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Comparisons of P-densities obtained from historical asset prices, option prices and risk transformations

Presented by Stephen Taylor

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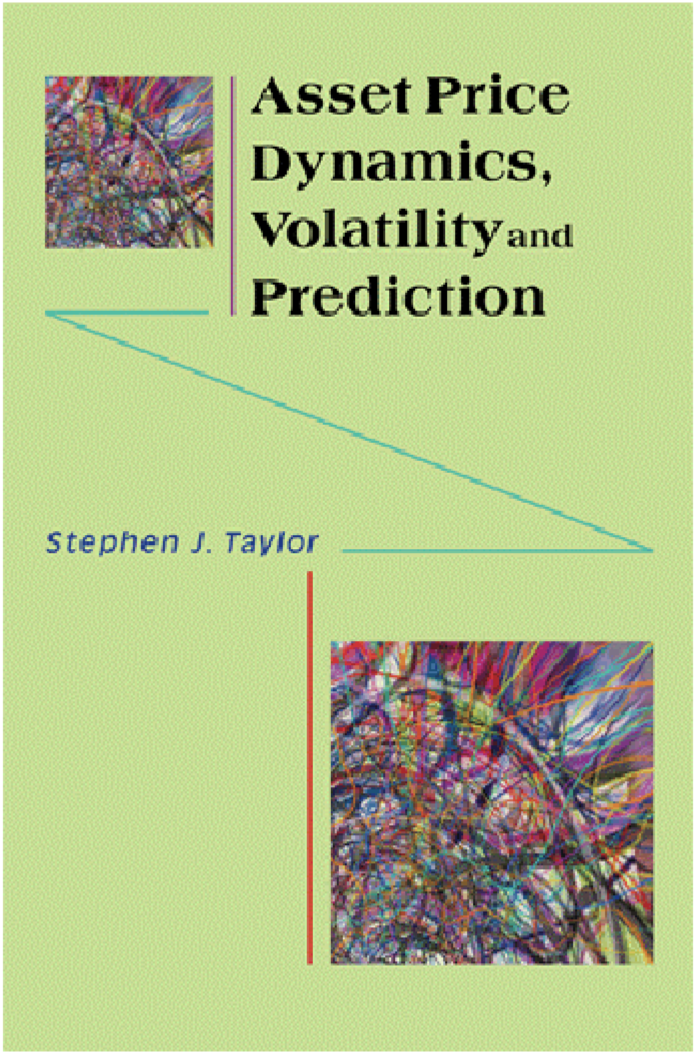

Available August/September 2005

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Or

Use Google to search for *Asset Price Dynamics*

P & Q

Q refers to probabilities, densities, etc., for a *risk-neutral* economy.

P refers to the *real-world*.

Some people use other adjectives: true, objective, subjective, physical, historical.

As we see later, we can interconnect:

a P-density,

a Q-density,

a representative utility function.

Context

- Option prices certainly contain useful information about future volatility.

- Sets of option prices, for the same expiry time T and for different strikes X, provide implied densities.

- Can these *risk-neutral* densities be transformed to provide *real-world* densities, that are more informative than a time series of asset prices?

- Related research by Bliss & Panigirtzoglou and by Anagnou, Bedendo, Hodges & Tompkins.

Today's talk covers:

- Two papers
    - One is "under revision", the other is preliminary
- Notation
- Theory
    - Risk-neutral densities
    - Transformations to real-world densities
    - Likelihood methods
- Data, initially U.K. and later U.S.
- Empirical results
    - Moments of densities
    - Calibration assessments
    - Likelihoods and incremental information
    - Measures of risk aversion

Study 1

Closed-form transformations from risk-neutral to real-world distributions

By: Liu, Shackleton, Taylor & Xu

PDF: at www.ssrn.com,
via http://ssrn.com/author=44638

<u>Contributions</u>

1. Closed-form real-world densities.
2. Likelihood methodology.

Applied to:

- Density estimation for the FTSE-100 index.
- With a four-week horizon.

Selected notation

S	The spot price now.
F	The futures price now.
T	The time until futures and options contracts cease trading.
$S_T = F_T$	Prices when trading of derivatives ends.
r	Risk-free interest rate.
X	Exercise (strike) price of an option.
$c(X)$	Price of a European call option.
σ	Volatility.
g	Risk-neutral density for S_T.
$\tilde{g}$	Real-world density for S_T.
G, $\tilde{G}$	Cumulative distribution functions.

Risk-neutral densities

A finite set of observed market prices $c_{market}(X_j)$ can not define a unique RND.

We estimate flexible densities using the spline method of Bliss & Panigirtzoglou.

More importantly, we estimate parametric RNDs from

$$\min_{\theta} \frac{1}{N} \sum_{j=1}^{N} (c_{market}(X_j) - c(X_j \mid \theta))^2 ,$$

with

$$c(X_j \mid \theta) = e^{-rT} \int_{X_j}^{\infty} (x - X_j) g(x \mid \theta) dx .$$

We consider two parametric families of RNDs.

A mixture of two lognormals

$$g_{MLN}(x|\theta) = w g_{LN}(x \mid F_1, \sigma_1, T) + (1-w) g_{LN}(x \mid F_2, \sigma_2, T)$$

with

$$g_{LN}(x \mid F, \sigma, T) = \frac{1}{x\sigma\sqrt{2\pi T}} \exp[-\tfrac{1}{2}(\frac{\log(x) - [\log(F) - \frac{1}{2}\sigma^2 T]}{\sigma\sqrt{T}})^2]$$

Parameter vector $\theta = (F_1, F_2, \sigma_1, \sigma_2, w)$.

Risk-neutral constraint : $wF_1 + (1-w)F_2 = F$.

Generalised beta (GB2)

$$g_{GB2}(x \mid a, b, p, q) = \frac{a}{b^{ap} B(p,q)} \frac{x^{ap-1}}{\left[1 + (\frac{x}{b})^a\right]^{p+q}}$$

Parameter vector $\theta = (a, b, p, q)$.

Risk-neutral constraint : $F = bB(p + a^{-1}, q - a^{-1}) \big/ B(p,q)$.

Two transformations to real-world densities (RWDs)

1. <u>CRRA utility functions</u>

The stochastic discount factor (pricing kernel) is

$$M(x) = e^{-rT} \frac{g(x)}{\tilde{g}(x)} .$$

When *M* is proportional to marginal utility, the RWD

for a power utility function, $\upsilon(x) \propto x^{1-\gamma}$, is

$$\tilde{g}(x) = \frac{x^{\gamma} g(x)}{\int_0^{\infty} y^{\gamma} g(y) dy} .$$

g	$\tilde{g}$
Lognormal	Lognormal, same σ
LN mixture	LN mixture, same σ_1, σ_2
GB2	GB2, same a, b

2. Statistical calibration

This method uses a calibration function C to transform the risk-neutral cumulative distribution function (c.d.f.), G into the real-world c.d.f. $\tilde{G}$, by

$$\tilde{G}(x) = C(G(x)).$$

The function C is a c.d.f.

Ideally:

- C is the real-world c.d.f. of $G(S_T)$,
- And then $\tilde{G}(S_T)$ has a uniform real-world distribution.

Following Fackler and King, we let the calibration function C be the Beta c.d.f. and then:

$$\tilde{g}(x) = \frac{G(x)^{j-1}(1-G(x))^{k-1}}{B(j,k)} g(x).$$

The transformation has parameters j and k.

- The RND & RWD coincide when $j = k = 1$.
- The implicit utility function is risk-averse if, and only if, $k \leq 1 \leq j$ with $j \neq k$.

Likelihood methods

Option expiry months, $i = 1, 2, 3,...$

Densities formed at times t_i.

Options expire at times t_i^*, at index levels $S_{T,i}$.

Non-overlapping densities, i.e. $t_i^* \le t_{i+1}$.

Hence the likelihood of observed index levels is the product of densities, and

$$\log(L(S_{T,1}, S_{T,2}, ..., S_{T,n} \mid \theta^*)) = \sum_{i=1}^{n} \log(\tilde{g}_i(S_{T,i} \mid \theta^*))$$

with θ^* any transformation parameter(s), γ or (j,k), that can be obtained by maximising L.

Data

- Seek densities for the FTSE-100 index.
- Densities are defined 4 weeks before options expire.
- Historical real-world densities are obtained by estimating and simulating asymmetric ARCH models.
- Expiry months:
  - Original dataset has 83, June 1993 to April 2000.
  - New dataset has 126, ending December 2003.
- Risk-neutral densities are estimated:
  - Originally from *intraday midquote* and *trade* option prices,
  - For the new data, from option *settlement* prices.

Results

Risk-neutral densities

- Negatively skewed. Crash phobia!?.
- Similar for GB2, lognormal mixtures and splines.
- Similar for midquote, trade and settlement prices.
- Exercise prices typically enclose 98% of the RND.

Transformation parameter estimates

- The CRRA parameter γ is near 4 for the shorter period but near 2 for the longer period – reflecting stock market falls after 2000.
- The calibration parameter j is near 1.40 (shorter period) and 1.25 (longer period), with k near 1.10.

Results are now for the original, shorter dataset

Moment comparisons

Of most interest for skewness. Median values are :

	RND	RWD-u	RWD-c
GB2	-0.75	-0.60	-0.56
Mix LN	-0.70	-0.59	-0.55

- Transformations reduce the level of skewness.
- Historical densities have median skewness –0.15.
- So, option densities are much more skewed than the historical densities.
- Crash anxieties are unduly pessimistic!?

Calibration

Let $u_i = G(S_{T,i})$ or $\tilde{G}(S_{T,i})$.

Sample c.d.f. $\hat{C}(u) =$ fraction of the $u_i \le u$.

Good calibration requires $\hat{C}(u) \cong u$, all u.

But all four sets of RNDs have $\hat{C}(u) = 0$ for $u \le 0.08$.

Historical and transformed densities are more satisfactorily calibrated.

The historical densities have the least value of

$$\max_{u} \left| \hat{C}(u) - u \right|.$$

Likelihoods

Maximum log-likelihood values for option methods, *relative to the historical densities*, using midquote option prices:

Transform…	RNDs	RWDs utility	RWDs calibration
Method			
Spline	−0.65	1.09	2.73
GB2	0.72	2.37	3.54
LN mixture	2.79	4.61	5.87

Likelihood-ratio tests

Option RNDs (null) versus option RWDs (alternative) :

- All eight test values reject the null at the 10% level (midquote or trade prices, GB2 or LN mixture densities, either transformation).

Encompassing densities

$$\tilde{g}_{combined}(x|\alpha) = \alpha g_{RND/RWD}(x) + (1-\alpha)\tilde{g}_{ARCH}(x)$$

- For RNDs, the average α is 0.57. All LR test values *reject* the null hypothesis $\alpha = 0$ at the 2% level.
- For RWDs, the average α is 0.71. All LR test values *accept* the null $\alpha = 1$ at the 5% level.

Empirical risk aversion

Within the representative agent framework, the utility function υ satisfies

$$M(x) = \lambda \frac{d\upsilon}{dx} = e^{-rT} \frac{g(x)}{\tilde{g}(x)}.$$

Empirical estimates of the pricing kernel M :

- Use options data for g and historical for $\tilde{g}$,
- Are shown as geometric means across all expiry months,
- Are generally decreasing functions of *x*, as required by theory $(\upsilon'' < 0)$.
- Are hence unlike U.S. estimates.

Conclusions

For our U.K. data:

Option prices provide densities that are more informative than historical densities, *before* any risk adjustments are made.

Two simple transformations provide *real-world* densities from option prices that are more accurate than the *risk-neutral* densities.

Figure 1

The historical density and three risk-neutral densities, obtained from midquote option prices, on March 21, 1997

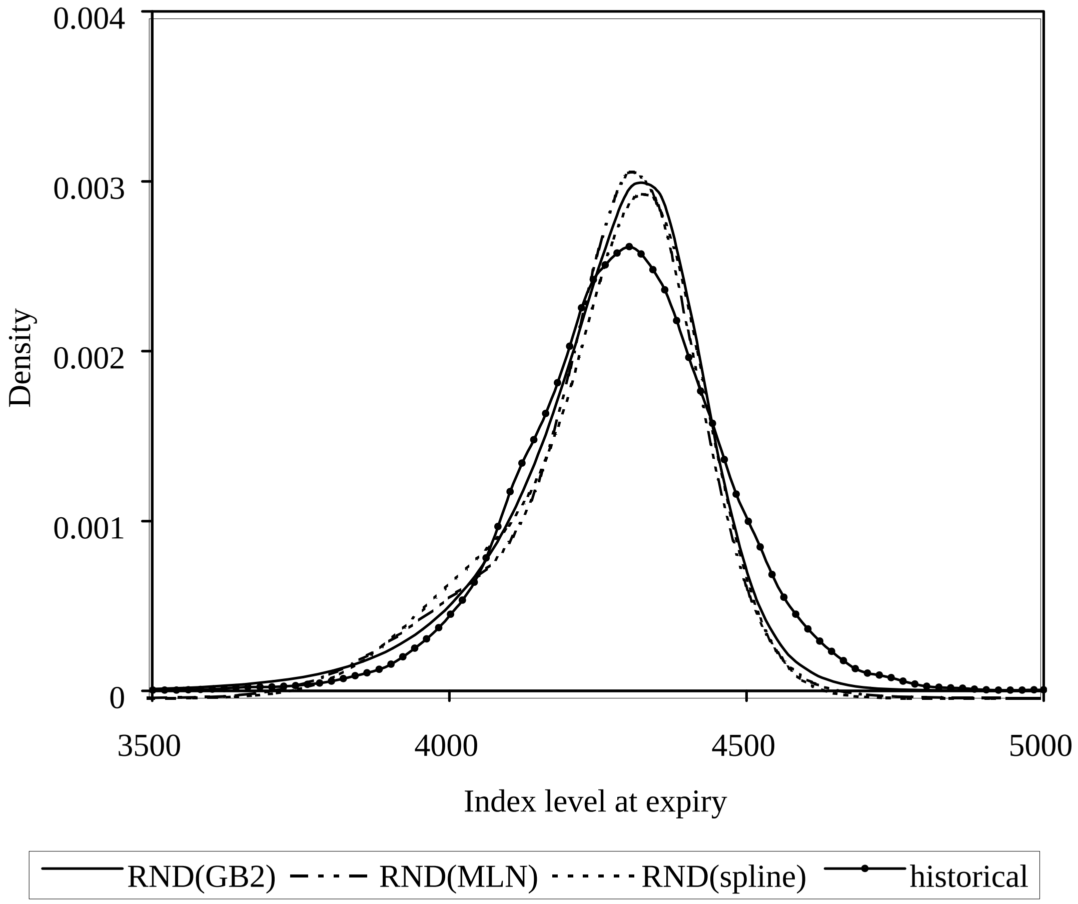


Figure 2

The risk-neutral GB2 density, from midquotes, and two corresponding real-world densities, on March 21, 1997

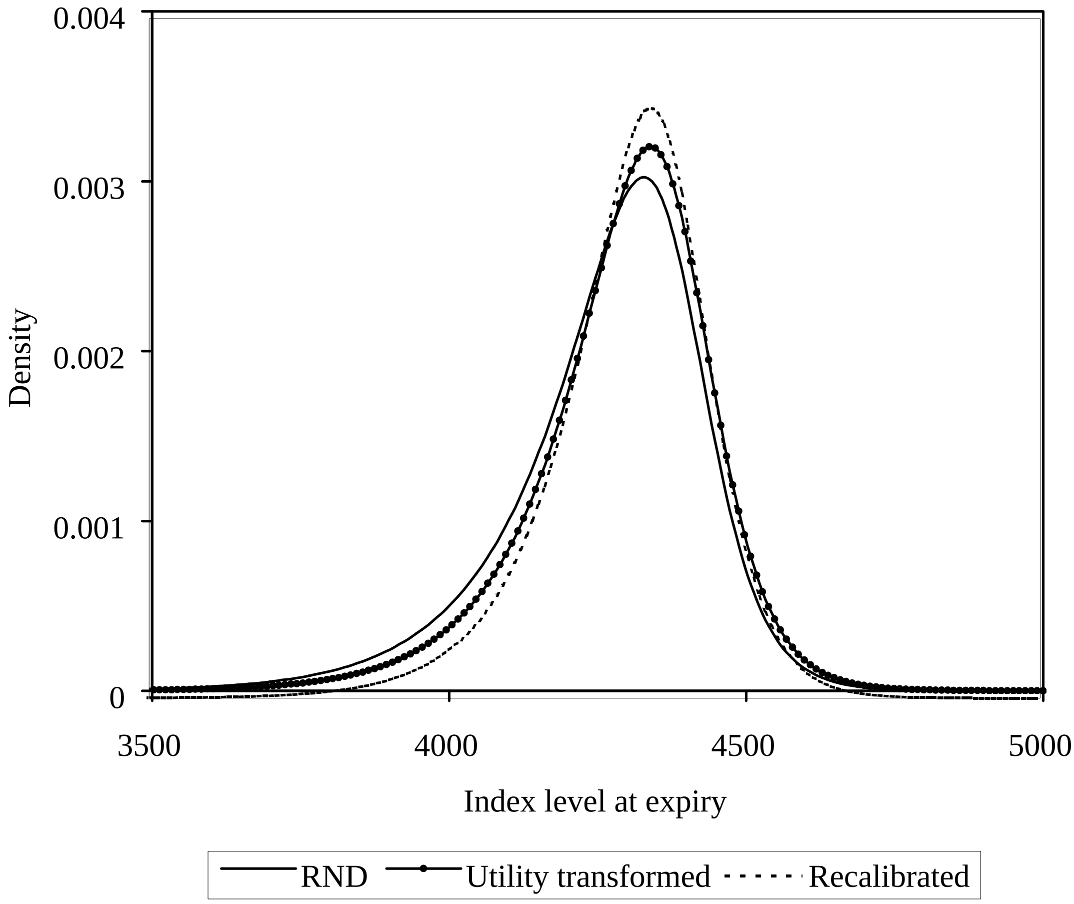


Figure 3

The risk-neutral lognormal mixture density, from midquotes, and two corresponding real-world densities, on March 21, 1997

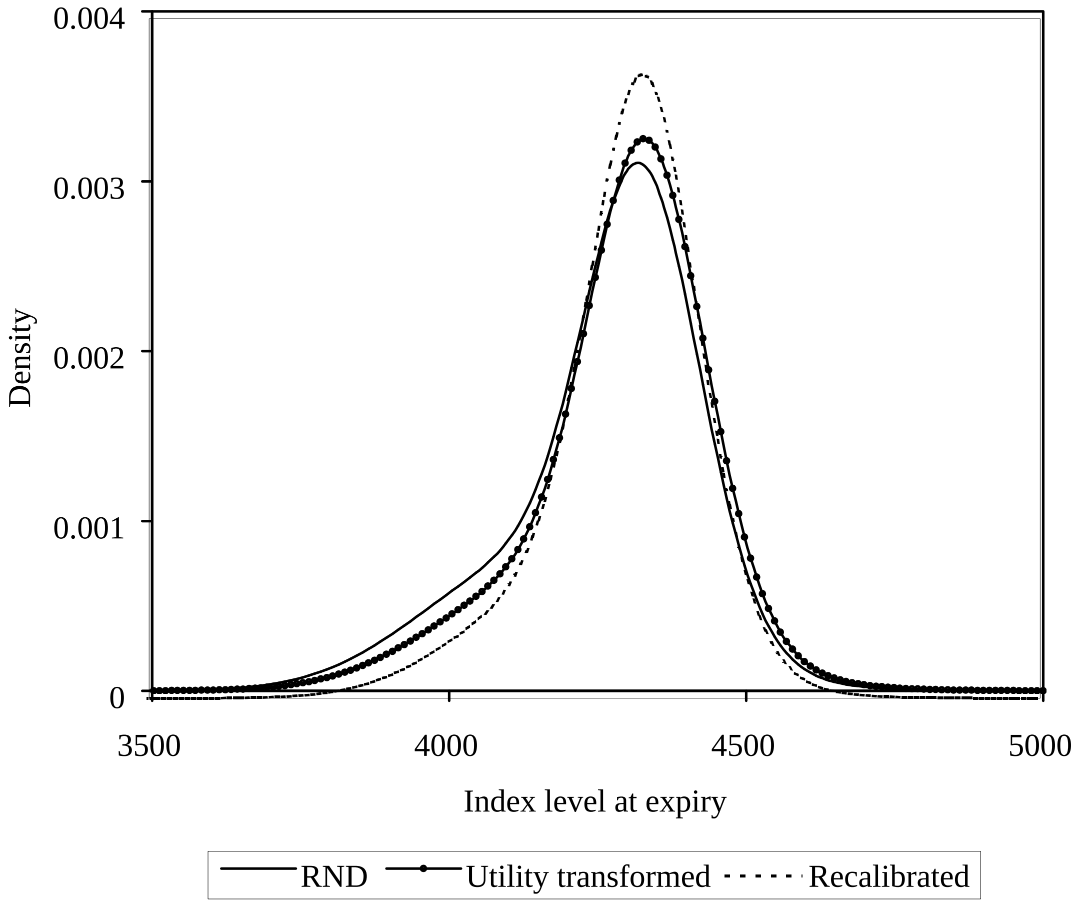


Figure 4

Cumulative distributions for cumulative probabilities obtained from GB2 RNDs and two sets of real-world densities

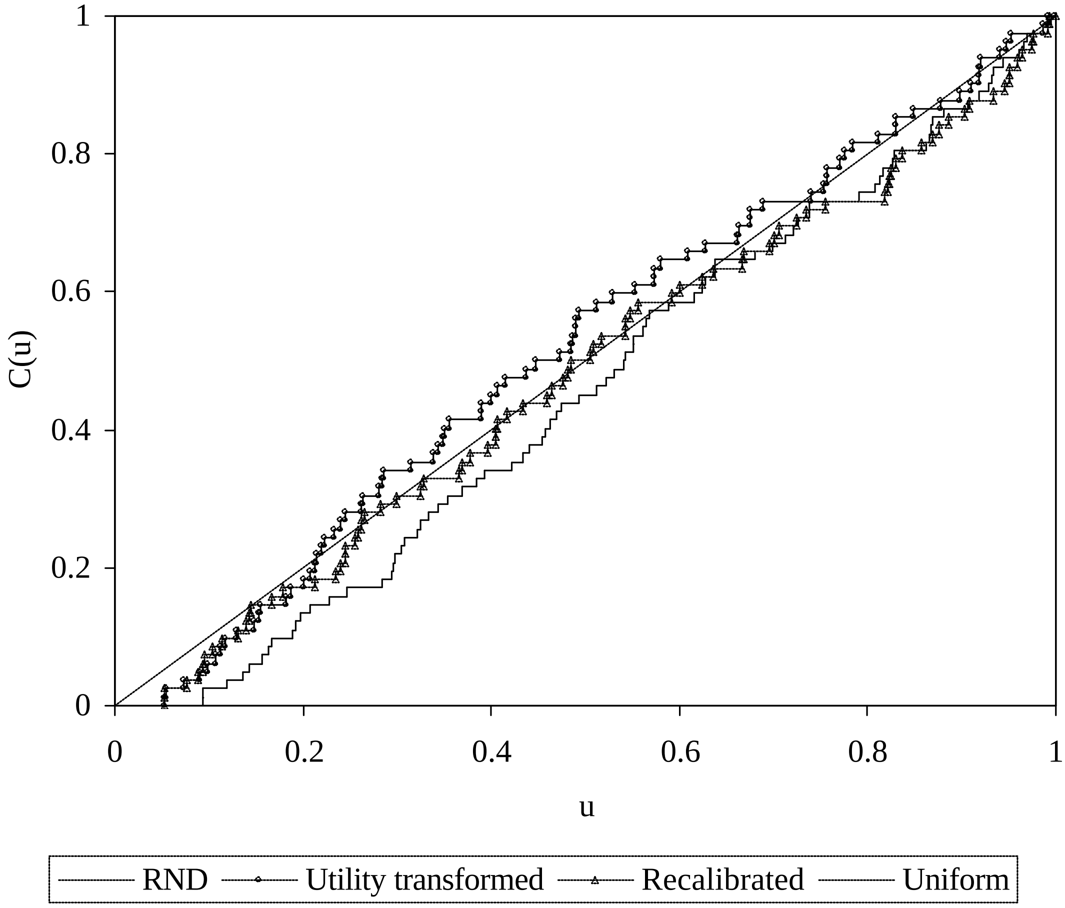


Figure 5

Cumulative distributions for cumulative probabilities obtained from historical densities

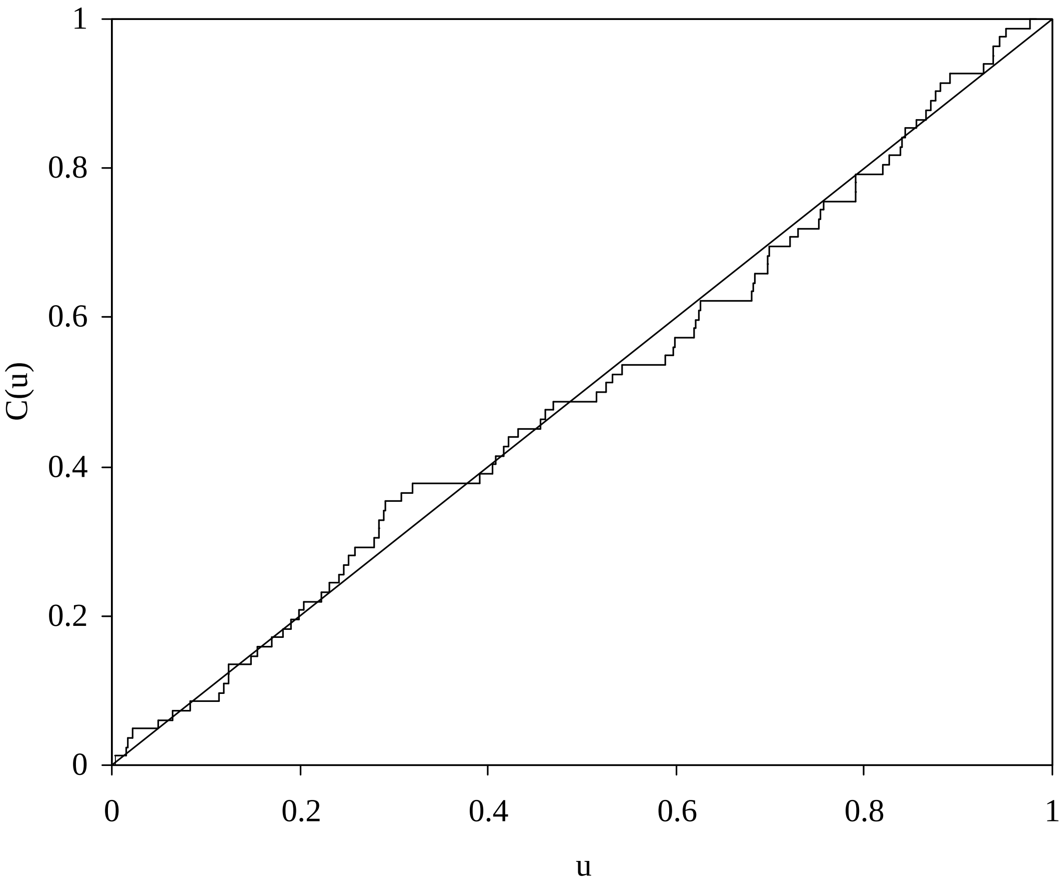
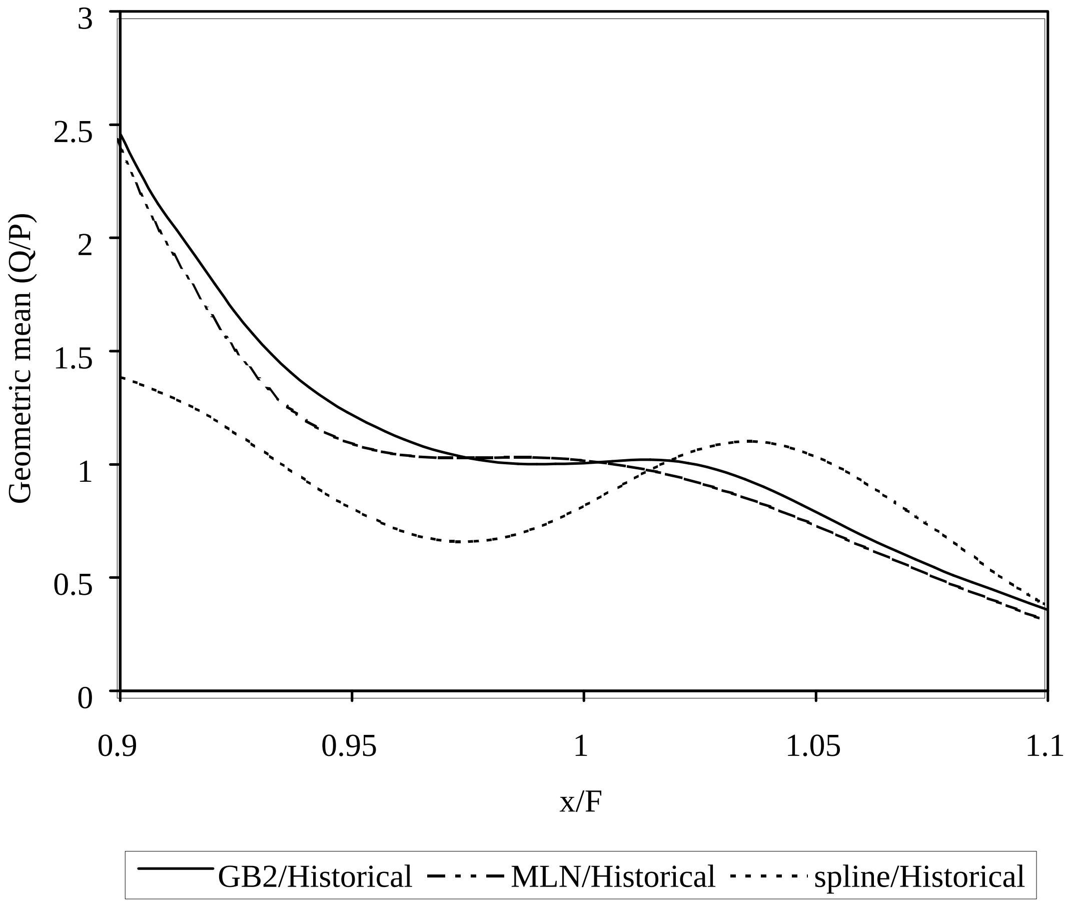


Figure 6

Empirical pricing kernels, averaged across all expiry months



Study 2

Density prediction for the S & P 500 index

By: Shackleton, Taylor & Yu

Preliminary: Paper to be completed in 2005

<u>Contributions</u>

1. Comparisons between densities estimated from option prices and from *high-frequency* index levels.

2. Densities obtained for *all horizons* from option prices.

Risk-neutral densities

Following Heston, suppose:

- the variance V follows a square-root process,
- the process for the futures prices, F, is continuous,
- the risk-neutral dynamics are:

$$dF = F\sqrt{V}dW,$$

$$dV = \kappa(\theta - V)dt + \xi\sqrt{V}dZ,$$

with

$$dWdZ = \rho dt.$$

Option prices can then be calculated from a "closed form" formula, that involves characteristic functions.

The initial variance V_0 and the parameters $\kappa, \theta, \xi, \rho$ can be estimated by matching a *matrix* of market prices $c_{market}(T_i, X_j)$ with theoretical Heston prices.

The risk-neutral density can then be deduced for any horizon *T*.

Data

- From 1990 to 2004.
- Daily and five-minute S & P 500 futures returns are used to estimate historical real-world densities from ARCH models. The data-frequency of observed returns for an ARCH model is matched with the density horizon.
- Risk-neutral densities are estimated from settlement prices for options on S & P 500 futures.
- Real-world densities, derived from option prices, are obtained by applying the calibration transformation mentioned earlier.

Horizons

1 day

1, 2, 4, 6, 8 & 12 weeks

Results (preliminary)

Likelihood comparisons

ARCH models versus Heston-RND

Horizon < 3 weeks: ARCH densities from high-frequency returns are best.

Horizon > 3 weeks: Heston-RND is best.

ARCH models versus calibrated Heston density

Horizon < 5 weeks: Calibrated Heston density best.

> 5 weeks: ARCH & calibrated densities are similar.

Encompassing densities

More weight is given to the calibrated Heston density than to the best ARCH density.

Diagnostic tests (5% significance level)

Kolmogorov-Smirnov test – All methods fail for the one-day horizon. Few failures at other horizons.

Berkowitz LR3 test –

- ARCH models usually pass.
- Heston RNDs fail for short horizons.
- Calibrated Heston densities pass for all horizons.