A Unified Framework for Portfolio Optimization and Asset Pricing

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E.Platen & D. Heath: Introduction to Quantitative Finance: A Benchmark Approach. Springer Finance (2006).

- **Capital asset pricing model (CAPM)**
- **Markowitz efficient frontier, Sharpe ratio**
- **Fair pricing**
- **Actuarial pricing**
- **Minimal Market Model**
- **Utility maximization**

Continuous Benchmark Model

- **traded uncertainty**

  Wiener processes W^1, W^2, $\dots$, W^d

- **primary security accounts**

$$dS_t^{(j)} = S_t^{(j)} \left(a_t^j \, dt + \sum_{k=1}^{d} b_t^{j,k} \, dW_t^k \right)$$

$t \in [0, T]$, $j \in \{0, 1, \dots, d\}$

- **savings account**

$$S_t^{(0)} = \exp\left\{\int_0^t r_s\, ds\right\}$$

- **market price of risk**

$$\theta_t = (\theta_t^1, \theta_t^2, \ldots, \theta_t^d)^\top$$

$$b_t\, \theta_t = [a_t - r_t\, \mathbf{1}]$$

Assumption 1 $b_t = [b_t^{j,k}]_{j,k=1}^d$ invertible.

- **market price of risk**

$$\theta_t = b_t^{-1}\,[a_t - r_t\,\mathbf{1}]$$

- jth **primary security account**

$$dS_t^{(j)} = S_t^{(j)}\left(r_t\,dt + \sum_{k=1}^{d} b_t^{j,k}\;\left[\theta_t^k\,dt + dW_t^k\right]\right)$$

Portfolios

- **strategy** $\delta = \{\delta_t = (\delta_t^0, \ldots, \delta_t^d)^\top,\, t \in [0,T]\}$

  predictable, S-integrable

- **self-financing portfolio** $\Longrightarrow$

$$dS_t^{(\delta)} = \sum_{j=0}^{d} \delta_t^j \, dS_t^{(j)}$$

- **logarithm of portfolio**

$$d\ln(S_t^{(\delta)}) = g_t^\delta\, dt + \sum_{k=1}^{d}\sum_{j=1}^{d} \pi_{\delta,t}^j\, b_t^{j,k}\, dW_t^k$$

- **growth rate**

$$g_t^\delta = r_t + \sum_{k=1}^{d}\sum_{j=1}^{d} \pi_{\delta,t}^j\, b_t^{j,k} \left(\theta_t^k - \frac{1}{2}\sum_{j=1}^{d} \pi_{\delta,t}^j\, b_t^{j,k}\right)$$

- **growth optimal portfolio**

Definition 2 ***GOP*** maximizes growth rate

$$g_t^{\delta_*} \geq g_t^\delta.$$

first order conditions $\Longrightarrow$

$$\pi_{\delta_*,t} = \left(b_t^{-1}\right)^\top \theta_t$$

- **GOP SDE**

$$dS_t^{(\delta_*)} = S_t^{(\delta_*)} \left(\left[r_t + |\theta_t|^2 \right] dt + |\theta_t| \, dW_t \right)$$

$\Longrightarrow$ **continuous benchmark model**

existence of equivalent risk neutral measure ***not*** required

Portfolio Selection

- **discounted portfolio**

$$\bar{S}_t^{(\delta)} = \frac{S_t^{(\delta)}}{S_t^{(0)}}$$

$$d\bar{S}_t^{(\delta)} = \psi_{\delta,t}^{\top} \{\theta_t \, dt + dW_t\}$$

with

$$\psi_{\delta,t}^{\top} = (\psi_{\delta,t}^1, \ldots, \psi_{\delta,t}^d) = \bar{S}_t^{(\delta)} \, \pi_{\delta,t}^{\top} \, b_t$$

- **discounted drift**

$$\alpha_t^{\delta} = \psi_{\delta,t}^{\top} \theta_t$$

- **aggregate diffusion coefficient**

$$\gamma_t^{\delta} = \sqrt{\psi_{\delta,t}^{\top} \psi_{\delta,t}}$$

Definition 3 *portfolio* **optimal** *if*

$$\gamma_t^{\tilde{\delta}} = \gamma_t^{\delta}$$
$$\alpha_t^{\tilde{\delta}} \geq \alpha_t^{\delta}$$

Assumption 4

$$|\theta_t| = \sqrt{\theta_t^\top \theta_t} > 0,$$

$$\pi^0_{\delta_*,t} \neq 1$$

- **Portfolio Selection Theorem**

$S^{(\delta)}$ optimal $\Longleftrightarrow$

$$d\bar{S}_t^{(\delta)} = \bar{S}_t^{(\delta)} \frac{1 - \pi^0_{\delta,t}}{1 - \pi^0_{\delta_*,t}} \; \theta_t^\top \, (\theta_t \, dt + dW_t)$$

with

$$\pi_{\delta,t} = \frac{1 - \pi^0_{\delta,t}}{1 - \pi^0_{\delta_*,t}} \, \pi_{\delta_*,t}$$

GOP interpreted as a *mutual fund*

- **market portfolio** $S^{(\delta_+)}$ investable wealth

Assumption 5 *Each investor forms optimal portfolio.*

$\Longrightarrow$

$$d\bar{S}_t^{(\delta_+)} = d\left(\sum_{\ell=1}^{n} \bar{S}_t^{(\delta_\ell)}\right)$$

$$= \bar{S}_t^{(\delta_+)} \frac{1 - \pi^0_{\delta_+,t}}{1 - \pi^0_{\delta_*,t}} \, \theta_t^\top \left(\theta_t \, dt + dW_t\right)$$

$\Longrightarrow$ **Market portfolio is optimal portfolio**

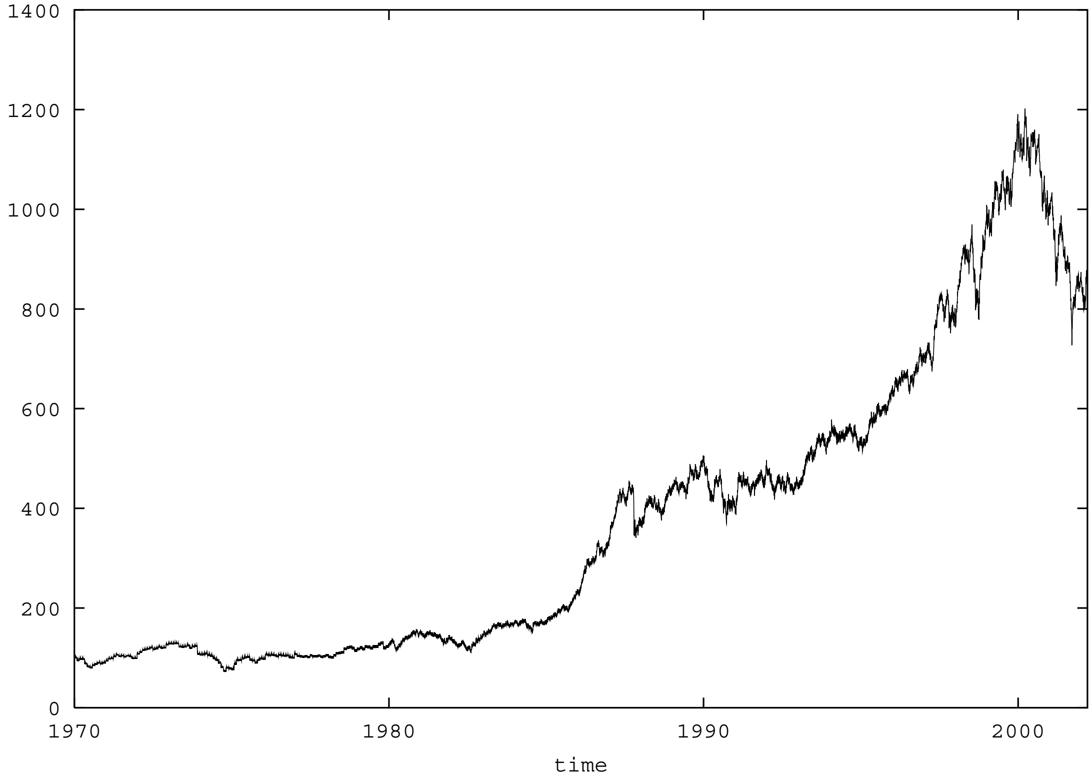

Figure 1: Discounted MSCI.

Capital Asset Pricing Model

- **risk premium** for $S^{(\delta)}$

$$p_\delta(t) = \sum_{k=1}^{d} \sum_{j=1}^{d} \pi^j_{\delta,t}\, b^{j,k}_t\, \theta^k_t$$

- **systematic risk parameter**

$$\beta_\delta(t) = \frac{\frac{d}{dt}\langle \ln(S^{(\delta)}), \ln(S^{(\delta_+)})\rangle_t}{\frac{d}{dt}\langle \ln(S^{(\delta_+)}), \ln(S^{(\delta_+)})\rangle_t}$$

if $S^{(\delta_+)}$ optimal portfolio $\Longrightarrow$

$$\beta_\delta(t) = \frac{p_\delta(t)}{p_{\delta_+}(t)}$$

Efficient Frontier

- **optimal portfolio** $S^{(\delta)}$

volatility

$$|b_\delta(t)| = \left| \frac{1 - \pi^0_{\delta,t}}{1 - \pi^0_{\delta_*,t}} \right| |\theta_t|$$

risk premium

$$p_\delta(t) = \left| \frac{1 - \pi^0_{\delta,t}}{1 - \pi^0_{\delta_*,t}} \right| |\theta_t|^2 = |b_\delta(t)| \, |\theta_t|$$

- **appreciation rate**

$$a_\delta(t) = r_t + p_\delta(t) = r_t + \sqrt{|b_\delta(t)|^2} \, |\theta_t|$$

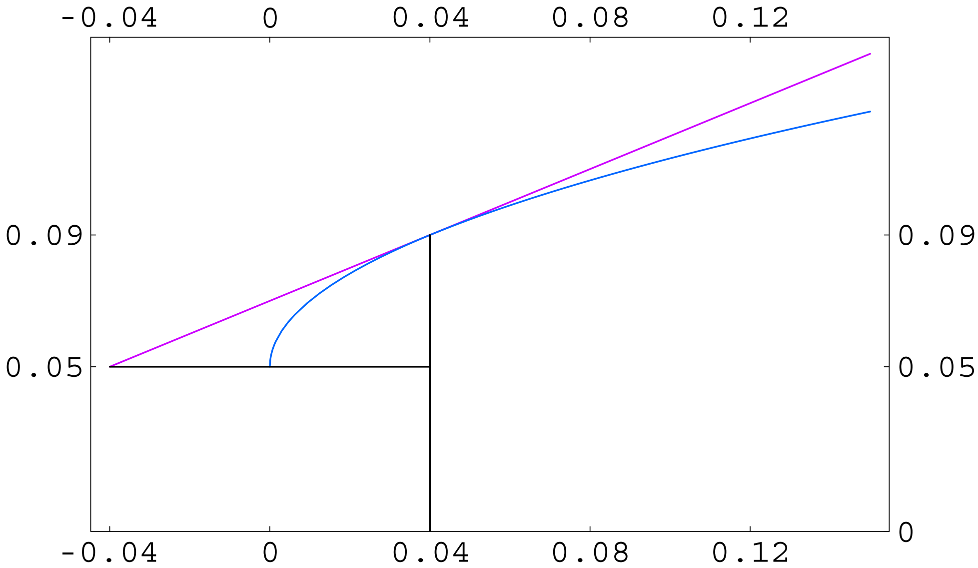

Figure 2: Efficient frontier.

- **Markowitz efficient frontier**

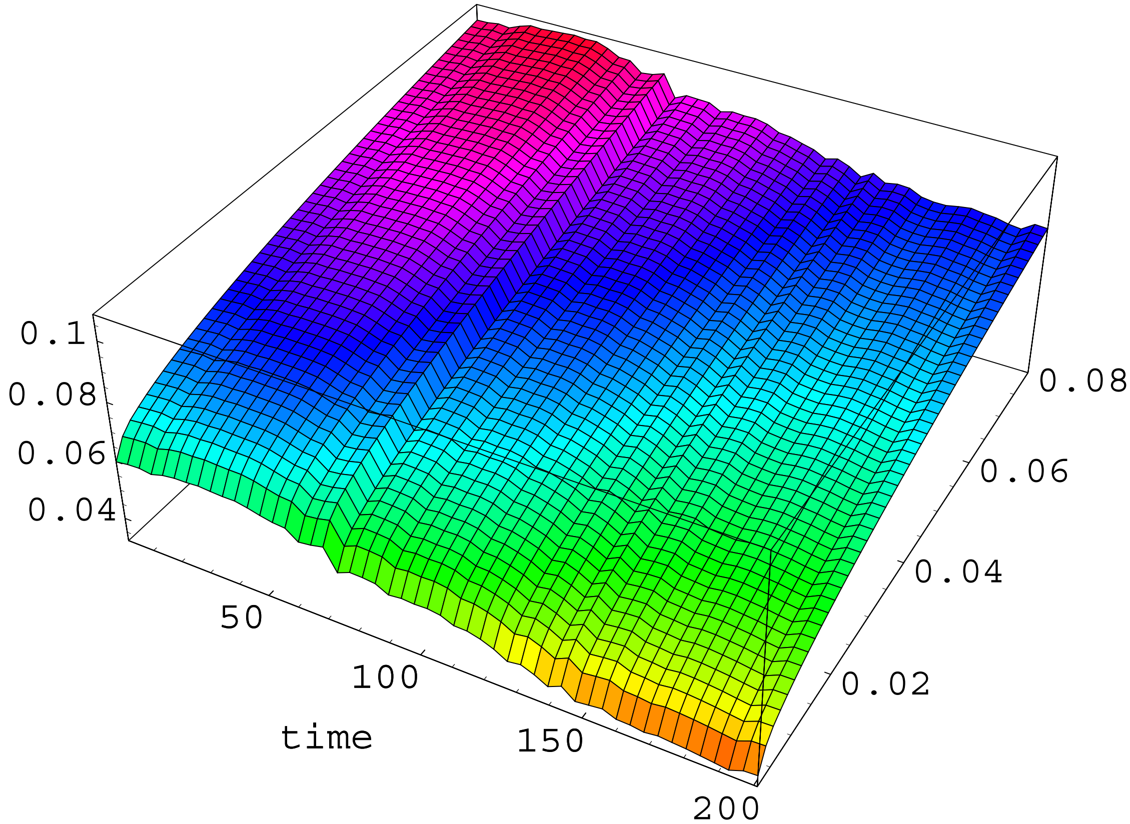

Figure 3: Efficient frontier over time.

- **Sharpe ratio**

$$s_\delta(t) = \frac{\text{risk premium}}{\text{aggregate volatility}} = \frac{p_\delta(t)}{|b_\delta(t)|} = \frac{\alpha_t^\delta}{\gamma_t^\delta}$$

$\Longrightarrow$ for any risky portfolio $S^{(\delta)}$

$$s_\delta(t) \leq |\theta_t|$$

equality for optimal portfolios

Benchmarked Portfolios

$$\hat{S}_t^{(\delta)} = \frac{S_t^{(\delta)}}{S_t^{(\delta_*)}}$$

$$d\hat{S}_t^{(\delta)} = -\sum_{k=1}^{d}\sum_{j=0}^{d} \delta_t^{(j)}\,\hat{S}_t^{(j)}\,\sigma_t^{j,k}\,dW_t^k$$

$(\underline{\mathcal{A}}, P)$-local martingale

$\Longrightarrow$

Nonnegative benchmarked portfolios are $(\underline{\mathcal{A}}, P)$-supermartingales

Definition:

Nonnegative portfolio $S^{(\delta)}$ permits **arbitrage** if for $S_0^{(\delta)} = 0$

$$P(S_T^{(\delta)} > 0) > 0$$

$\Longrightarrow$

Continuous benchmark models do not permit arbitrage

Equivalent local martingale measure may **not** exist

Fair Pricing

- **fair** if

  corresponding benchmarked value is an $(\underline{\mathcal{A}}, P)$-martingale

$$\hat{U}_{H_T}(t) = \frac{U_{H_T}(t)}{S_t^{(\delta_*)}} = E\left(\hat{U}_{H_T}(T) \,\middle|\, \mathcal{A}_t\right)$$

- **fair pricing formula**

$$U_{H_T}(t) = S_t^{(\delta_*)} \, E\left(\frac{H_T}{S_T^{(\delta_*)}} \,\middle|\, \mathcal{A}_t\right)$$

Risk Neutral Pricing

Candidate Radon-Nikodym derivative

$$\Lambda_t = \frac{S_0^{(\delta_*)}}{S_t^{(\delta_*)}} \frac{S_t^{(0)}}{S_0^{(0)}} = \frac{\hat{S}_t^{(0)}}{\hat{S}_0^{(0)}}$$

fair pricing formula

$$\begin{aligned} U_{H_T}(t) &= E\left(\frac{S_t^{(\delta_*)}}{S_T^{(\delta_*)}} H_T \,\middle|\, \mathcal{A}_t \right) = E\left(\left(\frac{S_t^{(\delta_*)}}{S_T^{(\delta_*)}} \frac{S_T^{(0)}}{S_t^{(0)}} \right) \frac{S_t^{(0)}}{S_T^{(0)}} H_T \,\middle|\, \mathcal{A}_t \right) \\ &= E\left(\frac{\Lambda_T}{\Lambda_t} \frac{S_t^{(0)}}{S_T^{(0)}} H_T \,\middle|\, \mathcal{A}_t \right) \end{aligned}$$

- **Actuarial pricing formula**

For H_T **independent** of the GOP

$$\begin{aligned} U_{H_T}(t) &= E\left(\frac{S_t^{(\delta_*)}}{S_T^{(\delta_*)}} \,\Big|\, \mathcal{A}_t\right) E\left(H_T \,\big|\, \mathcal{A}_t\right) \\ &= P(t,T)\, E\left(H_T \,\big|\, \mathcal{A}_t\right) \end{aligned}$$

Assumption 6 *Market portfolio is optimal portfolio in foreign currency.*

$\Longrightarrow$ **Market portfolio is GOP**

- **discounted GOP**

$$d\bar{S}_t^{(\delta_*)} = \bar{S}_t^{(\delta_*)} \, |\theta_t| \left(|\theta_t| \, dt + dW_t \right)$$

- **discounted GOP drift**

$$\alpha_t = \bar{S}_t^{(\delta_*)} \, |\theta_t|^2$$

- **time transformed squared Bessel process,** $\nu = 4$

$$d\bar{S}_t^{(\delta_*)} = \alpha_t \, dt + \sqrt{\bar{S}_t^{(\delta_*)} \, \alpha_t} \, dW_t$$

- **transformed time**

$$d\sqrt{\bar{S}_t^{(\delta_*)}} = \frac{3\,\alpha_t}{8\,\sqrt{\bar{S}_t^{(\delta_*)}}}\,dt + \frac{1}{2}\,\sqrt{\alpha_t}\,dW_t$$

$$\varphi_t - \varphi_0 = \int_0^t \alpha_s\,ds = 4\,\left\langle \sqrt{\bar{S}^{(\delta_*)}} \right\rangle_t$$

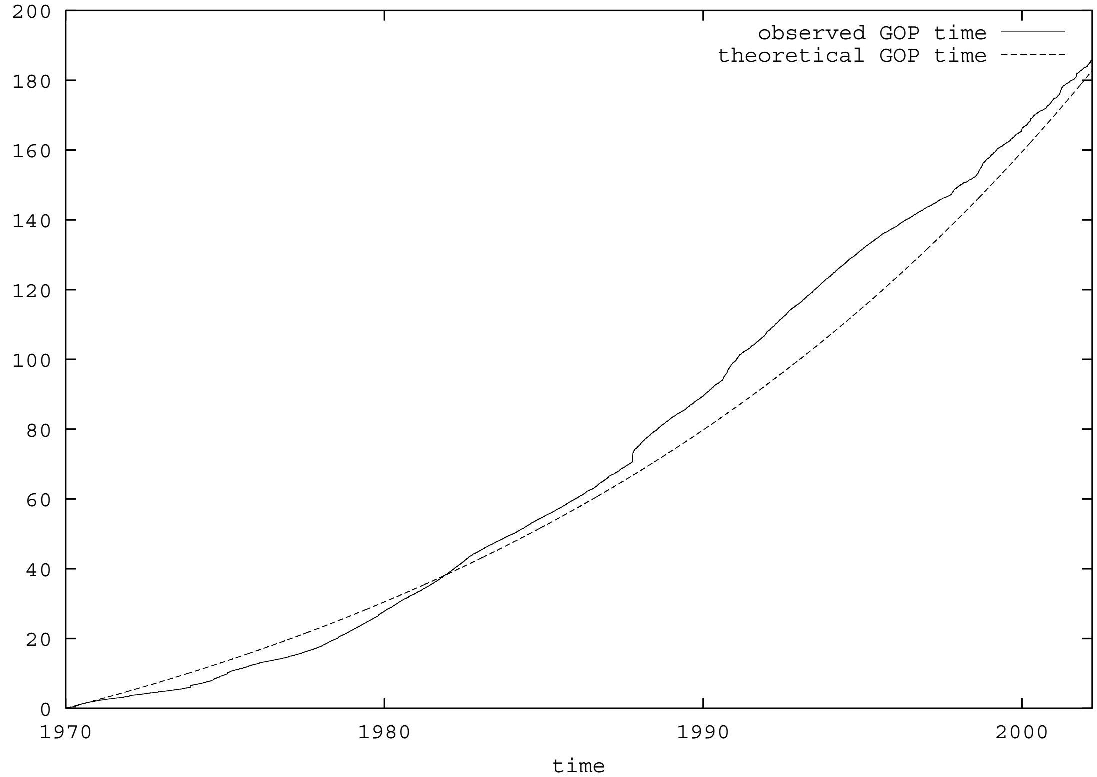

Figure 4: Observed and theoretical transformed time φ_t.

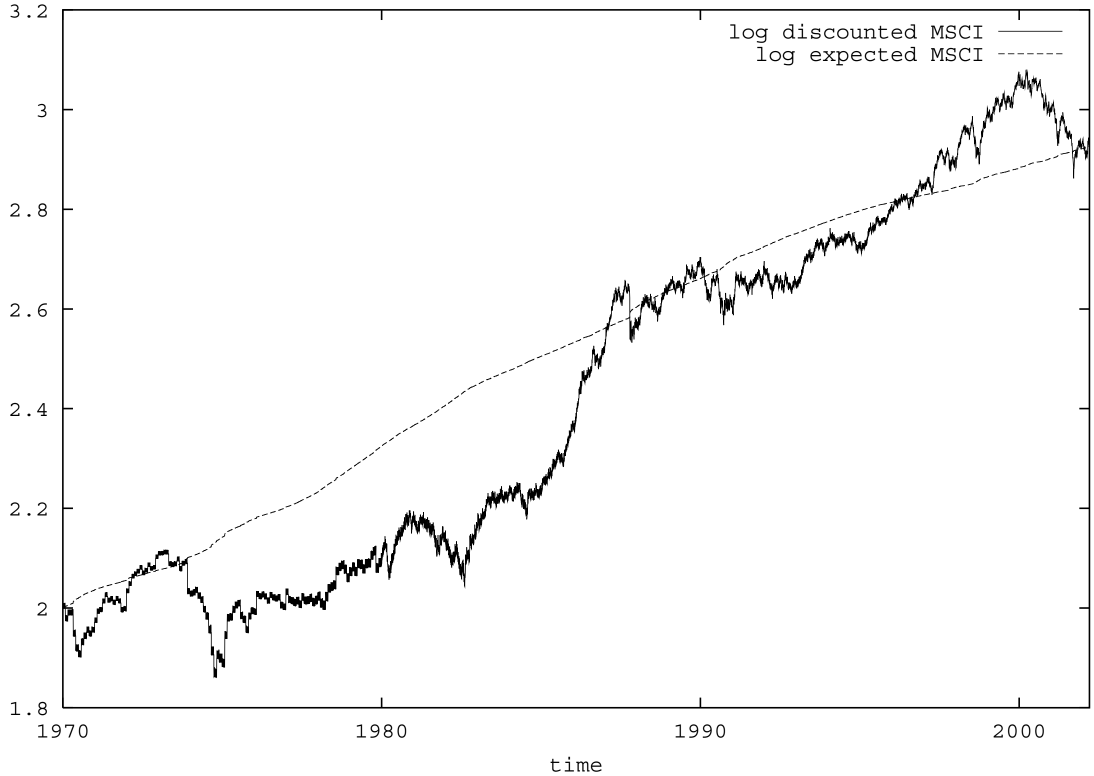

Figure 5: Logarithm of discounted and expected MSCI.

- **Minimal market model** (MMM)

  Platen (2001)

- **Discounted GOP** is

  squared Bessel process of dimension four

- Candidate Radon-Nikodym derivative

  is strict local martingale

$\Longrightarrow$ fair pricing

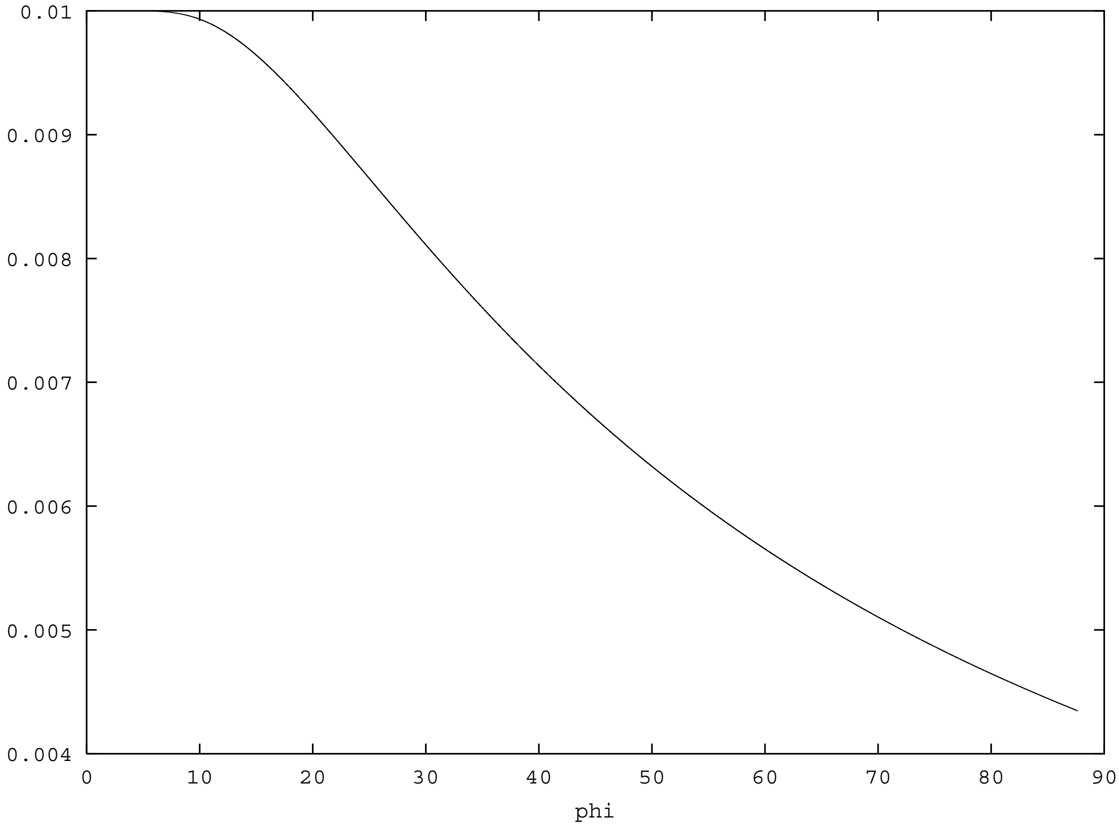

Figure 6: Expectation of the inverse of the squared Bessel process for $\nu = 4$.

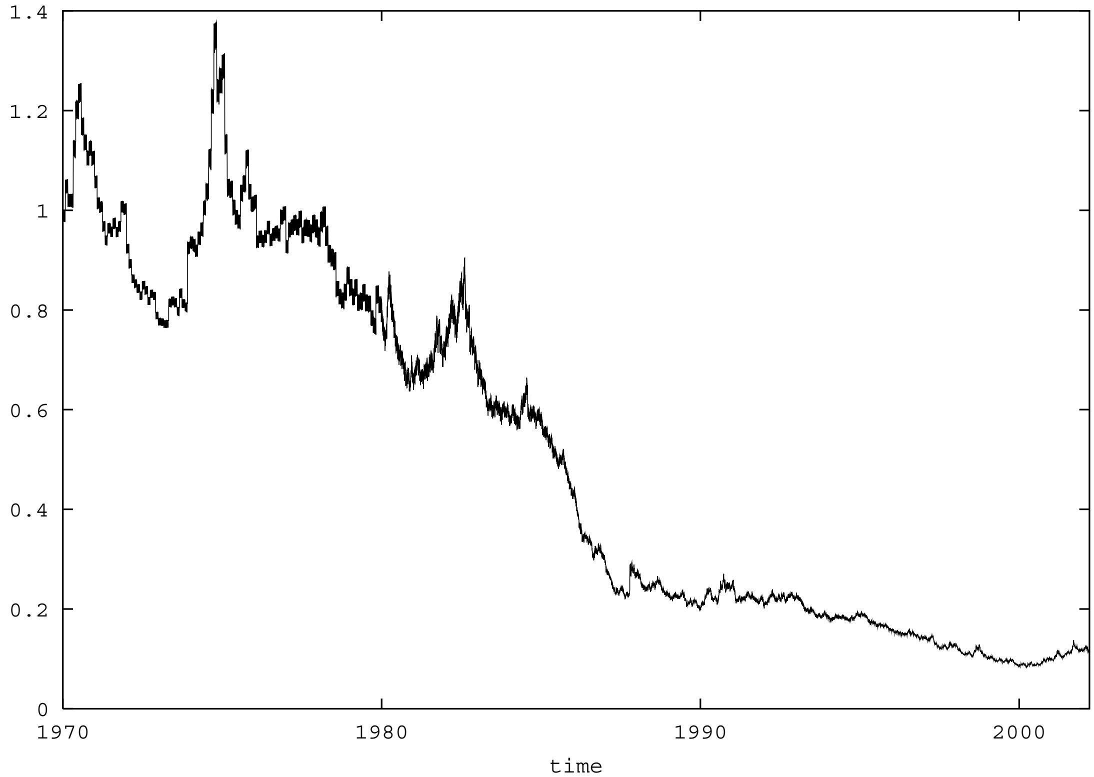

Figure 7: Radon-Nikodym derivative Λ_t.

Assumption 7 *discounted GOP drift differentiable*

$$\alpha_t = \alpha_0 \exp\left\{\int_0^t \eta_s \, ds\right\}$$

- **normalized GOP**

$$Y_t = \frac{\bar{S}_t^{(\delta_*)}}{\alpha_t} = \frac{1}{|\theta_t|^2}$$

- **square root process**

$$dY_t = (1 - \eta_t \, Y_t) \, dt + \sqrt{Y_t} \, dW_t$$

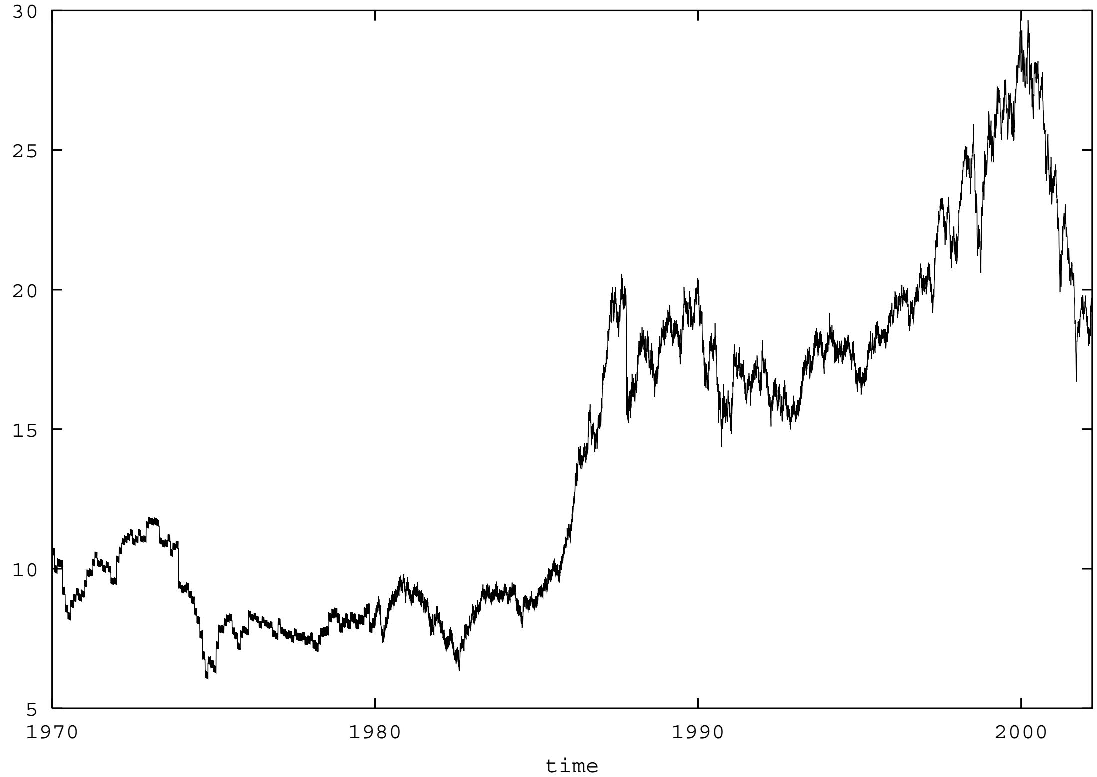

Figure 8: The normalized MSCI Y_t.

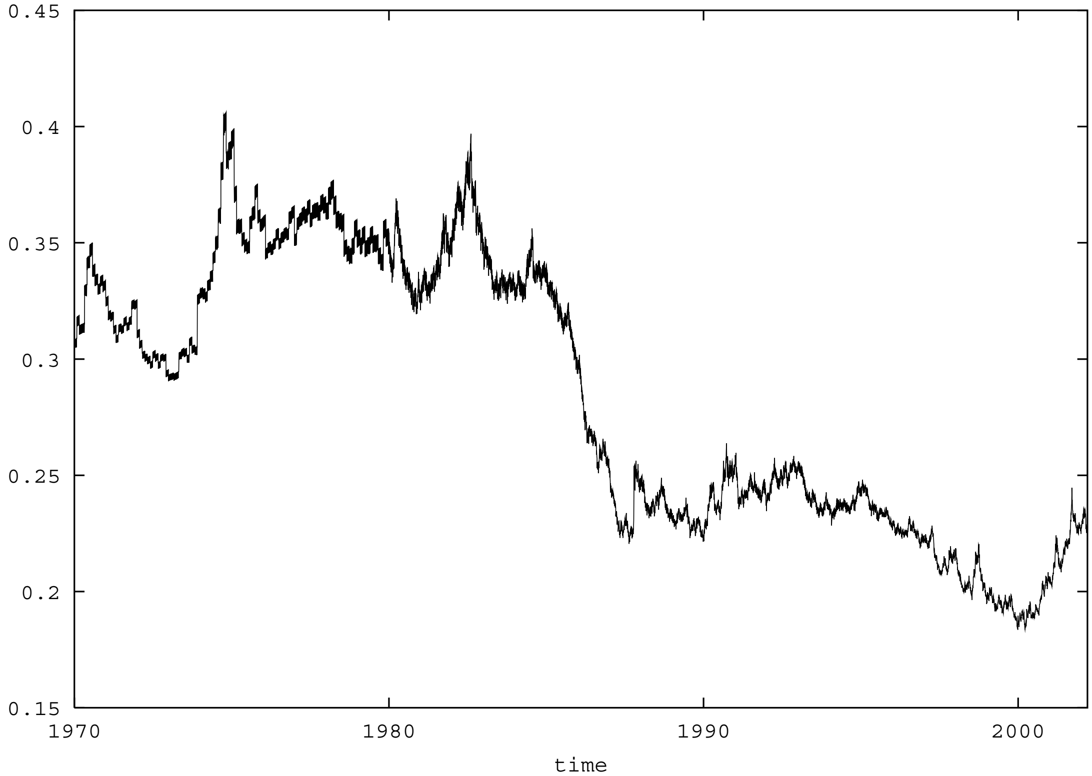

Figure 9: Volatility $|\theta_t|$ of MSCI.

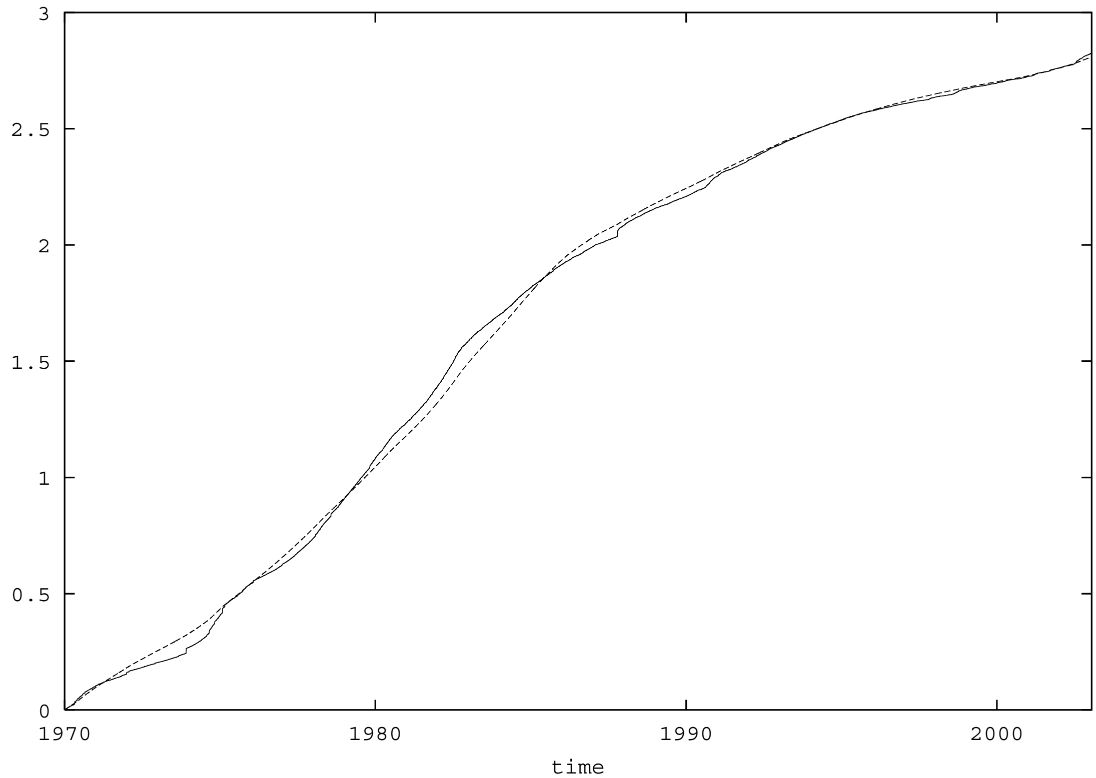

Figure 10: Integrated squared theoretical and realized volatility.

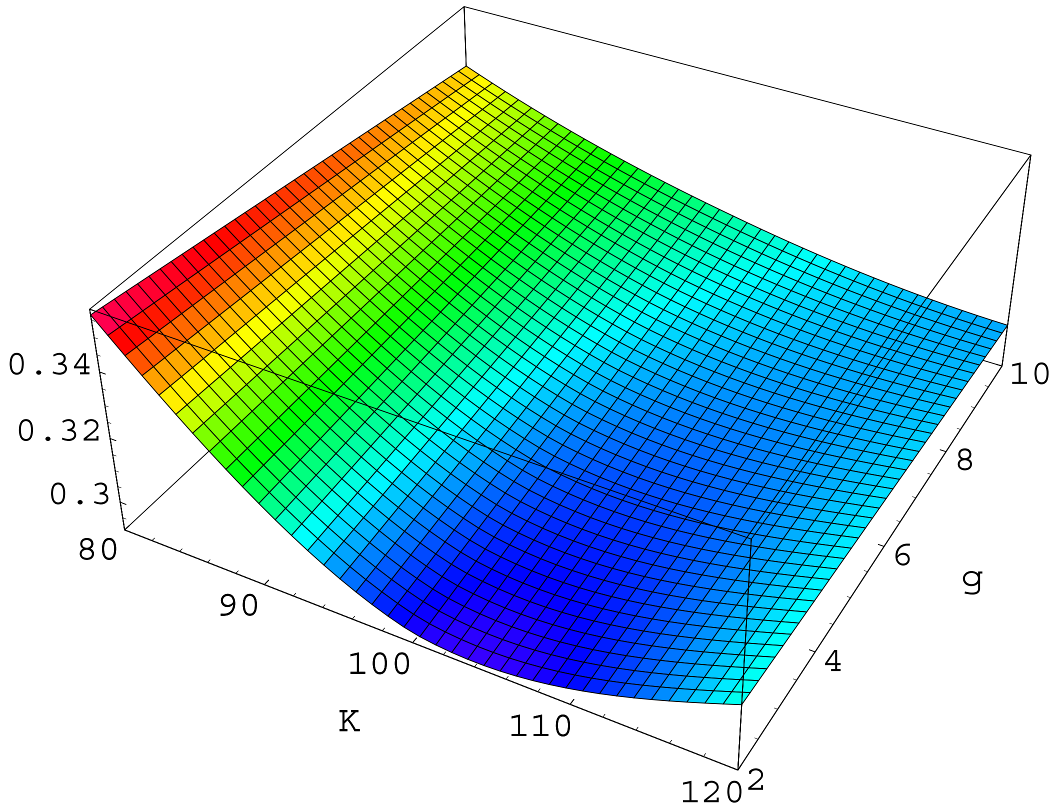

Figure 11: MMM implied volatility surface.

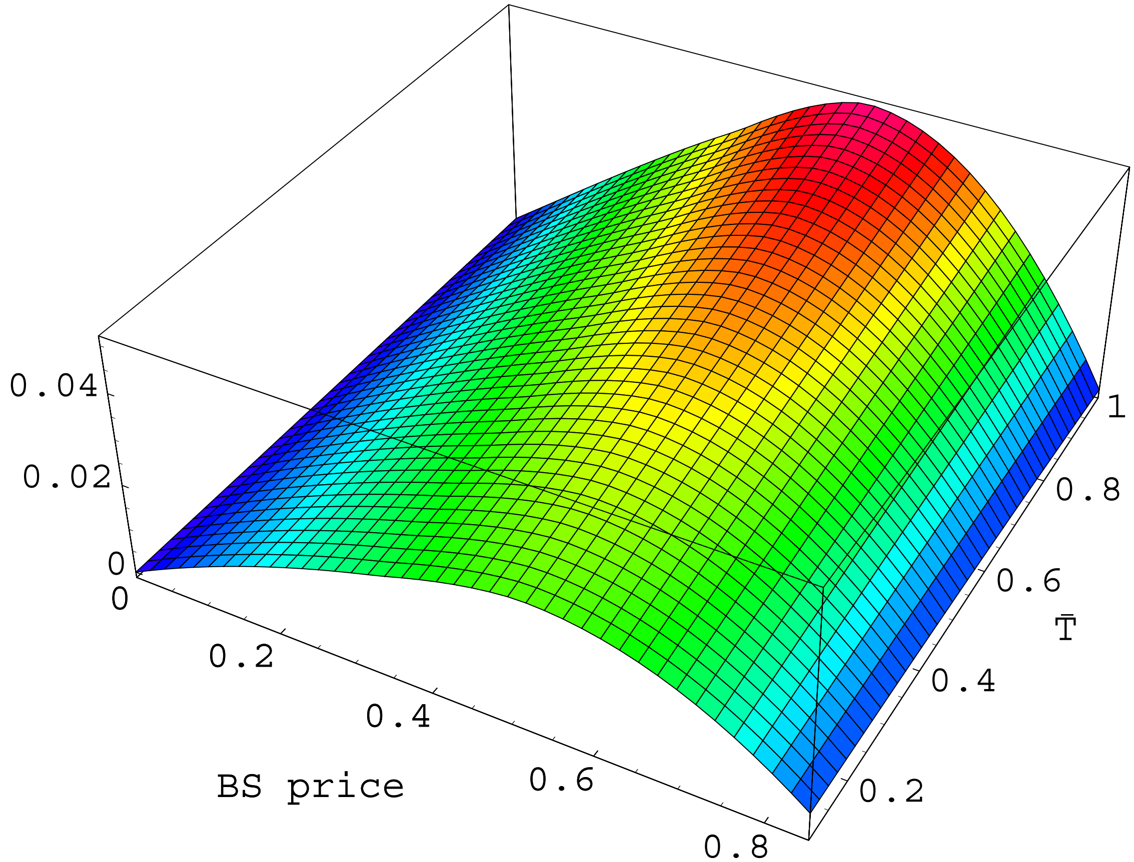

Figure 12: Differences between MMM and Black-Scholes prices for up-and-out binary options for different maturity values T.

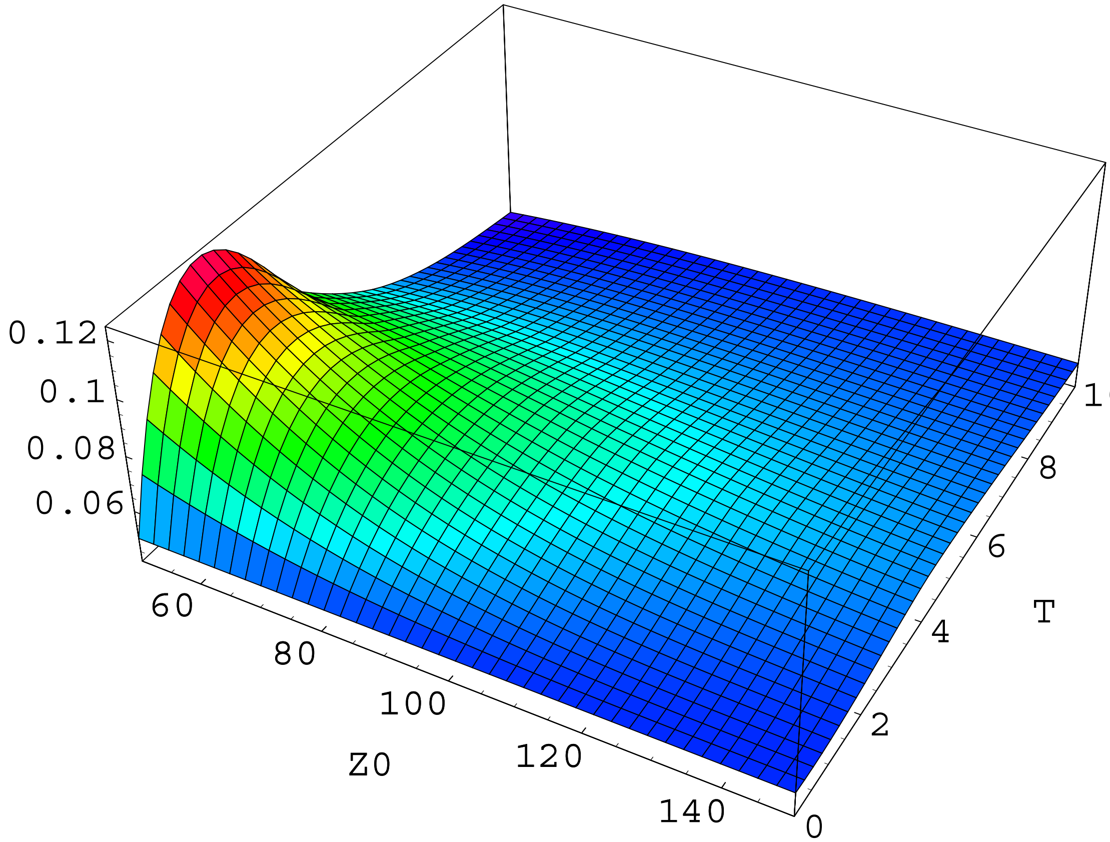

Figure 13: Forward rates as a function of Z_0 and T.

Currency Market

- **(i, j)th exchange rate**

$$X_t^{i,j} = \frac{S_{i,t}^{(\delta_*)}}{S_{j,t}^{(\delta_*)}}$$

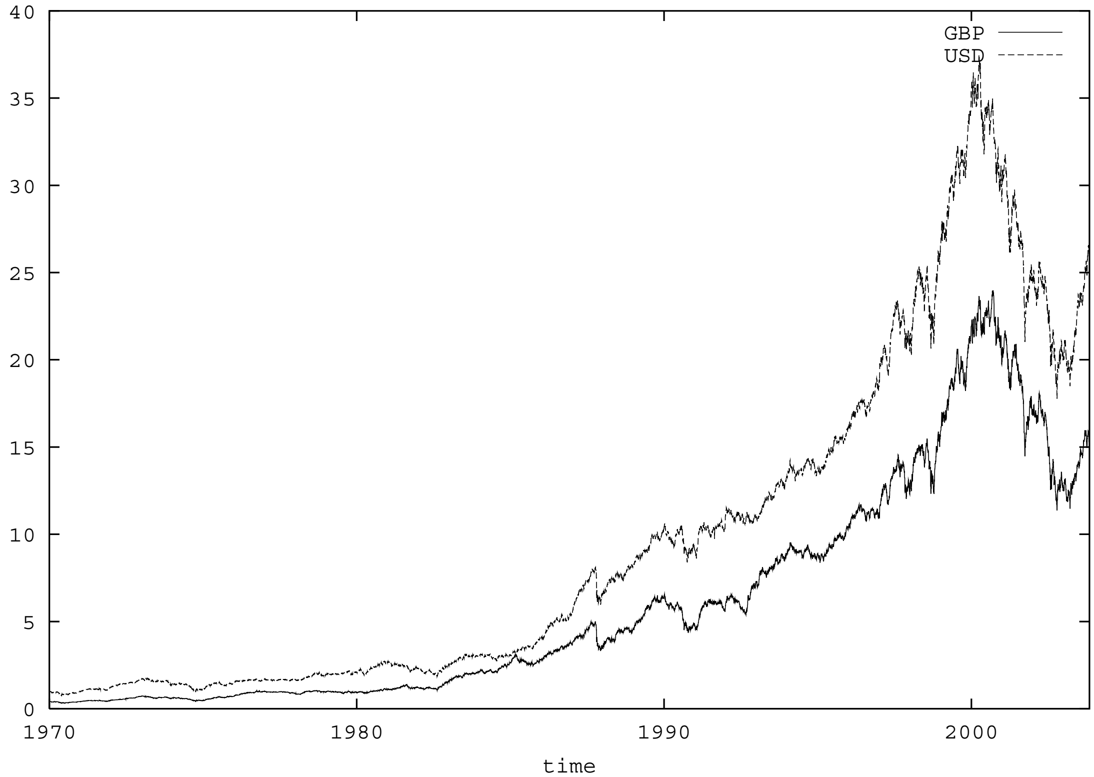

Figure 14: WSI denominated in GBP and USD.

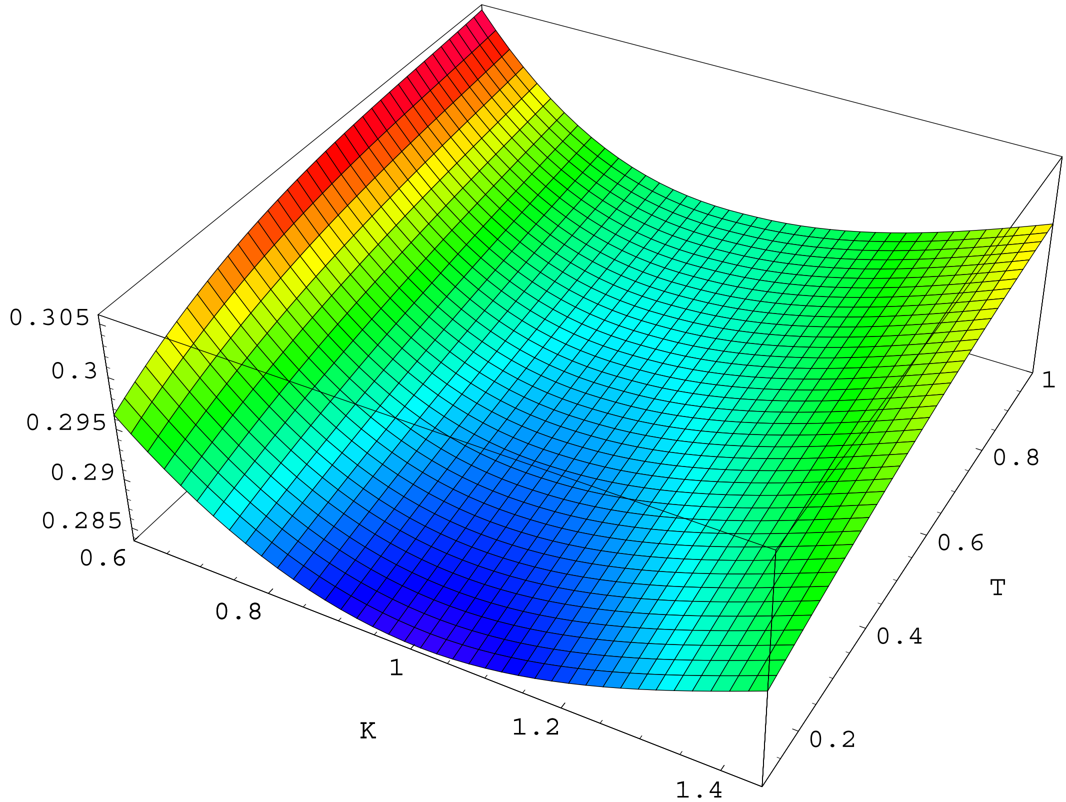

Figure 15: Implied call volatilities as function of strike K and time t.

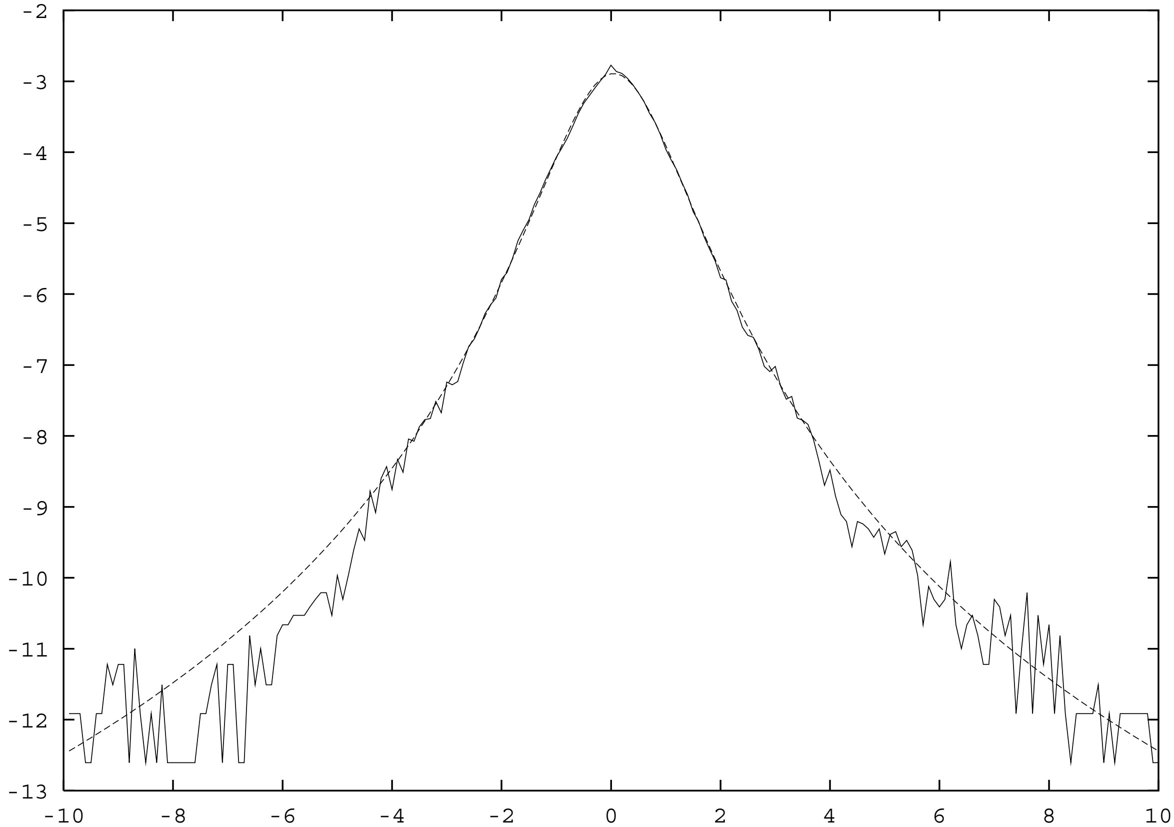

Figure 16: Log-histogram of WSI log-returns.

Country	Student t	Inverse Gaussian	Hyperbolic	Variance Gamma	Degrees of Freedom
Argentina	0.000000	137.566726	377.265316	414.030946	3.215953
Australia	0.000000	24.403096	54.527486	73.900060	4.618187
Austria	1.272478	10.533450	43.923596	63.696828	4.177998
Belgian	0.000000	15.755730	41.664530	59.922270	4.639238
Brazil	0.000000	132.348986	430.843576	444.117392	2.937178
Canada	0.000000	42.829996	80.784298	110.357598	4.927523
Denmark	0.000000	29.607334	72.807340	96.852868	4.328256
Finland	0.000000	130.807532	286.692740	326.546792	3.707350
France	0.303708	12.578282	42.737860	61.753332	4.329017
Germany	0.000000	17.945998	45.546996	64.739532	4.620013
Greece	0.000000	60.072066	120.786456	150.578024	4.340990
Hong Kong	0.000000	31.399542	84.191740	111.611116	4.080899
Hungary	0.002812	33.488642	102.407366	132.556744	3.827598
India	0.000000	218.752194	1096.862148	962.798700	2.283747
Indonesia	0.000000	54.595328	121.131360	148.098694	4.062652
Ireland	0.031796	16.660834	53.818850	76.062630	4.187040
Italy	0.002606	19.207820	60.267448	83.332082	4.172936
Japan	0.000000	24.017652	60.214094	81.351358	4.392711
Korea S.	0.000000	129.955438	386.626152	425.311040	3.265655
Malaysia	0.000000	56.525498	149.299592	189.659002	3.786499
Mexico	0.000000	440.818300	2132.850298	1746.118774	2.207160
Netherlands	0.000000	15.802518	41.848016	60.873070	4.611005
Norway	0.000000	27.920608	71.785758	96.835862	4.256095
Philippine	0.017290	52.048754	167.407546	199.781080	3.458544
Portugal	1.582056	13.129484	54.154638	76.914946	4.071114
Singapore	0.000000	30.656496	73.326620	99.354034	4.396040
Spain	0.000000	70.602362	139.884600	165.163122	4.206288
Sweden	0.000000	66.852560	130.642934	166.827332	4.468983
Switzerland	0.144462	15.390592	46.179620	67.726852	4.471172
Taiwan	0.000000	33.290522	82.900518	110.434774	4.224246
Thailand	0.000000	100.851126	282.814096	314.709524	3.298124
Turkey	0.000000	152.625500	493.285862	506.466162	2.893205
UK	0.000000	21.124390	47.980512	68.613654	4.868241
US	0.000000	31.352956	75.539480	102.259640	4.323809

Table 1: The L_n test statistic for log-returns of the WSI in different currencies.

Utility Maximization

- **utility function** $\quad U : [0, \infty) \to \Re$

$E\left(U\left(\bar{S}_T^\delta\right)\right) \to$ maximum over strictly positive, discounted fair portfolios

$\Longrightarrow$ **expected utility maximizing portfolio is optimal portfolio**

$$J_t^\delta = \frac{1 - \pi_{\delta_*}^0(t)}{1 - \pi_\delta^0(t)} = \frac{1}{1 - \frac{\hat{S}_t^0}{\hat{u}(t, \hat{S}_t^0)} \frac{\partial \hat{u}(t, \hat{S}_t^0)}{\partial \hat{S}_t^0}}$$

$$\hat{u}(t, \hat{S}_t^0) = E\left(U'^{-1}(\lambda\, \hat{S}_T^0)\, \hat{S}_T^0 \,\middle|\, \mathcal{A}_t\right)$$

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