A new approach to the modeling of default correlation

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1. Model specification

- N defaultable reference entities.
- Continuous time, finite state, Markov chain (ξ_t) with Q-matrix Q, generating a filtration F_t^ξ.
- Risk free discount factor

$$D(0,t) = e^{-\int_0^t r(\xi_u)du}. \tag{1}$$

- Default times τ^i,

$$q_t^i := P\left(\tau^i \geq t \mid F_t^\xi\right) := \exp\left(-\int_0^t \lambda^i(\xi_u)du\right). \tag{2}$$

.

$\Rightarrow$

$$Q_t^i(\xi_0) := P\left(\tau^i \geq t \mid \xi_0\right) = E\left[\exp\left(-\int_0^t \lambda^i(\xi_u)du\right)\right] \tag{3}$$

$$= \exp\left[(Q - \Lambda^i)t\right](\xi_0). \tag{4}$$

- This can be used to calibrate λ^i and Q to CDS quotes.

2. Default correlation

- The pairwise joint survival probability of name i and j is

$$\begin{aligned} Q_t^{ij}(\xi_0) &:= P(\tau^i \geq t, \tau^j \geq t \mid \xi_0) \\ &= \left(e^{(Q-\Lambda^i-\Lambda^j)t} \mathbf{1} \right)(\xi_0), \end{aligned}$$

- ...and the default correlation

$$\rho_T(\xi_t) = \frac{P_T^{ij}(\xi_t) - P_T^i(\xi_t) P_T^j(\xi_t)}{\sqrt{P_T^i(\xi_t)(1 - P_T^i(\xi_t))}\sqrt{P_T^j(\xi_t)(1 - P_T^j(\xi_t))}} \tag{5}$$

where

$$P_t^i(\xi_0) := P\left(\tau^i \leq t \mid \xi_0\right), \tag{6}$$

and

$$P_t^{ij}(\xi_0) := P(\tau^i \leq t, \tau^j \leq t \mid \xi_0). \tag{7}$$

3. Homogeneous, deterministic loss at default

- Assuming all losses at default are equal to 1,

$$L_t = \sum_{i=1}^{N} 1_{\{\tau^i \leq t\}} \tag{8}$$

- Conditional on F_t^{ξ}, L_t is a sum of independent $0-1$ random variables. By the Poisson limit theorem:

$$L_t \sim Po(\overline{\Lambda}_t), \tag{9}$$

where

$$\overline{\Lambda} := \sum_{i}^{N} (1 - q_t^i) = \sum_{i}^{N} \left(1 - e^{-\int_0^t \lambda^i(\xi_u)du}\right). \tag{10}$$

- Hence,

$$P(L_t \leq K) = \sum_{i=1}^{K} \frac{1}{i!} E\left[\overline{\Lambda}_t^i e^{-\overline{\Lambda}_t}\right]. \tag{11}$$

- Use Monte Carlo for MC to calculate (11).

- *Exact* solution

$$\overline{\phi}_t(u) = \frac{E\left[u^{L_t}\right]}{(1-u)}, \tag{12}$$

and

$$E\left[u^{L_t}\right] = E\left[\prod_{i=1}^{N}\left((1-q_t^i)u + q_t^i\right)\right]. \tag{13}$$

- N small - explicit expansion of the product in (13).
- N large - simulating the path of the chain works better.
- How to recover the loss distribution from $\overline{\phi}_t(u)$?
  1. Saddle point method,
  2. Numerical inversion of the pgf.

4. Name dependent stochastic recovery

- Assuming recovery rates R_i are mutually independent and independent of default times,

$$L_t = \sum_{i=1}^{N} l_i(1 - R_i)1_{\{\tau^i \le t\}}. \tag{14}$$

- Calculate the Laplace transform of the distribution

$$\overline{\varphi}_t(\alpha) := \int_0^\infty e^{-\alpha x} P(L_t \le x) dx \tag{15}$$

$$= \frac{E\left[e^{-\alpha L_t}\right]}{\alpha}, \tag{16}$$

where

$$E\left[e^{-\alpha L_t}\right] = E\left[\prod_{i=1}^{N}\left((1 - q_t^i)\zeta^i(\alpha) + q_t^i\right)\right], \tag{17}$$

and

$$\zeta^i(\alpha) = E\left[e^{-\alpha(1-R_i)l_i}\right]. \tag{18}$$

5. K^{th} to default basket

- What is a K^{th} to default basket?
- Default counting process

$$L_t := \sum_{i=1}^{N} 1_{\{L_t \leq t\}}. \tag{19}$$

- The risky **PV01**

$$PV01: \quad = \quad E\left[\sum_{i=1}^{M} \Delta_i D(0,T_i) 1_{\{\tau^K \geq T_i\}}\right] \tag{20}$$

$$= \quad \sum_{i=1}^{M} \Delta_i E\left[e^{-\int_0^{T_i} r(\xi_u)du} 1_{\{L_{T_i} \leq K-1\}}\right]. \tag{21}$$

- Using the Poisson approximation

$\rightarrow$

$$PV01 \approx \sum_{i=1}^{M} \sum_{j=1}^{K-1} \Delta_i E\left[e^{-\int_0^{T_i} r(\xi_u)du} \overline{\Lambda}_{T_i}^{j} e^{-\overline{\Lambda}_{T_i}}\right]. \tag{22}$$

- The **default leg** is given by

$$
\begin{aligned}
DL &:= -E\left[\int_0^T (1-R)D(0,u)d1_{\{\tau^K>u\}}\right] && (23)\\
&= (R-1)\int_0^T E\left[D(0,u)d1_{\{L_u\leq K-1\}}\right] && (24)\\
&= (R-1)E\left[D(0,T)1_{\{L_T\leq K-1\}}\right] && (25)\\
&+ (R-1)\int_0^T E\left[r(\xi_u)D(0,u)1_{\{L_u\leq K-1\}}\right]du. && (26)
\end{aligned}
$$

- The par-spread S is equal to

$$
S = \frac{DL}{PV01}. \tag{27}
$$

6. Synthetic CDOs

- What is a CDO?.
- The tranche **PV01** is equal to

$$PV01 = \sum_{j=1}^{M} \Delta_i E\left[\Phi(L_{T_j}) e^{-\int_0^{T_j} r_u du}\right], \tag{28}$$

where

$$\Phi(x) = \frac{1}{L^+ - L^-}\left[\left(L^+ - x\right)^+ - \left(L^- - x\right)^+\right]. \tag{29}$$

- PV01 as a portfolio of puts $P_t(K)$ on L_t.

$$P_t(K) \quad := \quad E\left[(K - L_t)^+ D(0,t)\right]. \tag{30}$$

- In the case of constant, homogenous recovery, the Poisson approximation yields

$$P_t(K) = \sum_{i=0}^{K} \frac{(K-i)}{i!} E\left[e^{-\int_0^t r(\xi_u)du} e^{-\overline{\Lambda}_t} \overline{\Lambda}_t^i\right]. \tag{31}$$

- For stochastic recovery...

$$\overline{\varphi_t}(\alpha) := \int_0^\infty e^{-\alpha x} E\left[L_t 1_{\{L_t \le x\}} e^{-\int_0^t r(\xi_u)du}\right] dx \quad (32)$$

$$= \frac{1}{\alpha}\varphi_t(\alpha), \quad (33)$$

where

$$\varphi_t(\alpha) = E\left[L_t e^{-\alpha L_t} e^{-\int_0^t r(\xi_u)du}\right] \quad (34)$$

$$= \sum_{j=1}^N L_j E\left[\frac{p_t^j \zeta^j(\alpha) e^{-\int_0^t r(\xi_u)du}}{p_t^j \zeta^j(\alpha) + 1 - p_t^j} \prod_{i=1}^N \left(p_t^i \zeta^i(\alpha) + 1 - p_t^i\right)\right]. \quad (35)$$

- Summing things up,...

$$PL = s \sum_{j=1}^M \frac{\Delta_j}{L^+ - L^-}\left[P_{T_j}(L^+) - P_{T_j}(L^-)\right]. \quad (36)$$

- The **default leg** equals expected present value of the tranche's losses

$$DL = E\left[\int_0^T e^{-\int_0^u r_u du} d\Xi(L_u)\right], \tag{37}$$

where $\Xi(x) = 1 - \Phi(x)$.

- Integrating by parts we can simplify (37) to

$$DL = -E\left[e^{-\int_0^T r(\xi_s)ds}\Phi(L_T)\right] - E\left[\int_0^T r(\xi_u)e^{-\int_0^u r(\xi_s)ds}\Phi(L_u)du\right]. \tag{38}$$

- **Remark**: Independent riskless rate???

7. Numerical results

Figure 1: **Accuracy of the Poisson approximation**, N=5, T=1

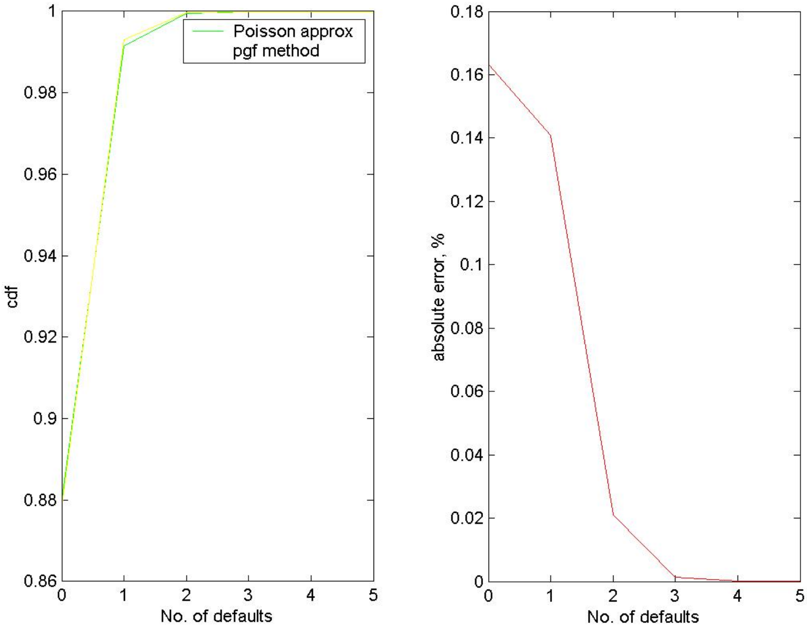

Figure 2: **Accuracy of the Poisson approximation**, N=15, T=1

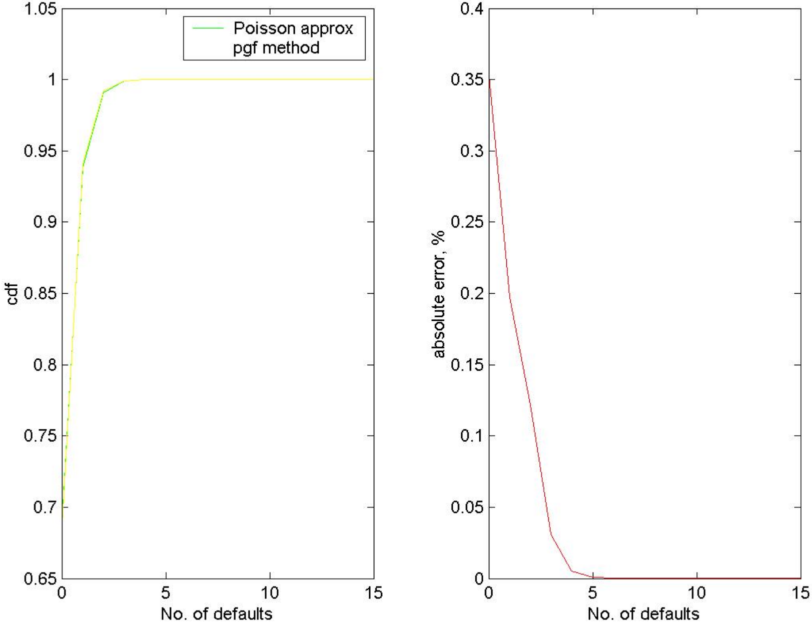

Figure 3: **Accuracy of the Poisson approximation**, N=30, T=1

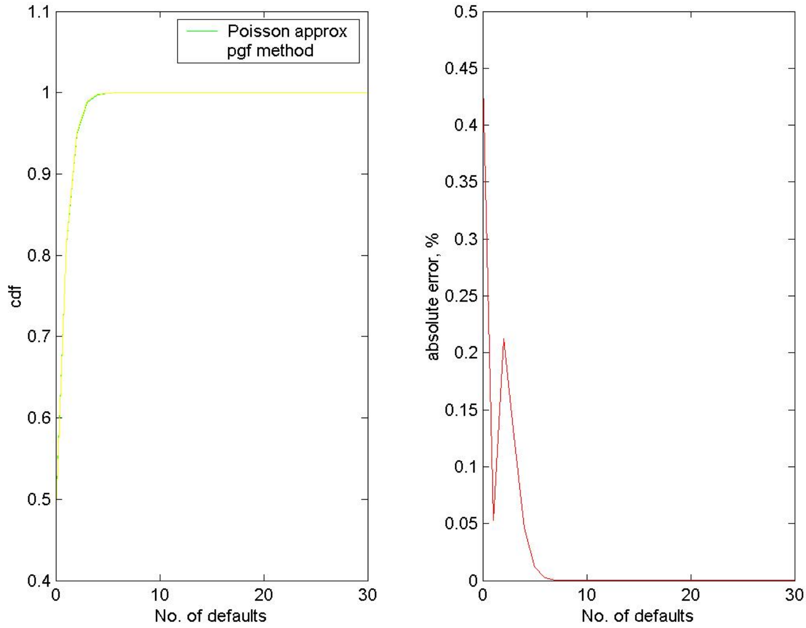

Figure 4: **Number of defaults density**, N=40, T=1, low λ^i

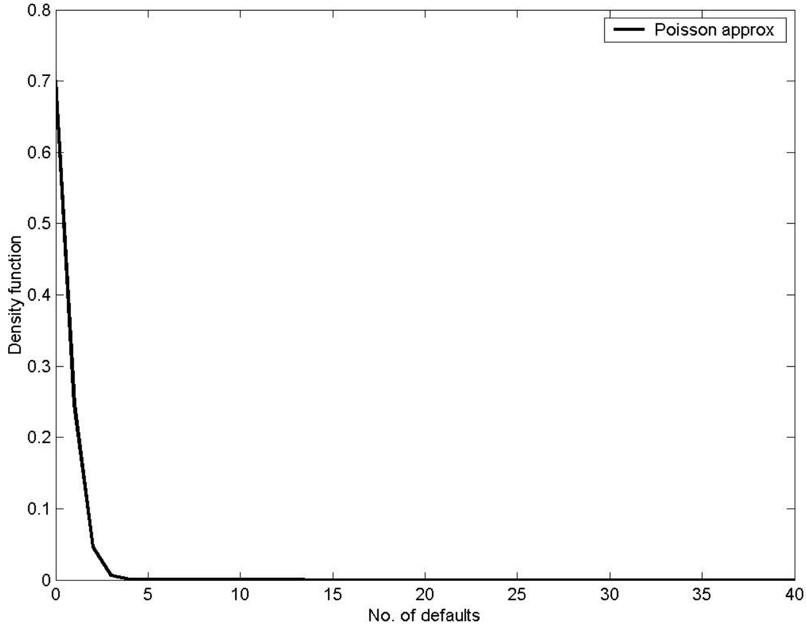

Figure 5: **Number of defaults density**, N=40, T=1, mid λ^i

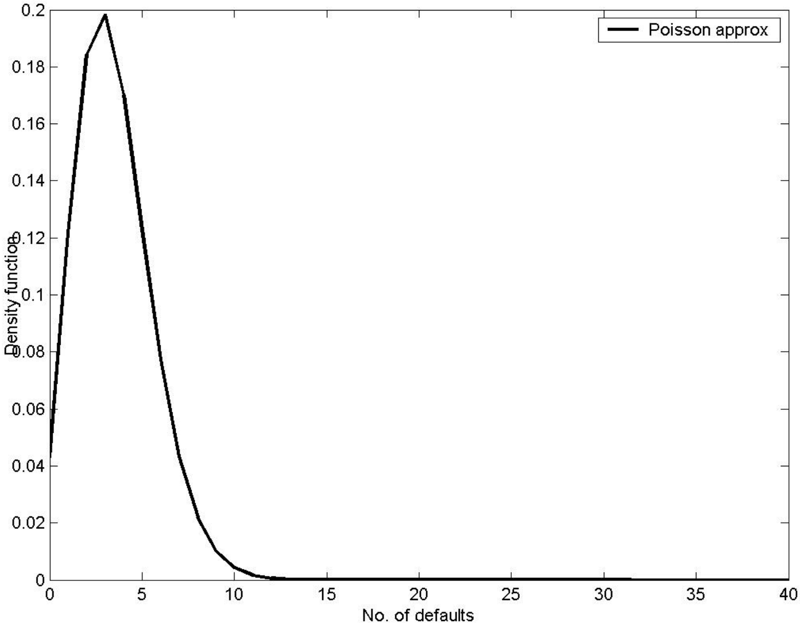

Figure 6: **Number of defaults density**, N=40, T=1, high λ^i

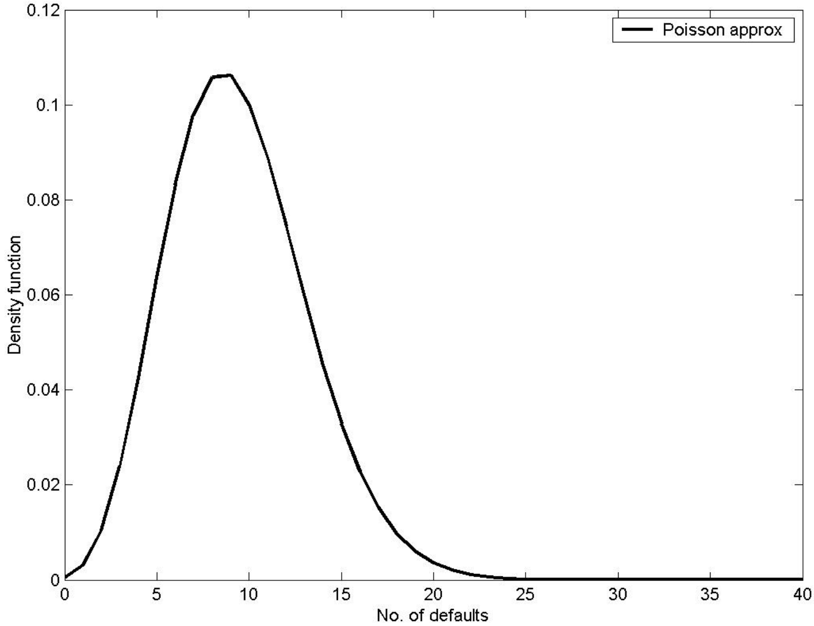

Figure 7: **Portfolio default density as a function of the chain's initial state,** N=40, T=1

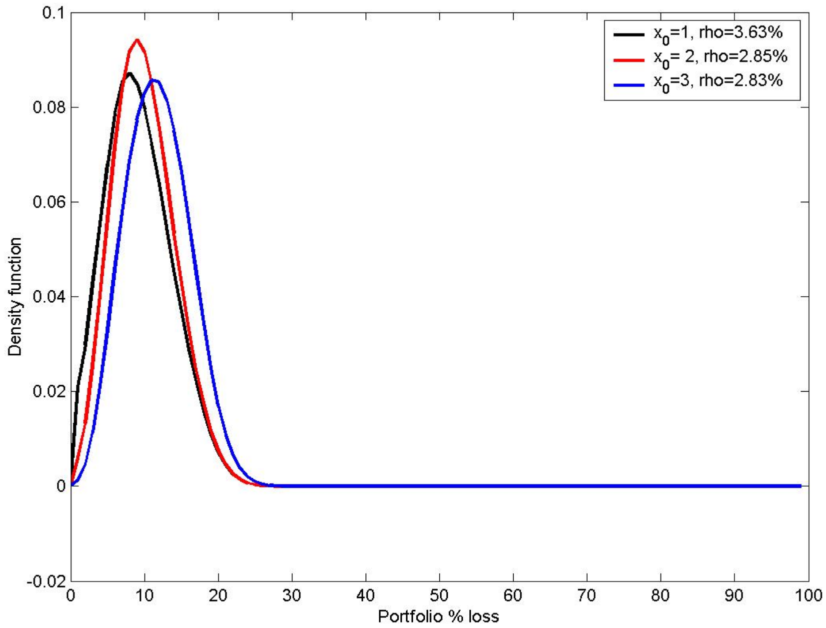

Figure 8: **Portfolio default density as a function of correlation**, N=100, T=1, $\xi = 1$