Portfolio Credit Derivatives with Markovian Default Interaction

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1. Introduction

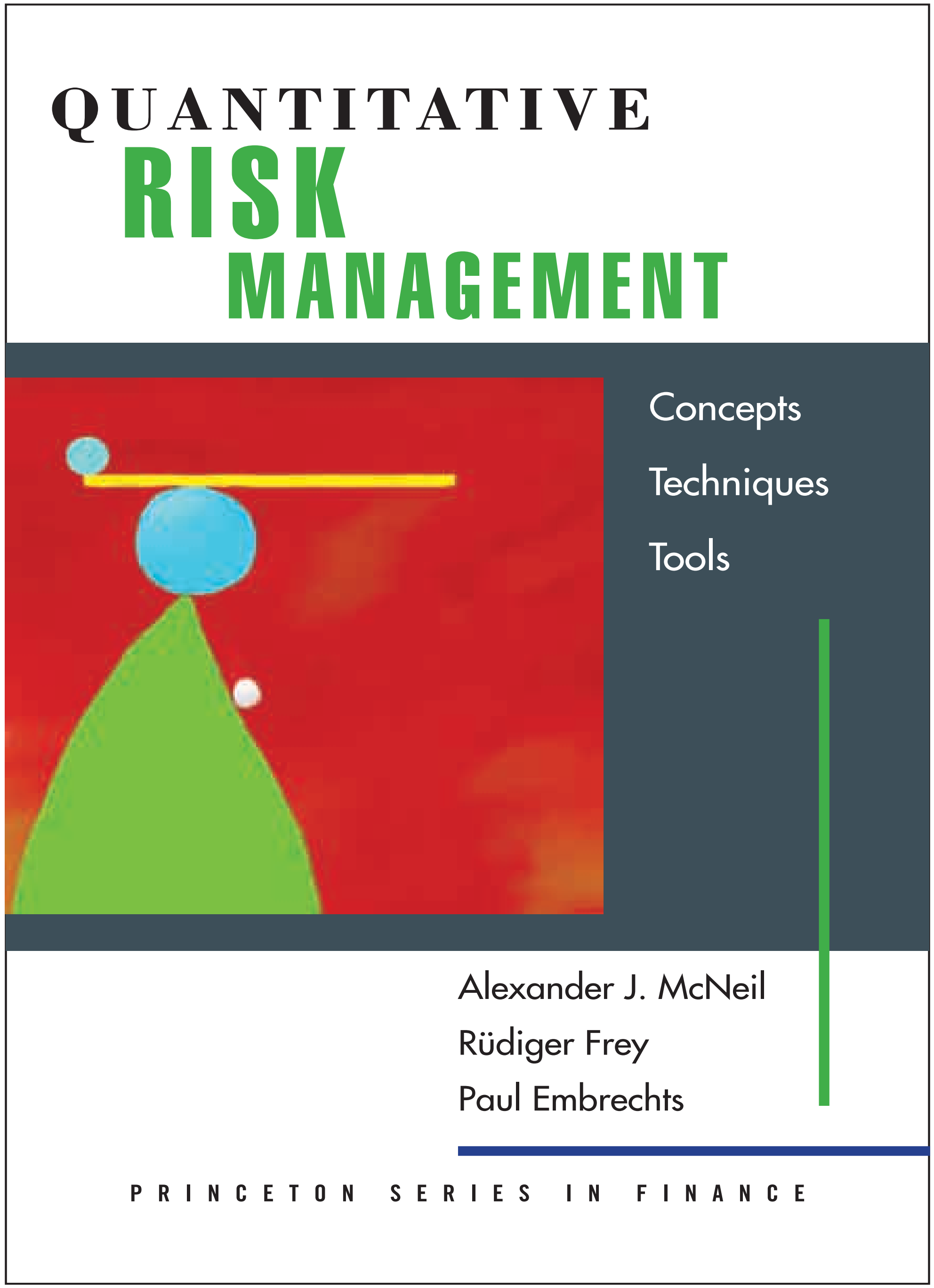
- *Portfolio credit derivatives* are securities whose payoff is contingent on credit events in a pool of firms (portfolio products).

- *Focus* of this talk: Pricing in portfolio credit derivatives in a Markovian model with interacting default intensities, which is an alternative to the market-standard Gauss copula model

- *Overview*

  - Introduction
  - Interacting intensities: a Markovian approach
  - Pricing (basket) default swaps in the Markov model
  - Pricing CDOs in the Markov model

Talk is based on Frey-Backhaus (2004); for background information see the forthcoming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Reduced Form Models for Credit Portfolios

- Models with *conditionally independent* defaults and stochastic intensity
- *Copula models* such as Li (2001), Schönbucher-Schubert (2001). Market standard since they allow for default contagion and are easily calibrated to defaultable term structure. Main drawback: unintuitive parametrization of dependence.
  Special case: *factor copula models* such as Laurent-Gregory (2003), Schönbucher (2003,2004), where contagion can be interpreted in terms of incomplete information.
- *Common shock models* such as Lindskog-McNeil(2001) or Kijima(2000)
- Models with *interacting intensities*

Basic Concepts and Notation

Consider m firms with default times τ_i and default indicator process $\mathbf{Y}_t = (Y_t(1), \ldots, Y_t(m))$ with $Y_t(i) = I_{\{\tau_i \le t\}}$.

- $\overline{F}_i(t) = P(\tau_i \ge t)$ survival function of obligor i; joint survival function: $\overline{F}(t_1, \ldots, t_m) = P(\tau_1 \ge t_1, \ldots, \tau_m \ge t_m)$.
- *ordered default times* denoted by $T_0 < T_1 < \ldots < T_m$. $\xi_n \in \{1, \ldots, m\}$ gives identity of the firm defaulting at time T_n
- *Filtrations.* $\mathcal{H}_t^i = \sigma(\{Y_{s,i} : s \le t\})$, $\mathcal{H}_t = \mathcal{H}_t^i \vee \cdots \vee \mathcal{H}_t^m$. $\{\mathcal{H}_t\}$ is *internal filtration* of $(\mathbf{Y}_t)$ (only information about default history).

Default intensities. An $\{\mathcal{H}_t\}$-adapted process $(\lambda_{t,i})$ is called the *default intensity* of τ_i (wrt $\{\mathcal{H}_t\}$) if $Y_i(t) - \int_0^{\tau_i \wedge t} \lambda_{s,i} ds$ is an $\{\mathcal{H}_t\}$-martingale. Default intensities determine the law of the default indicator process $(\mathbf{Y}_t)$.

Copula Models - the Market Standard.

Background on copulas A copula is a df C on $[0,1]^m$ with uniform margins. Copulas describe the *dependence structure* of a multivariate distribution with df F. If F has continuous margins $F_1, \ldots, F_m$ and $\mathbf{X} \sim F$ the copula C of $\mathbf{X}$ is the df of $(F_1(X_1), \ldots F_m(X_m))$, and we have *Sklars identity*

$$F(x_1, \ldots, x_m) = C(F_1(x_1), \ldots, F_m(x_m))\,.$$

Similarly, the survival function of $\mathbf{X}$ can be written as $\overline{F}(x_1, \ldots, x_m) = \hat{C}\left(\overline{F}_1(x_1), \ldots, \overline{F}_m(x_m)\right)$, where $\hat{C}$ is defined by $\hat{C}(u_1, \ldots, u_m) = C(1-u_1, \ldots, 1-u_m)$; $\hat{C}$ is called *survival copula*

Example. Gauss copula C_P^{Ga} is the copula of $\mathbf{X} \sim N(\mathbf{0}, P)$ where P is a correlation matrix.

Copula models.

Copula models are specified in terms of marginal distribution and survival copula (denoted C) of $(\tau_1, \ldots, \tau_m)$. Hence survival function of default times is given by

$$\overline{F}(t_1, \ldots, t_m) = C\left(\overline{F}_1(t_1), \ldots, \overline{F}_m(t_m)\right), \qquad (1)$$

Specifying dependence structure C and marginal distribution $\overline{F}_i$ separately is useful for *calibration*. Model is calibrated to given term structure of (single-name) CDS spreads by specifying $\overline{F}_i$; calibration of dependence structure (i.e. C) can then be done independently.

Exchangeable Gauss-Copula Model. Here $X_i = \sqrt{\rho}V +$ $+\sqrt{1-\rho}\epsilon_i$, for 'asset correlation' $\rho \in (0,1)$ and $V, (\epsilon_i)_{1 \le i \le m}$ iid standard normal rvs. Set $U_i = \Phi(X_i)$, so that $\mathbf{U} \sim C_P^{\text{Ga}}$. Survival function of $\tau_1, \ldots, \tau_m$ given by

$$\overline{F}(t_1, \ldots, t_m) = P\left(U_1 \le \overline{F}_1(t_1), \ldots, U_m \le \overline{F}_m(t_m)\right).$$

2. Models with Interacting Intensities

(joint work with Jochen Backhaus).

Basic idea. Default contagion is *explicitly* modelled. Default intensity is modelled as function $\lambda_i(t, \mathbf{Y}_t)$ of time *and* and of default-state $\mathbf{Y}_t$ of portfolio at time t. (Extension to stochastic state variables possible).

Advantage. Intuitive and explicit parametrization of dependence between defaults; Markov process techniques available for analysis and simulation of the model.

Disadvantage. Calibration to term structure of defaultable bonds or CDSs more difficult than with copula models, as marginal distribution of default times typically not available in closed form.

Related work

- Davis -Lo (2001): Default contagion, uses Markov chains.
- Jarrow-Yu(2001): only very special types of interaction; model is studied using Cox process techniques. Extensions by Yu(2004), who provides proper model construction and studies default correlations.
- Kusuoka(1999) and Bielecki - Rutkowski (2002) study mathematical aspects of the model
- Collin-Dufresne-Goldstein-Hugonnier(2003): analytical evaluation of certain derivatives using a particular change of measure.
- Giesecke-Weber(2002/03): application of interacting particle systems literature to default contagion.
- Herbertsson(2005); Markovian approach using phase-type distributions.

Construction via Markov Chains.

A model with interacting intensities is conveniently defined as *time-inhomogeneous Markov chain* with state space $S = \{0,1\}^m$ and transition rate functions (from $\mathbf{y}$ to $\mathbf{x}$)

$$\lambda(t, \mathbf{y}, \mathbf{x}) = \begin{cases} I_{\{y_i=0\}}\lambda_i(t, \mathbf{y}), & \text{if } \mathbf{x} = \mathbf{y}^i \text{ for some } i \in \{1\ldots, m\}, \\ 0 & \text{else,} \end{cases} \tag{2}$$

where $\mathbf{y}^i \in S$ is obtained from $\mathbf{y} \in S$ by flipping ith coordinate.

Interpretation. The chain can jump only to neighbouring states, which differ from the current state $\mathbf{Y}_t$ by exactly one default; if $Y_i(t) = 0$, the probability of a jump in $[t, t+h)$ to state $\mathbf{Y}_t^i$ (default of firm i), is $\approx h\lambda_i(t, \mathbf{Y}_t)$.

Model properties

- The *generator* of $(\mathbf{Y}_t)$ equals

$$G_{[t]}f\,(\mathbf{y}) = \sum_{i=1}^{m} I_{\{y_i=0\}}\lambda_i(t,\mathbf{y})\big(f(t,\mathbf{y}^i) - f(t,\mathbf{y})\big).$$

- $Y_i(t) - \int_0^{t\wedge\tau_i} \lambda_i(s,\mathbf{Y}_s)ds$ is an $\{\mathcal{H}_t\}$-martingale by the Dynkin formula, so that $\lambda_i(s,\mathbf{Y}_s)$ is in fact the $\{\mathcal{H}_t\}$-default intensity.

- Denote by $p(t,s,\mathbf{x},\mathbf{y}) = P_{(t,\mathbf{x})}(\mathbf{Y}_s = \mathbf{y})$, $s \geq t$, the transition probabilities of the chain. They satisfy the Kolmogorov forward- and backward equation, here an ODE system.

Kolmogorov equations

Backward equation.

$$\frac{\partial p(t,s,\mathbf{x},\mathbf{y})}{\partial t} + \sum_{i=1}^{m} (1 - x_i)\,\lambda_k(t,\mathbf{x})(p(t,s,\mathbf{x}^i,\mathbf{y}) - p(t,s,\mathbf{x},\mathbf{y})) = 0\,,$$

$$p(s,s,\mathbf{x},\mathbf{y}) = \delta_{\mathbf{y}}(\mathbf{x})\,. \quad \text{(terminal condition)}$$

Forward equation.

$$\begin{aligned}\frac{\partial}{\partial s} p(t,s,\mathbf{x},\mathbf{y}) &= \sum_{k=1}^{m} (y(k))\lambda_k\left(s,\mathbf{y}^k\right) p(t,s,\mathbf{x},\mathbf{y}^k) \\ &\quad - \sum_{k=1}^{m} (1 - y(k))\lambda_k\,(s,\mathbf{y})\,p(t,s,\mathbf{x},\mathbf{y}),\ s > t,\end{aligned}$$

$$p(t,t,x,y) = \delta_{\mathbf{x}}(\mathbf{y}). \quad \text{(initial condition)}$$

Modelling Default Intensities

The default intensities $\lambda_i(t, \mathbf{y})$ are essential ingredient of the model. Some examples:

- Jarrow-Yu (01): primary-secondary framework.
- Frey-Backhaus (04): homogenous group model with

$$\lambda_i(t, \mathbf{Y}_t) = h\left(t, M(\mathbf{Y}_t)\right), \text{ where } M(\mathbf{y}) = \sum_{i=1}^{m} y_i \,. \tag{3}$$

  In exchangeable models $\lambda_i(t, \mathbf{Y}_t)$ is necessarily of this form. Moreover, natural interpretation in terms of *mean-field interaction*. Extension to model with several groups possible.

- Yu (04) claims that $h(t, l) = 0.01 + 0.001 \cdot I_{\{l>0\}}$ is a reasonable model for European telecom bonds.

Properties of Homogenous-Group Model

- The process $M_t := M(\mathbf{Y}_t)$ is itself a Markov chain with generator

$$G^M_{[t]} f\,(l) = (1-l)h\,(t,l)\left(f(l+1) - f(l)\right). \tag{4}$$

- State space of (M_t) is $S^M := \{0, 1, \dots m\}$ so that $|S^M| = m+1$ (instead of 2^m).
- Kolmogorov equations for (M_t) available in closed form.
- $P(Y_i(T) = 1) = 1/mE(M_T)$ and $P(Y_i(T) = 1, Y_j(T) = 1) = m^{-2}E(M_T(M_T - 1))$ etc.
- Limit results for large portfolios available.

Conditional Expectations

Following results useful for (semi)analytic pricing of credit derivatives.

Proposition. The density of τ_{i_0} equals

$$P(\tau_{i_0} \in dt) = \sum_{\mathbf{y}: y(i_0)=0} \lambda_{i_0}(t, \mathbf{y}) P(\mathbf{Y}_t = \mathbf{y}) \,. \text{ Moreover,} \tag{5}$$

$$P\big(\mathbf{Y}_t = \mathbf{y} \big| \tau_{i_0} = t\big) = y(i_0) P(\tau_{i_0} \in dt)^{-1} \lambda_{i_0}(t, \mathbf{y}^{i_0}) P(\mathbf{Y}_t = \mathbf{y}^{i_0}) \,. \tag{6}$$

Proof is based on Markov property.

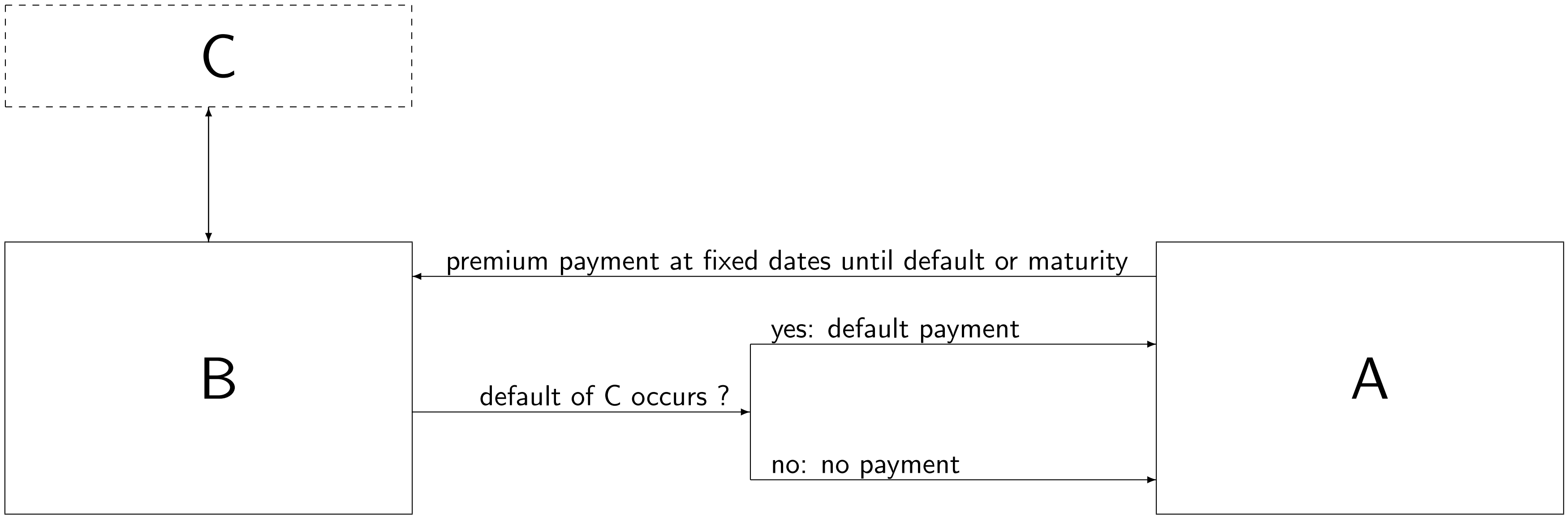
Corollary. In homogeneous-group model with default intensity $h(t, l)$

$$P(\tau_{i_0} \in dt) = m^{-1} \sum_{k=0}^{m-1} h(t, k) P(M_t = k)(m - k) \text{ and} \tag{7}$$

$$P\big(M_t = l \,\big|\, \tau_{i_0} = t\big) = \frac{(m - l + 1) h(t, l - 1) P(M_t = l - 1)}{\sum_{k=0}^{m-1} (m - k) h(t, k) P(M_t = k)} \,.$$

3. (Basket) Default Swaps in the Markov model

Credit Default Swap (CDS). Workhorse of market for credit derivatives. Three parties involved: reference entity C; protection buyer A; protection seller B.



(Basket) Default Swaps

Ordinary CDS.

- Premium payments are due at times $0 < t_1 < \cdots < t_N$. If $\tau_C > t_k$ A pays in t_k a premium of size $x(t_k - t_{k-1})$, where x denotes the *swap spread*; after τ_C premium payments stop. No initial payments. (accrued payments are neglected for simplicity)
- *Default payment.* If $\tau_C < t_N = T$ B pays A the LGD δ of C at τ_C.
- *Fair swap spread x^*.* Since there are no initial payments x^* is chosen such that value at $t = 0$ of default payments equals value of premium payments. x^* is the quantity which is quoted on the market.

kth-to-default swap. Here default payment is triggered by credit events in a portfolio (the basket). Premium payments as before. If kth default time $T_k < t_N$ there is a default payment whose size depends on identity ξ_k of defaulting firm.

Pricing Results for Default Swaps

Goal. Provide (semi)analytic pricing formulas which can be evaluated using Kolmogorov equations.

Throughout we assume that model has been set up under equivalent martingale measure P and that risk-free interest rate r and LGD δ_i are deterministic; $D(t) = \exp(-\int_0^t r(s)ds$ is default-free discount factor.

CDSs.

- Premium payments are terminal value claims of the form $H = g(\mathbf{Y}_{t_k})$ with $g(\mathbf{y}) = x^*(t_k - t_{k-1})(1 - y_i) \Rightarrow$ pricing via backward equation.
- Price of the default payments given by

$$E\left(\delta_{i_0} D(\tau_{i_0}) I_{\{\tau_{i_0} \le T\}}\right) = \delta_{i_0} \int_0^T D(t) P(\tau_{i_0} \in dt)\, dt\,, \qquad (8)$$

  which can be evaluated numerically using (5) or (7).

Basket Default Swaps

Default payments of kth-to-default swap. By definition

$$V^{\text{def}} := \sum_{j=1}^{m} \delta_j \, E\Big(D(\tau_j) I_{\{\tau_j \le T\}} I_{\{M_{\tau_j}=k\}}\Big).$$ Now

$$E\Big(D(\tau_j) I_{\{\tau_j \le T\}} I_{\{M_{\tau_j}=k\}}\Big) = \int_0^T D(t) P\big(M_t = k \mid \tau_j = t\big) P(\tau_j \in dt)\, dt.$$

Hence we get in homogeneous group model

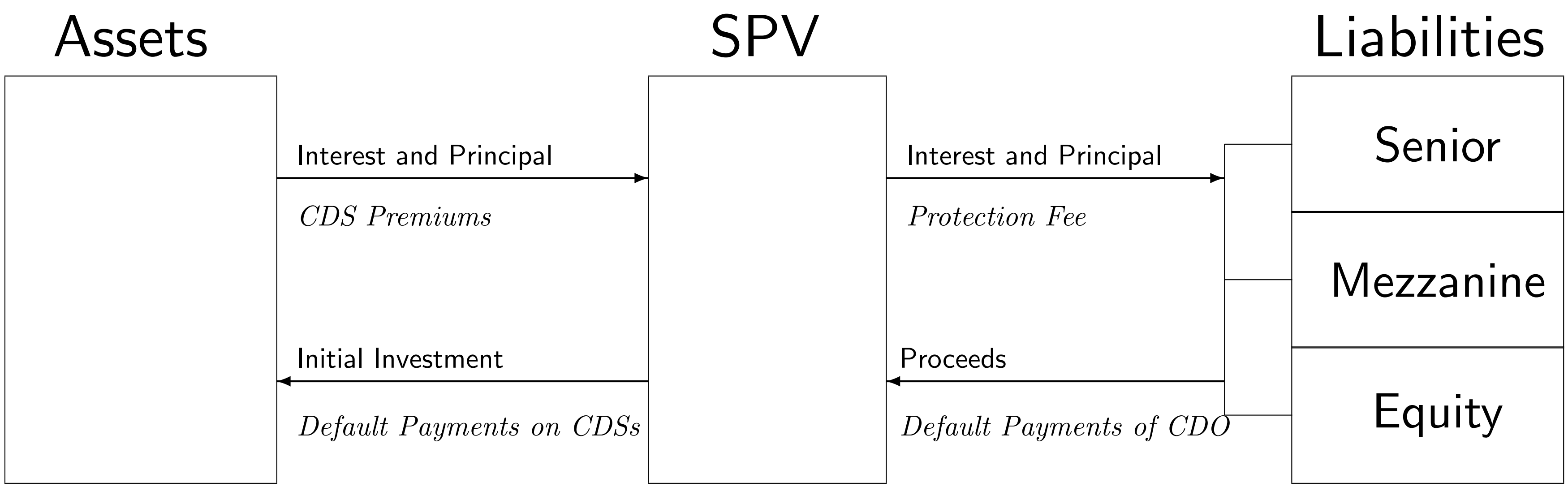
$$V^{\text{def}} := \sum_{j=1}^{m} \delta_j \int_0^T D(t) \frac{(m-k+1)}{m} h(t, k-1) P(M_t = k-1) dt,$$

which can be evaluated by Kolmogorov. Similar formula also for non-homogenous case.

Premium payments can be evaluated using $\{T_k \le t\} = \{M_t \ge k\}$.

4. CDOs: Pricing and Model Calibration

Basic Structure of CDOs



Payments in a CDO structure; above arrow: asset-based structure; below arrow: *synthetic CDO*.

Payoff of Tranches

Consider portfolio of m loans with nominal N_i, relative LGD δ_i and *default-indicator process* $\mathbf{Y}_t$. *Cumulative loss* of the portfolio in t given by $L_t = \sum_{i=1}^{m} \delta_i N_i Y_i(t)$.

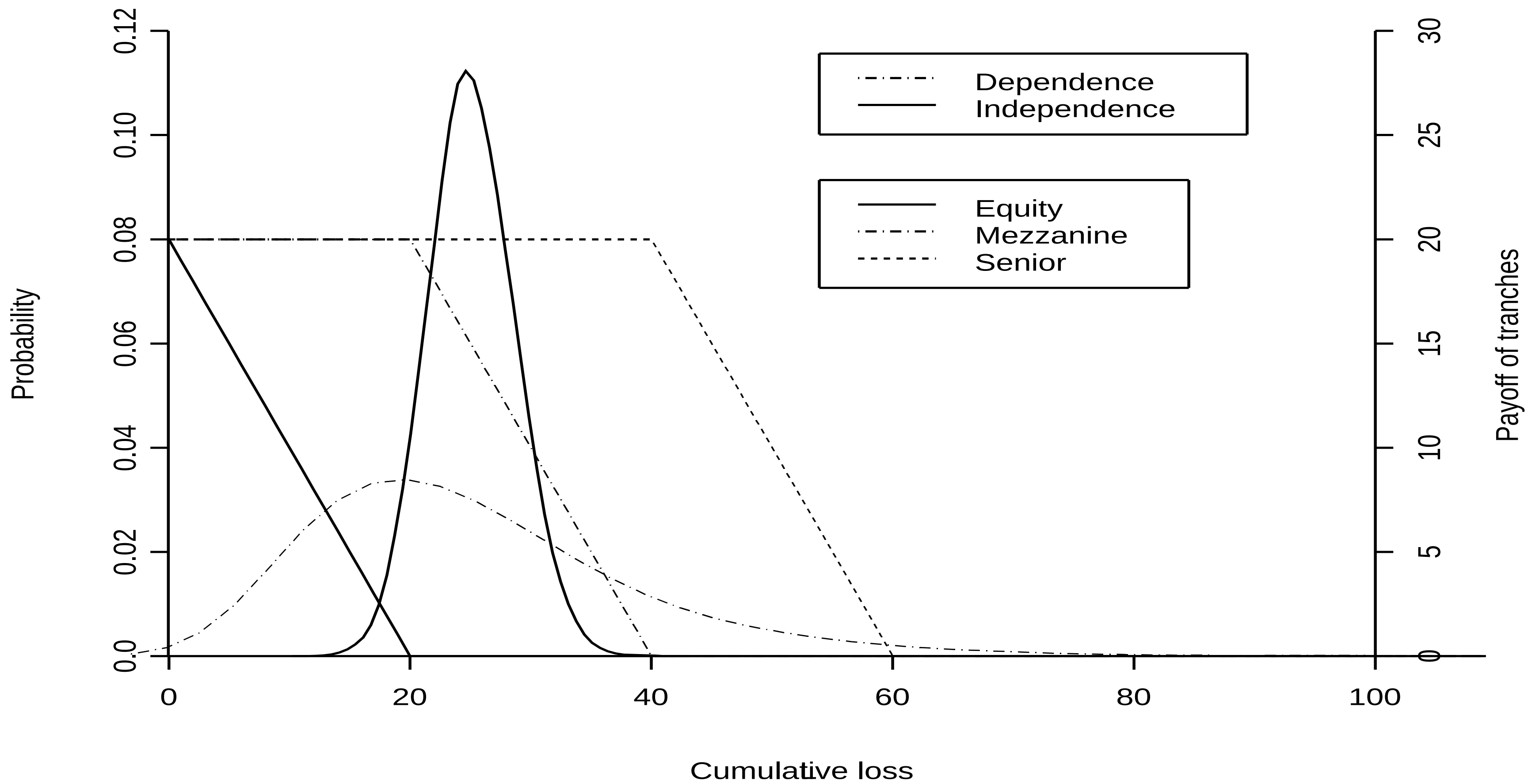
CDO-tranches. Maturity T. We have k tranches, characterized by attachment points $0 = K_0 < K_1 < \cdots < K_k \leq \sum_{i=1}^{m} N_i$. The *notional* of tranche κ at time t is given by

$$N_\kappa(t) = f_\kappa(L_t) \text{ with } f_\kappa(l) = (K_\kappa - l)^+ - (K_{\kappa-1} - l)^+$$

Note that $f_\kappa(l)$ corresponds to a *put spread*.

Stylized CDO. Payoff of tranche κ given by $N_\kappa(T)$, the value of the notional at maturity. Real CDOs more complicated, as there is intermediate income.

Default Correlation and CDO Tranches



Payoff of stylized CDO with 3 tranches and attachment points 20 40 and 60 with two different loss distributions overlayed. Properties carry over to more realistic contracts.

Synthetic CDOs: Payment Description

Consider CDO with attachment points $K_0 < \cdots < K_k$ and notional of tranche κ given by $N_\kappa(t) = (K_\kappa - L_t)^+ - (K_{\kappa-1} - L_t)^+$; define *cumulative loss of tranche* κ as $L_\kappa(t) := N_\kappa(0) - N_\kappa(t)$.

Default payments of CDO. Default payment of tranche κ at nth default time $T_n < T$ given by $\Delta L_\kappa(T_n) = (L_\kappa(T_n) - L_\kappa(T_{n-1}))$, i.e. by the part of cumulative loss at T_n which falls in the layer $[K_{\kappa-1}, K_\kappa]$.

Protection fee or premium payments. Holder of tranche κ receives periodic premium payments at $0 < t_1 < \cdots < t_N = T$ of size $x_\kappa^{\text{CDO}}(t_n - t_{n-1})N_\kappa(t_n)$. No initial payments. x_κ^{CDO} is called the (fair) CDO spread.

Synthetic CDOs: Pricing

Using partial integration we obtain for the value of the default payments of tranche κ

$$V^{\text{def}} = E\Big(\int_0^T D(t)dL_\kappa(t)\Big) = D(T)E\big(L_\kappa(T)\big) + \int_0^T rD(t)E\big(L_\kappa(t)\big)dt.$$

As $L_\kappa(t)$ is a function of L_t this can be computed by one-dimensional integration if we know the distribution of L_t. Premium payments can also be expressed in terms of L_t.

For deterministic LGD L_t is a function of $\mathbf{Y}_t$ resp M_t. Hence in homogeneous group model computation via Kolmogorov equations possible; otherwise Monte Carlo must be used.
Computation in Gauss-copula model can also be based on above representation.

Explaining Market Quotes for CDOs

Financial industry has developed CDS-indices for a variety of sectors; spreads for CDO-tranches on these indices are available.

Observed CDS spreads. Consider the 5 year iTraxx EUR from August 4, 2004. The (average) index level was 42 bp. Assuming a constant recovery rate of 40% this leads to a marginal default probability $P(\tau \leq t) = 1 - e^{-\lambda t}$ with $\lambda = 0.007$ and we obtain a 5-year default probability of 3.44%.

Observed spreads for CDO tranches. On the market we observed the following tranche prices[1].

$[0,3]^*$	[3,6]	[6,9]	[9,12]	[12,22]
27.6 %	1.68%	0.70 %	0.43 %	0.20%

*The [0,3] tranche quoted on upfront + 5% per year.

[1] Taken from Hull/White(2004)

Implied Tranche- and Base Correlation

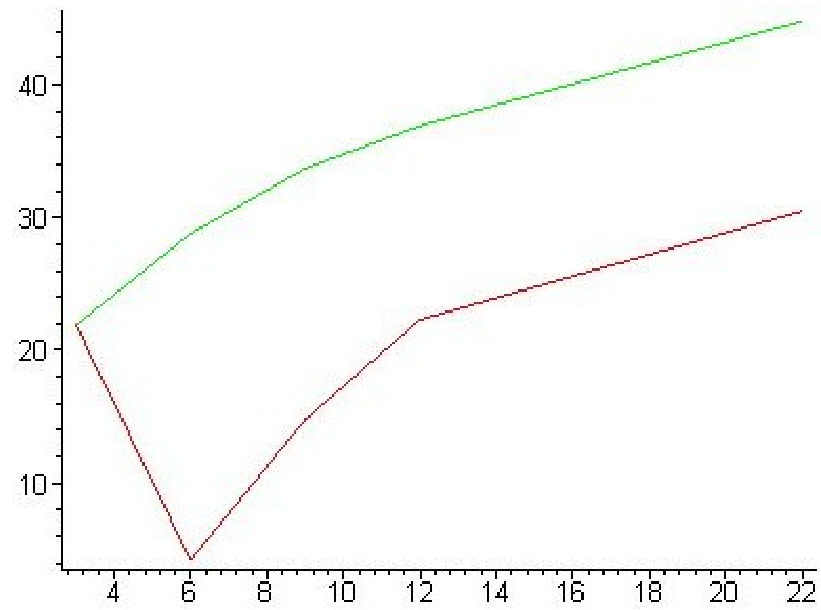
Attempts to express CDO prices in terms of correlation parameter of *Gauss copula model*, similarly as option prices are expressed using implied volatilities

- Implied *tranche correlation* is the value of ρ in a homogeneous Gauss-copula model leading to the observed tranche price (generally not uniquely defined for mezzanine tranches).
- Implied *base correlation* is the value of ρ explaining the price of an *equity tranche* with the corresponding attachment point ([0,3], [0,6], [0,9], . . .). Base correlation is unique. Moreover, (hypothetical) prices of equity tranches can be computed recursively from observed prices of CDO tranches.

Implied Tranche Correlation and Base Correlation (2)

In our example we have the following values for the tranche and base correlation.

	Tranche Correlation	Base Correlation
[0,3]	21.9%	21.9%
[3,6]	4.2%	28.8%
[6,9]	14.8%	33.7%
[9,12]	22.3%	36.9%
[12,22]	30.5%	44.8%



This is a typical pattern of tranche and base correlation, called *base correlation skew*. In particular model based on Gauss copula cannot explain all prices simultaneously.

Explaining Base Correlation Skews

Goal. Find a single model that explains (approximately) prices of all tranches or reproduces at least qualitative behavior of observed CDO prices. Needed for instance for pricing CDOs with non-standard attachment points.

Idea. Markov model does yields base correlation skews if interaction between intensities is increasing and *convex* in number of defaults l, leading to *infrequent but large clusters* of defaults. We use

$$h(t,l) = \lambda_0 \left\{ 1 + \lambda_1 \cdot [e^{\lambda_2 l/m} - e^{\lambda_2 \bar{\mu}_t}]^+ \right\}$$

Here $\lambda_0, \lambda_1, \lambda_2$ are parameters which should be calibrated to observed CDS and CDO spreads; $\bar{\mu}_t$ is a deterministic function giving the expected proportion of defaults.

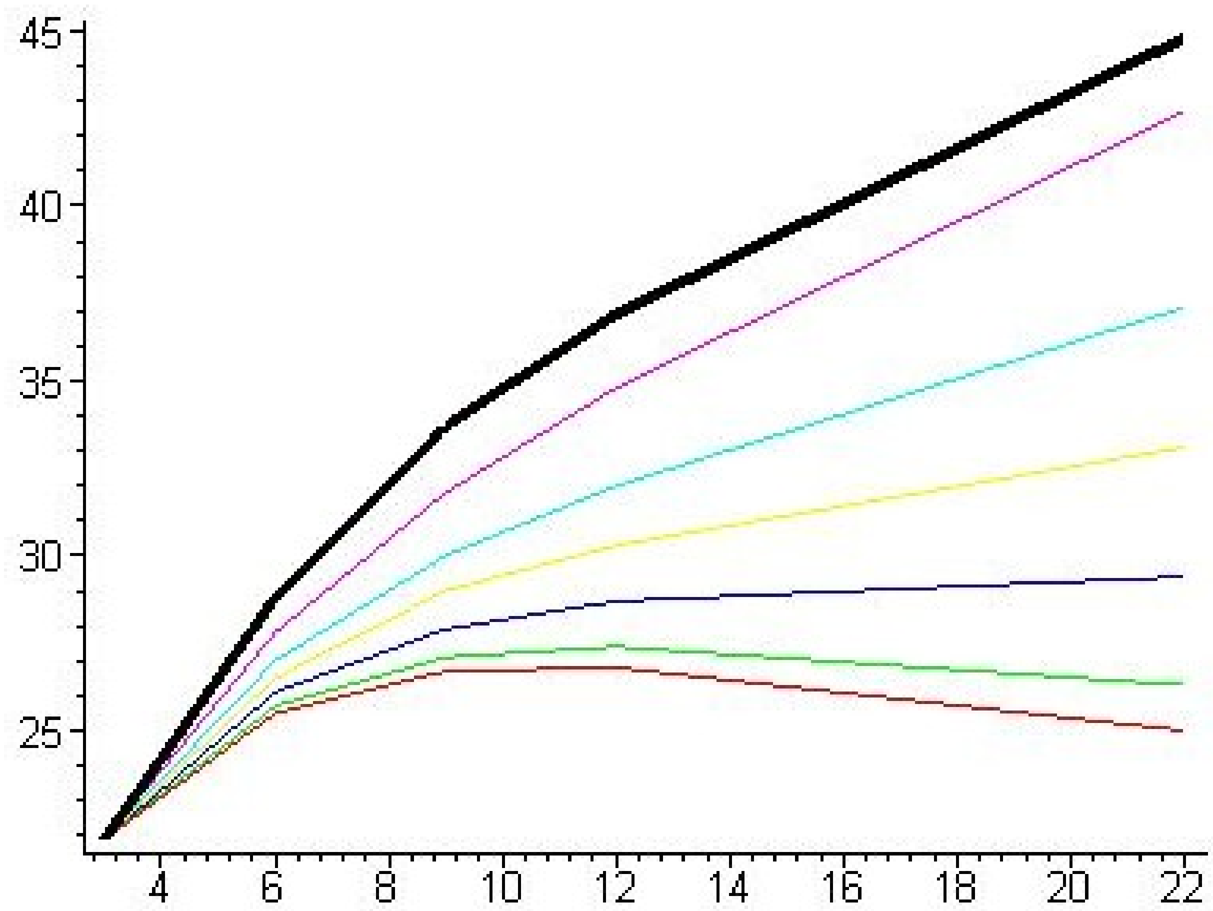
Numerical results

We priced the observed CDO tranches by this model. increasing the parameter λ_2 and calibrating the other parameters to the observed marginal 5-year default probability and the price of the equity tranche.

			[0,3]	[3,6]	[6,9]	[9,12]	[12,22]
Observed			27.6%	1.68%	0.70%	0.43%	0.20%
$\lambda_2 = 0.5$	$\lambda_1 = 230$	$\lambda_0 = 0.00427$	27.6%	2.30%	1.12%	0.58%	0.16%
$\lambda_2 = 1$	$\lambda_1 = 112$	$\lambda_0 = 0.00428$	27.6%	2.28%	1.11%	0.58%	0.16%
$\lambda_2 = 2$	$\lambda_1 = 53$	$\lambda_0 = 0.00429$	27.6%	2.23%	1.08%	0.57%	0.18%
$\lambda_2 = 3$	$\lambda_1 = 33.4$	$\lambda_0 = 0.00430$	27.6%	2.17%	1.04%	0.56%	0.19%
$\lambda_2 = 4$	$\lambda_1 = 23.5$	$\lambda_0 = 0.00432$	27.6%	2.10%	1.00%	0.54%	0.20%
$\lambda_2 = 6$	$\lambda_1 = 14$	$\lambda_0 = 0.00436$	27.6%	1.99%	0.92%	0.51%	0.21%

- Model prices come much closer to observed prices.
- Tentative interpretation of parameters: λ_0 responsible for default probability; product $\lambda_1\lambda_2$ responsible for default correlation/price of equity tranche; λ_2 responsible for base correlation skew.

The Resulting Base Correlations



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