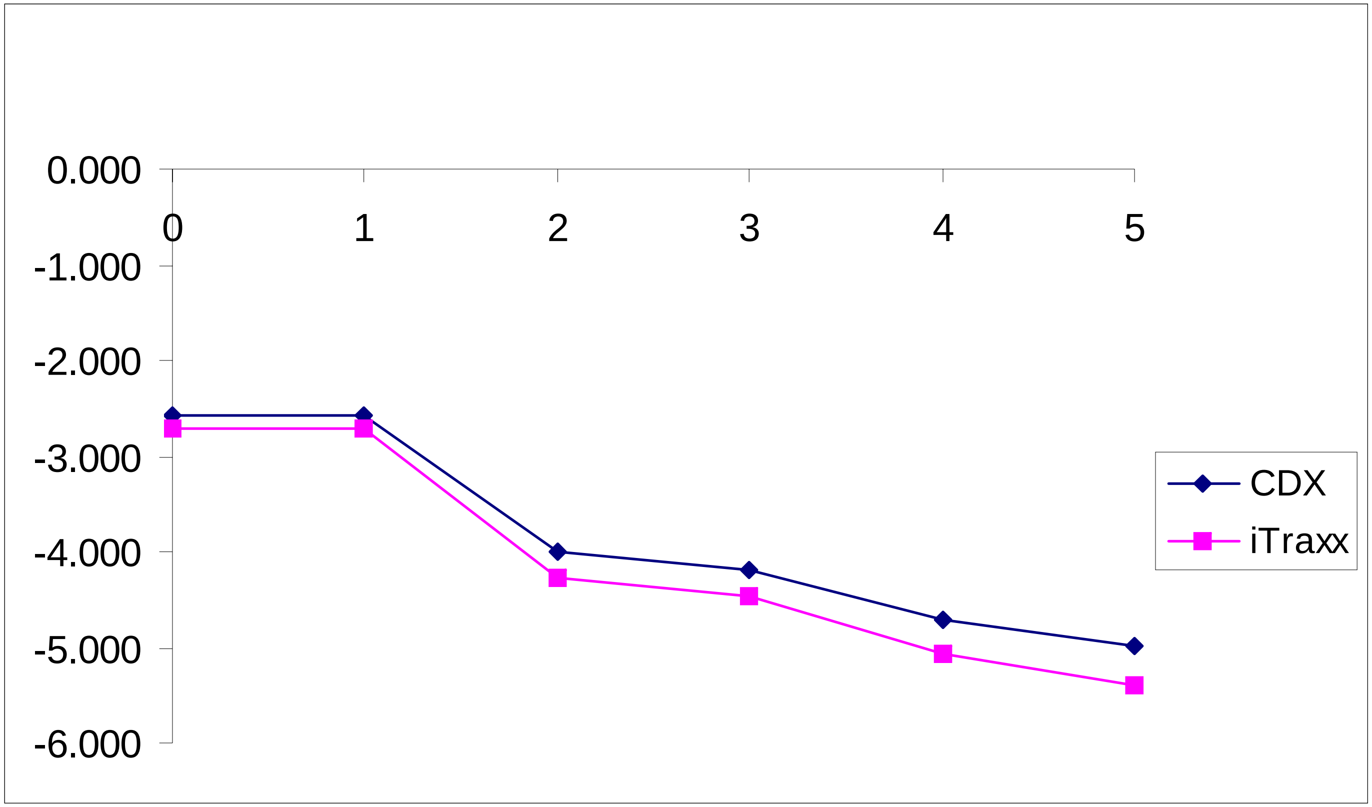
Gaussian Copula Model

Range		Correlation						Average for Market
Start	End	0.0	0.1	0.2	0.3	0.4	0.5	
0%	3%	61.5	47.5	38.0	30.0	22.9	16.5	40.7
3%	7%	229.1	389.1	424.3	423.8	407.2	381.1	319.7
7%	10%	0.4	70.1	141.4	184.1	208.2	221.2	122.9
10%	15%	0.0	11.8	48.4	85.1	115.4	134.4	45.9
15%	30%	0.0	0.4	5.4	17.2	32.3	50.2	13.1

Calibration

To calibrate to spreads in the market we use a numerical procedure to choose a piecewise linear barrier that is consistent with market data

Barrier when hazard rate is constant and consistent with average 5-year CDS for CDX and iTraxx



CDX Results for Constant Hazard Rate

Quasi Structural Model

Range		Correlation						Average for Market
Start	End	0.0	0.1	0.2	0.3	0.4	0.5	
0%	3%	65.1	49.3	36.8	26.2	17.0	8.7	40.7
3%	7%	224.3	432.8	505.1	508.1	478.2	430.4	319.7
7%	10%	0.4	61.9	157.8	231.6	262.4	257.4	122.9
10%	15%	0.0	7.6	47.0	95.0	145.8	197.1	45.9
15%	30%	0.0	0.1	3.9	17.3	37.6	61.8	13.1

Gaussian Copula Model

Range		Correlation						Average for Market
Start	End	0.0	0.1	0.2	0.3	0.4	0.5	
0%	3%	64.7	50.4	40.4	32.1	24.8	18.0	40.7
3%	7%	211.4	380.8	424.0	428.0	413.8	388.8	319.7
7%	10%	0.3	65.6	136.9	181.2	206.5	219.7	122.9
10%	15%	0.0	10.7	46.2	82.7	113.8	134.7	45.9
15%	30%	0.0	0.3	5.1	16.5	31.4	49.0	13.1

iTraxx Results for Constant Hazard Rate

Quasi Structural Model

Range		Correlation						Average for Market
Start	End	0.0	0.1	0.2	0.3	0.4	0.5	
0%	3%	43.0	33.9	24.9	16.7	9.4	2.8	26.9
3%	6%	59.5	223.4	315.0	345.0	334.7	302.0	160.5
6%	9%	0.1	28.2	92.7	150.3	194.9	210.7	67.9
9%	12%	0.0	3.4	29.5	67.3	95.5	133.5	42.2
12%	22%	0.0	0.1	4.2	17.7	38.8	61.2	19.7

Gaussian Copula Model

Range		Correlation						Average for Market
Start	End	0.0	0.1	0.2	0.3	0.4	0.5	
0%	3%	42.4	33.5	26.2	19.7	13.9	8.5	26.9
3%	6%	55.3	202.5	263.2	284.4	286.1	276.9	160.5
6%	9%	0.1	32.6	83.9	121.5	144.3	153.6	67.9
9%	12%	0.0	5.5	30.3	58.3	83.4	106.4	42.2
12%	22%	0.0	0.3	5.4	17.0	31.4	46.3	19.7

Taking Account of Uncertain Correlation

- One explanation for the market data is that correlation is uncertain
- As an approximation we can allow for uncertain correlation by
  - Choosing a number of different correlations
  - Assigning a probability to each one
  - Calculating a probability weighted PV of payoffs and payments
  - Both Gaussian copula and the structural models can fit market data well when correaltion distribution has two or three points.

Conclusions

- There is a quasi-analytic way of implementing the factor-based version of the structural model in Hull and White (2001)
- Model gives similar prices to Gaussian copula model when spreads are low
- May be useful alternative to Gaussian copula for CDOs when spreads are high
- May be useful in valuing more elaborate structures such as options on CDO tranches