Mortgage Valuation and Optimal Refinancing:
An Intensity Based, Equilibrium Approach with Endogenous Mortgage Rates

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Mortgages in the U.S. are Big Business

- Approximate number of mortgages: 50,000,000
- Approximate average initial principal: $200,000
- Approximate total value: $10,000,000,000,000
- Prepayment time is analogous to the default time of a security
- Defaultable securities have received considerable research attention recently
- So why not more mathematical research applied to mortgages?

Valuation of Mortgage Contracts: Expected Present Value of the Cash Flow

Two main variations in the literature:

- Empirical, reduced form approach – build a model of the cash flow, with the prepayment time a specified random variable, possibly depending on some economic factors. Then use Monte Carlo simulation to estimate the expectation of the discounted cash flow.
- Option based, structural approach – the decision to refinance is viewed like an American option holder's decision to exercise early, and so the mortgage contract is valued like an American option.

Selected Literature (Empirical, Reduced Form Approach

- E. Schwartz and W. Torous, Prepayment and the valuation of mortgage backed securities, 1989.
- E. Schwartz and W. Torous, Prepayment, default, and the valuation of mortgage pass-through securities, 1992.
- Y. Deng, Mortgage termination: an empirical hazard model with a stochastic term structure, 1997.
- Y. Deng, J. Quigley, and R. Van Order, Mortgage termination, heterogeneity, and the exercise of mortgage options, 2000.
- T. Kariya and M. Kobayashi, Pricing mortgage backed securities, 2000.
- T. Kariya, S. Pliska, and F. Ushiyama, A 3-factor valuation model for mortgage backed securities, 2002.
- J. Kau, D. Keenan, and A. Smurov, Reduced form mortgage valuation, 2004.

Selected Literature (Option Based, Structural Approach)

- K. Dunn and J. McConnell, A comparison of alternative models for pricing GNMA mortgage-backed securities, 1981.
- K. Dunn and J. McConnell, Valuation of mortgage-backed securities, 1981.
- R. Stanton, Rational prepayment and the valuation of mortgage-backed securities, 1995.
- H. Nakagawa and T. Shouda, Valuation of mortgage-backed securities based on unobservable prepayment cost processes, 2004.
- A. Kalotay, D. Yang, and F. Fabozzi, An option-theoretic prepayment model for mortgages and mortgage-backed securities, 2004.

Typical Implementation (Kalotay, Yang, and Fabozzi)

1. Using a "bullet" mortgage par yield curve and volatility, build an interest rate lattice.

2. Starting with the final scheduled cash flows of the mortgage, the value of the mortgage is computed for each node in the lattice, working backwards like with American options.

3. At each node the value of the existing mortgage is compared with the value of a newly refinanced mortgage (assumed to be par plus the refinancing cost); if the latter is less, then the value of the existing mortgage is replaced by the value of the new one.

Option Based Approach with Endogenous Mortgage Rates

Here discrete time, finite horizon models are considered, with time equal to the age of the contract. The mortgage rates are solved recursively, so they depend upon the age of the mortgage contract.

- R. Stanton and N. Wallace, Mortgage choice: what's the point?, 1997.
- K. Dunn and C. Spatt, The effect of refinancing costs and market imperfections on the optimal call strategy and the pricing of debt contracts, 1999.
- F. Longstaff, Optimal recursive refinancing and the valuation of mortgage-backed securities, 2002.

Remarks About the Option-Based Literature

- Models and assumptions often unclear, with limited use of dynamic programming or other mathematics.
- The analogy to an American option is flawed, because while this might be reasonable from the standpoint of the lender, the borrower has the opportunity to make **multiple** decisions.
- The borrower has a stochastic control problem, not simply an optimal stopping problem.
- All the models in the literature are fundamentally different from the one developed here.

The Main Ideas of This Paper

- Work with and by Yevgeny Goncharov (former Ph.D. student) used an intensity-based approach to value mortgage contracts.
- An intensity process models the mortgagor's exogenously-specified prepayment behavior.
- If the mortgagor is a representative agent in a competitive market, then mortgage rates are endogenous.
- Here we focus on the complementary problem: given exogenous mortgage rates, what is the mortgagor's optimal refinancing behavior?
- We then proceed to consider the equilibrium problem: the mortgagor acts optimally and the mortgage rates are endogenous.
- All done in a discrete-time framework.

Mortgage Theory 101: Some Notation

N = number of payments per contract

n = number of remaining payments $(= 0, 1, \ldots, N)$

m = mortgage rate at contract initiation

$c = c(m)$ = coupon payment (constant, arrears)

$P(n, m)$ = principal balance with n remaining payments

Some Basic Equations – Time Value of Money

$$P(n-1,m) = (1+m)P(n,m) - c$$

$$P(n,m) = \frac{1}{1+m}c + \frac{1}{1+m}P(n-1,m)$$
$$= \frac{1}{1+m}c + \frac{1}{1+m}\left[\frac{1}{1+m}c + \frac{1}{1+m}P(n-2,m)\right]$$

$$= \ldots$$

$$= \left[\frac{1}{1+m} + \left(\frac{1}{1+m}\right)^2 + \ldots + \left(\frac{1}{1+m}\right)^k\right]c + \left(\frac{1}{1+m}\right)^k P(n-k,m)$$
$$= \frac{c}{m}\left[1-(1+m)^{-k}\right] + \left(\frac{1}{1+m}\right)^k P(n-k,m)$$

Fully Amortized Mortgages*: $P(0, m) = 0$

$$P(n, m) = \frac{c}{m}\left[1 - (1+m)^{-n}\right]$$

$$\Longleftrightarrow c = \frac{mP(n, m)}{1 - (1+m)^{-n}}$$

In particular, since $P(N, m)$ is the initial principal, the contracted coupon payment is:

$$c = \frac{mP(N, m)}{1 - (1+m)^{-N}}$$

*assumed from now on

More Notation

r_t = one-period riskless short rate from $t-1$ to t

$r = \{r_t; t = 1, 2, \ldots\}$ = short rate process

M_t = mortgage rate for N-period mortgages contracted at t

$M = \{M_t; t = 0, 1, \ldots\}$ = mortgage rate process

$K = K(P)$ = transaction cost for refinancing

The Mortgagor's Problem: Given the stochastic interest rate process (r, M), choose the refinancing schedule that provides the best cash flow.

Assumptions: The mortgagor lives forever, does not default, and intends to eventually pay off the mortgage. Moreover, only contracted amounts can be paid, but the mortgage can be refinanced whenever desired and as many times as desired.

Modeling Approach: Markov Decision Chain

Assumption: (r, M) is a time-homogeneous Markov chain.

State Variables:

- (r_t, M_t) = state of Markov chain
- m = contracted mortgage rate
- n = number remaining contracted payments
- P = current principal balance

Computational Difficulty: too many state variables

Conceptual Difficulty: measure of performance?

Solution: we digress

Mortgage Theory 102: Default-Free Prepayment Model

$(\Omega, \mathcal{F}, Q, \mathcal{F}_t)$ = risk-neutral probability space

τ = prepayment time, $\tau = 1, 2, \ldots, N-1, N$

Prepayment intensity process:

$$\gamma = \{\gamma_t; t = 1, 2, \ldots, N-1\}$$

$$\gamma_t = Q(\tau = t | \tau \geq t, \mathcal{F}_{t-1})$$

V = initial value (for bank) of mortgage = ?

Mortgage Value: Risk Neutral Valuation

$$V = E_{r_1}\Big[\frac{c}{1+r_1} + \frac{c}{(1+r_1)(1+r_2)} \quad + \quad \ldots$$

$$\ldots \quad + \quad \frac{c}{(1+r_1)\ldots(1+r_{\tau\wedge N})} + \frac{P_{\tau\wedge N}}{(1+r_1)\ldots(1+r_{\tau\wedge N})}\Big]$$

$$= E_{r_1}\Big[\frac{c}{1+r_1} + \frac{c(1-\gamma_1)}{(1+r_1)(1+r_2)} + \ldots + \frac{c(1-\gamma_1)\ldots(1-\gamma_{N-1})}{(1+r_1)\ldots(1+r_N)} \quad +$$

$$\frac{P(N-1,m)\gamma_1}{1+r_1} + \frac{P(N-2,m)(1-\gamma_1)\gamma_2}{(1+r_1)(1+r_2)} + \ldots + \frac{P(1,m)(1-\gamma_1)\ldots(1-\gamma_{N-1})\gamma_N}{(1+r_1)\ldots(1+r_N)}\Big]$$

$$= E_{r_1}\Big[\sum_{i=1}^{N} \frac{c+\gamma_i P(N-i,m)}{1+r_1} \prod_{j=2}^{i} \frac{1-\gamma_{j-1}}{1+r_j}\Big], \qquad \text{where} \quad \prod_{j=2}^{1} \equiv 1$$

Mortgage Value: A Better Formula

Using the basic equation $c = (1+m)P(n,m) - P(n-1,m)$:

$$V = \frac{1}{1+r_1} E_{r_1}\Big[(1+m)P(N,m) - P(N-1,m) + \gamma_1 P(N-1,m)$$

$$+ \sum_{i=2}^{N-1} \Big((1+m)P(N-i+1,m) - P(N-i,m) + \gamma_i P(N-i,m)\Big) \prod_{j=2}^{i} \frac{1-\gamma_{j-1}}{1+r_j}$$

$$+ \quad \Big((1+m)P(1,m) - P(0,m)\Big) \prod_{j=2}^{N} \frac{1-\gamma_{j-1}}{1+r_j}\Big]$$

$$= \quad \ldots \quad \text{lots of algebra} \quad \ldots$$

$$= \frac{1+m}{1+r_1} P(N,m) + \frac{1}{1+r_1} E_{r_1}\Big[\sum_{i=1}^{N-1} (m - r_{i+1}) P(N-i,m) \prod_{j=1}^{i} \frac{1-\gamma_j}{1+r_{j+1}}\Big]$$

Note: It is like a swap!

ENDOGENOUS MORTGAGE RATES

By competitive market considerations:

$$V = P(N, m),$$

so this gives us an equation for m:

$$0 = (m - r_1)P(N, m) + E_{r_1}\Big[\sum_{i=1}^{N-1}(m - r_{i+1})P(N - i, m)\prod_{j=1}^{i}\frac{1 - \gamma_j}{1 + r_{j+1}}\Big]$$

- This is a non-linear equation in m (with $P(N, m)$ factored out).
- We can suppose $m = m(r_1)$ for some strictly positive function $m(\cdot)$.
- Note that the solution m will be some weighted average of future r_i's.

Back to the Markov Decision Chain

Assumption: r = time homogeneous Markov chain and $M_{t-1} = m(r_t)$ for some specified function $m(\cdot)$

State variables:

- r = short rate
- m = contracted mortgage rate
- n = remaining contracted payments
- P = principal balance

Measure of performance: value of cash flow to mortgagor

$v(n, P, m, r)$ = minimum expected discounted value of the cash flow

Deriving the Dynamic Programming Equation

If the mortgagor continues when n payments remain:

$$v(n, P, m, r) = \frac{1}{1+r}[c + E_r v(n-1, P(n-1, m), m, R)]$$

$$= \frac{1}{1+r}\left[\frac{mP}{1-(1+m)^{-n}} + E_r v\Big(n-1, \frac{(1+m)^n - 1 - m}{(1+m)^n - 1} P, m, R\Big)\right] \equiv A$$

If the mortgagor refinances when n payments remain:

$$v(n, P, m, r) = K(P) + v(N, P, m(r), r) \equiv B$$

Hence

$$v(n, P, m, r) = \min\{A, B\}$$

We still have computational difficulties:

- Still too many state variables
- P is continuous

Simplifying Assumption: $K(P) = kP$

Consequence:

$$v(n, P, m, r) = Pf(n, m, r),$$

where the function f is the solution of the new dynamic programming equation

$$f(n, m, r) = \min \Big\{ k + f(N, m(r), r),$$

$$\frac{1+m}{(1+r)[(1+m)^n - 1]} \Big[m(1+m)^{n-1} + [(1+m)^{n-1} - 1] E_r f(n-1, m, R) \Big] \Big\}$$

Note this is not a standard kind of dynamic programming equation, and so solution existence, uniqueness, etc., are in doubt.

DP Solution Existence, Uniqueness, and Computation

Since $f(N, m(r), r) \neq k + f(N, m(r), r)$:

$$f(n,m,r) = \min\Big[\frac{m(1+m)^n}{(1+r)[(1+m)^n - 1} + \frac{(1+m)[(1+m)^{n-1} - 1]}{(1+r)[(1+m)^n - 1]} E_r f(n-1, m, R),$$

$$k + \frac{m(r)[1+m(r)]^N}{(1+r)[(1+m(r))^N - 1]} + \frac{(1+m(r))[(1+m(r))^{N-1} - 1]}{(1+r)[(1+m(r))^N - 1]} E_r f(N-1, m(r), R)\Big]$$

Since for all $n \geq 1$, all r, and all m:

$$\frac{(1+m)[(1+m)^{n-1} - 1]}{(1+r)[(1+m)^n - 1]} < \frac{1}{1+r} < \frac{1}{1+r_{\min}} < 1,$$

we see that the "optimal return operator" is a contraction, which means:

- The DP equation always has a solution, which is unique
- The "Successive Approximations" algorithm converges to a solution
- The "Policy Improvement" algorithm converges to a solution
- The DP solution gives the optimal refinancing strategy

Success: We Have a Decent Model

Shortcoming: How do we choose the mortgage rate function $m(\cdot)$?

Two Answers:

- Maybe a statistical procedure (regression?) would work.
- We can seek to obtain the mortgage rate function by solving an **economic equilibrium problem**.

The Economic Equilibrium Problem

The basic ideas:

- The mortgagor is a representative agent who, given the mortgage rates, refinances in an optimal fashion.
- Given the mortgagor's prepayment behavior, the mortgage rates are set endogenously in accordance with a competitive market.
- An economic equilibrium occurs when both requirements are simultaneously satisfied.
- To obtain a solution, start with an arbitrary mortgage rate function $m(\cdot)$, solve the DP for the optimal refinancing time τ, solve for the corresponding, competitive mortgage rate function, solve the DP again for a (possibly) new τ, solve for (possibly) new mortgage rates, etc.
- We hope this converges.

Example: r is a 4-State Markov Chain

$N = 5$ (think periods = years)

$k = .03$ (3% refinancing fee)

States: r = 2%, 3%, 4%, and 5%

Markov chain transition matrix:

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Note: The Markov decision chain has $4 \times 4 \times 5 = 80$ states.

Initial Value of a Mortgage Held to Term

$$V = cE_{r_1}\Big[\sum_{n=1}^{N} \frac{1}{(1+r_1)\dots(1+r_n)}\Big]$$

$$= \frac{mP(N,m)}{1-(1+m)^{-N}} E_{r_1}\Big[\sum_{n=1}^{N} \frac{1}{(1+r_1)\dots(1+r_n)}\Big]$$

In a competitive market one has $V = P(N,m)$ so

$$\frac{1-(1+m)^{-N}}{m} = E_{r_1}\Big[\sum_{n=1}^{N} \frac{1}{(1+r_1)\dots(1+r_n)}\Big]$$

Solving for $m = m(r_1)$:

Short rate r_1	2%	3%	4%	5%
Mortgage rate $m(r_1)$	2.4733%	3.0773%	3.9061%	4.5201%

Optimal Refinancing States

n	r	m	Original r	Possible?
5	3%	4.5201%	5%	no
4	3%	4.5201%	5%	no
5	2%	4.5201%	5%	no
4	2%	4.5201%	5%	no
3	2%	4.5201%	5%	no
2	2%	4.5201%	5%	YES
5	2%	3.9061%	4%	no
4	2%	3.9061%	4%	no
3	2%	3.9061%	4%	YES

Iteration #1

The optimal refinancing strategy for iteration #1:

- Starting with $r = 5\%$, refinance if and only if $r = 2\%$ at end of the third period
- Starting with $r = 4\%$, refinance if and only if $r = 2\%$ at end of the second period
- Starting with $r = 3\%$ or $r = 2\%$, do not refinance.

Next step:

- Using this refinancing strategy, solve for the endogenous mortgage rates.

Mortgage Value if Refinanced at τ

$$V = cE_{r_1}\Big[\sum_{n=1}^{\tau\wedge N}\frac{1}{(1+r_1)\dots(1+r_n)}\Big] + E_{r_1}\Big[\frac{P_{\tau\wedge N}}{(1+r_1)\dots(1+r_{\tau\wedge N})}\Big]$$

$$= \frac{mP(N,m)}{1-(1+m)^{-N}}E_{r_1}\Big[\frac{1-(1+m)^{-(N-\tau)}}{(1+r_1)\dots(1+r_{\tau\wedge N})}m^{-1}+\sum_{n=1}^{\tau\wedge N}\frac{1}{(1+r_1)\dots(1+r_n)}\Big]$$

In a competitive market $V = P(N,m)$, so $m = m(r_1)$ satisfies:

$$1 = \frac{m}{1-(1+m)^{-N}}E_{r_1}\Big[\frac{1-(1+m)^{-(N-\tau)}}{(1+r_1)\dots(1+r_{\tau\wedge N})}m^{-1}+\sum_{n=1}^{\tau\wedge N}\frac{1}{(1+r_1)\dots(1+r_n)}\Big]$$

Short rate r_1	2%	3%	4%	5%
Iteration #1 $m(r_1)$	2.4733%	3.0773%	3.9061%	4.5201%
Iteration #2 $m(r_1)$	2.4733%	3.0773%	3.9820%	4.5465%

Iteration #2

Next Step:

- Using these new mortgage rates, solve for the optimal refinancing strategy

The result:

- Same optimal refinancing strategy as before.

Conclusion:

- We converged to an equilibrium solution.

Short rate r_1	2%	3%	4%	5%
Optimal value $f(5, m(r), r)$	1.00000	1.00000	1.00194	1.00063

Remarks About the Equilibrium Problem

- The optimal value exceeds one by the expectation of the discounted transaction cost(s)
- The mortgagors can end up with an optimal value strictly greater than one because they are greedy and myopic.
- This economic market is like a game (mortgagors versus the market):
    1. The market sets the mortgage rates.
    2. The mortgagors choose optimal refinancing strategies.
    3. The markets adjust due to competitive forces.
    4. The mortgagors revise their refinancing strategies.
    5. Etc.

Existence of Equilibrium Solution

Idea: Solve for equilibrium solution all at once with a **New Markov Decision Chain**, a MDC which is known to have a solution.

The Old MDC:

- Time periods: the natural unit of time
- States: (n, m, r)
- One-period actions: either refinance or continue
- One-period costs: k if you refinance; a function of (n, m, r) if you continue
- Markov transition probabilities: based on dynamics of r

The New Markov Decision Chain

- Time periods: one refinance cycle
- States: all values of the short rate r at contract initiation
- One-period actions: all stopping times $\tau \leq N$
- One-period costs: the expected discounted value of the Old MDC's one-period costs up through time τ
- Markov transition probabilities:

* Given (r, τ), one knows the probability distribution of the next state r

* The new transition probabilities will include the discount factors from the old MDC

* The resulting Markov transition matrices will be strictly substochastic

Some Economic Conclusions

Given the optimizing behavior of mortgagors and the dynamics of riskless interest rates, the mortgage rates are endogenous.

The equilibrium concept implies greedy, myopic mortgagors.

- In the short run they choose refinancing strategies that minimize the cost of the loan.
- In the long run they might cause mortgage rates to change, thereby ending up worse off.

Mortgagors should not be so eager to refinance, just because mortgage rates have dropped. Sure, the periodic payments will be smaller, but:

- The payments will be stretched over more periods.
- By waiting the mortgage rate might drop even more.
- If mortgage rates declined, then the short rate probably dropped too, which means discount factors changed adversely.

Some Future Research Topics

Compute some realistic solutions (360 months, ...)

Does the naïve algorithm always converge?

Incorporate exogenous reasons for mortgage prepayment.

- Job transfer
- Retirement
- Etc.

Allow the mortgagor to pay more than the contracted amount.

What about income taxes?

Continuous time models?